COLOURING, PACKING AND THE CRITICAL PROBLEM

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Introduction

SECTION 1 of this paper considers a matroid conjecture of Welsh [**16**] which is based on Gallai's theorem on the stability and covering numbers of graphs [**6**].

The chromatic number $\chi(M)$ of a loopless matroid M is the least positive integer at which the characteristic polynomial of M is positive [**15**]. In Section 2 we show that for all matroids without loops, $\chi(M)-1$ does not exceed the maximum size of a cocircuit of M. For regular matroids sharper bounds on χ are proved which resemble the bounds of Brooks [**2**] and Szekeres/Wilf [**13**] for graphs. One such bound improves on a result of Lindström [**10**]. A key lemma of this section is used in Section 3 to generalize another result of Lindström [**10**] by giving an upper bound on the critical exponent of a matroid representable over $GF(q)$.

Brylawski [**3**] and Heron [**8**] showed independently that a binary matroid is affine if and only if it is a disjoint union of cocircuits. We show that if M is representable over $GF(q)$ and M is a disjoint union of cocircuits, then M is affine. The converse is only true when $q=2$.

The terminology used here for matroids and graphs will in general follow Welsh [**15**]. In any unexplained context M will denote an arbitrary matroid having rank function ρ and ground set S. We shall sometimes denote the restriction of M to $S \setminus T$ by $M \setminus T$ or, if $T=\{x_1, x_2, \ldots, x_m\}$, by $M \setminus x_1, x_2, \ldots, x_m$. Likewise the contraction of M to $S \setminus T$ will sometimes be written M/T or $M/x_1, x_2, \ldots, x_m$. The simple matroid associated with a matroid M will be denoted by $\bar{M}$ and $\mathscr{C}^*(M)$ will denote the set of cocircuits of M.

If r is a non-negative real number, then $[r]$ and $\{r\}$ will denote respectively the greatest integer not exceeding r and the least integer not less than r. The set of integers, the set of positive integers and the set of positive real numbers will be denoted by $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{R}^+$ respectively.

1. Cocircuit coverings and packings

Throughout this section all graphs and matroids considered have no loops.

If G is a graph, denote by $\beta_v(G)$ the maximum size of a set of mutually non-adjacent vertices of G and let $\alpha_v(G)$ be the minimum size of a set U of vertices such that every edge of G has at least one endpoint in U.

A well-known result of Gallai [6] states:

(1.1) If G has n vertices, then $\alpha_v(G)+\beta_v(G)=n$.

In a 2-connected graph G the set of edges incident with a particular vertex is a cocircuit in the cycle matroid of G. For an arbitrary matroid M on a set S we define $\beta(M)$ to be the maximum size of a set of pairwise disjoint cocircuits of M and $\alpha(M)$ to be the minimum size of a set of cocircuits of M whose union is S. Clearly if M is the cycle matroid of a 2-connected graph G, then $\alpha(M)\leq\alpha_v(G)$ while $\beta(M)\geq\beta_v(G)$. Moreover, one can easily find examples in which strict inequality holds in both of these statements.

Welsh [16] has made the following conjecture based on (1.1).

(1.2) If M is an arbitrary connected matroid, then $\alpha(M)+\beta(M)\leq \rho(M)+1$; or equivalently:

(1.3) If M has k components, then $\alpha(M)+\beta(M)\leq\rho(M)+k$.

It is routine to check that (1.3) holds for uniform matroids, projective and affine geometries and complete graphs and their duals as well as for all matroids of rank less than six.

The following assertion is well-known (see, for example, [15, p. 37]).

(1.4) A circuit and a cocircuit of a matroid cannot have exactly one common element.

A consequence of this is that if $\bar{M}$ is the simple matroid associated with an arbitrary matroid M, then $\alpha(M)=\alpha(\bar{M})$ and $\beta(M)=\beta(\bar{M})$.

It follows immediately from the definition of α that for all matroids M

(1.5) $\alpha(M)=\min\{k\in\mathbf{Z}^+\colon M$ has hyperplanes $H_1, H_2, \ldots, H_k$ such that $\bigcap_{i=1}^{k} H_i=\emptyset\}$.

Therefore

(1.6) $\alpha(M)\leq\rho(M)$.

Furthermore, from [8, p. 42], we have:

(1.7) If M is a simple connected matroid, then $\alpha(M)=\rho(M)$ if and only if M has rank less than three or M is a projective geometry.

The next two statements are easily checked.

(1.8) For all matroids M, $\beta(M)\leq\rho(M)$.

