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Extensions of Tutte's Wheels-and-Whirls Theorem

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Tutte's wheels-and-whirls theorem states that if M is a 3-connected matroid and, for every element e, both the deletion and the contraction of e destroy 3-connectivity, then M is a wheel or a whirl. We prove some extensions of this theorem, one of which states that if M is 3-connected and has both a wheel and a whirl minor, then either M has only seven elements or there is some element the deletion or contraction of which maintains 3-connectivity and leaves a matroid with both a wheel and a whirl minor.

1. INTRODUCTION

Suppose that M and N are 3-connected matroids such that N is a proper minor of M, $|E(N)| \geqslant 4$, and if N is a wheel or a whirl, then M has no larger wheel or whirl as a minor. Then Seymour's Splitter Theorem [5] asserts that M has an element that can be deleted or contracted from M to give a matroid N_1 that retains the two properties of being 3-connected and of having a minor isomorphic to N.

Sometimes one is interested in N_1 retaining even more of the properties of M. This paper is concerned with just such a situation. Suppose that the 3-connected matroid M has a minor isomorphic to the rank-3 wheel W_3

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and, in addition, M is nonbinary. Our main result is that, provided $|E(M)| \geqslant 8$, M had a single-element deletion or a single-element contraction that retains the three properties of being 3-connected, having a W_3 minor, and being nonbinary. This result has as a sequence a chain theorem for 3-connected matroids that extends the chain theorem implicit in Tutte's wheels-and-whirls theorem. As part of the proof of the main theorem, we shall establish another extension of Tutte's theorem. (This extension can also be viewed as an extension of Seymour's Splitter Theorem, although we shall derive it as a consequence of the splitter theorem.)

The recent matroid literature contains a number of other generalizations of Tutte's theorem. (See, for example, [2, 3, 7].) A survey of such results and of their role in the development of matroid structure theory can be found in Seymour [6].

We assume familiarity with matroid theory; for an introduction, see Welsh [10]. In Section 2, we give the necessary basic definitions and facts. In Section 3, we prove a "top-down" chain theorem, Theorem 3.1, which will be used in the proof of the main result. In Section 4, we prove the extensions of Tutte's wheels-and-whirls theorem, Theorem 4.1 and Corollary 4.3.

2. PRELIMINARIES

Let M be a matroid on E with (Whitney) rank function r. M^* denotes the *dual* of M, with rank function r^*, where $r^*(A) = |A| - r(E) + r(E-A)$, for $A \subseteq E$. A *loop* (*coloop*) of M is a 1-element circuit (cocircuit). Two distinct elements $e, f \in E$ are *parallel* (*in series*) in M if $\{e, f\}$ is a circuit (cocircuit). The *parallel* (*series*) *class* of the element e is the set containing e together with all elements parallel to (in series with) e. A *triangle* (*triad*) of M is a 3-element circuit (cocircuit).

For $X \subseteq E$, $M \backslash X$ (M/X) denotes the matroid obtained from M by deleting (contracting) X. Given matroids N and M on sets $E(N)$, $E(M)$, respectively, N is a *minor* of M if $E(N) \subseteq E(M)$ and $E(M) - E(N)$ can be partitioned into sets X and Y such that $N = M \backslash X/Y$. M is said to *have an N minor* if $N \cong N'$, for some minor N' of M.

To *simplify* (*cosimplify*) a matroid M means to delete all loops (coloops) and delete (contract) all but one element from each parallel (series) class. For $e \in E$, $M - e$ ($M | e$) is used to denote the matroid obtained by cosimplifying $M \backslash e$ (simplifying M/e). Note that $M - e$ and $|e$ are defined only up to isomorphism in the sense that the element left from each series class of $M \backslash e$ (parallel class of M/e) is not specified.

Given integers n and m with $0 \leqslant n \leqslant m$, U_m^n denotes the *uniform* matroid on m elements in which every n-element subset is a base. Given a graph

G, $M(G)$ denotes the usual polygon matroid on the edge-set of G. Let $n \geqslant 3$ be an integer, and let H_n be the simple graph on $n+1$ nodes in which n of the nodes form a polygon P and the remaining node is adjacent to each node of P. H_n has $2n$ edges and is called a *wheel*. The edges of P are called *rim* edges and the remaining edges are called *spokes*. The matroid $M(H_n)$ is also called a *wheel* and is denoted W_n. The *whirl* matroid, $\mathscr{W}_n$, is obtained from W_n by declaring $E(P)$ to be independent and leaving the remaining independent sets the same. Define $\mathscr{W}_2$ to be the matroid U_4^2. The terms *rim* and *spoke* will be used in the obvious way in $\mathscr{W}_n$, when $n \geqslant 3$.

