CONSTRUCTING INTERNALLY 4-CONNECTED BINARY MATROIDS

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Abstract. In an earlier paper, we proved that an internally 4-connected binary matroid with at least seven elements contains an internally 4-connected proper minor that is at most six elements smaller. We refine this result, by giving detailed descriptions of the operations required to produce the internally 4-connected minor. Each of these operations is top-down, in that it produces a smaller minor from the original. We also describe each as a bottom-up operation, constructing a larger matroid from the original, and we give necessary and sufficient conditions for each of these bottom-up moves to produce an internally 4-connected binary matroid. From this, we derive a constructive method for generating all internally 4-connected binary matroids.

1. Introduction

A chain theorem says that every matroid with a certain type of connectivity contains a proper minor with the same type of connectivity that can be obtained by deleting or contracting a bounded number of elements. The most famous example of a chain theorem is due to Tutte [8], his well-known "Wheels-and-Whirls Theorem". It says that every non-empty 3-connected matroid contains a 3-connected proper minor that is obtained by removing at most two elements.

We have proved a chain theorem for internally 4-connected binary matroids [1]. Every such matroid that has at least seven elements contains an internally 4-connected proper minor that is obtained by removing at most six elements.

Like the Wheels-and-Whirls Theorem, our result can be refined. Tutte actually proved that every non-empty 3-connected matroid that is not a wheel or a whirl has a 3-connected single-element deletion or contraction. The bound of two elements is required only for the exceptional classes of wheels and whirls. A similar phenomenon can be seen in our chain theorem. Almost every internally 4-connected binary matroid contains an internally 4-connected proper minor that is at most three elements smaller. The bound of six elements is needed only for one dual pair of matroids. Apart from this

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pair, a bound of four elements holds and even this is attained only in a few exceptional classes. In particular, the analogous classes to wheels and whirls in our chain theorem are the classes of quartic ladders.

For $n \geq 3$, a *planar quartic ladder* is a graph with vertex set $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ that consists of two disjoint cycles, $\{u_1u_2, u_2u_3, \ldots, u_nu_1\}$ and $\{v_1v_2, v_2v_3, \ldots, v_nv_1\}$, and two matchings $\{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ and $\{u_1v_n, u_2v_1, \ldots, u_nv_{n-1}\}$. A *Möbius quartic ladder* consists of a Hamiltonian cycle $\{v_0v_1, v_1v_2, \ldots, v_{2n-2}v_0\}$ along with the set of edges $\{v_iv_{i+n-1}, v_iv_{i+n} : 1 \leq i \leq n\}$ where all subscripts are interpreted modulo $2n-1$. For $n = 3$, the Möbius and planar quartic ladders coincide with K_5 and the *octahedron*, $K_{2,2,2}$, respectively. The *cube* is the dual of the octahedron. A *terrahawk* is the graph, $\mathcal{T}$, that is obtained from the cube by adjoining one new vertex and adding edges from this vertex to each of the four vertices that bound a face of the cube (see the left-hand diagram in Figure 1). Clearly $M^*(\mathcal{T}) \cong M(\mathcal{T})$ and $\mathcal{T}$ has both the cube and the octahedron as minors. We shall later refer to the *Wagner graph* (see the right-hand diagram in Figure 1). It is an example of a Möbius *cubic* ladder (see, for example, [5, Fig. 12.5, p. 463]).

FIGURE 1. The terrahawk, and the Wagner graph.

The refinement of our chain theorem is as follows.

Theorem 1.1. *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$. Then M contains an internally 4-connected proper minor M' such that $|E(M)| - |E(M')| \leq 3$, unless M or its dual is the cycle matroid of a planar or Möbius quartic ladder, or a terrahawk, or the cube. If M or M^* is the cycle matroid of a planar or Möbius quartic ladder or a terrahawk, then M contains an internally 4-connected proper minor M' such that $|E(M)| - |E(M')| = 4$. If M or M^* is the cycle matroid of the cube, then M contains an internally 4-connected proper minor M', namely $M(K_4)$, such that $|E(M)| - |E(M')| = 6$.*

As it happens, our chain theorem can be refined even further. In Theorem 1.2, we give a detailed analysis of the operations required to produce M' from M when $|E(M)| - |E(M')|$ is two or three. This theorem is proved in Section 3. The proof is essentially contained in [1], although extracting it requires some very careful reading of that paper.

Tutte's Wheels-and-Whirls Theorem is a top-down theorem: it describes how the proper minor M' can be produced from M, by deleting or contracting a single element, or, if M is a wheel or a whirl, by moving to the next smallest wheel or whirl. This top-down theorem has bottom-up consequences. We know that a single-element extension or coextension of a 3-connected matroid (with at least three elements) will also be 3-connected, unless the new element is a loop, a coloop, or is in a series or parallel pair (see [5, Proposition 8.2.7]). By combining this fact with Tutte's Theorem, we produce a constructive method for generating all 3-connected matroids. We start with the set $\mathcal{M}^{(3)} = \{U_{1,3}, U_{2,3}\}$, since every 3-connected matroid with at least three elements has either $U_{1,3}$ or $U_{2,3}$ as a minor. We perform the following recursive procedure: for $i > 3$, let $\mathcal{M}^{(i)}$ be defined so that M is in $\mathcal{M}^{(i)}$ if and only if there is a matroid N such that either

(i) $N \in \mathcal{M}^{(i-1)}$, and M is a single-element extension or coextension of N, where the new element is not in a circuit or cocircuit of M of size at most two; or
(ii) $N \in \mathcal{M}^{(i-2)}$, and both M and N are wheels or both are whirls, and $|E(M)| - |E(N)| = 2$

It follows immediately by combining the Wheels-and-Whirls Theorem with the characterization of 3-connected single-element extensions and coextensions that $\mathcal{M}^{(i)}$ is exactly the set of all i-element 3-connected matroids.

Geelen and Zhou [2, p.539] observed that: "For binary matroids, internal 4-connectivity is certainly the most natural variant of 4-connectivity and it would be particularly useful to have an inductive construction for this class." Our main theorem (Theorem 1.4) is a bottom-up version of Theorem 1.2 that gives us exactly such a construction.

To prove Theorem 1.4, we must characterize when the bottom-up moves produce internally 4-connected matroids. This is exactly analogous to characterizing when a single-element extension or coextension of a 3-connected matroid will be 3-connected. In Section 4, we reverse each of the operations (1)–(7) in Theorem 1.2. This gives us a number of operations which build a binary matroid M, starting from the internally 4-connected binary matroid N. We give necessary and sufficient conditions for M to be internally 4-connected. With this information in hand, we can prove Theorem 1.4, and thus describe a constructive method for generating all internally 4-connected binary matroids.

Before we can state our theorems, we need two more definitions. A *quasi rotor* with *central triangle* $\{4, 5, 6\}$ is a tuple

$$(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$$

where $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ are disjoint triangles, $\{2, 3, 4, 5\}$ and $\{5, 6, 7, 8\}$ are cocircuits, and $\{3, 5, 7\}$ is a triangle (see [1, p. 146]). A *bowtie* (T_1, T_2, C^*) consists of two disjoint triangles, T_1 and T_2, and a 4-element cocircuit C^* that is contained in their union.

The following theorem, which we prove in Section 3, is the detailed top-down chain theorem. Throughout the statement of Theorem 1.2, if some subset of the variables $\{1, 2, \ldots, 11, a, b, c\}$ is used to label elements of a matroid, it is assumed that distinct labels are applied to distinct elements.

Theorem 1.2. *Let M be an internally 4-connected binary matroid with $|E(M)| \geq 7$ such that no single-element deletion or contraction of M is internally 4-connected. Then M has a proper internally 4-connected minor N such that, up to duality, one of the following occurs.*

(1) *M has an $M(K_4)$-restriction with triangles $\{1, 2, 3\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$, and $\{3, 4, 5\}$, and M contains cocircuits $\{1, 3, 5, 7\}$ and $\{2, 3, 4, 8\}$, and $N = M\backslash 3, 6$.*
(2) *M has triangles $\{1, 2, 3\}$ and $\{3, 4, 5\}$ and cocircuits $\{2, 3, 4, 6\}$ and $\{1, 3, 5, 7\}$, and $N = M\backslash 1, 4$.*
(3) *M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{2, 3, 4, 5\}, \{5, 6, 7, 8\}, \{3, 5, 7\})$ as a quasi rotor, triangles $\{6, 8, 10\}$ and $\{2, 4, 11\}$, and $N = M\backslash 3, 4/5$.*
(4) *M has triangles $\{1, 2, 3\}$, $\{a, b, c\}$, and $\{4, 5, 6\}$, and has cocircuits $\{1, 2, b, c\}$ and $\{4, 5, a, c\}$. Moreover, either*
   (i) *$N = M/c\backslash b$; or*
   (ii) *M has a triangle $\{7, 8, 9\}$ and a cocircuit $\{a, b, 7, 8\}$, and $N = M/a, b, c$.*
(5) *M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as a bowtie, $\{2, 5, 7\}$ as a triangle, and $\{1, 2, 7, 8\}$ as a cocircuit. Moreover, either*
   (i) *$N = M/4\backslash 6$; or*
   (ii) *M has $\{5, 6, 7, 9\}$ as a cocircuit and $N = M\backslash 1/8$; or*
   (iii) *M has $\{3, 4, 11\}$ as a triangle and $\{4, 6, 10, 11\}$ as a cocircuit and $N = M\backslash 1/8$.*
(6) *M has $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ as a bowtie, $\{2, 5, 7\}$ as a triangle, and $\{1, 2, 7, 8\}$ and $\{5, 6, 7, 9\}$ as cocircuits. Moreover M has a 4-circuit $\{7, 8, 9, b\}$ and triads $\{a, b, 8\}$ and $\{b, c, 9\}$, and $N = M/8, 9\backslash b$.*
(7) *M has bowties $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ and $(\{2, 5, 7\}, \{3, 4, 11\}, \{2, 3, 4, 5\})$ and cocircuits $\{1, 2, 7, 8\}$ and $\{4, 6, 10, 11\}$, and $N = M\backslash 3, 6, 7$.*
(8) *M is $M(K_5)$ or $M(K_{3,3})$, or the cycle matroid of a cube, and N is $M(K_4)$.*
(9) *M is the cycle matroid of (respectively) a planar quartic ladder, a Möbius quartic ladder, or the terrahawk, and N has four fewer elements than M and is the cycle matroid of (respectively) a quartic planar ladder, a quartic Möbius ladder, or the cube.*

Moreover, if $|E(M)| \leq 11$, then, up to duality, M is isomorphic to $M(K_5)$ or $M(K_{3,3})$, and (8) holds. If $|E(M)| = 12$, then, up to duality, M is isomorphic to the cycle matroid of the cube or the Wagner graph, or M is isomorphic to one of D_1, D_2, or D_3. If M is isomorphic to the cycle matroid of the cube, then (8) holds. If M is isomorphic to the cycle matroid of the

Wagner graph, then (3) *holds for* M^* *and* N^*, *where* $N = M(K_{3,3})$. *If* M *is isomorphic to* D_1 *or* D_2, *then* (1) *holds, where* $N = \widetilde{K}_5^*$. *If* M *is isomorphic to* D_3, *then* (4) *holds, where* $N = \widetilde{K}_5$.

The matroid $\widetilde{K}_5$, which is discussed in more detail in Section 2, is the unique 3-connected binary extension of $M^*(K_{3,3})$. To describe the matroids D_1, D_2, and D_3 from Theorem 1.2, we use the notion of grafts, introduced by Seymour [7]. A *graft* is a pair $(G, \{\gamma_1, \gamma_2, \ldots, \gamma_n\})$ where G is a graph and each γ_i is a subset of $V(G)$. The *incidence matrix* of $(G, \{\gamma_1, \gamma_2, \ldots, \gamma_n\})$ is the matrix that is obtained from the 0-1 vertex-edge incidence matrix of G by adjoining a new column for each γ_i. This column, which we label γ_i, has a 1 in each row corresponding to a vertex in γ_i and a 0 in every other row. The matroid $M(G, \{\gamma_1, \gamma_2, \ldots, \gamma_n\})$ is the vector matroid over $GF(2)$ of the incidence matrix of $(G, \{\gamma_1, \gamma_2, \ldots, \gamma_n\})$. We shall call $M(G, \{\gamma_1, \gamma_2, \ldots, \gamma_n\})$ a *graft matroid* and refer to $\gamma_1, \gamma_2, \ldots, \gamma_n$ as *graft elements*, or *hyperedges*. Seymour [7] deals only with the case that $n = 1$ (see also [5, p. 386]). In this case we write $(G, \{\gamma_1\})$ as (G, γ_1). Seymour also requires that $|\gamma_1|$ is even, since otherwise γ_1 is a coloop of $M(G, \gamma_1)$. We shall also impose this restriction, as we will use grafts to illustrate connected extensions of graphs. We show, in the relevant cases, that all graft elements that we consider are incident with an even number of vertices. When we represent a graft having a single graft element γ, we do so by colouring the vertices in γ, and leaving the other vertices uncoloured. When we represent a graft with two graft elements, then one of them corresponds to coloured vertices, and the other corresponds to the vertices contained in boxes. Figure 2 shows graft representations of D_1, D_2, and D_3.

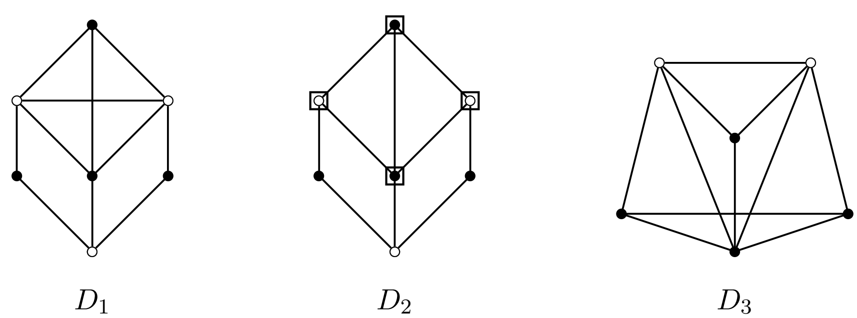


FIGURE 2. Graft representations of D_1, D_2, and D_3.

Since the statement of our main theorem is extremely long, we first present a simplified version of it.

Corollary 1.3. *Let* $\mathcal{M}^{(6)}$ *be* $\{M(K_4)\}$. *For* $i > 6$, *let* $M^{(i)}$ *be defined so that* $M_0 \in \mathcal{M}^{(i)}$ *if and only if* M_0 *is an internally* 4*-connected binary matroid, and there is a matroid* N_0, *such that for some pair* (M, N) *in* $\{(M_0, N_0), (M_0^*, N_0^*)\}$, *one of the following holds.*

(i) *M has N as a minor, where $N \in \mathcal{M}^{(i-k)}$ for some $k \in \{1, 2, 3\}$ such that $|E(M)| - |E(N)| = k$; or*
(ii) *$N \in \mathcal{M}^{(i-4)}$, and either $N = M(K_4)$ and $M = M(K_5)$, or N is the cycle matroid of a cube, and M is a terrahawk, or N and M are cycle matroids of planar or Möbius quartic ladders, and M has four more elements than N; or*
(iii) *$N \in \mathcal{M}^{(i-6)}$, and $N = M(K_4)$, while M is the cycle matroid of a cube.*

Then $\mathcal{M}^{(i)}$ is exactly the set of all i-element internally 4-connected binary matroids.

The next theorem is our main result. It describes a construction that will generate every internally 4-connected binary matroid in a minor-closed class, and produce only internally 4-connected binary matroids. Note that each of the operations (I)–(VII) in Theorem 1.4 is the reverse of the corresponding operation (1)–(7) in Theorem 1.2.