(1.9) If M is simple and connected, then

(i) $\beta(M)=\rho(M)$ if and only if M is a coloop; and

(ii) if $\beta(M)=\rho(M)-1$, then $\alpha(M)=2$.

Combining (1.6) and (1.8) we get trivially:

(1.10) For all matroids M, $\alpha(M)+\beta(M)\leq 2\rho(M)$.

A corollary of the next result sharpens this bound.

(1.11) Proposition. *If M is a simple matroid having no coloops, then*

$$\left\{\frac{\alpha(M)}{2}\right\}+\beta(M)\leq\rho(M).$$

Proof. Let $\{C_1^*, C_2^*, \ldots, C_\beta^*\}$ be a maximal set of pairwise disjoint cocircuits of M. Then since M has no coloops, for all i in $\{1, 2, \ldots, \beta\}$, we may choose distinct elements x_i and y_i from C_i^*. Let $X=\{x_1, x_2, \ldots, x_\beta\}$ and $Y=\{y_1, y_2, \ldots, y_\beta\}$. Then by (1.4), X and Y are independent flats of M. Thus M has $\rho-\beta$ hyperplanes whose intersection is X and $\rho-\beta$ hyperplanes whose intersection is Y. But X and Y are disjoint and so $2(\rho-\beta)\geqslant\alpha$.

(1.12) COROLLARY. *If M is simple and connected and $\rho(M)>2$, then*

$$\alpha(M)+\beta(M)\leqslant\left[\frac{3\rho(M)-1}{2}\right].$$

For binary matroids we argue in terms of the cocircuit space to prove the following:

(1.13) PROPOSITION. *Let M be a binary connected matroid, then*

$$\alpha(M)+\left[\frac{\beta(M)}{2}\right]\leqslant\rho(M).$$

Proof. Let $\{C_1^*, C_2^*, \ldots, C_\beta^*\}$ be a maximal set of pairwise disjoint cocircuits of M. This set may be extended to a basis $\{C_1^*, C_2^*, \ldots, C_\beta^*, D_1^*, D_2^*, \ldots, D_m^*\}$ of the cocircuit space of M where $\{D_1^*, D_2^*, \ldots, D_m^*\}\subseteq\mathscr{C}^*(M)$. Now $\beta+m=\rho(M)$ and we may suppose that $m\geqslant 1$ since otherwise, by (1.9)(i), $\beta(M)=1$ and the proposition holds by (1.6).

Clearly $S=\left(\bigcup_{i=1}^{\beta} C_i^*\right)\cup\left(\bigcup_{j=1}^{m} D_j^*\right)$. If $\bigcup_{j=1}^{m} D_j^*=S$, then the proposition holds. Thus assume that $S\backslash\left(\bigcup_{j=1}^{m} D_j^*\right)$ is non-empty, intersecting each of $C_1^*, C_2^*, \ldots, C_n^*$ but not intersecting $\bigcup_{i=n+1}^{\beta} C_i^*$. For each i in $\{1, 2, \ldots, n\}$, choose x_i from $C_i^*\cap\left(S\backslash\left(\bigcup_{j=1}^{m} D_j^*\right)\right)$. If $n=1$, then $C_1^*, D_1^*, D_2^*, \ldots, D_m^*$ cover S and the proposition holds. Otherwise, since M is connected, there is a cocircuit $E_{1,2}^*$ of M containing x_1 and x_2. Now $\{E_{1,2}^*, C_2^*, \ldots, C_\beta^*, D_1^*, D_2^*, \ldots, D_m^*\}$ is a basis of the cocircuit space of M and therefore $E_{1,2}^*\cup\left(\bigcup_{j=1}^{m} D_j^*\right)\supseteq C_1^*$. Likewise, $\{C_1^*, E_{1,2}^*, C_3^*, \ldots, C_\beta^*, D_1^*, D_2^*, \ldots, D_m^*\}$ is a basis of the cocircuit space of M and hence $E_{1,2}^*\cup\left(\bigcup_{j=1}^{m} D_j^*\right)\supseteq C_2^*$. A similar argument shows that for all k such that $1\leqslant k\leqslant[n/2]$, there is a cocircuit $E_{2k-1,2k}^*$ such that $E_{2k-1,2k}^*\cup\left(\bigcup_{j=1}^{m} D_j^*\right)\supseteq C_{2k-1}^*\cup C_{2k}^*$. We conclude that $m+\{n/2\}\geqslant\alpha(M)$ and hence that $\alpha(M)+\left[\frac{\beta(M)}{2}\right]\leqslant\rho(M)$.