A matroid is *binary* if it is representable over $GF(2)$. By Tutte's characterization of binary matroids [8], M is binary if and only if M has no whirl minor.

A bipartition $\{A, B\}$ of E is a (Tutte) *k-separation* [9], for some positive integer k, if $|A| \geqslant k \leqslant |B|$, and $r(A)+r(B) \leqslant r(E)+k-1$. M is *n-connected*, for some integer $n \geqslant 2$, if M has no k-separation for all $0<k<n$. A k-separation is called *minimal* if $\min\{|A|, |B|\}=k$.

The next five basic lemmas will be used implicitly throughout the paper. Their proofs are left to the reader. Assume M is a matroid on E.

LEMMA 2.1. *For a bipartition $\{A, B\}$ of E,*

$$r(A)+r(B)-r(E)=r^*(A)+r^*(B)-r^*(E)=r(A)+r^*(A)-|A|.$$

Thus, connectivity is invariant under duality.

LEMMA 2.2. *If $\{A, B\}$ is a minimal k-separation of a k-connected matroid with $|A|=k$, then A is either a circuit and coindependent or a cocircuit and independent.*

LEMMA 2.3. *If $\{A, B\}$ is a nonminimal k-separation of M and X is a circuit or cocircuit with $X \cap B=\{x\}$, then $\{A \cup x, B-x\}$ is a k-separation of M.*

LEMMA 2.4. *If N is a 3-connected matroid with at least four elements and $\{e, f\}$ is a circuit of M, then M has an N minor if and only if $M \backslash e$ has an N minor.*

LEMMA 2.5. *Let N be a 3-connected minor of M and let $\{A, B\}$ be a k-separation of M, $k \leqslant 2$. Then* $\min\{|A \cap E(N)|, |B \cap E(N)|\} \leqslant k-1$.

Two pairs of sets $\{A, B\}$ and $\{C, D\}$ *cross* if each of the sets $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$ is nonempty. The following lemma is due to Bixby.

LEMMA 2.6 [1]. *Assume M is 3-connected and let $e \in E$. Then every 2-separation of $M \backslash e$ crosses every 2-separation of M/e, and one of $M \backslash e$, M/e has no nonminimal 2-separation; moreover, either $M-e$ or $M \backslash e$ is 3-connected.*

The next lemma follows from Lemmas 2.3 and 2.6.

LEMMA 2.7 [2]. *Assume M is 3-connected and elements x, y, z, w are distinct such that $\{x, y, z\}$ is a triangle and either $\{x, y, w\}$ or $\{x, y, z, w\}$ is a cocircuit. Then $M \backslash z$ has no nonminimal 2-separation.*

3. A TOP-DOWN CHAIN THEOREM

In this section we prove the following result.

THEOREM 3.1. *Let N and M be 3-connected matroids with $|E(N)| \geqslant 4$, and let $e \in E(M)$ be such that $M \backslash e$ has an N minor and M/e has no N minor. Then either $M \backslash e$ has no nonminimal 2-separation, and hence $M-e$ is 3-connected, or, for some element f, either $M \backslash f$ or M/f is 3-connected, is nonbinary, and has an N minor.*

In order to prove Theorem 3.1, we use the following strengthened version of a theorem of Truemper [7], which appears in [2].