Theorem 1.4. *Let $\mathcal{M}$ be a minor-closed class of binary matroids that contains at least one internally 4-connected matroid with at least six elements. Define $\mathcal{M}^{(6)}$ to be $\{M(K_4)\}$. For $i > 6$, let $\mathcal{M}^{(i)}$ be the set of binary matroids such that $M_0 \in \mathcal{M}^{(i)}$ if and only if $M_0 \in \mathcal{M}$, and there is a matroid N_0 such that for some pair (M, N) in $\{(M_0, N_0), (M_0^*, N_0^*)\}$, one of the statements* (i)–(iv) *holds. Then, for $i \geq 6$, the set of i-element internally 4-connected members of $\mathcal{M}$ is exactly $\mathcal{M}^{(i)}$.*

(i) *$i = 12$, and M is the cycle matroid of a cube, while $N = M(K_4)$; or*
(ii) *$N \in \mathcal{M}^{(i-4)}$, and either $N = M(K_4)$ and $M = M(K_5)$, or N is the cycle matroid of a cube, and M is a terrahawk, or N and M are cycle matroids of planar or Möbius quartic ladders, and M has four more elements than N; or*
(iii) *M is a simple single-element extension of N by the element e, where $N \in \mathcal{M}^{(i-1)}$ and $r(M) = r(N)$, and, if $i > 7$, there is no triad T^* of N such that $e \in \mathrm{cl}_M(T^*)$; or*
(iv) *either $i = 9$, and $M = M(K_{3,3})$, while $N = M(K_4)$, or M and N are as described in one of the statements* (I)–(VII) *below, and $N \in \mathcal{M}^{(i-k)}$, where $k = |E(M)| - |E(N)|$, so $k \in \{2, 3\}$.*

(I) *$|E(N)| \geq 8$, and N has $\{1, 2, 4, 5\}$ as a circuit and $\{1, 5, 7\}$ and $\{2, 4, 8\}$ as triads, but N has no triad $\{a, b, c\}$ such that $\{1, 2, a, b\}$ or $\{2, 4, a, b\}$ is a circuit; M is obtained from N by extending with the elements 3 and 6 so that $\{3, 4, 5\}$ and $\{2, 4, 6\}$ are triangles.*
(II) *$|E(N)| \geq 8$ and N has $\{3, 5, 7\}$ and $\{2, 3, 6\}$ as triads, but N has no triad $\{a, b, c\}$ such that $\{3, 2, a, b\}$ or $\{3, 5, a, b\}$ is a circuit; M is obtained from N by extending with the elements 1 and 4 so that $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are triangles.*
(III) *N has $\{2, 6, 7, 8\}$ as a cocircuit and $\{6, 8, 10\}$, $\{7, 8, 9\}$, $\{1, 2, 7\}$, and $\{2, 6, 11\}$ as triangles; M is obtained from N by adding the element*

5 in series with 2, and then extending by the elements 3 and 4 so that $\{3,5,7\}$ *and* $\{4,5,6\}$ *are triangles.*

(IV) *N has* $\{1,2,3\}$ *and* $\{4,5,6\}$ *as triangles and*
   (i) $|E(N)| \geq 8$ *and N has* $\{1,2,a,4,5\}$ *as a cocircuit; M is obtained from N by adding the element b in parallel to a, and then coextending by the element c so that* $\{1,2,b,c\}$ *is a cocircuit.*
   (ii) *N has* $\{7,8,9\}$ *as a triangle and* $\{1,2,4,5,7,8\}$ *as a cocircuit, but N has no 4-cocircuit containing a pair in* $\{\{1,2\},\{4,5\},\{7,8\}\}$ *and an element in* $\{3,6,9\}$*, and N has no triangle* $\{x,y,z\}$ *such that each of* $\{y,z,1,2\}$*,* $\{x,z,4,5\}$*, and* $\{x,y,7,8\}$ *is a cocircuit; M is obtained from N by adding the element a as a coloop, and then coextending by the elements b and c so that* $\{a,b,7,8\}$ *and* $\{a,c,4,5\}$ *are circuits.*

(V) $|E(N)| \geq 8$ *and*
   (i) *N has* $\{1,2,3\}$ *and* $\{2,5,7\}$ *as triangles and* $\{1,2,7,8\}$ *as a cocircuit, but N has no 4-cocircuit containing* $\{2,3,5\}$*; M is obtained from N by adding the element 6 in parallel with 5, and then coextending by the element 4 so that* $\{2,3,4,5\}$ *is a cocircuit; or*
   (ii) *N has* $\{2,5,7\}$ *and* $\{4,5,6\}$ *as triangles and has* $\{2,3,4,5\}$ *as a cocircuit. Moreover, either N has* $\{5,6,7,9\}$ *as a cocircuit, or N has* $\{3,4,11\}$ *as a triangle and* $\{4,6,10,11\}$ *as a cocircuit. In addition, N has no 4-cocircuit* $\{2,7,a,b\}$ *such that* $\{a,b,c\}$ *or* $\{2,3,a\}$ *is a triangle; M is obtained from N by extending by the element 1 so that* $\{1,2,3\}$ *is a triangle and then coextending by the element 8 so that* $\{1,2,7,8\}$ *is a cocircuit.*

(VI) *N has* $\{1,2,3\}$*,* $\{2,5,7\}$*, and* $\{4,5,6\}$ *as triangles and* $\{2,3,4,5\}$*,* $\{a,1,2,7\}$*, and* $\{c,5,6,7\}$ *as cocircuits, and N does not have* $\{a,c,7\}$ *as a triangle; M is obtained from N by adding 8 and 9 in series with a and c, respectively, and then extending by the element b so that* $\{b,7,8,9\}$ *is a circuit.*

(VII) *N has* $\{1,2,4,11\}$ *as a circuit and* $\{1,2,8\}$*,* $\{2,4,5\}$*, and* $\{4,10,11\}$ *as triads, but N has no triad* $\{8,u,v\}$ *such that* $\{2,5,8,u\}$ *is a circuit, and has no triad* $\{10,w,x\}$ *so that* $\{4,5,10,w\}$ *is a circuit; M is obtained from N by extending by the elements 3, 6, and 7, so that* $\{1,2,3\}$*,* $\{2,5,7\}$*, and* $\{4,5,6\}$ *are triangles.*

2. Preliminaries

The matroid terminology used here will follow Oxley [5]. A *quad* in a matroid is a 4-element set that is both a circuit and a cocircuit. The property that a circuit and a cocircuit in a matroid cannot have exactly one common element will be referred to as *orthogonality*. It is also well-known ([5, Theorem 9.1.2]) that, in a binary matroid, a circuit and cocircuit meet in an even number of elements.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X)+r(E-X)-r(M)$. Equivalently, $\lambda_M(X) = r(X)+r^*(X)-|X|$. For a positive integer k, a subset X or a partition $(X, E-X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $|X|, |E-X| \geq k$. If n is an integer exceeding one, a matroid is *n-connected* if it has no k-separations for all $k < n$. This definition has the attractive property that a matroid is n-connected if and only if its dual is. Moreover, this matroid definition of n-connectivity is relatively compatible with the graph notion of n-connectivity when n is 2 or 3. For example, if G is a graph with at least four vertices and with no isolated vertices, $M(G)$ is a 3-connected matroid if and only if G is a 3-connected simple graph. But the link between n-connectivity for matroids and graphs breaks down for $n \geq 4$. In particular, a 4-connected matroid with at least six elements cannot have a triangle. Hence, for $r \geq 3$, neither $M(K_{r+1})$ nor $PG(r-1, 2)$ is 4-connected. For this reason, other types of 4-connectivity have been investigated in which certain 3-separations are allowed. In particular, a matroid is *internally 4-connected* if it is 3-connected, and whenever (X, Y) is a 3-separation, either $|X| = 3$ or $|Y| = 3$.

A k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. A k-separation $(X, E-X)$ is *minimal* if $|X| = k$ or $|E-X| = k$. It is well known (see, for example, [5, Corollary 8.2.2]) that if M is k-connected having $(X, E-X)$ as a k-separation with $|X| = k$, then X is a circuit or a cocircuit of M.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The intersection of two fully-closed sets is fully-closed, and the full closure of X is the intersection of all fully closed sets that contain X. One way to obtain $\mathrm{fcl}(X)$ is to take $\mathrm{cl}(X)$, and then $\mathrm{cl}^*(\mathrm{cl}(X))$ and so on until neither the closure nor coclosure operator adds any new elements of M. The full closure operator enables one to define a natural equivalence on exactly 3-separating partitions as follows. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of a 3-connected matroid M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$.

A subset S of a 3-connected matroid M is a *fan* in M if $|S| \geq 3$ and there is an ordering $(s_1, s_2, \ldots, s_n)$ of S such that $\{s_1, s_2, s_3\}, \{s_2, s_3, s_4\}, \ldots, \{s_{n-2}, s_{n-1}, s_n\}$ alternate between triangles and triads beginning with either. We call $(s_1, s_2, \ldots, s_n)$ a *fan ordering* of S. A 4-element fan will often be called just a 4-fan. We think of a fan as being sequential. A matroid M is *$(4, 4, S)$-connected* if M is 3-connected and, if (X, Y) is a 3-separation where $|X| \leq |Y|$ and $|X| > 3$, then X is X is a 4-fan.

A 3-separation (X, Y) of a 3-connected matroid M is a *$(4, 3)$-violator* if $|X|, |Y| \geq 4$. Evidently M is internally 4-connected if and only if it has no

$(4,3)$-violators. It is well known and easy to check that if (X,Y) is a $(4,3)$-violator in a 3-connected binary matroid, and $|X| = 4$, then X is either a quad or a 4-fan.

We shall require the some basic properties of graft matroids. In a graft, we say that a set of edges E' *spans a hyperedge* if, in the matroid of the graft, the hyperedge is in a circuit with a subset of E'. It is worth noting that any hyperedge that is incident with an even number of vertices in each component of a graph is spanned by the edges of that graph. To see this, recall that a connected graph contains a path between each pair of vertices. Thus, for a component containing $2k$ vertices incident with a hyperedge, we may assign each vertex to a unique pair and obtain k paths, $P_1, P_2, \dots, P_k$, in this component, each between a pair of vertices incident with the hyperedge. Let E' be $E(P_1) \triangle \cdots \triangle E(P_k)$. Then E' is a forest in G and, by considering the binary matrix representation of this graft, it is easy to see that E' together with the hyperedge is a circuit in the matroid of the graft. Conversely, it is impossible for a hyperedge with an odd number of vertices in a component to be contained in a circuit.

A *hyperplane in a graph* $G = (V,E)$ is a set of edges $E - B$, where B is a bond. For a graph, G, we say that subgraph H is *induced* by an edge set E' if $V(H)$ is the set of endpoints of all edges in E' and $E(H) = E'$. Then a hyperplane in $M(G, \{\gamma_1, \gamma_2, \dots, \gamma_n\})$ is a set E' of edges that form a hyperplane of G together with the set $\Gamma' \subseteq \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of all of the hyperedges that are spanned by this set of edges; that is, each component induced by E' contains an even number of vertices incident with each hyperedge in Γ'. We state the complement of this result as the following lemma.

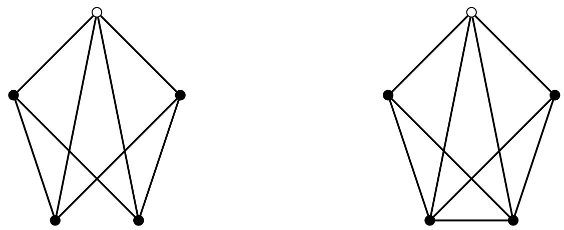
Lemma 2.1. *Let $(G, \{\gamma_1, \gamma_2, \dots, \gamma_n\})$ be a graft. Let D be a set $E_D \cup \Gamma_D$, where $E_D \subseteq E(G)$ and $\Gamma_D \subseteq \{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Then D is a cocircuit of $M(G, \{\gamma_1, \gamma_2, \dots, \gamma_n\})$ if and only if E_D is a bond of G and each component induced by $E(G) - E_D$ contains an even number of vertices incident with γ_i if and only if $\gamma_i \notin \Gamma_D$.*

The proof of Theorem 1.2 uses the following result of Qin and Zhou [6, Theorem 1.3].

Theorem 2.2. *Let M be an internally 4-connected binary matroid with no minor isomorphic to any of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$. Then either M is isomorphic to the cycle matroid of a planar graph, or M is isomorphic to F_7 or F_7^*.*

Before stating the next theorem, we need to introduce some small internally 4-connected binary matroids. The matroid $\widetilde{K}_5$, which has the graft representation shown in the right-hand picture in Figure 3, is the complement in $PG(3,2)$ of $U_{2,3} \oplus U_{2,2}$.

The matroid $M(K_{3,3})$ has a unique non-regular internally 4-connected single-element extension N_{10}. This matroid, which is self-dual, is the graft matroid $M(K_{3,3}, \gamma)$, where γ consists of the vertex set of some



$M^*(K_{3,3})$ $\widetilde{K}_5$

FIGURE 3. Graft representations of $M^*(K_{3,3})$ and $\widetilde{K}_5$.

4-cycle of $K_{3,3}$. The matroid T_{12}, which was discovered by Kingan [3], is represented over $GF(2)$ by the matrix A_{12} shown below. From this, we can see that T_{12} is self-dual. Furthermore, Kingan showed that T_{12} has a transitive automorphism group. Hence it has a unique single-element deletion and a unique single-element contraction, which we denote by $T_{12}\backslash e$ and T_{12}/e, respectively.

$$A_{12} = \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

The next result is due to Zhou [9].

Theorem 2.3. *Let M be an internally 4-connected binary matroid with no minor isomorphic to $\widetilde{K}_5$ or $\widetilde{K}_5^*$. Then M is non-regular if and only if M is isomorphic to F_7, F_7^*, N_{10}, T_{12}, $T_{12}\backslash e$, or T_{12}/e.*

Oxley [4, Theorem 2.1] determined all the 3-connected simple graphs with no minor isomorphic to the 5-wheel $\mathcal{W}_5$. The next result is an immediate corollary of that theorem (see [4, Table I]). We shall use it here to prove the two subsequent results.

Theorem 2.4. *Let G be a graph. Then G is internally 4-connected having no $\mathcal{W}_5$-minor if and only if G is isomorphic to K_4, $K_{3,3}$, K_5, the cube, or the octahedron.*

Lemma 2.5. *No internally 4-connected regular matroid has exactly eleven elements.*

Proof. Assume that M is a counterexample to the lemma. Since M is regular, by Seymour's decomposition theorem [7], $M \cong R_{10}$, or M or M^* is graphic. Thus, by duality, we may assume that M is graphic, say

$M \cong M(G)$. Then $M(G)$ has no $M(\mathcal{W}_5)$-minor because, as one can easily check, there is no 11-edge internally 4-connected graph that is obtained from $\mathcal{W}_5$ by adding an edge or splitting a vertex. The lemma follows by Theorem 2.4 since none of the graphs listed there has exactly eleven edges. $\square$

Lemma 2.6. *Let G be a 12-edge graph. Then G is internally 4-connected if and only if G is the cube, the octahedron, or the Wagner graph.*

Proof. It is straightforward to check that each of the graphs listed is internally 4-connected. Now assume that G is internally 4-connected. By Theorem 2.4, we may also assume that G has a 5-wheel minor H with vertex set $\{a, b, c, d, e, f\}$, where H has $abcdea$ as a cycle and f is adjacent to every other vertex. Suppose that G has a simple 3-connected minor H' that is obtained from H by adding an edge. By symmetry, we may assume that this edge is ac. Now $M(H')$ has two disjoint fans, (ac, ab, bc, bf) and (ae, ef, de, df, cd), and it is easy to check that no graph obtained from H' by splitting a vertex or adding an edge is internally 4-connected. Thus G is not internally 4-connected, a contradiction. By Seymour's Splitter Theorem [7], we may now assume that G has a simple 3-connected minor H' that is obtained from H by splitting the vertex f into vertices f_1 and f_2. Suppose first that H' is planar. Then we may assume that the set $N(f_1)$ of neighbors of f_1 is $\{a, b, c, f_2\}$ and that $N(f_2)$ is $\{d, e, f_1\}$. Then $M(H')$ has $(ae, af_1, ab, bf_1, bc, cf_1, cd)$ and (df_2, de, ef_2, ae) as fans, and it is easy to check that no graph obtained by splitting a vertex or adding an edge to H' is internally 4-connected, a contradiction. We deduce that H' is non-planar. By symmetry, we may assume that $N(f_1)$ is $\{a, c, f_2\}$ and $N(f_2)$ is $\{b, d, e, f_1\}$. Because of the fan (ae, ef_2, de, df_2, cd) in $M(H')$, we can see that no edge can be added to H' to produce an internally 4-connected graph. Thus G is obtained by a splitting a vertex. The only vertex with degree more than three is f_2, so we split this vertex into f_3 and f_4. Since $M(G)$ has no 4-fans, neither f_3 nor f_4 is adjacent to both d and e. Thus, up to isomorphism, $N(f_3)$ and $N(f_4)$ are $\{e, f_1, f_4\}$ and $\{b, d, f_3\}$. Then it is not difficult to check that G is the Wagner graph. $\square$

Finally, we consider necessary and sufficient conditions for the binary matroid M to be internally 4-connected when M is a single-element extension of an internally 4-connected matroid.

Lemma 2.7. *Let N be an internally 4-connected binary matroid with at least seven elements, and let M be a single-element binary extension of N by the element e. Then M is internally 4-connected if and only if M is simple, $r(M) = r(N)$, and there is no triad, T^*, of N such that $e \in \mathrm{cl}_M(T^*)$.*

Proof. If M is not simple, or if $r(M) > r(N)$, then M is not 3-connected, and therefore not internally 4-connected. If N has a triad T^* such that $e \in \mathrm{cl}_M(T^*)$, then $T^* \cup e$ is 3-separating in M, and as M has at least eight elements, M is not internally 4-connected.

This completes the proof of the "only if" direction. Therefore we assume that $r(M) = r(N)$, that M is simple, and that there is no triad of N that spans e in M. Certainly M is 3-connected since M is simple having the same rank as N [5, Proposition 8.2.7]. Suppose that M has a $(4,3)$-violator (X, Y) where $e \in X$. Then $(X - e, Y)$ is a 3-separation of N, so $X - e$ is a triangle or a triad of $M\backslash e$. If $r_{M\backslash e}(X - e) < r_M(X)$, then $\lambda_N(X - e) < \lambda_M(X) = 2$, a contradiction. Thus $e \in \mathrm{cl}_M(X - e)$ and we see that, since M is binary and simple, $X - e$ is not a triangle. Thus $X - e$ is a triad T^* of N. $\square$

3. Proving the detailed chain theorem

In this section, we prove Theorem 1.2 by mining the work done in [1].

Proof of Theorem 1.2. Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$, and assume that no single-element deletion or contraction of M is internally 4-connected.

1.2.1. *If $|E(M)| \geq 13$, and M is neither a 2-element coextension of the octahedron, nor a 2-element extension of the cube, then one of the cases (1)–(9) holds.*

Proof. Throughout the proof of (1.2.1), every cited lemma or theorem comes from [1]. By Theorem 6.1, if M has a quasi rotor

$$(\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{2,3,4,5\}, \{5,6,7,8\}, \{3,5,7\}),$$

then one of the following three things happens: either (3) holds; or we can relabel $1,2,3,4,5,6$ as $6,4,5,3,2,1$, and see that 5(i) holds; or we can relabel $2,3,4,5,6,7,8,9$ as $8,7,1,2,3,5,4,6$ and see that 5(i) holds. Assume, then, that M has no quasi rotor. If M has an $M(K_4)$-restriction with triangles $\{1,2,3\}$, $\{1,5,6\}$, $\{2,4,6\}$, and $\{3,4,5\}$, then we apply 7.5, Lemma 7.9, and Lemma 7.10, together with the symmetry of pair $\{1,4\}$ with $\{2,5\}$, and conclude that either $M\backslash 3,6$ or $M\backslash 1,4$ is internally 4-connected. In the first case (1) holds, and in the second, (2) holds. Therefore we make the following assumption.

Assumption 1. *M has no quasi rotor and no $M(K_4)$-restriction.*

If M is 4-connected, then Theorem 2.7 says that M has a single-element deletion or contraction that is internally 4-connected. Therefore, we can apply duality and assume that M has a triangle, T. By Theorem 5.1, there is an element e in T such that $M\backslash e$ is $(4,4,S)$-connected. Since $M\backslash e$ is not internally 4-connected, there is a 4-fan, $\{a,b,c,d\}$, in $M\backslash e$, where $\{a,b,c\}$ is a triangle and $\{b,c,d\}$ is a triad. As M has no 4-element fans, $\{b,c,d,e\}$ is a cocircuit of M. By orthogonality, $T - e$ contains an element of $\{b,c,d\}$. By symmetry, there are two possibilities:

(A) T contains d or
(B) T contains b.