(1.14) COROLLARY. *Let M be a binary connected matroid of rank greater than one. Then*

$$\alpha(M)+\beta(M)\leq\left[\frac{4\rho(M)+1}{3}\right].$$

Proof. Combine Propositions 1.11 and 1.13.

The next two results give bounds on α and β in terms of rank, maximum circuit size and maximum cocircuit size. For an arbitrary loopless matroid M, let $\Delta(M)=\max\{|C^*|:\ C^*\in\mathscr{C}^*(M)\}$, and let

$$\Delta^*(M)=\begin{cases}0, \text{ if } M \text{ is free;}\\ \Delta(M^*), \text{ otherwise.}\end{cases}$$

(1.15) PROPOSITION. $\left\{\frac{\rho(M)}{\Delta(M)}\right\}\leq\beta(M)\leq\rho(M)+1-\left\{\frac{\Delta^*(M)}{2}\right\}.$

Proof. Let $\{C_1^*, C_2^*, \ldots, C_\beta^*\}$ be a maximal set of pairwise disjoint cocircuits of M. Then $\rho\left(\bigcup_{i=1}^{\beta} C_i^*\right)=\rho(M)$ and so $\sum_{i=1}^{\beta}|C_i^*|\geq\rho(M)$. But $|C_i^*|\leq\Delta(M)$ for all $1\leq i\leq\beta$. Thus $\beta(M)\Delta(M)\geq\rho(M)$. This gives the lower bound on β.

The upper bound is certainly satisfied if $\Delta^*(M)=0$. If $\Delta^*(M)>0$, then let C be a circuit of M of maximum size. Add $\rho(M)-\Delta^*(M)+1$ elements to C to get a subset B' of S which contains a base of M. Clearly B' intersects each of $C_1^*, C_2^*, \ldots, C_\beta^*$. Moreover, if $B'\cap C_i^*$ intersects C, then by (1.4), it contains at least two elements of C. Therefore $\Delta^*(M)=|C|\geq\sum_{i=1}^{\beta}|C_i^*\cap C|\geq 2\,|\{i:\ 1\leq i\leq\beta \text{ and } C_i^*\cap C\neq\varnothing\}|$. Thus $\Delta^*(M)\geq 2(\beta(M)-(\rho(M)-\Delta^*(M)+1))$ and the upper bound on β follows.

One lower bound on α comes from the obvious relation $\alpha(M)\Delta(M)\geq|S|$. Another bound which is sometimes better and sometimes worse than this is:

(1.16) PROPOSITION. $\alpha(M)\geq\max_{\varnothing\neq A\subseteq S}\left\{\frac{|A|}{\rho^*(A)+1}\right\}.$

Proof. Consider the hypergraph H having S as its set of vertices and $\mathscr{C}^*(M)$ as its set of edges. The covering number [**1**, p. 448] of this hypergraph is precisely $\alpha(M)$ and the result follows from [**1**, Theorem 1, p. 449].

Another upper bound on $\beta(M)$ may also be obtained from [**1**, p. 449] by noting that $\beta(M)$ is precisely the strong stability number [**1**, p. 448] of the dual hypergraph of H.

2. Colouring

By analogy with graphs, Welsh [**15**, p. 262] has defined the *chromatic number* $\chi(M)$ of a loopless matroid M by $\chi(M) = \min\{j \in \mathbf{Z}^+ : P(M; j) > 0\}$. For graphic matroids $\chi(M)$ clearly equals $\pi(M)$ where $\pi(M) = \min\{j \in \mathbf{Z}^+ : P(M; j+k) > 0 \text{ for all } k = 0, 1, 2, \ldots\}$. However, in general, π may exceed χ (see [**15**, p. 264]). Indeed given any positive integer n, the binary projective geometry $PG(n+2, 2) = M$ satisfies $\pi(M) - \chi(M) > n$ (see [**8**, p. 99]).

The next result gives an upper bound on $\pi(M)$ which is always at least as good as a bound of Heron [**7**, Lemma 3.15]. For an arbitrary loopless matroid M, let $\mu(M) = \max\{r \in \mathbf{R}^+ : P(M; r) = 0\}$.

(2.1) THEOREM. *If M is a loopless matroid, then* $\mu(M) \leqslant \max_{C^* \in \mathscr{C}^*(\bar{M})} |C^*|$ *and so*

$$\pi(M) \leqslant 1 + \max_{C^* \in \mathscr{C}^*(\bar{M})} |C^*| \leqslant 1 + \max_{C^* \in \mathscr{C}^*(M)} |C^*|.$$

The main part of the proof of this theorem is contained in Lemmas 2.6 and 2.7 which use the following four basic properties of the characteristic polynomial of a matroid M (see [**15**, p. 263]). Let e be an element of M.