THEOREM 3.2. *Let M be a 3-connected matroid and let N be a 3-connected minor of M with $|E(N)| \geqslant 4$. Then one of the following holds:*

(a) *There is some element $x \in E(M)-E(N)$ such that $M \backslash x$ or M/x is 3-connected and has N as a minor.*

(b) *There is some element $x \in E(M)-E(N)$ such that N is a minor of both $M \backslash x$ and M/x, and there are distinct elements $y, z \in E(M)-E(N)$ and $n, m, p \in E(N)$ such that $\{x, y, z\}$ is a triad (triangle) and $\{x, y, n\}$, $\{y, z, m\}$, $\{x, z, p\}$ are triangles (triads). Moreover, $M \backslash \{x, y, z\}$ $(M/\{x, y, z\})$ is 3-connected and has N as a minor. Also, $M \backslash a$ (M/a) is 3-connected, for each $a \in \{n, m, p\}$.*

(c) *For every element $x \in E(M)-E(N)$, exactly one of $M \backslash x$, M/x has N as a minor; moreover, there is an element y such that $M \backslash x/y$ or $M/x \backslash y$ is 3-connected and has N as a minor.*

We will also use the following lemma due to Oxley.

LEMMA 3.3 [4]. *Let M be a 3-connected nonbinary matroid such that for some element e, both $M \backslash e$ and M/e are binary. Then $M \cong U_4^2$.*

Proof of Theorem 3.1. We proceed by induction on $|E(M)|$. Let $\tilde{N}$ be a minor of $M\backslash e$ that is isomorphic to N. By assumption, $\tilde{N}$ is not a minor of M/e. If $M\backslash e$ has no nonminimal 2-separation, then we are finished, so assume not. Then, applying Theorem 3.2 to the minor $\tilde{N}$ of M, we see that (c) fails when $x=e$. Therefore either (a) or (b) holds. Suppose first that (b) holds, and let $\{x, y, z, n, m, p\}$ be as given in (b). Suppose $\{x, y, z\}$ is a triad. Since $\tilde{N}$ is a minor of $M\backslash\{x, y, z\} = M\backslash\{y, z\}/x$, the element $e \neq x$. Similarly, $e \neq y$ and $e \neq z$. Now since $\tilde{N}$ is not a minor of M/e but is a minor of $M\backslash e$, we have that $\tilde{N}$ is a minor of $M\backslash\{x, y, z, e\} = M\backslash\{y, z, e\}/x$. As n is an element of $\tilde{N}$ but e is not, $e \neq n$. By Theorem 3.2, $M\backslash n$ is 3-connected, Moreover, since $\tilde{N}$ is a minor of $M\backslash\{y, e\}/x \cong M\backslash\{n, e\}/x$, $M\backslash n$ has an N minor that does not contain e. A similar argument shows that if $\{x, y, z\}$ is a triangle, then M/n is 3-connected and has an N minor that does not contain e.

We conclude that if (b) holds, then there exists an element f distinct from e such that, for $\tilde{M} = M\backslash f$ or M/f, $\tilde{M}$ is 3-connected and $\tilde{M}\backslash e$ has an N minor. But the same conclusion is true if (a) holds, where the fact that $f \neq e$ follows in this case because neither $M\backslash e$ nor M/e is 3-connected and has an N minor. Note that, as M/e has no N minor, $\tilde{M}/e$ has no N minor.

If $\tilde{M}$ is nonbinary, then we are finished, so assume $\tilde{M}$ is binary. Then, by induction, $\tilde{M}\backslash e$ has no nonminimal 2-separation.