If (A) holds, then M contains a bowtie. We first consider the following case.

Case 1. *M has no bowties.*

Therefore, for every element in a triangle of M whose deletion produces a $(4, 4, S)$-connected matroid, (A) does not hold, so (B) does. This means that there is a triangle $T = \{3, 4, 5\}$ in M, such that $M\backslash 4$ is $(4, 4, S)$-connected, and $\{1, 2, 3\}$ and $\{2, 3, 4, 6\}$ are a triangle and a cocircuit in M respectively. By Theorem 9.1 and Lemma 9.5, there is a cocircuit $\{1, 3, 5, 7\}$ in M. Although M need not be graphic, it will be convenient to use graph diagrams to keep track of some of the circuits and cocircuits in M. For example, Figure 4 shows the triangles and cocircuits in $\{1, 2, 3, 4, 5, 6, 7\}$. In this figure and the other figures in this proof, the edges incident with circled vertices make up a cocircuit.

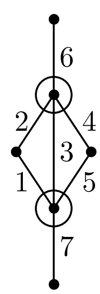


FIGURE 4. Structure diagram for (2).

It also follows from Theorem 9.1 that $M\backslash 1, 4$ is internally 4-connected, so (2) holds. Therefore we can consider the next case.

Case 2. *M has a bowtie.*

Let $(\{1, 2, 3\}, \{4, 5, 6\}, \{2, 3, 4, 5\})$ be a bowtie of M, as shown in Figure 5.

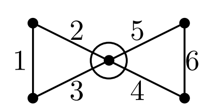


FIGURE 5. A bowtie.

We first consider the following subcase.

Case 2.1. *M has no bowtie (T_1, T_2, C^*) containing triangle $\{1, 2, 3\}$ or triangle $\{4, 5, 6\}$, unless C^* meets this triangle in $\{2, 3\}$ or $\{4, 5\}$, respectively.*

Neither (i) nor (ii) in Lemma 6.3 holds, so we can assume that M has a triangle $\{2,5,7\}$ and cocircuit $\{1,2,7,8\}$, as shown in Figure 6(i). We will now show that one of the three cases from Lemma 10.3 holds. These cases are as follows.

10.3(a) M has a cocircuit $\{5,6,7,9\}$ where $9 \notin \{1,2,\ldots,8\}$ as shown in Figure 6(ii); or

10.3(b) $M/4\backslash 6$ is internally 4-connected; or

10.3(c) M has a triangle $\{3,4,11\}$ and a cocircuit $\{4,6,10,11\}$, where $10, 11 \notin \{1,\ldots,8\}$, as depicted in Figure 6(iii).

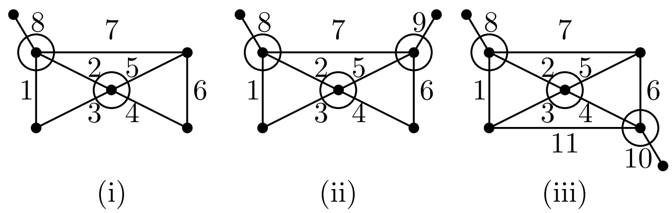


FIGURE 6. Structure diagrams for (5). Note (iii) is also the diagram for (7).

If $M\backslash 1$ has a unique fan, then Theorem 10.3 immediately implies that 10.3(a), 10.3(b), or 10.3(c) holds. Therefore we assume that $M\backslash 1$ has two fans. Then $M\backslash 1$ has two distinct triads, S_1 and S_2, such that $S_1 \cup 1$ and $S_2 \cup 1$ are cocircuits of M. We may as well assume $S_1 = \{2,7,8\}$. Suppose 3 is not in S_2. By orthogonality with triangle $\{1,2,3\}$, we know that $2 \in S_2$. If $7 \in S_2$, then $S_1 \triangle S_2$ is a series pair in M, a contradiction. So orthogonality with the triangle $\{2,5,7\}$ implies that $5 \in S_2$. By orthogonality with $\{4,5,6\}$, we have that 4 or 6 is in S_2. Now

$$r_M(\{1,2,\ldots,7\}) + r^*_M(\{1,2,\ldots,7\}) - |\{1,2,\ldots,7\}| \leq 4+5-7 = 2,$$

which contradicts the fact that M is internally 4-connected. Evidently, 3 is in the triad of a 4-fan of $M\backslash 1$. Assume that 3 is not in a triangle of a fan of $M\backslash 1$. Then $\{1,3,a,b\}$ is a cocircuit of M and $\{a,b,c\}$ is a triangle, so $(\{1,2,3\},\{a,b,c\},\{1,3,a,b\})$ is a bowtie that contains $\{1,2,3\}$, but $\{1,3,a,b\} \cap \{1,2,3\}$ is $\{1,3\}$, contradicting the assumption in Case 2.1. Therefore 3 is in the triangle of a 4-fan in $M\backslash 1$. As M has no $M(K_4)$-restriction, this triangle meets cocircuit $\{2,3,4,5\}$ in elements 3 and 4, thus we have a triangle $\{3,4,9\}$ and cocircuit $\{1,3,9,11\}$ in M. By symmetry of 7 and 1 now, we may relabel the elements to obtain the case that 10.3(a) holds. This completes the proof that 10.3(a), 10.3(b), or 10.3(c) holds.

If 10.3(b) holds, then (5)(i) holds, so we assume not. If 10.3(c) holds, then by Lemma 10.11, we let $N = M\backslash 3,6,7$ and (7) holds, or $N = M\backslash 1/8$ and (5)(iii) holds, or, up to symmetry, M has a 4-element cocircuit containing $\{1,3,11\}$ or $\{5,6,7\}$. Therefore we will assume that $\{5,6,7,9\}$ is a 4-cocircuit. This means that we can assume that 10.3(a) holds. We summarize our current assumptions in the following statement.

Assumption 2. *M has a bowtie $(\{1,2,3\},\{4,5,6\},\{2,3,4,5\})$, a triangle $\{2,5,7\}$, and cocircuits $\{1,2,7,8\}$ and $\{5,6,7,9\}$.*

First we consider the case when $M\backslash 1$ has more than one 4-fan. Statement (ii) in Lemma 10.7 does not arise, by the assumption in Case 2.1. Since 2 and 5 are in a triangle, 2 and 4 are not in a triangle, or else M has an $M(K_4)$-restriction consisting of $\{2,4,5,6,7\}$ and the element that is in a triangle with 2 and 4. Similarly, 3 is not in a triangle with 5. Now we may deduce from Lemma 10.7 that there are elements $11, 12 \notin \{1,\ldots,9\}$ such that $\{3,4,11\}$ is a triangle of M, and $\{1,3,11,12\}$ is a cocircuit.

If $M\backslash 11$ has a unique 4-fan, then we can relabel $1,2,3,4,5,7,8,11,12$ as $7,5,2,3,4,6,9,1,8$, and we will have exactly the same structure as in Assumption 2, except that $M\backslash 1$ will have a unique 4-fan. Therefore we assume that $M\backslash 11$ has more than one 4-fan. By again applying Lemma 10.7, and arguing as in the previous paragraph, we deduce that there is an element $10 \notin \{1,\ldots,9,11,12\}$ such that $\{4,6,10,11\}$ is a cocircuit. Therefore the hypotheses of Lemma 10.15 apply.

Lemma 10.15 tells us that $M.\{1,\ldots,12\}$ is the cycle matroid of the octahedron. If statement (iii) in Lemma 10.15 holds, then M is the terrahawk, and the dual of (9)(iii) holds. If (ii) is true, then, up to relabelling, M has a 4-element circuit $\{7,8,9,b\}$ and triads $\{8,b,c\}$ and $\{a,b,9\}$, where $|\{1,2,\ldots,12,a,b,c\}| = 15$. This structure is shown in Figure 7.

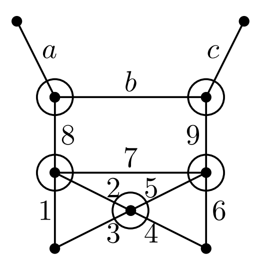


FIGURE 7. Structure diagram for (6).

In this case $M\backslash b/8,9$ is internally 4-connected, and (6) holds. Therefore we assume that statements (ii) and (iii) in Lemma 10.15 do not hold. A close reading of the proof of Lemma 10.15 shows that this implies $|E(M)| = 14$ and that M is a 2-element coextension of the octahedron. This contradicts the hypotheses of Lemma 1.2.1, so now we make the following assumption.

Assumption 3. *$M\backslash 1$ has a unique 4-fan.*

If $M\backslash 1,5$ is internally 4-connected, then we relabel $2,3,4,5,7,8$ as $3,2,6,4,5,7$, and we see that (2) holds. If $M\backslash 1/8$ is internally 4-connected, then 5(ii) holds. Therefore we make the following assumption.

Assumption 4. *Neither $M\backslash 1/8$ nor $M\backslash 1,5$ is internally 4-connected.*

Next we assume that M has triangles $\{6,9,10\}$ and $\{1,8,11\}$ and a cocircuit $\{1,3,11,12\}$, where $|\{1,2,\ldots,12\}| = 12$. This means that we can apply Lemma 10.8, and deduce that $M\backslash 11/12$ is internally 4-connected. By relabeling $1,2,3,4,5,7,8,11,12$ as $7,5,2,3,4,6,1,8,9$ we see that (5)(ii) holds. Hence we will assume that this structure does not exist in M. This means that the hypotheses of Lemma 10.6 hold, but that statements (i) and (iii) in that Lemma do not apply. Therefore statement (ii) in Lemma 10.6 holds, so M has a triangle $\{3,4,11\}$.

By relabeling $1,2,3,4,5,6,7,8,9,11$ as $10,3,4,5,2,7,1,9,8,6$, we can apply Lemma 10.9, and deduce that M has a 4-element cocircuit containing $\{1,3,11\}$ or $\{4,6,11\}$ and an element not in $\{1,2,\ldots,9,11\}$. By symmetry, we will assume that $\{4,6,10,11\}$ is a cocircuit of M. Thus the hypotheses of Lemma 10.13 hold. We have assumed no single-element deletion of M is internally 4-connected, so statement (i) of Lemma 10.13 does not hold. If $M\backslash 11/10$ is internally 4-connected, then we can relabel $2,3,4,5,6,7,9,10,11$ as $4,3,2,5,7,6,9,8,1$, and conclude that (5)(ii) holds. Therefore we assume $M\backslash 11/10$ is not internally 4-connected. We have already assumed that $M\backslash 1/8$ is not internally 4-connected. Therefore statement (ii) in Lemma 10.13 does not hold. If $M\backslash 1,11,5$ is internally 4-connected, then by swapping the labels on 1 and 7, 3 and 5, and 6 and 11, we see that (7) holds. Therefore we assume that statement (iii) in Lemma 10.13 does not hold, and deduce that M has a 4-cocircuit $\{1,3,11,12\}$. Thus the hypotheses of Lemma 10.15 again hold, and we can again deduce that either (9)(iii) or (6) holds, or M is a 2-element coextension of the octahedron.

This completes the analysis in Case 2.1, so we consider the following case.

Case 2.2. *M has a bowtie $(\{1,2,3\},\{a,b,c\},\{1,2,b,c\})$ and another bowtie $(\{4,5,6\},\{a,b,c\},\{4,5,a,c\})$ where $|\{1,2,\ldots,6,a,b,c\}| = 9$.*

This structure is illustrated in Figure 8.

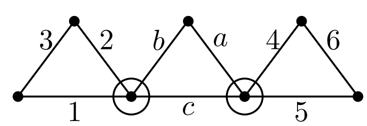


FIGURE 8. Structure diagram for (4).

If M has $(\{7,8,9\},\{a,b,c\},\{7,8,a,b\})$ as a bowtie, then by relabeling a as c, b as a, and c as b, we can apply Lemma 8.3, and see that $N = M/a,b,c$ is internally 4-connected. In this case (4)(ii) holds. Therefore we assume there is no such bowtie. By relabeling $1,2,3,a,b,c$ as $3,2,1,c,a,b$, we can apply Lemma 8.4. A close reading of the proof of Lemma 8.4 reveals that statement (i) cannot apply, or else there is a bowtie $(\{7,8,9\},\{a,b,c\},\{7,8,a,b\})$. Certainly statement (ii) cannot apply, as M has no quasi rotor. If $M/c\backslash b$ is internally 4-connected, then (4)(i) holds, so we assume that statement (iii) does not hold. Therefore, without loss of generality, M has $\{2,b,7\}$ as a triangle and $\{7, 8, a, b\}$ as a cocircuit, where $|\{1, 2, \ldots, 8, a, b, c\}| = 11$. We can assume that every bowtie in M sits inside a larger string of bowties, as otherwise we reduce to Case 2.1. Therefore the hypotheses of Theorem 11.1 apply. A careful reading of the proof of Theorem 11.1 shows that if M has an internally 4-connected minor N such that $|E(M)| - |E(N)| \leq 3$, then one of (1)–(7) applies. Therefore M is isomorphic to the cycle matroid of a quartic planar ladder or quartic Möbius ladder, and hence (9)(i) or (9)(ii) holds.

This concludes the proof of (1.2.1). $\square$

Because of (1.2.1), we can now assume that either $|E(M)| \leq 12$, or, up to duality, M is a 2-element extension of the cube. The next lemma deals with one of these cases.

1.2.2. *If M is a 2-element extension of the cube, then case* (1) *holds.*

Proof. Let us assume that $M\backslash a\backslash b$ is equal to the cycle matroid of the cube. Let G be obtained from two four-vertex cycles $v_1v_2v_3v_4v_1$ and $v_5v_6v_7v_8v_5$ by adding edges v_1v_5, v_2v_6, v_3v_7, and v_4v_8 to obtain a cube. Assume that $M\backslash a\backslash b = M(G)$. For convenience, we let $M_a = M\backslash b$ and $M_b = M\backslash a$. Since we have assumed no single-element deletion of M is internally 4-connected, neither M_a nor M_b is internally 4-connected.

Lemma 2.7 (henceforth, all citations refer to results in this paper) implies that the single-element extension, M_a, of $M(G)$ by the element a is internally 4-connected if and only if a is not a loop or coloop in M_a, and there is no triad, T^*, of $M(G)$ such that a is in the closure of T^* in M_a. Certainly a is not a loop or coloop in M_a, as M has no loops and no cocircuits of size at most two. Therefore there is a triad, T^*_a, of $M(G)$ such that $a \in \mathrm{cl}_{M_a}(T^*_a)$. Similarly, there is a triad, T^*_b, of $M(G)$ such that $b \in \mathrm{cl}_{M_b}(T^*_b)$. If T^*_a is a triad in M, then $(T^*_a \cup a, E(M) - (T^*_a \cup a))$ is a $(4, 3)$-violator in M. Therefore T^*_a, and by the same argument T^*_b, is not a triad in M.

Each triad of $M(G)$ consists of the set of edges incident with a vertex, and the automorphism group of G is transitive on triads. So up to symmetry, a — considered as a hyperedge in the graft $(G, \{a, b\})$ — is incident with $\{v_2, v_4\}$ or $\{v_1, v_2, v_4, v_5\}$. Suppose first that a is the edge v_2v_4. Then M_a has two 4-fans, so b is incident with an even number of vertices of G including v_1 and v_3. Since b is also in the closure of a triad, T^*_b, up to isomorphism, b is incident with $\{v_1, v_3\}$ or $\{v_1, v_2, v_3, v_6\}$. In the first case, M has an $M(K_4)$-restriction, and it is easy to see that (1) holds. In the latter, $M\backslash\{v_3v_4\}$ is internally 4-connected.

Therefore we assume that a is incident with $\{v_1, v_2, v_4, v_5\}$. By a similar argument, we can assume that b is incident with more than two vertices. As M is internally 4-connected, b is incident with v_1, thus we may assume, without loss of generality, that b is incident with $\{v_1, v_2, v_3, v_6\}$. Then $M\backslash\{v_1v_5\}$ is internally 4-connected. This contradiction completes the proof of (1.2.2). $\square$

By combining (1.2.1) and (1.2.2) and exploiting duality, we can now assume that $|E(M)| \leq 12$. The next result restricts the number of options further.

1.2.3. *If $|E(M)| \leq 12$, then, up to duality, one of the following statements holds.*

(i) *M is isomorphic to $M(K_5)$ or $M(K_{3,3})$; or*
(ii) *M is isomorphic to the cycle matroid of the Wagner graph or the cube; or*
(iii) *$|E(M)| = 12$ and M has a $\widetilde{K}_5$-minor.*

Proof. As F_7 and F_7^* are a single-element extension and coextension, respectively, of the internally 4-connected matroid $M(K_4)$, we know that M is neither of these. Then, by Theorem 2.2, either M is isomorphic to the cycle matroid of a planar graph, or M or its dual has a minor isomorphic to $M(K_{3,3})$ or $M(K_5)$. We first assume that M is planar graphic. Consider the case that M has no $M(\mathcal{W}_5)$-minor. Then, by Theorem 2.4, M or its dual is isomorphic to the cycle matroid of the cube. Therefore we assume that M has a $M(\mathcal{W}_5)$-minor. As $\mathcal{W}_5$ contains ten edges, and is not internally 4-connected, it follows that $|E(M)| > 10$. Then Lemmas 2.5 and 2.6 imply that M is the cycle matroid of a cube, an octahedron, or the Wagner graph. None of these options is possible, as M is planar graphic, and has an $\mathcal{W}_5$-minor.