(2.2) If M has a loop, then $P(M; \lambda) = 0$.

(2.3) If e is a coloop of M, then $P(M; \lambda) = (\lambda - 1)P(M \backslash e; \lambda)$.

(2.4) If e is neither a loop nor a coloop of M, then $P(M; \lambda) = P(M \backslash e; \lambda) - P(M/e; \lambda)$.

Note that if e is a loop of M, then $M \backslash e \cong M/e$. Hence combining (2.2) and (2.4) we get:

(2.5) If e is not a coloop of M, then $P(M; \lambda) = P(M \backslash e; \lambda) - P(M/e; \lambda)$.

(2.6) LEMMA. *Let $\{x_1, x_2, \ldots, x_m\}$ be a coindependent set in a matroid M. Then*

$$\begin{aligned} P(M; \lambda) = {} & P(M \backslash x_1, x_2, \ldots, x_m; \lambda) \\ & + \sum_{j=2}^{m} \sum_{i=1}^{j-1} P(M \backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}/x_i, x_j; \lambda) \\ & - \sum_{i=1}^{m} P(M \backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m/x_i; \lambda). \end{aligned}$$

Proof. By induction on m. The result is immediate for $m = 0$. Now assume that the proposition is true for $m-1$. Then as x_m is not a coloop of $M \backslash x_1, \ldots, x_{m-1}$, $P(M \backslash x_1, \ldots, x_{m-1}; \lambda) = P(M \backslash x_1, \ldots, x_{m-1}, x_m; \lambda) - P(M \backslash x_1, \ldots, x_{m-1}/x_m; \lambda)$ by (2.5). Moreover if $1 \leqslant i \leqslant m-1$, then x_m is not a coloop of $M \backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m-1}/x_i$

and therefore

$$\begin{aligned}P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m-1}/x_i; \lambda) &= P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m-1}, x_m/x_i; \lambda)\\ &\quad - P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m-1}/x_i, x_m; \lambda).\end{aligned}$$

It follows by induction that the proposition is true for all non-negative integers m.

(2.7) LEMMA. *Let $\{x_1, x_2, \ldots, x_k\}$ be a cocircuit of a matroid M. Then*

$$\begin{aligned}P(M; \lambda) &= (\lambda - k)P(M\backslash x_1, \ldots, x_k; \lambda)\\ &\quad + \sum_{j=2}^{k}\sum_{i=1}^{j-1} P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}/x_i, x_j; \lambda).\end{aligned}$$

Proof. Taking $m = k-1$ in Lemma 2.6 we note that x_k is a coloop of $M\backslash x_1, \ldots, x_{k-1}$ and so $P(M\backslash x_1, \ldots, x_{k-1}; \lambda) = (\lambda - 1)P(M\backslash x_1, \ldots, x_{k-1}, x_k; \lambda)$. Furthermore x_k is not a coloop of $M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}/x_i$ and so

$$\begin{aligned}P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}/x_i; \lambda) &= P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k/x_i; \lambda)\\ &\quad - P(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}/x_i, x_k; \lambda).\end{aligned}$$

To complete the proof, notice that x_i is a coloop of $N = M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$. Therefore $N/x_i \cong N\backslash x_i$ and so the characteristic polynomials of these two matroids are equal.

Proof of Theorem 2.1. As $\mu(M) = \mu(\bar{M})$, to prove this theorem it suffices to show that for all loopless matroids M, $\mu(M) \leq \max_{C^* \in \mathscr{C}^*(M)} |C^*|$. We prove this proposition by induction on $|S|$. If $|S| = 1$, then the result is immediate. Assume now that the proposition holds for all matroids on sets of fewer than n elements and let M be a loopless matroid on a set of size n. Let $\{x_1, x_2, \ldots, x_k\}$ be a cocircuit of M. Then we can suppose $k < n$. By (2.2) and Lemma 2.7,

$$\textbf{(2.8)}\quad \begin{aligned}\mu(M) \leq \max\{&k, \mu(M\backslash x_1, x_2, \ldots, x_k),\\ &\mu'(M\backslash x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}/x_i, x_j)\\ &\qquad (2 \leq j \leq k, 1 \leq i \leq j-1)\}\end{aligned}$$

where

$$\mu'(N) = \begin{cases}\mu(N), & \text{if } N \text{ has no loops;}\\ 0, & \text{otherwise.}\end{cases}$$

Now $k \leq \max_{C^* \in \mathscr{C}^*(M)} |C^*|$, and, by the induction assumption, all the other terms on the right-hand side of (2.8) are bounded above by $\max_{C^* \in \mathscr{C}^*(M)} |C^*|$. Therefore this number is an upper bound on $\mu(M)$. The required result follows by induction.