Suppose first that $\tilde{M} = M\backslash f$. Let $\{A, B\}$ be a nonminimal 2-separation of $M\backslash e$; assume $f \in A$. Since $\tilde{M}\backslash e$ has no nonminimal 2-separation, $|A| = 3$; and in $M\backslash e$, we have that $r(A) + r^*(A) = 4$. Since f is in no triad of M, $A = \{a, b, f\}$ is a triangle and either $\{a, b\}$ or $\{a, b, f\}$ is a cocircuit of $M\backslash e$. If $\{a, b\}$ is a cocircuit of $M\backslash e$, then $M\backslash e/a$ has an N minor, implying, since $\{b, f\}$ is a circuit of $M\backslash e/a$, that $M\backslash\{e, b\}/a$ has an N minor. But $\{a, b, e\}$ is a cocircuit of M, implying $M\backslash\{e, b\}/a = M\backslash\{a, b\}/e$, contradicting the fact that M/e has no N minor. Therefore, $\{a, b, f\}$ is a triad and a triangle of $M\backslash e$. This implies $M\backslash e$ is nonbinary, and thus, M is nonbinary. By Lemma 3.3, since $M\backslash f$ is binary, M/f is nonbinary. But then $M/f\backslash b$ is nonbinary since $\{b, a\}$ is a circuit of M/f; thus, $M\backslash b$ is nonbinary. Since $M\backslash\{e, f\}$ has an N minor and $\{a, b\}$ is a cocircuit of $M\backslash\{e, f\}$, $M\backslash\{e, f\}/a$ has an N minor; but $M\backslash\{e, f\}/a \cong M\backslash\{e, b\}/a$, implying $M\backslash b$ has an N minor. Next we show that $M\backslash b$ is 3-connected. As $\{a, b, f\}$ is a triad of $M\backslash e$, but f is in no triad of M, $\{a, b, f, e\}$ is a cocircuit of M. Thus, by Lemma 2.7, $M\backslash b$ has no nonminimal 2-separation; so suppose b is in a triad of M. Since $\{a, b, f\}$ is a triangle of M, any triad containing b must also contain a or f. But f is in no triad; thus there is some element $g \neq f$ such that $\{a, b, g\}$ is a triad of M. But then, since $M\backslash b$ is nonbinary, M/a is nonbinary implying, since $\{b, f\}$ is a circuit of M/a, that $M\backslash f$ is nonbinary, a contradiction. Thus, $M\backslash b$ is 3-connected, is nonbinary, and has an N minor, as desired.

Now assume $\tilde{M} = M/f$. Let $\{A, B\}$ be a nonminimal 2-separation of $M \backslash e$; assume $f \in A$. Since $\tilde{M} \backslash e$ has no nonminimal 2-separation, $|A| = 3$; and, in $M \backslash e$, $r(A) + r^*(A) = 4$, implying, since f is in no triangle of M, $r(A) = 3$ and $r^*(A) = 1$. Let $A = \{a, b, f\}$. Then every triple of $\{a, b, e, f\}$ is a triad of M. Thus, $\{\{a, e\}, E(M) - \{b, a, e\}\}$, $\{\{e, f\}, E(M) - \{b, e, f\}\}$, and $\{\{a, f\}, E(M) - \{b, a, f\}\}$ are 2-separations of $M \backslash b$. Since no bipartition of $E(M) - b$ can cross each of these 2-separations, Lemma 2.6 implies that M/b is 3-connected. M/b has an N minor, since b is in series in $M \backslash e$. Since M/f is binary, Lemma 3.3 implies $M \backslash f$ is nonbinary, from which it follows that M/b is nonbinary, as desired. ∎

COROLLARY 3.4. *Let N be a 3-connected minor of a 3-connected binary matroid M, such that $|E(N)| \geq 4$, and let $e \in E(M) - E(N)$. Then either $M \backslash e$ or M/e has no nonminimal 2-separation and has an N minor.*

Proof. If both $M \backslash e$ and M/e have N minors, then the result follows from Lemma 2.6. Otherwise, it follows from Theorem 3.1 and duality. ∎

4. NONBINARY MATROIDS WITH WHEEL MINORS

In this section we prove the following extension of Tutte's wheels-and-whirls theorem to the case where M is a 3-connected matroid with both a wheel and a whirl minor. This is the main result of the paper.

THEOREM 4.1. *If M is a 3-connected nonbinary matroid with a W_3 minor, then M has a 3-connected nonbinary minor $\tilde{M}$ such that $W_3 \cong \tilde{M} \backslash e$ or $\tilde{M}/e$ for some element e.*

An alternative statement of this theorem is that if M is a 3-connected matroid having both W_3 and $\mathcal{W}_2$ minors, then M has a 3-connected minor M that has both W_3 and $\mathcal{W}_2$ minors and has exactly seven elements. Since no matroid on fewer than seven elements has both W_3 and $\mathcal{W}_2$ minors, these two minors are packed into $\tilde{M}$ as efficiently as possible. It is natural to ask whether Theorem 4.1 can be extended to wheels W_n with $n \geq 4$; that is, does a 3-connected matroid with both W_n and $\mathcal{W}_2$ minors have a $(2n+1)$-element minor $\tilde{M}$ having both W_n and $\mathcal{W}_2$ minors? We remark that such a minor $\tilde{M}$ will necessarily be 3-connected. In Section 5, we answer this question negatively for $n \geq 5$. The case when $n = 4$ remains open.