Now we will assume that M is not planar graphic, and that (by switching to M^* as necessary) M contains an $M(K_{3,3})$-minor or an $M(K_5)$-minor. Consider the case that M is regular. Then, as R_{10} has $M(K_{3,3})$ as a single-element deletion, and $M(K_{3,3})$ is internally 4-connected, it follows that M is not isomorphic to R_{10}. Thus, by Seymour's decomposition theorem [7], M is graphic or cographic. Since M has $M(K_{3,3})$ or $M(K_5)$ as a minor, it follows that M is graphic. Assume that M has no $M(\mathcal{W}_5)$-minor. By Theorem 2.4 and the fact that M has an $M(K_{3,3})$-minor or an $M(K_5)$-minor, M is isomorphic to $M(K_{3,3})$ or $M(K_5)$. Therefore we assume that M has an $M(\mathcal{W}_5)$-minor. This minor must be proper. Hence, Lemmas 2.5 and 2.6 imply that M is the cycle matroid of the Wagner graph.

Now we assume M is non-regular. Assume that M has no minor isomorphic to $\widetilde{K}_5$ or $\widetilde{K}_5^*$. Then, by Theorem 2.3, M is isomorphic to one of F_7, F_7^*, N_{10}, T_{12}, $T_{12}\backslash e$, or T_{12}/e. Since M has a proper minor isomorphic to $M(K_{3,3})$ or $M(K_5)$, it follows that M is not isomorphic to F_7 or F_7^*. Moreover, N_{10} and T_{12} have the internally 4-connected single-element deletions $M(K_{3,3})$ and $T_{12}\backslash e$ respectively. Therefore, we can assume that, up to duality, M is isomorphic to $T_{12}\backslash e$. In this case M has a single-element deletion isomorphic to the internally 4-connected matroid $M^*(K_5)$.

Therefore we assume that, up to duality, M has a minor isomorphic to $\widetilde{K}_5$. Note that M is not equal to $\widetilde{K}_5$, since then it would have a single-element deletion isomorphic to $M^*(K_{3,3})$. Since M has no internally 4-connected single-element deletions or contractions, and $\widetilde{K}_5$ has ten elements, it follows that M has twelve elements. This completes the proof of (1.2.3). □

Note that (1.2.3) implies that if $|E(M)| \leq 11$, then, up to duality, M is isomorphic to $M(K_5)$ or $M(K_{3,3})$, and (8) holds, so the case in Theorem 1.2 where $|E(M)| \leq 11$ is now proved. Therefore we will assume that $|E(M)| = 12$. If M is the cycle matroid of the Wagner graph, then it is not difficult to see that M^* has a quasi rotor, and (3) holds. Hence, by (1.2.3), we can assume that M has a $\widetilde{K}_5$-minor or a $\widetilde{K}_5^*$-minor.

1.2.4. *If $|E(M)| = 12$, and M has either a $\widetilde{K}_5$-minor or a $\widetilde{K}_5^*$-minor, then, up to duality, M is isomorphic to D_1, D_2, or D_3. If M is isomorphic to D_1 or D_2, then* (1) *holds, where $N = \widetilde{K}_5^*$. If M is isomorphic to D_3, then* (4) *holds, where $N = \widetilde{K}_5$.*

Proof. By duality, we will assume that M has a $\widetilde{K}_5^*$-minor. Seymour's splitter theorem [7] implies that M contains a 3-connected single-element deletion or contraction, call it M', which is itself a single-element extension or coextension of $\widetilde{K}_5^*$. Note that M' is not internally 4-connected. Since $\widetilde{K}_5$ has no triad, Lemma 2.7 implies that every 3-connected extension of $\widetilde{K}_5$ is internally 4-connected. Thus every 3-connected coextension of $\widetilde{K}_5^*$ is internally 4-connected, so M' is a single-element extension of $\widetilde{K}_5^*$.

In the following case-analysis, we use the software package MACEK, developed by Petr Hliněný. The MACEK package is available to download, along with supporting documentation. The current website is `http://www.fi.muni.cz/~hlineny/MACEK`, and the interested reader is invited to download MACEK and use it to confirm the details of the case analysis. We shall represent $\widetilde{K}_5^*$ as the matroid of the graft (G,γ) shown in Figure 9. We rename the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_1v_6, v_1v_7, v_3v_7, and v_5v_7 of G as 2, 5, -3, -1, 1, 3, 4, -4, and -2. This labeling accords with that used for $\widetilde{K}_5^*$ in the MACEK library of well-known matroids. Note that $\{1,2,3,4,5,\gamma\}$ is a basis of $M(G,\gamma)$. We refer to the graft matroid $M(G,\gamma)$ as N, so that $N \cong \widetilde{K}_5^*$. Let ε be the unique element in $E(M') - E(N)$, and let δ be the unique element of $E(M) - E(M')$.

By Lemma 2.7, ε is in the closure of a triad T^* of N. It follows from [5, Lemma 10.3.13] that interchanging the labels on every pair of edges meeting a degree-2 vertex in G gives another graft representation of N. By exploiting such symmetries, we may assume that T^* is $\{5,-3,-4\}$ or $\{-1,-3,\gamma\}$. Up to isomorphism, there are six choices for ε, which we shall view as a new graft element adjoined to (G,γ). In these six cases, we rename ε as a, b, c, d, e, or f respectively. Let a be the edge v_2v_7 and let M_a be obtained from N by adding a. Likewise, let M_b, M_c, M_d, M_e, and M_f be the single-element extensions of N obtained by adding b, c, d, e, and f, respectively, where b and d are the edges v_2v_4 and v_3v_5, while c, e, and f are the hyperedges $\{v_2,v_3,v_4,v_7\}$, $\{v_2,v_3,v_6,v_7\}$, and $\{v_2,v_3,v_4,v_5,v_6,v_7\}$. Note that c is in a

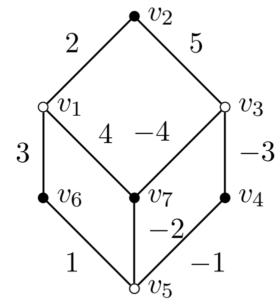


FIGURE 9. A graft (G, γ) such that $M(G, \gamma) \cong \widetilde{K}_5^*$.

circuit with $\{5, -3, -4\}$ in M_c, that $\{e, -3, \gamma\}$ is a circuit in M_e, and that $\{f, -1, -3, \gamma\}$ is a circuit in M_f. Adding an element in a triangle with -3 and -4 gives a matroid isomorphic to M_a. Adding an element in a triangle with -1 and γ gives a matroid isomorphic to M_e. Therefore M' is isomorphic to one of M_a, M_b, M_c, M_d, M_e, or M_f.

Consider the case when M' is isomorphic to M_a. Then M' has a graft representation (H, γ), where H is obtained from G by adding the edge v_2v_7. Suppose first that M is an extension of M'. Then M can be represented by a graft obtained from (H, γ) by adding an edge or hyperedge, δ. As M' has $\{a, 2, 4, 3\}$ and $\{a, -4, 5, -3\}$ as 4-fans, and M is internally 4-connected, δ is incident with an even number of vertices of H including v_1 and v_3. If δ is incident with only these two vertices, then M is isomorphic to D_1, and by relabeling $2, 3, 4, 5, -3, -4, a, \delta$ as $1, 7, 5, 2, 8, 4, 6, 3$, we see that (1) holds. Therefore we assume that δ is incident with more than two vertices. As $M\backslash a$ is not internally 4-connected, Lemma 2.7 implies δ is in the closure of a triad of N that is not a triad of $M(H, \gamma)$. Up to symmetry, this triad is $\{2, 5, \gamma\}$ or $\{2, -1, -4\}$. It follows, as δ is incident with an even number of vertices, that δ is incident with $\{v_1, v_2, v_3, v_4, v_6, v_7\}$, $\{v_1, v_2, v_3, v_5, v_6, v_7\}$, or $\{v_1, v_2, v_3, v_7\}$. In the first case, consider $M/\{-2\}$ and, in the second, consider $M/\{-3\}$. A straightforward check establishes that both of the last two matroids are internally 4-connected, a contradiction, as no single-element deletion or contraction of M is internally 4-connected. Evidently, δ is incident with $\{v_1, v_2, v_3, v_7\}$. Then M is isomorphic to D_2. Since $M|\{2, 4, 5, -4, a, \delta\}$ is $M(K_4)$, and $\{2, 3, 4, \delta\}$ and $\{5, -3, -4, \delta\}$ are cocircuits, by relabeling $2, 3, 4, 5, -3, -4, a, \delta$ as $1, 7, 5, 4, 8, 2, 6, 3$, we see that (1) holds.

Next assume that M^* is an extension of M_a^*. It is not difficult to check that the latter matroid has (H', a), as depicted in Figure 10, as a graft representation.

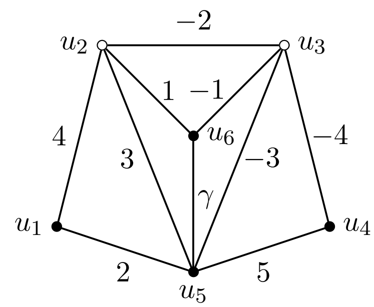


FIGURE 10. A graft representation, (H', a), of M_a^*, where a is incident with coloured vertices.

In the following argument, it is necessary to contract a hyperedge from a graft representation. This is accomplished by deleting a vertex from the underlying graph that is incident with the hyperedge. Then each other edge or hyperedge that was incident with that vertex is now incident with the symmetric difference of its original incidences and the original incidences of the contracted hyperedge. Since M has no 4-fan and is internally 4-connected, δ is incident with an even number of vertices including u_1 and u_4. Up to symmetry, δ is incident with $\{u_1, u_4\}$, $\{u_1, u_2, u_3, u_4\}$, $\{u_1, u_3, u_4, u_5\}$, $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, or $\{u_1, u_2, u_4, u_6\}$. First we assume that δ is incident with $\{u_1, u_4\}$. Then M^* is isomorphic to D_3. Moreover, $N^* = M^*/a\backslash\delta$, and by relabelling $1, 5, -1, -2, -3, -4, \gamma, a, \delta$ as $4, 2, 5, 6, 3, 1, a, c, b$, respectively, we see that (4) holds for M^* and N^*. Therefore we assume that δ is incident with $\{u_1, u_2, u_3, u_4\}$, $\{u_1, u_3, u_4, u_5\}$, $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, or $\{u_1, u_2, u_4, u_6\}$ In the first case, M^*/a is internally 4-connected, a contradiction. In the second and third cases, $M^*\backslash\{-3\}$ is internally 4-connected and $M^*\backslash 3$ is internally 4-connected, respectively. In the last case, $\{2, 4, a, \delta\}$ is a quad in M, which contradicts the fact that M is internally 4-connected. This completes the analysis in the case that $M' = M_a$. Henceforth we will assume that M has no minor isomorphic to M_a.

We assume that M' is isomorphic to M_b, which may be represented as graft (H, γ), where H is obtained from G by adding edge $b = v_2v_4$. Suppose M is an extension of M'. Since $\{b, 5, -3, -4\}$ is a 4-fan of M_b and M is internally 4-connected, δ may be represented as an edge or hyperedge incident with an even number of vertices of H including v_3. Since $M\backslash b$ is not internally 4-connected, δ is in the closure of a triad of N that is not a triad of M_b. Up to symmetry, this triad is $\{-1, -3, \gamma\}$ or $\{3, -2, -3\}$ and, up to symmetry, δ is incident with $\{v_3, v_5\}$, $\{v_2, v_3, v_6, v_7\}$, $\{v_2, v_3, v_4, v_5, v_6, v_7\}$, $\{v_1, v_3, v_4, v_6\}$, $\{v_1, v_3, v_4, v_5, v_6, v_7\}$, or $\{v_3, v_4, v_5, v_7\}$.

Then, respectively, $M/3$, $M/\{-1\}$, $M/4$, $M/1$, or $M/1$ is internally 4-connected, or $\{\delta, -1, -4, 2\}$ is a 4-fan of M. In any case we have a contradiction, so we assume that M^* is an extension of M_b^*, which has graft representation $(H', 4)$, as shown in Figure 11. As M is internally 4-connected, δ is incident with an even number of vertices of H' including u_1. Up to symmetry, keeping in mind that δ is parallel with another element in M^*/b, we know that δ is incident with $\{u_1, u_6\}$, $\{u_1, u_4\}$, $\{u_1, u_2, u_3, u_5\}$, $\{u_1, u_3, u_4, u_5\}$, $\{u_1, u_2, u_4, u_6\}$, $\{u_1, u_4, u_5, u_6\}$, $\{u_1, u_2, u_5, u_6\}$, or $\{u_1, u_3, u_5, u_6\}$. Then, respectively, M_a is a minor of $M^*\backslash 1$, or $M^*\backslash 1$ is internally 4-connected, or $\{5, -3, b, \delta\}$ is a quad of M, or $M^*\backslash 3$ is internally 4-connected, or $M^*\backslash 1$ is internally 4-connected, or $M^*\backslash 2$ is internally 4-connected, or $M^*\backslash\{-1\}$ is internally 4-connected, or $M^*\backslash 1$ is internally 4-connected; all contradictions. Evidently, M has no minor isomorphic to M_b.

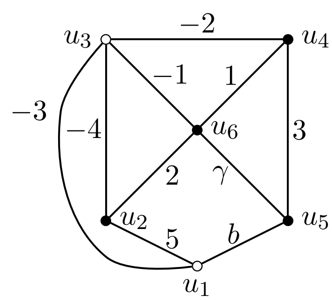


FIGURE 11. Graft representation $(H', 4)$ of M_b^*, where 4 is incident with coloured vertices.

Suppose M' is isomorphic to M_c, which may be represented as graft $(G, \{\gamma, c\})$, where c is incident with $\{v_2, v_3, v_4, v_7\}$. Assume M is an extension of M'. Since $\{5, -3, -4, c\}$ is a quad of M_c and M is internally 4-connected, δ may be represented as an edge or hyperedge incident with an even number of vertices of G including v_3, but δ does not have the same incidences as c. Since $M\backslash c$ is not internally 4-connected, δ is in the closure of a triad of N that is not a triad of M_c. Up to symmetry, this triad is $\{2, 5, \gamma\}$, $\{1, 4, 5\}$, or $\{2, -1, -4\}$. Since $\{1, 4, 5\}$ is isomorphic to $\{5, -3, -4\}$, we do not need to consider the case that δ is in a triangle with two elements in $\{1, 4, 5\}$, as such an extension is isomorphic to M_a or M_b. Thus, up to symmetry, δ is incident with $\{v_1, v_2, v_3, v_5, v_6, v_7\}$, $\{v_3, v_4, v_6, v_7\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$, or $\{v_1, v_2, v_3, v_4, v_5, v_7\}$. Then, respectively, $M/4$, $M/2$, $M/2$, or $M/2$ is internally 4-connected, a contradiction. Suppose then that M^* is an extension of M_c^*, which has graft representation $(H', \{5, -4\})$, as shown in Figure 12. We know that $M\backslash c$ has δ in a series pair. Up to symmetry, δ is in series with -1, -2, γ, -3, -4, 2, or 4. As M is internally 4-connected, δ is incident with an even number of vertices of

H' including u_1. Up to symmetry, we know that δ is incident with $\{u_1, u_3\}$, $\{u_1, u_6\}$, $\{u_1, u_2, u_3, u_4\}$, $\{u_1, u_2, u_3, u_6\}$, $\{u_1, u_2, u_4, u_5\}$, $\{u_1, u_2, u_4, u_6\}$, or $\{u_1, u_2, u_5, u_6\}$. Then, respectively, $M^*\backslash\{-3\}$, $M^*\backslash\{-4\}$, $M^*\backslash 2$, $M^*\backslash 4$, $M^*\backslash 2$, $M^*\backslash 2$, or $M^*\backslash 4$ is internally 4-connected, a contradiction. Therefore we can assume that M has no minor isomorphic to M_c.

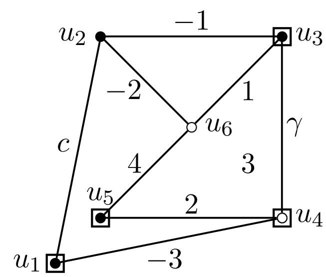


FIGURE 12. Hypergraph representation $(H', \{5, -4\})$ of M_c^*, where 5 is incident with coloured vertices and -4 is incident with boxed vertices.

Suppose M' is isomorphic to M_d, which may be represented as graft (H, γ), where H is obtained from G by adding edge $d = v_3v_5$. Suppose M is an extension of M'. Since $\{d, -3, -1, \gamma\}$ is a 4-fan of M_d and M is internally 4-connected, δ may be represented as an edge or hyperedge incident with an even number of vertices of H including v_4, but δ does not have the same incidences as γ. Since $M\backslash d$ is not internally 4-connected, δ is in the closure of a triad of N that is not a triad of M_d. We assume that M has no minor isomorphic to M_a, M_b, or M_c, thus the triad contains γ and is, up to symmetry, $\{2, 5, \gamma\}$ or $\{-1, -3, \gamma\}$. Since these are both triads of M_d, by Lemma 2.7, M is not internally 4-connected, a contradiction. Suppose then that M^* is an extension of M_d^*, which has graft representation $(H', \{5, -3\})$, as shown in Figure 13. We know that $M\backslash d$ has δ in a series pair. Up to symmetry, δ is in series with 5, 4, -3, -4, or γ in $M\backslash d$. As M is internally 4-connected, by Lemma 2.7, δ is not in the closure of triad $\{-2, -4, d\}$ or triad $\{-1, -3, d\}$ of M_d^*, so δ is not parallel with $-3, -4$, or γ in M^*/d. Evidently, δ is parallel with 5, 4, 2, or -1 in M^*/d, so δ is incident with $\{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\{u_3, u_4, u_5, u_6\}$, or $\{u_1, u_4, u_5, u_6\}$ in H. Then $M^*\backslash 2$ is internally 4-connected, a contradiction. Therefore we can assume that M has no minor isomorphic to M_d.

Suppose M' is isomorphic to M_e, which may be represented as graft $(G, \{\gamma, e\})$, where e is incident with $\{v_2, v_3, v_6, v_7\}$. Suppose M is an extension of M'. Since $\{e, -3, \gamma, -1\}$ is a 4-fan of M_e and M is internally 4-connected, δ may be represented as an edge or hyperedge incident with an even number of vertices of G including v_4, but δ does not have the same

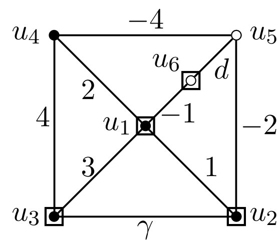


FIGURE 13. Hypergraph representation $(H', \{5, -3\})$ of M_d^*, where 5 is incident with coloured vertices and -3 is incident with boxed vertices.

incidences as γ. Since $M\backslash e$ is not internally 4-connected, δ is in the closure of a triad of N that is not a triad of M_e. We assume that M has no minor isomorphic to M_a, M_b, M_c, or M_d, thus the triad contains γ and is, up to symmetry, $\{2, 5, \gamma\}$ or $\{1, 3, \gamma\}$. Since M' also has a graft representation obtained from $(G, \{\gamma, e\})$ by relabeling the edges of the cycle $v_1v_2v_3v_4v_5v_6v_1$ in G as 5, 2, -1, -3, 3, and 1, respectively, and changing the incidences of e to $\{v_2, v_5, v_6, v_7\}$, we see that these two triads are actually isomorphic to one another, thus we restrict our attention to $\{2, 5, \gamma\}$. Now, δ is not the edge v_1v_3, as $M\backslash e$ is not isomorphic to M_d. By combining these restrictions, up to symmetry, δ is incident with $\{v_1, v_4, v_6, v_7\}$, $\{v_3, v_4, v_6, v_7\}$, or $\{v_1, v_2, v_3, v_4, v_6, v_7\}$. Then $M\backslash\gamma$, $M/2$, or $M/2$ is internally 4-connected, respectively. Therefore we assume that M^* is an extension of M_e^*, which has graft representation $(H', \{5, -3\})$, as shown in Figure 14. We know that M^*/e has δ in a parallel pair. Up to symmetry, δ is parallel with -3, γ, 1, 4, 2, -4, or 5 in M^*/e. As M is internally 4-connected, Lemma 2.7 implies that δ is not in the closure of triad $\{-3, \gamma, e\}$, so δ is not parallel with -3 or γ in M^*/d. Evidently, δ is parallel with 1, 4, 2, -4, or 5 in M^*/d, but M contains no parallel pair, so $M^*\backslash 1$, $M^*\backslash 1$, $M^*\backslash 2$, $M^*\backslash\{-1\}$, or $M^*\backslash 2$ is internally 4-connected, a contradiction. Evidently, M has no minor isomorphic to M_e.

Suppose M' is isomorphic to M_f, which may be represented as graft $(G, \{\gamma, f\})$, where f is incident with $\{v_2, v_3, v_4, v_5, v_6, v_7\}$. Suppose M is an extension of M'. Since $\{-1, -3, \gamma, f\}$ is a quad of M_f and M is internally 4-connected, δ may be represented as an edge or hyperedge incident with an even number of vertices of G including v_4, but δ does not have the same incidences as γ or f. Since $M\backslash e$ is not internally 4-connected, δ is in the closure of a triad of N that is not a triad of M_f. We assume that M has no minor isomorphic to a matroid in $\{M_a, M_b, M_c, M_d, M_e\}$, thus, up to symmetry, δ is in a quad with $\{2, 5, \gamma\}$ in $M\backslash f$. Then δ is incident with $\{v_1, v_2, v_3, v_4, v_6, v_7\}$ in G, and $M\backslash 1$ is internally 4-connected, a contradiction. Suppose then that M^* is an extension of M_f^*, which has graft

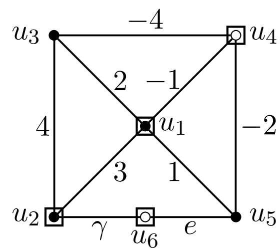


FIGURE 14. Hypergraph representation $(H', \{5, -3\})$ of M_e^*, where 5 is incident with coloured vertices and -3 is incident with boxed vertices.

representation (H', γ), as shown in Figure 15. This representation of M_f^* displays the symmetries of the matroid, including the symmetry between f and -1, so we know that M^*/f has δ in a parallel pair and $M^*/\{-1\}$ has δ in a parallel pair, or else contracting one of these two elements in M^* is internally 4-connected. Up to symmetry, δ is incident with $\{u_3, u_5\}$, $\{u_4, u_6\}$, $\{u_1, u_3, u_5\}$, or $\{u_3, u_4, u_5, u_6\}$ and $M^*\backslash\{-4\}$ is internally 4-connected, or M has $\{f, -1, -3, \gamma\}$ as a quad, or $M^*\backslash 4$ is internally 4-connected, or $M^*\backslash\{-1\}$ is internally 4-connected, respectively, contradicting either the fact that M is internally 4-connected, or the assumption that M has no internally 4-connected single-element deletion or contraction. □

Now Theorem 1.2 follows without difficulty from (1.2.1), (1.2.2), (1.2.3), and (1.2.4). □

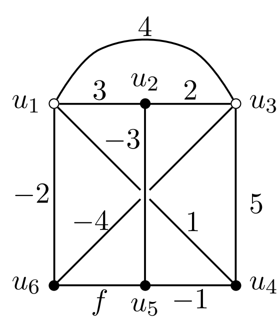


FIGURE 15. Graft representation (H', γ) of M_f^*.

4. CONSTRUCTIONS

In this section we consider what happens when we reverse the operations that produce N from M in statements (1)–(7) of Theorem 1.2. In each case we assume that N is an internally 4-connected binary matroid, and that M is a binary matroid produced from N by reversing the operations. We arrive at a set of necessary and sufficient conditions for M to be internally 4-connected. We start with the operations that involve adding three elements. The next lemma concerns the operation in case (3).

Lemma 4.1. *Let N be a binary internally 4-connected matroid having a 4-cocircuit $\{2,6,7,8\}$ and triangles $\{6,8,10\}$, $\{7,8,9\}$, $\{1,2,7\}$, and $\{2,6,11\}$. Let M be the binary matroid that is obtained by adding the element 5 in series with 2, and then extending by the elements 3 and 4 so that $\{3,5,7\}$ and $\{4,5,6\}$ are triangles. Then M is internally 4-connected.*

Proof. The construction of M ensures that it is connected. Moreover, M has $\{5,6,7,8\}$ and $\{2,3,4,5\}$ as cocircuits and has $\{1,2,3\}$ and $\{2,4,11\}$ as circuits. We show next that M is 3-connected. Assume that M contains a parallel pair. This parallel pair must contain 3 or 4. We consider the first case, as the second yields to an identical argument. Let $\{3,x\}$ be a circuit of M. By orthogonality with the cocircuit $\{2,3,4,5\}$, x is 2, 4, or 5. It cannot be 2 or 5, since $\{2,3\}$ and $\{3,5\}$ are contained in triangles. Therefore $\{3,4\}$ is a circuit of M. It is also a circuit of $M/5$, and so are $\{4,6\}$ and $\{3,7\}$. Thus $\{6,7\}$ is a circuit in $M/5$. Since N is simple, this means that $\{5,6,7\}$ is a triangle of M that meets the cocircuit $\{5,6,7,8\}$ in three elements. This is impossible, so M is simple.

Assume that M contains a series pair. This pair must contain 5. Therefore it meets one of the triangles $\{4,5,6\}$ and $\{3,5,7\}$ in a single element, violating orthogonality.

Let (X,Y) be a 2-separation of M. Since M is simple and cosimple, $|X|,|Y|\geq 3$. But N is 3-connected, so $(X-\{3,4,5\},Y-\{3,4,5\})$ is not a 2-separation of N. Therefore we can assume that $|X|\leq 4$. Since $r(X)+r^*(X)=|X|+1\leq 5$, either $r(X)\leq 2$ or $r^*(X)\leq 2$. In either case, we see that $|X|=3$, as X does not contain a parallel pair or series pair. Then $r(X)$ and $r^*(X)$ must both be equal to 2, and X must be both a triangle and a triad. This is impossible, as a circuit and a cocircuit of M meet in an even number of elements. Thus M is 3-connected.

To complete the proof, we need to show that M has no $(4,3)$-violators. Assume that M does have a $(4,3)$-violator (X,Y). We show next that

4.1.1. *if $|\{3,5,7\}\cap X|\geq 2$, then $(X\cup\{3,5,7\},Y-\{3,5,7\})$ is a $(4,3)$-violator.*

Clearly $(X\cup\{3,5,7\},Y-\{3,5,7\})\cong(X,Y)$. Thus (4.1.1) holds unless $Y-\{3,5,7\}$ is a triad, hence Y itself is a 4-fan. Consider the exceptional case. Then Y has a fan ordering (g_1,g_2,g_3,e) where $\{g_2,g_3,e\}$ is a triangle and e is $3,5$, or 7. Suppose $e=7$. Then the cocircuit $\{5,6,7,8\}$ implies that $\{g_2,g_3\}$ meets $\{6,8\}$. Thus the triad $\{g_1,g_2,g_3\}$ meets $\{6,8\}$ but avoids 5, thus it is a triad of $M/5$. As N has no series pair or coloop, it is also a triad of $M/5\backslash 3,4$, which is N. Hence 6 or 8 is in a triad of N, so N has a

4-fan; a contradiction. We deduce that $e \neq 7$. Suppose e is 3 or 5. Then the cocircuit $\{2,3,4,5\}$ implies that $\{g_1, g_2, g_3\}$ meets $\{2,4\}$ and avoids $\{3,5\}$. Thus either N has a 2-cocircuit, or N has a triad containing 2. Neither is possible. Hence $e \notin \{3,5\}$. We conclude that (4.1.1) holds.

We show next that

4.1.2. *M has no $(4,3)$-violator (X,Y) with $\{3,4,5,6,7\} \subseteq X$.*

Suppose M does have such a $(4,3)$-violator. Then $(X \cup 8, Y - 8) \cong (X,Y)$, so we may assume that $8 \in X$ unless Y is a 4-fan having a triad T^* containing 8. In the exceptional case, as $5 \in X$, it follows that 8 is in both a triangle and a triad of N; a contradiction. Thus we may indeed assume that $8 \in X$. Then $X \supseteq \{3,4,5,6,7,8\}$. Thus $(X - \{3,4,5\}, Y)$ is a 3-separation of N. As $\{6,7,8\}$ is neither a triangle nor a triad of N, it follows that $|X - \{3,4,5\}| \geq 4$. Hence we have a $(4,3)$-violator of N; a contradiction. Therefore (4.1.2) holds.

By (4.1.1), we may assume that M has a $(4,3)$-violator (X,Y) with $\{3,5,7\} \subseteq X$. By (4.1.1) and the symmetry between $\{3,5,7\}$ and $\{4,5,6\}$, if $|X \cap \{4,5,6\}| \geq 2$, then $(X \cup \{4,6\}, Y - \{4,6\})$ is a $(4,3)$-violator of M. This contradicts (4.1.2), so $\{6,4\} \subseteq Y$.

Suppose $2 \in X$. Then $(X \cup 4, Y - 4) \cong (X,Y)$ and using (4.1.1) and (4.1.2) we get a contradiction unless 4 is in a triad contained in Y. This exceptional case does not arise, otherwise N has a 2-cocircuit. Hence $2 \in Y$.

Now $(X,Y) \cong (X - 5, Y \cup 5) \cong (X - 5 - 3, Y \cup 5 \cup 3)$. As M has no triad containing 3 but avoiding 5, since $M/5\backslash 3,4$ has no series pair or coloop, it follows that both $(X - 5, Y \cup 5)$ and $(X - 5 - 3, Y \cup 5 \cup 3)$ are $(4,3)$-violators of M. Using (4.1.1) and (4.1.2), we now get a contradiction. We conclude that M is internally 4-connected. □

The next lemma considers the case that M^* is obtained from N^* by reversing the operations in (4)(ii).

Lemma 4.2. *Let N be a binary internally 4-connected matroid. Let $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$ be triads of N and $\{1,2,4,5,7,8\}$ be a 6-circuit. Let M be the binary matroid that is obtained from N by adding the element a as a coloop, and then coextending by the elements b and c so that $\{a,b,7,8\}$ and $\{a,c,4,5\}$ are circuits. Then M is internally 4-connected if and only if*

- (A) *there is no 4-circuit of N containing a pair in $\{\{1,2\},\{4,5\},\{7,8\}\}$ and an element in $\{3,6,9\}$; and*
- (B) *there is no triad $\{x,y,z\}$ of N such that each of $\{y,z,1,2\}$, $\{x,z,4,5\}$, and $\{x,y,7,8\}$ is a circuit.*

Proof. The construction of M guarantees that it is simple, connected and has $\{a,b,c\}$ as a triad. Observe that $\{b,c,1,2\}$ is a circuit of M since it is the symmetric difference of the circuits $\{1,2,4,5,7,8\}$, $\{a,b,7,8\}$, and $\{a,c,4,5\}$.

First we prove the "only if" direction. Assume that M is internally 4-connected but that either (A) or (B) fails. We start by assuming that there is a 4-circuit of N containing 1, 2, and 6. The other cases are symmetric. By orthogonality with the triad $\{4,5,6\}$, the circuit containing $\{1,2,6\}$ contains 4 or 5. We will assume that $\{1,2,4,6\}$ is a circuit of N. Then $\{1,2,4,6\} \triangle \{1,2,b,c\} = \{4,6,b,c\}$ is a circuit of M, and so is $\{1,2,4,6\} \triangle \{7,8,a,b\} \triangle \{1,2,4,5,7,8\} = \{5,6,a,b\}$. Thus $\{4,5,6,a,b,c\}$ is spanned by $\{4,5,6,b\}$ in M. Since it is spanned by $\{4,5,a,b\}$ in M^*, it follows that $\lambda_M(\{4,5,6,a,b,c\}) \leq 4+4-6 = 2$. This leads to a $(4,3)$-violator in M, contradicting our assumption that M is internally 4-connected. Symmetric arguments show that (A) must hold.

Therefore (B) fails. Then it is easy to see that $\{a,b,c,x,y,z\}$ is spanned by $\{a,b,c,x\}$ in M and by $\{a,b,x,y\}$ in M^*. Thus we obtain an identical contradiction. This completes the proof of the "only if" direction.

For the "if" direction, we assume that (A) and (B) hold. Because $M\backslash a,b,c$ is 3-connected and M has $\{a,b,c\}$ as a triad, M has no 2-cocircuits. We show next that M is 3-connected. Let (X,Y) be a 2-separation of M. Then (X,Y) is non-minimal. Without loss of generality, $|X \cap \{a,b,c\}| \geq 2$. Thus $(X \cup \{a,b,c\}, Y - \{a,b,c\})$ is a non-minimal 2-separation of M. Hence we may assume that $\{a,b,c\} \subseteq X$. Now $r(X) + r^*(X) - |X| = 1$. Moreover, $|X| \leq 4$ otherwise $(X - \{a,b,c\}, Y)$ is a 2-separation of $M\backslash a,b,c$. If $|X| = 3$, then $\{a,b,c\}$ is a circuit of M contradicting the fact that M is binary. Thus $|X| = 4$ and either $r(X)$ or $r^*(X)$ is at most two. Thus X contains a series or parallel pair, and we have a contradiction. We conclude that M is 3-connected.

Now suppose that (X,Y) is a $(4,3)$-violator of M. Assume first that $|X \cap \{a,b,c\}| = 2$. Then $(X \cup \{a,b,c\}, Y - \{a,b,c\})$ is a 3-separation of M. If $|Y - \{a,b,c\}| = 3$, then Y is a 4-fan of M and $Y - \{a,b,c\}$ is a triangle. Thus Y contains a triad of M meeting $\{a,b,c\}$ in a single element. Hence $M\backslash a,b,c$ has a 2-cocircuit; a contradiction. We conclude that $|Y - \{a,b,c\}| \geq 4$. Thus we may suppose that $X \supseteq \{a,b,c\}$.

If $|X| \geq 7$, then $(X - \{a,b,c\}, Y)$ is a $(4,3)$-violator for N; a contradiction. Hence $|X| \leq 6$. We show next that

4.2.1. *M has no triangle T with $|T \cap \{a,b,c\}| \geq 2$.*

Assume that M has such a triangle. Since $\{a,b,c\}$ is a cocircuit of M and M is binary, $\{a,b,c\}$ is not a triangle. By symmetry, we may assume that T contains $\{a,b\}$. Then $T \triangle \{a,b,7,8\}$ is a triangle of M and hence of N. This triangle meets the triad $\{7,8,9\}$ of N, so N has a 4-fan; a contradiction. Thus (4.2.1) holds.

If $|X| = 4$, then it follows from (4.2.1) that X is a quad of M. This is a contradiction, as it contains the triad $\{a,b,c\}$. Now assume that $|X| = 5$. It follows easily from (4.2.1) that X cannot contain a triangle. It is routine to verify that X must contain a quad, and a single element that is the coclosure of that quad. Since $\{a,b,c\}$ is not contained in a quad, we can assume without loss of generality that a is the single element in X that is not in the quad. By taking the symmetric difference of $\{a, b, c\}$ with $X - a$, we see that $X - \{b, c\}$ is a disjoint union of cocircuits in M. Therefore $X - \{a, b, c\}$ contains a cocircuit of N with at most two elements. This contradiction implies that $|X| = 6$.

We observe that $r(X) + r^*(X) = 8$. If $r(X) = 3$, then $M|X \cong M(K_4)$ and so M has a triangle containing at least two elements of $\{a, b, c\}$; a contradiction to (4.2.1). We deduce that $r(X) \geq 4$. We show next that

4.2.2. $r(X) = 4$

If not, then $r(X) = 5$, so $r^*(X) = 3$. Then $M^*|X \cong M(K_4)$. As $\{a, b, c\}$ is a triangle of $M^*|X$, there is a triad of M^* contained in X that meets $\{a, b, c\}$ in two elements. Therefore M has a triangle contained in X that meets $\{a, b, c\}$ in two elements. This contradiction to (4.2.1) shows that (4.2.2) holds.