The proof of Theorem 4.1 requires some preliminaries. The following is a version of Seymour's splitter theorem.

THEOREM 4.2 [5]. *Let N be a 3-connected proper minor of a 3-connected matroid M such that $|E(N)| \geqslant 4$ and if $N \cong W_k$ ($\mathcal{W}_k$), then M has no W_{k+1} ($\mathcal{W}_{k+1}$) minor. Then M has a 3-connected minor $\tilde{M}$ such that, for some element e, $\tilde{M} \backslash e$ or $\tilde{M}/e$ is isomorphic to N.*

Now we prove a slight strengthening of Seymour's theorem.

COROLLARY 4.3. *Let M be a 3-connected matroid that is not a wheel or a whirl; let N be a 3-connected proper minor of M such that $|E(N)| \geqslant 4$ and if $N \cong W_3$ ($\mathcal{W}_2$), then M has no W_4 ($\mathcal{W}_3$) minor. Then M has a 3-connected minor M such that, for some element e, $\tilde{M} \backslash e$ or $\tilde{M}/e$ is isomorphic to N.*

Proof. By Theorem 4.2, we may assume N is a wheel (whirl). Let k be the smallest integer for which M has a W_k ($\mathcal{W}_k$) minor and M also has a 3-connected minor $\tilde{M}$ and an element e such that $\tilde{M} \backslash e \cong W_k$ ($\mathcal{W}_k$) or $\tilde{M}/e \cong W_k$ ($\mathcal{W}_k$). By duality, assume $\tilde{M} \backslash e \cong W_k$ ($\mathcal{W}_k$). It suffices to show $k \leqslant 4$ ($k \leqslant 3$). Suppose not.

For every rim element y of W_k ($\mathcal{W}_k$), there is a spoke element x such that $W_k/y \backslash x \cong W_{k-1}$ ($\mathcal{W}_k/y \backslash x \cong \mathcal{W}_{k-1}$), and hence is 3-connected. By the choice of k, $M/y \backslash x$ is not 3-connected. It follows that, in $\tilde{M}$, e must be in a triangle with y and some element $z \neq x$.

Suppose there are two nonadjacent rim elements, y, y', such that $\{y, e, z\}$ and $\{y', e, z'\}$ are distinct triangles. Then, without loss of generality, we may assume $y \neq z'$, implying, by circuit elimination, that $\{y', y, z', z\}$ contains a circuit that contains y. Since $k > 4$ ($k > 3$), this circuit must be $\{y, z', z\}$, and z', z must be the two spokes adjacent to y. But then, $\{y', y, z', z\}$ also contains a circuit containing y', a contradiction. Thus, no two such nonadjacent rim elements and distinct triangles exist, implying $k \leqslant 4$. If N is a wheel, we are finished. Assume $N \cong \mathcal{W}_4$, and, for each of the two pairs y, y' of nonadjacent rim elements, $\{y, y', e\}$ is a triangle. Then by circuit elimination, the set of rim elements contains a circuit, a contradiction. ∎

LEMMA 4.4. *Let M be a 3-connected nonbinary matroid that has a W_3 minor, and let $e \in E(M)$. Then either M has a W_3 minor that does not contain e or M has a 7-element 3-connected nonbinary minor that has a W_3 minor.*

Proof. If M has a wheel minor W_k, $k \geqslant 4$, then clearly M has a W_3 minor that does not contain e. Otherwise, by Theorem 4.2, M has a 3-connected minor $\tilde{M}$ such that $W_3 \cong \tilde{M} \backslash x$ or $\tilde{M}/x$, for some x. By duality, assume $\tilde{M} \backslash x \cong W_3$. If $\tilde{M}$ is nonbinary, then we are finished. Otherwise, $\tilde{M} \cong F_7$, the Fano matroid, and $\tilde{M}$ certainly has a W_3 minor that does not contain e. ∎

LEMMA 4.5. *If M is a 3-connected nonbinary matroid with rank and corank at least 3 and M has no $\mathcal{W}_3$ minor, then there is some element f such that both $M\backslash f$ and M/f are nonbinary.*