Let $X - \{a, b, c\} = \{x, y, z\}$. Next we show that

4.2.3. *$\{x, y, z\}$ is a triad of $M\backslash a, b, c$.*

As $(\{x, y, z\}, Y)$ is a 3-separation of $M\backslash a, b, c$, it follows that $\{x, y, z\}$ is a triangle or a triad of $M\backslash a, b, c$. Assume the former. As $r(X) = 4$, there is a circuit C of $M|X$ other than $\{x, y, z\}$. As M is binary, $|C \cap \{a, b, c\}| = 2$. Thus, by (4.2.1), $|C| = 4$, so $|C \cap \{x, y, z\}| = 2$. Then $C \triangle \{x, y, z\}$ is a triangle of M containing two of a, b, and c; a contradiction to (4.2.1). We conclude that (4.2.3) holds.

Since $\{a, b, c\}$ is a triad of M and $\{x, y, z\}$ is a triad of $M\backslash a, b, c$, by symmetry and using symmetric difference, we may assume that either $\{x, y, z\}$ or $\{x, y, z, a\}$ is a cocircuit of M.

4.2.4. *$\{x, y, z\}$ is a cocircuit of M.*

Assume not. Then we can assume that $\{x, y, z, a\}$ is a cocircuit of M. As $r(X) = 4$, the matroid $M|X$ has at least two circuits C_1 and C_2. Clearly $|C_i \cap \{a, b, c\}| = 2$ for each i. If $C_1 \cap \{a, b, c\} = C_2 \cap \{a, b, c\}$, then $C_1 \triangle C_2$ is the disjoint union of circuits contained in $\{x, y, z\}$. As M is binary, and $\{x, y, z, a\}$ is a cocircuit, each circuit in $\{x, y, z\}$ contains exactly two elements, so M contains a parallel pair; a contradiction. Thus $C_1 \cap \{a, b, c\} \neq C_2 \cap \{a, b, c\}$. Now $|C_1 \triangle C_2| \leq 5$, so $C_1 \triangle C_2$ is a circuit of $M|X$. Moreover, $|(C_1 \triangle C_2) \cap \{a, b, c\}| = 2$. The circuits C_1, C_2, and $C_1 \triangle C_2$ imply that $M|X$ has circuits D_{ab} and D_{ac} meeting $\{a, b, c\}$ in $\{a, b\}$ and $\{a, c\}$, respectively. Each of these circuits has even intersection with $\{x, y, z, a\}$. Since $|D_{ab} \triangle D_{ac}| \geq 3$, it follows that $|D_{ab}|$ or $|D_{ac}|$ is 3. This leads to an immediate contradiction with (4.2.1). We conclude that (4.2.4) holds.

Since $\{a, b, c\}$ and $\{x, y, z\}$ are triads of M and $r(X) = 4$, after a possible relabelling, we deduce that $\{a, b, x, y\}, \{a, c, x, z\}$, and $\{b, c, y, z\}$ are circuits of M. We also know that $\{a, b, 7, 8\}$ is a circuit of M. Thus either $\{x, y\} = \{7, 8\}$ or $\{x, y\} \cap \{7, 8\} = \emptyset$. In the first case, since $\{x, y, z\}$ and $\{7, 8, 9\}$

are triads of M, we deduce that $\{x, y, z\} = \{7, 8, 9\}$ and $z = 9$. Then $\{a, c, 4, 5\} \triangle \{a, c, x, z\}$, which equals $\{4, 5, x, 9\}$ is a circuit of $M\backslash\{a, b, c\}$; a contradiction to (A). We deduce that $\{x, y\} \cap \{7, 8\} = \emptyset$. By symmetry, $\{x, z\} \cap \{4, 5\} = \emptyset$ and $\{y, z\} \cap \{1, 2\} = \emptyset$. Thus $\{x, y, 7, 8\}$, $\{x, z, 4, 5\}$, and $\{y, z, 1, 2\}$ are circuits of N, contradicting (B). □

The next lemma concerns the case when M is constructed from N using the reverse of the operations in case (6).

Lemma 4.3. *Let N be an internally 4-connected binary matroid with triangles $\{1, 2, 3\}$, $\{2, 5, 7\}$, and $\{4, 5, 6\}$ and cocircuits $\{2, 3, 4, 5\}$, $\{a, 1, 2, 7\}$, and $\{c, 5, 6, 7\}$. Let M be the binary matroid obtained from N by adding 8 and 9 in series with a and c, respectively, and then extending by the element b so that $\{b, 7, 8, 9\}$ is a circuit. Then M is internally 4-connected if and only if $\{a, c, 7\}$ is not a triangle of N.*

Proof. We first prove the "only if" direction. Assume that $\{a, c, 7\}$ is a triangle of N. Then there is a circuit C of M such that $\{a, c, 7\} \subseteq C \subseteq \{a, c, 7, 8, 9\}$. By orthogonality with the triads $\{a, b, 8\}$ and $\{b, c, 9\}$, we see that $\{a, c, 7, 8, 9\}$ is a circuit of M. Taking the symmetric difference with the circuit $\{b, 7, 8, 9\}$, we deduce that $\{a, b, c\}$ is a disjoint union of circuits in M. By again using orthogonality with $\{a, b, 8\}$ and $\{b, c, 9\}$, we see that $\{a, b, c\}$ is a triangle of M. Therefore $\{8, a, b, c\}$ is a 4-fan, and M is not internally 4-connected.

To prove the "if" direction, we assume that $\{a, c, 7\}$ is not a triangle of N. Certainly M is connected and has $\{a, b, 8\}$ and $\{b, c, 9\}$ as triads. If M has a parallel pair, then it must contain b, but neither 8 nor 9. Then orthogonality with either $\{a, b, 8\}$ or $\{b, c, 9\}$ is violated. So M is simple. If M contains a series pair, it must contain 8 or 9, but it cannot contain b. Orthogonality with the circuits $\{b, 7, 8, 9\}$ and $\{2, 5, 7\}$ means that $\{8, 9\}$ is a series pair of M. Then $\{8, 9\}$, $\{a, 8\}$, and $\{c, 9\}$ are series pairs of $M\backslash b$, so $\{a, c\}$ is a series pair of N. This contradiction shows M is simple and cosimple.

Let (X, Y) be a 2-separation of M. Then $|X|, |Y| \geq 3$. Without loss of generality, two of $8, a$, and b are in X, so we may assume that all three are. Let $Z = \{b, 8, 9\}$. If $9 \in X$, then $((X \cup c) - Z, Y - c)$ is a 2-separation of N. Therefore $9 \in Y$. Now $(X - Z, Y - Z)$ is a 2-separation of N unless $|X - Z| = 1$. In the exceptional case, $X = \{a, b, 8\}$ and $r(X) + r^*(X) = 4$. As $r^*(X) = 2$, we deduce that $r(X) = 2$, so X contains a circuit C. As $\{b, c, 9\}$ is a cocircuit, C does not contain b. Then $C = \{8, a\}$; a contradiction. We conclude that M is 3-connected.

We now show that

4.3.1. *none of $a, b, c, 8$, and 9 is in a triangle of M.*

Take $x \in \{a, b, c, 8, 9\}$ and suppose that T is a triangle of M containing x. As N is simple, T is not a triangle of $M\backslash b$. Thus $b \in T$. As M is binary with the cocircuits $\{a, b, 8\}$ and $\{b, c, 9\}$, this triangle meets $\{a, 8\}$ and $\{c, 9\}$. If 8 or 9 is in T, then, by orthogonality, T meets $\{1,2,7\}$ or $\{5,6,7\}$, so $|T| \geq 4$; a contradiction. Thus $T = \{a,b,c\}$. Then $\{a,b,c\} \triangle \{b,7,8,9\}$ is a circuit, $\{a,c,7,8,9\}$, of M, so $\{a,c,7\}$ contains a circuit of N, contradicting our assumption. Thus (4.3.1) holds.

Now suppose that (X,Y) is a $(4,3)$-violator of M. We show first that

4.3.2. *if* (X,Y) *is a* $(4,3)$*-violator of* M*, then neither* X *nor* Y *contains* $\{b,8,9\}$*.*

Assume that $\{b,8,9\} \subseteq X$. Then $(X,Y) \cong (X \cup 7, Y-7)$. If $(X \cup 7, Y-7)$ is not a $(4,3)$-violator of M, then Y is a 4-fan of M. As $|E(N)| \geq 9$, this implies that $(X - \{b,8,9\}, Y)$ is a $(4,3)$-violator of N. Therefore $(X \cup 7, Y-7)$ is a $(4,3)$-violator of M. The same argument shows that $(X \cup \{7,a\}, Y - \{7,a\})$, and $(X \cup \{7,a,c\}, Y - \{7,a,c\})$ are $(4,3)$-violators of M. Since

$$((X - \{b,8,9\}) \cup \{7,a,b\}, Y - \{7,a,b\})$$

is not a $(4,3)$-violator of N, it follows that $X = \{b,8,9\}$. This contradicts the fact that (X,Y) is a $(4,3)$-violator of M. Thus (4.3.2) holds.

Let (X,Y) be a $(4,3)$-violator of M. We assume that $\{8,b\} \subseteq X$ and $9 \in Y$. Assume that $c \in X$. Then $(X,Y) \cong (X \cup 9, Y-9)$. It follows from (4.3.2) that $(X \cup 9, Y-9)$ is not a $(4,3)$-violator of M. Thus Y is a 4-fan of M whose triad T^* contains 9. By orthogonality with $\{b,7,8,9\}$, we have that $7 \in T^*$. Thus T^* meets the circuits $\{2,5,7\}$ and

$$\{1,3,4,6,7\} = \{1,2,3\} \triangle \{2,5,7\} \triangle \{4,5,6\}$$

in at least two elements. Therefore T^* has at least four elements, a contradiction. We deduce that $c \in Y$.

If $7 \in X$, then, as (4.3.2) implies that $(X \cup 9, Y-9)$ is not a $(4,3)$-violator, Y is a 4-fan whose triangle contains 9; a contradiction to (4.3.1). Thus $7 \in Y$. Then $(X-b, Y \cup b)$ is a $(4,3)$-violator of M unless X is a 4-fan whose triad contains b. In the exceptional case, by orthogonality with the circuit $\{b,7,8,9\}$, this triad contains 8, so the triangle contained in X contains 8 or b, contradicting (4.3.1). Thus $(X-b, Y \cup b)$ is indeed a $(4,3)$-violator of M. Then $(X - \{b,8\}, Y \cup \{b,8\})$ is a $(4,3)$-violator contradicting (4.3.2), unless 8 is in the triangle of a 4-fan, contradicting (4.3.1).

From the last paragraph and symmetry, we deduce that neither $\{8,b\}$ nor $\{9,b\}$ is contained in X. It remains to consider the case when $\{8,9\} \subseteq X$ and $b \in Y$. As M has $\{7,8,9,b\}, \{7,2,5\}$, and $\{7,1,3,4,6\}$ as circuits, it follows by orthogonality that

4.3.3. *M has no triad containing* 7*.*

Suppose that $a \in X$. By (4.3.2), $(X \cup b, Y-b)$ is not a $(4,3)$-violator. Thus Y is a 4-fan with b in its triad T^*. Then T^* meets $\{7,8,9\}$, so $7 \in T^*$, contradicting (4.3.3). Thus we may assume that $a \in Y$ and, by symmetry, $c \in Y$. Then $(X-8, Y \cup 8)$ is a $(4,3)$-violator, reducing to a previous case, unless X is a 4-fan with 8 in its triad. In the exceptional case, this triad contains 7 or 9, so, by (4.3.3), the triad contains 9. Thus the triangle of this fan contains 8 or 9, which contradicts (4.3.1). □

The next lemma corresponds to case (7) in Theorem 1.2.

Lemma 4.4. *Let N be a binary internally 4-connected matroid with $\{1,2,4,11\}$ as a circuit and $\{1,2,8\}$, $\{2,4,5\}$, and $\{4,10,11\}$ as triads. Let M be the binary matroid obtained from N by extending by the elements 3, 6, and 7, so that $\{1,2,3\}$, $\{2,5,7\}$, and $\{4,5,6\}$ are triangles. Then M is internally 4-connected if and only if the following conditions hold.*

(A) *N has no triad $\{8,u,v\}$ such that $\{2,5,8,u\}$ is a circuit; and*
(B) *N has no triad $\{10,w,x\}$ such that $\{4,5,10,w\}$ is a circuit.*

Proof. Assume that condition (A) fails. It is easy to see, using orthogonality, that $\{8,u,v\}$ is a triad of M. Moreover, $\{7,8,u\} = \{2,5,7\} \triangle \{2,5,8,u\}$ is a triangle of M, so $\{v,u,7,8\}$ is a 4-fan in M. A similar argument shows that if (B) fails, then $\{x,w,6,10\}$ is a 4-fan in M. This completes the "only if" direction of the proof.

We assume (A) and (B) hold. Since $r(M) = r(N)$, we observe that M is 3-connected provided M has no parallel pairs. If M has a parallel pair, then it contains 3, 6, or 7. Consider the case that 3 is in a parallel pair with the element x. If $x = 6$, then $\{1,2,6\}$ is a circuit, and by symmetric difference with $\{4,5,6\}$, so is $\{1,2,4,5\}$. This circuit meets the cocircuit $\{4,10,11\}$ of N in a single element, so $x \neq 6$. Similarly, if $x = 7$, then $\{3,4,11\} = \{1,2,3\}\triangle\{1,2,4,11\}$ and $\{2,5,3\}$ are circuits, so $\{2,4,5,11\}$ is a circuit of N that meets $\{1,2,8\}$ in a single element. Thus x is neither 6 nor 7, so $\{1,2,x\}$ is a triangle of N that meets the triad $\{2,4,5\}$ in a single element. Very similar arguments show that if 6 or 7 is in in a parallel pair, then $\{6,7\}$ must be a circuit of M. In this case $\{2,5,7\} \triangle \{4,5,6\} = \{2,4,6,7\}$ is a disjoint union of circuits, so $\{2,4\}$ contains a circuit of M. This contradiction shows that M has no parallel pairs, and is therefore 3-connected.

Now let (X,Y) be a (4,3)-violator of M. Let $Z = \{3,6,7\}$. As N is internally 4-connected, $|X-Z| \leq 3$ or $|Y-Z| \leq 3$. Assume that $|X-Z| \leq 2$. As $|X| \geq 4$, it follows that $|X-Z| = 1, 1$. Note that $r(X)+r^*(X) = |X|+2$. Since M has no parallel pairs, $r(X) \geq 3$, so $r^*(X) < |X|$. Thus X contains a cocircuit of M. As $r(M) = r(N)$, we deduce that $X-Z$ contains a cocircuit of N, a contradiction.

We may now assume that $|X-Z| = 3$. Likewise, $|Y-Z| \geq 3$. Hence

$$2 \leq r(X-Z) + r(Y-Z) - r(N) \leq r(X) + r(Y) - r(M) \leq 2.$$

Thus $r(X-Z) = r(X)$ and $r(Y-Z) = r(Y)$. Now $X-Z$ is a triangle or triad of N. In the former case, we contradict the fact that M is binary as $r(X) = 2$ and $|X| \geq 4$. Thus $X-Z$ is a triad T^* of N, and $3 = r(X-Z) = r(X)$.

Assume that $X-Z$ is not a triad in M. Then there is a cocircuit C^* of M such that $X - Z \subseteq C^* \subseteq X$. As $X-Z$ must be independent in N, it is independent in M, and therefore spans C^*. Assume that x and y are distinct elements in $C^* - (X-Z)$. Since M is simple, $(X-Z)\cup x$ and $(X-Z)\cup y$

are circuits of M, or else they contain triangles of M that are contained in C^*. Thus $\{x, y\}$ contains a circuit of M, which is impossible. So if $X - Z$ is not a triad of M, then there is a single element $x \in Z$ such that $(X - Z) \cup x$ is a circuit and a cocircuit. But x is contained in a triangle that meets Z in exactly x. By using orthogonality, and taking the symmetric difference of this triangle with $(X - Z) \cup x$, we see that $X - Z$ spans an element in N. Thus N has a 4-fan, which is impossible. We deduce from this that $X - Z$ is a triad of M.

Since $|X| \geq 4$, one of 3, 6, and 7 is contained in a triangle T such that $T - Z \subseteq X - Z$. By orthogonality between T and the cocircuits $\{1, 2, 3, 8\}$, $\{3, 4, 10, 11\}$, and $\{2, 3, 4, 5\}$, it follows that $X - Z$ must contain at least one element from $\{1, 2, 4, 5, 8, 10, 11\}$. Assume that $2 \in X - Z$. By orthogonality between $X - Z$ and the triangles $\{2, 5, 7\}$ and $\{1, 2, 3\}$, we see that $X - Z = \{1, 2, 5\}$. But $\{1, 2, 8\}$ and $\{2, 4, 5\}$ are also triads of N, and this implies that N has a series pair, a contradiction. Therefore $2 \notin X - Z$. Similarly, if $4 \in X - Z$, then the triangles $\{4, 5, 6\}$ and $\{3, 4, 11\}$ imply that $X - Z = \{4, 5, 11\}$. As $\{4, 10, 11\}$ and $\{2, 4, 5\}$ are triads of N, this leads to an impossible situation.

Therefore $2, 4 \notin X - Z$. If $1 \in T$, then orthogonality between $X - Z$ and $\{1, 2, 3\}$ implies that $2 \in X - Z$, contradicting our conclusion. Similarly, if $11 \in X - Z$, then the triangle $\{3, 4, 11\}$ implies $4 \in X - Z$. Thus $1, 11 \notin X - Z$. The triangles $\{2, 5, 7\}$ and $\{4, 5, 6\}$ lead to the conclusion that $5 \notin X - Z$.

By orthogonality with the cocircuit $\{2, 3, 4, 5\}$, we now see that T does not contain 3. Suppose it contains 6. By orthogonality with the cocircuit $\{4, 6, 10, 11\}$, it must contain 10. Thus 10 is in a triad $\{10, w, x\}$ of N, where $\{6, 10, w\}$ is a triangle of M. As $\{6, 10, w\} \triangle \{4, 5, 6\} = \{4, 5, 10, w\}$ is a circuit of N, we have violated (A). A similar argument shows that if 7 is in T, then there is a triad $\{8, u, v\}$ and a circuit $\{2, 7, 8, u\}$ of N. This completes the proof. $\square$

We now move on to the operations in Theorem 1.2 that involve the removal of two elements. We will make repeated use of the following two observations.