Proof. We proceed by induction on $|E(M)|$. By Theorem 4.2 and duality, we may assume that there is some element e such that M/e is 3-connected and nonbinary. Clearly M/e has corank at least 3. If M/e has rank at least 3, then we are finished by induction, so assume M/e has rank 2. Since M/e is 3-connected, it has no parallel elements, implying $M/e \cong U_n^2$, for some $n \geq 5$. Thus, $M\backslash f$ is nonbinary, for all elements $f \neq e$. If, for some such f, $M\backslash f$ is 3-connected, then, since $M\backslash f$ has rank 3, we can assume by induction that $M\backslash f$ has corank 2 and thus $M\backslash f \cong U_5^3$. It follows that, for $g \notin \{e, f\}$, M/g is nonbinary and hence, as $M\backslash g$ is also nonbinary, the lemma holds. Therefore we can assume that $M\backslash f$ is 2-separable. Since $M\backslash f/e$ is 3-connected, e and f are in a triad, $\{e, f, g\}$. But then, since $M\backslash f$ is nonbinary, M/g must be nonbinary. Thus, $M\backslash g$ and M/g are both nonbinary, as desired. ▌

LEMMA 4.6. *If M is 3-connected and $M\backslash e \cong \mathcal{W}_3$, then there is some element f distinct from e such that either both $M\backslash f$ and M/f are nonbinary or $M\backslash f \cong W_3$.*

Proof. Let $E(M\backslash e)$ be as labeled on the Euclidean representation for $\mathcal{W}_3$ in Fig. 1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. For all y in Y, M/y is nonbinary. Thus we may assume that, for all such y, $M\backslash y$ is binary; otherwise the lemma holds.

Clearly X spans e. Let C_e be the circuit contained in $X \cup e$. If $|C_e| = 3$, then $C_e = \{x_1, x_2, e\}$, say. But $\{x_1, x_2, y_1\}$ is also a circuit of $M\backslash y_3$. Thus, as $M\backslash y_3$ is binary, $\{e, y_1\}$ is a disjoint union of circuits of $M\backslash y_3$ and hence of M, a contradiction. Thus, $|C_e| = 4$. Therefore, a binary representation of $M\backslash y_1$ is

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & y_2 & y_3 & e \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and this represents W_3. ▌

Proof of Theorem 4.1. We proceed by induction on $|E(M)|$. First we reduce to the case where there is some element f such that both $M\backslash f$ and M/f are nonbinary. If M has no $\mathcal{W}_3$ minor, then the existence of such an element follows from Lemma 4.5. Otherwise, by Corollary 4.3, M has a 3-connected minor $\tilde{M}$ such that $\mathcal{W}_3 \cong \tilde{M}\backslash e$ or $\tilde{M}/e$, for some e. Now

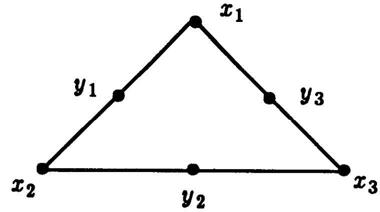

FIG. 1. $\mathcal{W}_3$.

applying Lemma 4.6 and duality, we deduce that either the theorem holds or, for some element f, both $\tilde{M}\backslash f$ and $\tilde{M}/f$ are nonbinary, and hence both $M\backslash f$ and M/f are nonbinary.

By Lemma 4.4 and duality, we may assume $M\backslash f$ both is nonbinary and has a W_3 minor. If $M-f$ is 3-connected, then we are finished, by induction. Assume not. Then, by Lemma 2.6, $M \mid f$ is 3-connected. If M/f has a W_3 minor, then we are finished, by induction, so assume not. Now by Theorem 3.1, there is some element g such that either $M\backslash g$ or M/g is 3-connected, is nonbinary, and has a W_3 minor. By induction, the theorem is proved. ∎

To conclude this section, we note that the following corollary along with the restatement of the main result given in the abstract follows without difficulty from Theorems 4.1 and 4.2.