Lemma 4.5. *Let M be a connected binary matroid and N be a 3-connected minor of M with $|E(M) - E(N)| \leq 2$ and $|E(N)| \geq 4$. Then M is 3-connected provided it has no 2-circuit or 2-cocircuit meeting $E(M) - E(N)$.*

Proof. Let (X, Y) be a 2-separation of M. Then $(X \cap E(N), Y \cap E(N))$ is a 2-separation of N provided $|X \cap E(N)|, |Y \cap E(N)| \geq 2$. But N is 3-connected so, without loss of generality, $|X \cap E(N)| \leq 1$. If $|X| = 2$, then (X, Y) is a minimal 2-separation of M and X is a 2-circuit or a 2-cocircuit of M meeting $E(M) - E(N)$. Thus we may assume that $|X \cap E(N)| = 1$ and $|X| = 3$, so $E(M) - E(N) \subseteq X$. Hence $r(X) + r^*(X) = 4$. If $r(X)$ or $r^*(X)$ is 1, then X contains a 2-circuit or a 2-cocircuit of M meeting $E(M) - E(N)$. Thus we may assume that $r(X) = r^*(X) = 2$. Hence X

contains both a circuit and a cocircuit. Since M is binary, X is not both a circuit and a cocircuit, so X contains a 2-circuit or a 2-cocircuit of M meeting $E(M) - E(N)$. $\square$

Lemma 4.6. *Let N be a 3-connected matroid with at least four elements and let e be an element of N. Add an element f in parallel to e and coextend the resulting matroid by an element g to give a binary matroid M in which $\{e, f, g\}$ is a triangle and neither $\{e, g\}$ nor $\{f, g\}$ is a cocircuit. Then M is 3-connected.*

Proof. By construction, M is connected. By the last lemma, if M is not 3-connected, then M has a 2-element subset V that is either a circuit or a cocircuit. Now $N = M\backslash f/g$ and N is simple and cosimple. Thus either V is a 2-circuit containing f or a 2-cocircuit containing g. As M has $\{e, f, g\}$ as a triangle but neither $\{e, g\}$ nor $\{f, g\}$ as a cocircuit, the second possibility does not occur. Thus V is a 2-circuit containing f. Hence V is a 2-circuit of M/g, so $V = \{e, f\}$ contradicting the fact that $\{e, f, g\}$ is a circuit of M. $\square$

The next lemma concerns the reversal of the operations in (1).

Lemma 4.7. *Let N be an internally 4-connected binary matroid with at least eight elements, such that $\{1, 2, 4, 5\}$ is a circuit and $\{1, 5, 7\}$ and $\{2, 4, 8\}$ are triads. Let M be the binary matroid obtained from N by extending with the elements 3 and 6 so that $\{3, 4, 5\}$ and $\{2, 4, 6\}$ are triangles. Then M is internally 4-connected if and only if N has no triad $\{a, b, c\}$ such that $\{1, 2, a, b\}$ or $\{2, 4, a, b\}$ is a 4-circuit.*

Proof. Assume that $\{a, b, c\}$ is a triad, and $\{1, 2, a, b\}$ is a 4-circuit in N. It is easy to see that $\{a, b, c\} \cap \{1, 2, 3, 4, 5, 6\} = \emptyset$, and therefore orthogonality implies that $\{a, b, c\}$ is a triad of M. But $\{1, 2, a, b\} \triangle \{3, 4, 5\} \triangle \{1, 2, 4, 5\} = \{3, a, b\}$ is a triangle, so $\{c, a, b, 3\}$ is a 4-fan. Similarly, if $\{2, 4, a, b\}$ is a circuit, then $\{c, a, b, 6\}$ is a 4-fan of M. Therefore the "only if" direction holds.

Assume that there is no such triad $\{a, b, c\}$. By taking symmetric differences, we deduce that $\{1, 2, 3\}$ and $\{1, 5, 6\}$ are circuits of M. Moreover, by orthogonality, $\{1, 3, 5, 7\}$ and $\{2, 3, 4, 8\}$ are cocircuits of M. As $r(M) = r(N)$, it follows by Lemma 4.5 that M is 3-connected unless 3 or 6 is in a 2-circuit of M. If 3 is in a 2-circuit of M, then this circuit violates orthogonality with $\{2, 3, 4, 8\}$ or $\{1, 3, 5, 7\}$. If 6 is in a 2-circuit with a, then $\{a, 2, 4, 8\}$ is a 4-fan of N, a contradiction. Hence M is indeed 3-connected.

Let (X, Y) be a $(4, 3)$-violator of M. Suppose first that $\{3, 6\} \subseteq X$. As $(X - \{3, 6\}, Y)$ is not a $(4, 3)$-violator of N, we have that $|X| \leq 5$. If $|\operatorname{fcl}(X)| > 5$, then there is a subset X' such that $X \subseteq X' \subseteq \operatorname{fcl}(X)$, where $|X'| = 6$, and $(X', E(M) - X')$ is a $(4, 3)$-violator of M. As $|E(N)| \geq 8$, this means that $(X' - \{3, 6\}, E(M) - X')$ is a $(4, 3)$-violator of N. As this is impossible, it follows that $|\operatorname{fcl}(X)| = 4, 5$. By replacing X with $\operatorname{fcl}(X)$ as required, we can assume that X is fully closed. Thus X contains no element in $\{1,2,4,5\}$, otherwise it contains all of them, and $|X| \geq 6$, a contradiction. Furthermore, neither 3 nor 6 is in a cocircuit that is contained in X, as this would violate orthogonality with $\{1,2,3\}$ or $\{2,4,6\}$. As X is a 4- or 5-element 3-separating set in M and neither 3 nor 6 is in a cocircuit contained in X, a simple analysis of possible separators shows that $X = \{3,a,b,c,6\}$, where $\{3,a,b\}$ and $\{b,c,6\}$ are triangles and $\{a,b,c\}$ is a triad. Then $\{3,a,b\} \triangle \{1,2,3\}$ is a circuit, $\{1,2,a,b\}$, and $\{a,b,c\}$ is a triad, contradicting our assumption.

We may now assume that exactly one element in $\{3,6\}$ is in X. Let $Z = \{3,6\}$. As $(X-Z, Y-Z)$ is not a $(4,3)$-violator of N, we may assume that $|X| = 4$ and $3 \in X$ or $6 \in X$. Then X is a quad or a 4-fan of M. As N is cosimple, neither 3 nor 6 is in a triad of M. Thus, if X is a 4-fan, then 3 or 6 in its triangle, T, but not its triad, T^*. A striaghtforward orthogonality argument shows that T^* does not contain any element in $\{1,2,3,4,5,6\}$. Therefore the symmetric difference of T with either $\{1,2,3\}$ or $\{2,4,6\}$, gives a 4-element circuit, which, together with T^*, contradicts the assumptions of our lemma. We deduce that X is a quad. If X contains 3, then, by orthogonality, X meets $\{1,2\}$ and $\{4,5\}$, thus $\mathrm{fcl}(X)$ contains $\{1,2,3,4,5,6\}$, and $(\{1,2,4,5\}, E(N) - \{1,2,4,5\})$ is a $(4,3)$-violator in N. If X contains 6, then, by orthogonality, X meets $\{2,4\}$ and $\{1,5\}$, thus it has two elements in each of $\{2,3,4,8\}$ and $\{1,3,5,7\}$, so $|X|$ exceeds four, a contradiction. □

The next lemma deals with the case that (2) holds.

Lemma 4.8. *Let N be an internally 4-connected binary matroid with at least eight elements that has $\{3,5,7\}$ and $\{2,3,6\}$ as triads. Let M be the binary matroid obtained from N by extending with the elements 1 and 4 so that $\{1,2,3\}$ and $\{3,4,5\}$ are triangles. Then M is internally 4-connected if and only if N has no triad $\{a,b,c\}$ such that $\{3,2,a,b\}$ or $\{3,5,a,b\}$ is a circuit.*

Proof. It is easy to verify that if N has a triad $\{a,b,c\}$, as in the statement of the lemma, then $\{c,a,b,1\}$ or $\{c,a,b,4\}$ is a 4-fan. To prove the "if" direction, we assume that N has no such fan.

By orthogonality, $\{1,3,5,7\}$ and $\{2,3,4,6\}$ are cocircuits of M. Suppose 1 is in a 2-circuit of M. Then this circuit is $\{1,3\}$, $\{1,5\}$, or $\{1,7\}$. The first possibility does not occur by orthogonality with $\{2,3,4,6\}$. The second and third do not occur, or else N has $\{6,2,3,5\}$ or $\{6,2,3,7\}$ as a 4-fan. We conclude that M has no 2-circuit containing 1. A similar argument shows that 4 is not in any parallel pair. Therefore Lemma 4.5 implies that M is 3-connected.

Let (X,Y) be a $(4,3)$-violator of M and let Z be $\{1,4\}$. We first assume that $\{1,4\} \subseteq X$. As $(X-Z, Y)$ is not a $(4,3)$-violator of N, it follows that $|X| = 4,5$. If $|\mathrm{fcl}(X)| > 5$, then there is a set X' such that $X \subseteq X' \subseteq \mathrm{fcl}(X)$, where $|X'| = 6$ and $(X', E(M) - X')$ is a 3-separation of M. As $|E(N)| \geq 8$, it follows that $(X' - Z, E(M) - X')$ is a $(4,3)$-violator of N, which is impossible. Therefore $|\operatorname{fcl}(X)| = 4, 5$, and by replacing X with $\operatorname{fcl}(X)$ as necessary, we assume that X is fully closed. If 2, 3, or 5 is in X, then X contains $\{1,2,3,4,5,6,7\}$; a contradiction. Thus $\{2,3,5\} \subseteq Y$, and therefore $Z \subseteq \operatorname{cl}_M(Y)$. If $r(X-Z) < r(X)$, then $r(X-Z) + r(Y \cup Z) - r(M) < r(X) + r(Y) - r(M) = 2$, so $(X - Z, Y \cup Z)$ is a 2-separation of M. Therefore $r(X - Z) = r(X)$. This implies that $|X - Z| > 2$, since X cannot not have rank 2 in M, as M is simple. Therefore $|X - Z| = 3$, and as $(X - Z, Y \cup Z)$ is a 3-separation of M, we see that $X - Z$ is a triad of M. As $X - Z$ spans 1 and 4 in M, orthogonality tells us that there are triangles contained in $(X - Z) \cup 1$ and $(X - Z) \cup 4$ that contain 1 and 4 respectively. Orthogonality with the cocircuits $\{1,3,5,7\}$ and $\{2,3,4,6\}$ implies that $X - Z$ contains 6 and 7. But neither $\{1,6,7\}$ nor $\{4,6,7\}$ is a triangle in M, by orthogonality with the same cocircuits. Therefore, if x is the element in $X - (Z \cup \{6,7\})$, then $\{1,6,x\}$ or $\{1,7,x\}$ is a triangle. In the first case, we have a contradiction to orthogonality with $\{2,3,4,6\}$. In the second, $\{1,7,x\} \triangle \{1,2,3\} = \{2,3,7,x\}$ is a circuit of N. As $\{6,7,x\}$ is a triad, we have contradicted the hypotheses of the lemma.

Now we can assume that if (X, Y) is a $(4,3)$-violator of M, then neither side of the separation contains $\{1,4\}$. Let (X, Y) be a $(4,3)$-violator of M, and assume that $1 \in X$ and $4 \in Y$. As $(X - Z, Y - Z)$ is not a $(4,3)$-violator of N, either $|X| = 4$ or $|Y| = 4$. By symmetry, we can assume the former. Assume that $X \neq \operatorname{fcl}(X)$. Let $X' \subseteq \operatorname{fcl}(X)$ be such that $|X'| = 5$, and $(X', E(M) - X')$ is a 3-separation in M. As $|E(M)| \geq 10$, it is certainly a $(4,3)$-violator of M. Therefore $4 \notin X'$, by our earlier conclusion. Then $(X' - 1, E(M) - (X' \cup 4))$ is a $(4,3)$-violator of N. As this is impossible, it follows that X is fully closed. If 1 is in a cocircuit in X, then this cocircuit contains 2 or 3. As X is closed, X contains the triangle $\{1,2,3\}$. Thus X is a 4-fan in M with $\{1,2,3\}$ as its triangle and $\{2,3,c\}$ as its triad. The triangle $\{3,4,5\}$ implies that $c \in \{4,5\}$. But $4 \in Y$, so $c = 5$. Hence N has $\{2,3,5\}$ and $\{2,3,6\}$ as cocircuits, and hence has a series pair. This contradiction shows 1 is not in a cocircuit in X, thus X is 4-fan $\{1, x_1, x_2, x_3\}$, where $\{1, x_1, x_2\}$ is a triangle. By orthogonality with the cocircuit $\{1,3,5,7\}$, without loss of generality, x_1 is in $\{3,5,7\}$. If x_1 is 3 or 5, then, by orthogonality of $\{x_1, x_2, x_3\}$ with triangle $\{3,4,5\}$ and the fact that X is closed, $\{3,4,5\} \subseteq X$, so $\{3,4,5\}$ is a triad and a triangle, a contradiction. Evidently, $\{1,7,x_2,x_3\}$ is a 4-fan, so $\{1,7,x_2\} \triangle \{1,2,3\}$ is 4-circuit $\{2,3,7,x_2\}$ and $\{7,x_2,x_3\}$ is a triad, contradicting the assumption of the lemma. □

Next we consider the case that (4)(i) holds in Theorem 1.2.

Lemma 4.9. *Let N be an internally 4-connected binary matroid with at least eight elements and with $\{1,2,3\}$ and $\{4,5,6\}$ as triangles and $\{1,2,a,4,5\}$ as a 5-cocircuit. Let M be the binary matroid obtained from N by adding the element b in parallel to a, and then coextending by the element c so that $\{1,2,b,c\}$ is a cocircuit. Then M is internally 4-connected.*

Proof. By construction and orthogonality, M is connected, and $\{1,2,3\},\{4,5,6\}$, and $\{a,b,c\}$ are triangles. Moreover, orthogonality tells us that $\{1,2,4,5,a,b\}$ is a cocircuit of M, so $\{4,5,a,c\} = \{1,2,4,5,a,b\} \triangle \{1,2,b,c\}$ is a disjoint union of cocircuits in M. The only 2-cocircuits in M must contain c, and M has no coloops, so $\{4,5,a,c\}$ is a cocircuit of M. As neither $\{a,c\}$ nor $\{b,c\}$ is a cocircuit of M, it follows, by Lemma 4.6, that M is 3-connected. We show first that

4.9.1. *no element in $\{a,b,c\}$ is in a triad of M.*

Suppose there is a triad T^* that meets $\{a,b,c\}$. Then $|T^* \cap \{a,b,c\}| = 2$. As $N = M/c\backslash b$ is cosimple, $c \in T^*$. Then, for some $C^* \in \{\{1,2,b,c\},\{a,c,4,5\}\}$, we see that $T^* \triangle C^*$ is a triad of N containing $1,2,4$, or 5, so N contains a 4-fan, a contradiction. Thus (4.9.1) holds.

Let (X,Y) be a $(4,3)$-violator of M. Suppose that $\{b,c\} \subseteq X$. As $\mathrm{fcl}(X)$ contains no set X' such that $(X', E(N)-X')$ is a $(4,3)$-violator of N, we may assume that $X = \mathrm{fcl}(X)$ and that $|X| = 4$ or $|X| = 5$. Thus $a \in X$. By (4.9.1), none of a, b, or c is in a triad. Hence X is not a fan so X consists of a quad Q and an element w in its closure where $\{a,b,c\} \subseteq Q \cup w$ and $w \in \{a,b,c\}$. As $Q \cup w$ is fully closed and does not contain $\{1,2,3\}$ or $\{4,5,6\}$, by orthogonality, it must avoid both these triangles. Since $\{a,b,c\}$ and $(Q-\{a,b,c\}) \cup w$ are both triangles, it follows that $w \neq c$, by orthogonality with the cocircuits $\{1,2,b,c\}$ and $\{4,5,a,c\}$. But Q must meet each of the cocircuits $\{1,2,b,c\}$ and $\{4,5,a,c\}$ in at least two elements, so Q contains $\{a,b,c\}$; a contradiction.

We may now assume that no $(4,3)$-violator of M has b and c on the same side. Let (X,Y) be a $(4,3)$-violator of M where $b \in X$ and $c \in Y$. If $a \in X$, then $(X \cup c, Y - c)$ is a 3-separation of M, but not a $(4,3)$-violator, while if $a \in Y$, then $(X - b, Y \cup b)$ is a 3-separation but not a $(4,3)$-violator. We deduce that M has a 4-fan F meeting $\{a,b,c\}$ such that $F \cap \{a,b,c\}$ contains a single element z, and z is either b or c. By $z \neq c$, for otherwise $N = M/c\backslash b$ contains a parallel pair. Thus $z = b$ and the triangle in F must contain 1 or 2 by orthogonality with $\{1,2,b,c\}$. Now the triangle $\{a,b,c\}$ means that b is not in $\mathrm{cl}^*_M(F-b)$. Therefore $F - b$ is a triad of N that meets the triangle $\{1,2,3\}$. Hence N has a 4-fan; a contradiction. $\square$

The next lemma corresponds to case (5)(i).

Lemma 4.10. *Let N be an internally 4-connected binary matroid with at least eight elements, such that $\{1,2,3\}$ and $\{2,5,7\}$ are triangles and $\{1,2,7,8\}$ is a cocircuit. Let M be the binary matroid obtained from N by adding the element 6 in parallel with 5, and then coextending by the element 4 so that $\{2,3,4,5\}$ is a cocircuit. Then M is internally 4-connected if and only if N has no 4-cocircuit containing $\{2,3,5\}$.*

Proof. Assume that N contains a 4-cocircuit $\{2,3,5,x\}$. By orthogonality with the circuit $\{5,6\}$, we see that $\{2,3,5,6,x\}$ is a cocircuit in $M/4$, and hence in M. Symmetric difference with $\{2,3,4,5\}$ shows that $\{4,6,x\}$ is a disjoint union of cocircuits in M. If M contains a cocircuit with fewer than three elements, it is certainly not internally 4-connected, so assume that $\{4, 6, x\}$ is a triad. Orthogonality with $\{2, 3, 4, 5\}$ tells us that $\{4, 5, 6\}$ is a triangle, so $\{x, 4, 5, 6\}$ is a 4-fan of M, and therefore M is not internally 4-connected.

To prove the "if" direction, we assume that N has no such 4-cocircuit. By construction and orthogonality, M is connected having $\{1, 2, 3\}, \{2, 5, 7\}$, and $\{4, 5, 6\}$ as triangles. As $N = M/4\backslash 6$ and $\{2, 3, 4, 5\}$ is a cocircuit of M, it follows by Lemma 4.6 that M is 3-connected provided $\{4, 6\}$ is not a cocircuit. In the exceptional case, $\{2, 3, 5, 6\}$ is a cocircuit of M so $\{2, 3, 5\}$ is a cocircuit of N. Thus N has a 4-fan; a contradiction. Hence M is indeed 3-connected.

We show next that

4.10.1. *no element in* $\{4, 5, 6\}$ *is in a triad of* M.

Suppose there is a triad T^* that meets $\{4, 5, 6\}$. As N is cosimple, $4 \in T^*$, so $T^* \cap \{4, 5, 6\}$ is $\{4, 5\}$ or $\{4, 6\}$. In the first case, $T^* \triangle \{2, 3, 4, 5\}$ is a disjoint union of cocircuits in N that meets the triangle $\{1, 2, 3\}$. Therefore N is not internally 4-connected. Thus $T^* \cap \{4, 5, 6\} = \{4, 6\}$, and $T^* \triangle \{2, 3, 4, 5\}$ is a cocircuit $\{2, 3, 5, 6, a\}$ of M. Hence $\{2, 3, 5, a\}$ is a cocircuit of N, contradicting our assumption. This completes the proof of (4.10.1).

Let (X, Y) be a $(4, 3)$-violator of M. Suppose $\{4, 6\} \subseteq X$. As $\mathrm{fcl}(X)$ contains no set X' such that $(X', E(N) - X')$ is a $(4, 3)$-violator of N, we can assume that $X = \mathrm{fcl}(X)$ and that $|X| = 4$ or $|X| = 5$. Thus $5 \in X$. As X contains a triangle, but none of $4, 5$, or 6 is in a triad of M, the set X is not a fan. Thus $X = Q \cup w$ where Q is a quad of M and $w \in \{4, 5, 6\}$. As w is contained in two triangles, namely $\{4, 5, 6\}$ and $(Q - \{4, 5, 6\}) \cup w$, we see that $w \neq 4$, or else $N = M/4\backslash 6$ contains a parallel pair. Therefore $4 \in Q$. The cocircuit $\{2, 3, 4, 5\}$ implies that $2, 3$, or 5 is in Q. By using orthogonality with the circuits $\{1, 2, 3\}$ and $\{2, 5, 7\}$ and the fact that $Q \cup w$ is fully closed, we get that $Q \cup w$ contains $\{1, 2, 3, 4, 5, 6, 7\}$; a contradiction.

We may now assume that $4 \in X$ and $6 \in Y$. If $5 \in X$, then $(X \cup 6, Y - 6)$ is a 3-separation of M, while if $5 \in Y$, then $(X - 4, X \cup 4)$ is a 3-separation of M. The previous paragraph implies that no $(4, 3)$-violator of M contains 4 and 6 in the same side, so we deduce that M has a 4-fan F meeting $\{4, 5, 6\}$ in some element z of $\{4, 6\}$ where z is in the triangle T of F but not its triad T^*. If $z = 4$, then $M/4\backslash 6$ is not simple; a contradiction. Hence $z = 6$ so we may suppose that $T = \{6, a, b\}$ and $T^* = \{a, b, c\}$. Then $\{6, a, b\} \triangle \{4, 5, 6\}$ is a circuit $\{4, 5, a, b\}$ of M. Thus N has $\{5, a, b\}$ as a triangle and $\{a, b, c\}$ as a triad; a contradiction. □

In the next lemma, we deal with cases (5)(ii) and (5)(iii) simultaneously.

Lemma 4.11. *Let* N *be an internally* 4*-connected binary matroid with at least eight elements, such that* $\{2, 5, 7\}$ *and* $\{4, 5, 6\}$ *are triangles and* $\{2, 3, 4, 5\}$ *is a cocircuit. Furthermore, assume that either*

(i) *N has $\{5,6,7,9\}$ as a cocircuit; or*
(ii) *N has $\{4,6,10,11\}$ as a cocircuit and $\{3,4,11\}$ as a triangle.*

Let M be the binary matroid obtained from N by extending with the element 1 so that $\{1,2,3\}$ is a triangle and then coextending by the element 8 so that $\{1,2,7,8\}$ is a cocircuit. Then M is internally 4-connected if and only if N has no 4-cocircuit $\{2,7,a,b\}$ such that either $\{a,b,c\}$ or $\{2,3,a\}$ is a triangle.

Proof. Assume that $\{2,7,a,b\}$ is a 4-cocircuit of N. By orthogonality with the triangle $\{1,2,3\}$, we see that $\{1,2,7,a,b\}$ is a cocircuit in M. Symmetric difference between $\{1,2,7,a,b\}$ and $\{1,2,7,8\}$ shows that $\{8,a,b\}$ is a triad in M. Now if $\{a,b,c\}$ is a triangle, then $\{8,a,b,c\}$ is a 4-fan. If $\{2,3,a\}$ is a triangle, then the triad $\{8,a,b\}$ contains 2 or 3, and again we see that M has a 4-fan. This proves the "only if" direction. Therefore we assume that N has no such 4-cocircuit.

By construction and orthogonality, M is connected having $\{1,2,3\},\{2,5,7\}$, and $\{4,5,6\}$ as triangles and $\{2,3,4,5\}$ as a cocircuit. Furthermore, M has $\{5,6,7,9\}$ as a cocircuit or M has $\{3,4,11\}$ as a triangle and $\{4,6,10,11\}$ as a cocircuit. Since $N = M\backslash 1/8$, it follows by Lemma 4.5 that M is 3-connected provided M has no 2-circuit containing 1 and no 2-cocircuit containing 8. Suppose M has a 2-cocircuit $\{8,z\}$. By orthogonality, $z \notin \{1,2,\ldots,7\}$. By taking symmetric differences, we deduce that M has $\{1,2,7,z\}$ as a cocircuit, so N has $\{2,7,z\}$ as a triad. Hence N has a 4-fan; a contradiction. If M has a 2-circuit $\{1,a\}$, then, by orthogonality, a is in $\{2,7,8\}$. But $\{1,2\}$ is not a 2-circuit, as it is contained in the triangle $\{1,2,3\}$. If $\{1,8\}$ is a parallel pair in M, then 1 is a loop of $M/8$ that is contained in the triangle $\{1,2,3\}$. Finally, if $\{1,7\}$ is a circuit in M, then $\{1,7\} \triangle \{1,2,3\} \triangle \{2,5,7\} = \{3,5\}$ is a union of circuits in N. Therefore 1 is in no parallel pair in M, so M is 3-connected.

Let (X,Y) be a $(4,3)$-violator of M. Assume that $\{1,8\} \subseteq X$. As $\mathrm{fcl}(X)$ contains no set X' such that $(X', E(N)-X')$ is a $(4,3)$-violator of N, we may assume that $X = \mathrm{fcl}(X)$ and that $|X|$ is four or five. Then $2,3$, and 7 are all contained in Y, or else $\{1,2,3,4,5,6,7\} \subseteq X$, a contradiction.

Assume that $|X| = 4$. Note that $1 \in \mathrm{cl}_M(Y)$, because of the circuit $\{1,2,3\}$. This means that X is not a quad of M. Therefore X is a 4-fan, where 1 is in the triangle of X, but not the triad. Since 1 is in the triangle $\{1,2,3\}$ of $M/8$, it follows that $\{1,8\}$ is not contained in a triangle. Therefore 8 is in the triad of X, but not the triangle. Now the symmetric difference of $X-1$ and $\{1,2,7,8\}$ is a disjoint union of cocircuits in M that contains the triangle in X. This is impossible, so $|X| = 5$.

Since $1 \in \mathrm{cl}_M(Y)$, it follows that either X is a 5-fan, or a quad with a single element in its closure. The second case cannot happen, since $8 \in \mathrm{cl}^*_M(Y \cup 1)$ because of the cocircuit $\{1,2,7,8\}$. Therefore X is a 5-fan, and $X-1$ contains a single triangle. But 8 cannot be in this triangle because of orthogonality with $\{1,2,7,8\}$. This means that 1 and 8 are contained in a triangle of X, which means that 1 is a loop of $M/8$ that is contained in a triangle.

Because of this contradiction, we can now assume that whenever (X, Y) is a $(4,3)$-violator of M, neither X nor Y contains $\{1, 8\}$. Let (X, Y) be a $(4,3)$-violator, and assume that $1 \in X$ and $8 \in Y$. As $(X - 1, Y - 8)$ is a 3-separation of N, either $|X| = 4$ or $|Y| = 4$.

First assume that $|X| = 4$. If $X \neq \mathrm{fcl}(X)$, then there is a subset X' such that $X \subseteq X' \subseteq \mathrm{fcl}(X)$, where $|X'| = 5$ and $(X', E(M) - X')$ is a 3-separation in M. Since $|E(N)| \geq 8$ implies $|E(M)| \geq 10$, it follows that $(X', E(M) - X')$ is a $(4,3)$-violator of M, so $8 \notin X'$. Therefore $(X' - 1, E(M) - (X' \cup 8))$ is a $(4,3)$-violator of N, and we have a contradiction. Thus X is fully closed.

If 2 or 3 is in X, then both are in X, so X is a 4-fan, and two elements of $\{1, 2, 3\}$ are in a triad of M. As N is cosimple, 1 is not in this triad. Thus $\{2, 3\}$ is contained in a triad of N. But 2 is in a triangle of N, so N has a 4-fan; a contradiction. We may assume then that $\{2, 3\} \subseteq Y$. By orthogonality with $\{1, 2, 3\}$, the element 1 is not in a cocircuit in X. Thus X is a 4-fan with $\{1, a, b\}$ as its triangle and $\{a, b, c\}$ as its triad. Orthogonality with $\{1, 2, 7, 8\}$ implies $7 \in \{a, b\}$. We assume, without loss of generality, that $7 = a$. Then the triad $\{7, b, c\}$ contains 5, by orthogonality with $\{2, 5, 7\}$. Then $\{1, 5, 7\} \subseteq X$, so, as X is fully-closed, X contains 2, a contradiction.

Therefore we now assume that $|Y| = 4$. As in the previous paragraph, we can argue that Y is fully closed. If Y is a quad, then, by orthogonality with $\{1, 2, 7, 8\}$, we know that Y contains 2 or 7. Then, by orthogonality with $\{2, 5, 7\}$ and the fact that Y is closed, we deduce that Y contains the triangle $\{2, 5, 7\}$; a contradiction. Hence Y is a 4-fan. Assume 8 is in the triangle of X. By orthogonality with the cocircuit $\{1, 2, 7, 8\}$, we see that this triangle contains either 2 or 7. But then $N = M/8\backslash 1$ contains a parallel pair. Therefore 8 is in the triad of X, but not the triangle. Label X so that $X = \{8, a, b, c\}$, where $\{8, a, b\}$ is a triad, and $\{a, b, c\}$ is a triangle. Note that $\{2, 7\} \cap \{a, b\} = \emptyset$, since otherwise, by orthogonality, and the fact that Y is fully closed, we deduce that $\{2, 5, 7\} \subseteq Y$. In this case $\{2, 7, 8\}$ must be a triad, which contradicts the fact that $\{1, 2, 7, 8\}$ is a 4-cocircuit. Thus $\{1, 2, 7, a, b\} = \{1, 2, 7, 8\} \triangle \{8, a, b\}$ is a 5-cocircuit of M, and $\{2, 7, a, b\}$ is a 4-cocircuit in N. As $\{a, b, c\}$ is a triangle of N, this contradicts our assumption. $\square$

5. Proof of the main theorem

Proof of Theorem 1.4. Let $\mathcal{M}$ be a minor-closed class of binary matroids that contains at least one internally 4-connected matroid with at least six elements. Define $\mathcal{M}^{(6)}$ to be $\{M(K_4)\}$, and assume that $\mathcal{M}^{(6)}, \mathcal{M}^{(7)}, \mathcal{M}^{(8)}, \ldots$ are constructed as in the statement of the theorem. An obvious inductive argument shows that the members of $\mathcal{M}^{(i)}$ all have i elements, for every $i \geq 6$.

First let us assume that the matroid M is contained in some set $\mathcal{M}^{(i)}$, where $i \geq 6$. Then M is contained in $\mathcal{M}$, by construction and the fact that every internally 4-connected binary matroid with at least six elements has an $M(K_4)$-minor. Therefore we must show that M is internally 4-connected. If $i = 6$, then $M = M(K_4)$, so M is certainly internally 4-connected. Hence we assume that $i > 6$. Up to duality, there is a matroid N such that one of the statements (i)–(iv) in Theorem 1.4 holds. If M is, up to duality, the cycle matroid of K_5, a quartic ladder, the cube, or a terrahawk, then M is certainly internally 4-connected. Therefore we will assume that (iii) or (iv) holds. If (iii) holds and $i > 7$, then M is internally 4-connected by Lemma 2.7. If (iii) holds, and i is equal to 7, then M is a simple and cosimple single-element extension of $M(K_4)$ and is therefore 3-connected. Any 3-connected matroid with seven elements is also internally 4-connected, so in this case we are done. Therefore we assume that (iv) holds. If $M = M(K_{3,3})$ or $M^* = M(K_{3,3})$, then M is certainly internally 4-connected, so we assume this is not the case. Then M and N are as described in, respectively, (I), (II), (III), (IV)(i), (IV)(ii), (V)(i), (V)(ii), (VI), or (VII). In these cases M is internally 4-connected by, respectively, Lemmas 4.7, 4.8, 4.1, 4.9, the dual of 4.2, 4.10, 4.11, 4.3, or 4.4. This shows that $\mathcal{M}^{(i)}$ is contained in the set of i-element internally 4-connected matroids that belong to $\mathcal{M}$.

For the converse, assume that $M \in \mathcal{M}$ is internally 4-connected and $|E(M)| = i \geq 6$, but that M is not contained in $\mathcal{M}^{(i)}$. Assume that M has been chosen so that i is as small as possible subject to these conditions. If $|E(M)| = 6$, then M is isomorphic to $M(K_4)$, and M is contained in $\mathcal{M}^{(6)}$. Therefore $|E(M)| \geq 7$. Assume that $M\backslash e$ is internally 4-connected for some $e \in E(M)$. Then our assumption on i means that $M\backslash e$ is contained in $\mathcal{M}^{(i-1)}$. But M is a simple single-element extension of $M\backslash e$, and $r(M) = r(M\backslash e)$. Moreover, if $i > 7$, then Lemma 2.7 implies that there is no triad of $M\backslash e$ that contains e in its closure in M. Therefore statement (iii) in Theorem 1.4 applies, and M is in $\mathcal{M}^{(i)}$. This contradiction means that no single-element deletion of M is internally 4-connected. The dual argument shows that no single-element contraction of M is internally 4-connected. Therefore we can apply Theorem 1.2.

By Theorem 1.2, M has a proper minor N such that N is internally 4-connected. Our assumption on the minimality of i means that $N \in \mathcal{M}^{(i-k)}$, where $k = |E(M)| - |E(N)|$. If, up to duality, $M = M(K_{3,3})$ and $N = M(K_4)$, then M is in $\mathcal{M}^{(9)}$, by statement (iv) of Theorem 1.4. Similarly, if M is $M(K_5)$ or the cycle matroid of the cube, and $N = M(K_4)$, then M is in $\mathcal{M}^{(10)}$ or $\mathcal{M}^{(12)}$. If, up to duality, M is the cycle matroid of (respectively) a planar quartic ladder, a Möbius quartic ladder, or a terrahawk, and N is the cycle matroid of (respectively) a planar quartic ladder, a Möbius quartic ladder, or the cube, then statement (ii) in Theorem 1.4 holds, and M is contained in $\mathcal{M}^{(i)}$. Therefore neither (8) nor (9) holds in Theorem 1.2.

If $|E(M)| \leq 11$, then Theorem 1.2 states that, up to duality, M is isomorphic to $M(K_5)$ or $M(K_{3,3})$, and that (8) applies. Therefore we must assume that $|E(M)| \geq 12$, and therefore $|E(N)| \geq 9$. Assume that (1) holds in Theorem 1.2. Then M has an $M(K_4)$-restriction on the set $\{1,2,3,4,5,6\}$, where $\{1,2,3\}$, $\{1,5,6\}$, $\{2,4,6\}$, and $\{3,4,5\}$ are triangles, and $\{1,3,5,7\}$ and $\{2,3,4,8\}$ are cocircuits. Since $N = M\backslash 3,6$, we see that $\{1,2,4,5\}$ is a circuit of N, and $\{1,5,7\}$ and $\{2,4,8\}$ are triads. Moreover, $N \in \mathcal{M}^{(i-2)}$. Now M is obtained from N by extending with 3 and 6 so that $\{3,4,5\}$ and $\{2,4,6\}$ are triangles. Lemma 4.7 implies that there is no triad $\{a,b,c\}$ in N such that $\{1,2,a,b\}$ or $\{2,4,a,b\}$ is a cocircuit, or else M would not be internally 4-connected. Now M is in $\mathcal{M}^{(i)}$, as M and N are as described in (I).

Arguing in exactly the same way, we see that if M and N are as described in, respectively, (2), (3), (4)(i), (4)(ii), (5)(i), (5)(ii), (5)(iii), (6), or (7), then the hypotheses of, respectively, Lemmas 4.8, 4.1, 4.9, the dual of 4.2, 4.10, 4.11, 4.11, 4.3, or 4.4 hold. Therefore M and N must be as described in, respectively, (II), (III), (IV)(i), (IV)(ii), (V)(i), (V)(ii), (V)(ii), (VI), or (VII). In any case, M is contained in $\mathcal{M}^{(i)}$. This contradiction completes the proof of Theorem 1.4. $\square$

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