COROLLARY 4.7. *Let M be a 3-connected nonbinary matroid having a wheel minor. Then there is a chain $M_1, M_2, ..., M_n$ of 3-connected matroids, each a single-element deletion or single-element contraction of its successor, such that $M_n = M$, $M_1 \cong W_3$, and M_2 is nonbinary.*

5. FURTHER QUESTIONS

It is natural to ask whether Theorem 4.1 can be extended to wheels W_n with $n \geqslant 4$. That is, given a 3-connected nonbinary matroid M that has a W_n minor, does M have a $(2n+1)$-element nonbinary minor that is 3-connected and has a W_n minor? In this section we provide a class of examples to show that M does not always have such a minor, for $n \geqslant 5$. The case where $n = 4$ remains open,

Let $\mathbb{F}$ be a field of characteristic two having at least three elements, and let $p \in \mathbb{F} - \{0, 1\}$. Given an integer $n \geqslant 4$, let D_n be the matrix in Fig. 2, with columns as labeled, and let G_n be the graph in Fig. 3, with edges as labeled.

$$
\begin{array}{c|ccccccc|c}
x_1 \; x_2 \; x_3 \; \cdots \; x_{n-1} \; x_n \; a & y_1 & y_2 & y_3 & \cdots & y_{n-2} & y_{n-1} & y_n & b \\
 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
I_{n+1} & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\
\cline{2-9}
 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & p
\end{array}
$$

FIG. 2. D_n.

Let M_n be the matroid represented by D_n over $\mathbb{F}$. We wish to show that for $n \geqslant 5$, M_n is 3-connected, is nonbinary, and has a W_n minor, and that no proper minor of M_n has these three properties. Since $M_n \backslash \{y_1, y_3, y_4, ..., y_n\} / \{x_1, x_2, ..., x_{n-1}\} \cong U_4^2$, we conclude that M_n is nonbinary. To see that M_n has a W_n minor, note that $M_n \backslash b/a = M(G_n \backslash b/a) \cong W_n$. Moreover, $M_n \backslash b = M(G_n \backslash b)$ and $M_n/a = M(G_n/a)$. It follows that $M_n \backslash b$ and M_n/a are both binary and 3-connected. The fact that M_n is 3-connected now follows from the fact that $M_n \backslash b$ is 3-connected, together with the fact that b is not a loop, coloop, or parallel element of M_n.

Since M_n has $2n+2$ elements, it remains to show that, for each element z, both the deletion and the contraction of z yield a matroid that is either binary, is 2-separable, or has no W_n minor. By the symmetry of D_n, we see that M_n is self dual. Thus, it suffices to consider M_n/z, for each element z. We have already established that M_n/a is binary. And each element

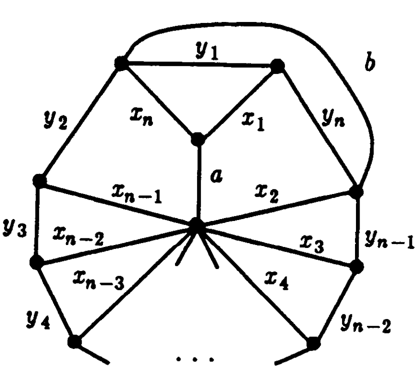

FIG. 3. G_n.

$z \in \{y_1, y_3, y_4, ..., y_{n-1}, x_1, x_2, ..., x_n\}$ is in a triangle, implying M_n/z is 2-separable. It remains to check the members of $\{b, y_2, y_n\}$. We shall show in this case that M_n/z has no W_n minor. Since M_n has corank $n+1$ and W_n has corank n, if M_n/z has a W_n minor, then it must be of the form $M_n/z\backslash w$, for some element w.

We leave it to the reader to establish the following facts:

(1) No member of $\{b, y_2, y_n\}$ is in a triangle of M_n.

(2) Element y_2 is in no 4-element circuit with y_n or with b. (Note: This is not true if $n=4$.)

Suppose $M_n/z\backslash w \cong W_n$, for some elements z and w, where $z \in \{b, y_n\}$. By (1) and (2), y_2 is in no triangle of M_n/z. Since every element of W_n is in a triangle, it must be that $w = y_2$. But $\{y_2, y_3, x_{n-1}\}$ is a triad of M_n, implying $M_n/z\backslash y_2$ has series elements, contradicting the 3-connectivity of W_n.

Now suppose $M/y_2\backslash w \cong W_n$, for some element w. Since neither y_n nor b is in a triangle in M_n/y_2, at least one of these two elements remains in no triangle in $M_n/y_2\backslash w$, a contradiction.

Thus we have established for each $z \in \{b, y_2, y_n\}$ that M/z has no W_n minor, completing the argument.

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