For regular matroids Lemma 2.7 may be used again to sharpen the bound given in Theorem 2.1. The set of simple restrictions of a matroid N will be denoted by $\mathscr{R}(N)$.

(2.9) THEOREM. *If M is a regular loopless matroid, then*

$$\chi(M) = \pi(M) \leqslant 1 + \max_{N \in \mathscr{R}(M)} \left(\min_{C^* \in \mathscr{C}^*(N)} |C^*| \right).$$

Proof. For all positive integers n, let $\mathbf{Z}_n$ denote the ring of integers modulo n. By [**4**, Theorem III] (see also [**3**, Theorem 12.4]), if N is a regular matroid, then

(2.10) $P(N; n)$ equals the number of nowhere-zero $\mathbf{Z}_n$-coboundaries on N.

Therefore $P(N; n) \geqslant 0$. The upper bound for $\pi(M)$ now follows easily by induction using Lemma 2.7.

To verify that $\chi(M) = \pi(M)$ we show that, since M is regular, if k is a positive integer and $P(M; k) > 0$, then $P(M; k+1) > 0$.

An easy consequence of a result of Tutte [**14**, 5.44] is that the following statements are equivalent for a regular matroid N (see [**9**, Proposition 1]):

(a) There is a nowhere-zero $\mathbf{Z}_n$-coboundary on N.

(b) There is a nowhere-zero $\mathbf{Z}$-coboundary on N with all values in $[1-n, n-1]$.

Using this equivalence and (2.10) the rest of the proof is straightforward.

This theorem generalizes Matula's upper bound [**11**, Theorem 14] on the chromatic number of a graph—a bound which sharpens that given by Szekeres and Wilf [**13**]. Another consequence of Theorem 2.9 is the following result of Lindström [**10**, Theorem 17].

(2.11) COROLLARY. *If M is a loopless regular matroid, then*

$$\chi(M) \leqslant 1 + \max_{p \in S} \left(\min_{p \in C^* \in \mathscr{C}^*(M)} |C^*| \right).$$

Proof. Let $\max_{N \in \mathscr{R}(M)} \left(\min_{C^* \in \mathscr{C}^*(N)} |C^*| \right) = \min_{C^* \in \mathscr{C}^*(M|T)} |C^*| = |C_N^*|$ where $C_N^* \in \mathscr{C}^*(M \mid T)$. Choose x in C_N^*. If $C_0^* \in \mathscr{C}^*(M)$ and $x \in C_0^*$, then $C_0^* \supseteq C_1^*$ where $C_1^* \in \mathscr{C}^*(M \mid T)$. Thus $|C_0^*| \geqslant |C_1^*| \geqslant |c_N^*|$. It follows that $|C_N^*| \leqslant \min_{x \in C^* \in \mathscr{C}^*(M)} |C^*| \leqslant \max_{p \in S} \left(\min_{p \in C^* \in \mathscr{C}^*(M)} |C^*| \right)$.

Theorem 2.9 is used again to prove the following analogue of Brooks' Theorem [**2**].

(2.12) THEOREM. *If M is a simple connected regular matroid, then $\pi(M) \leq \max_{C^* \in \mathscr{C}^*(M)} |C^*|$ unless M is an odd circuit or a coloop.*

Proof. By Theorem 2.1, $\pi(M) \leq 1 + \max_{C^* \in \mathscr{C}^*(M)} |C^*| = 1 + m$, say. Now suppose that M is simple, connected and regular and that $\pi(M) = 1 + m$. Then by Theorem 2.9, $\max_{N \in \mathscr{R}(M)} \left(\min_{C^* \in \mathscr{C}^*(N)} |C^*| \right) = m$. Thus for some $T \subseteq S$, all the cocircuits of $M | T$ have cardinality equal to $\max_{C^* \in \mathscr{C}^*(M)} |C^*|$. Therefore, no cocircuit of M intersects both T and $S \setminus T$. Thus, as M is connected, $S = T$ and so the cocircuits of M are equicardinal.

Murty [**12**] has shown that a simple connected binary matroid having equicardinal cocircuits is either a coloop, a circuit or a binary projective or affine space. From this it is easy to show that M is either an odd circuit or a coloop as required.

3. The critical problem

If A is a subset of $V(n, q)$, the n-dimensional vector space over $GF(q)$, then a k-tuple $(f_1, f_2, \ldots, f_k)$ of linear functionals on $V(n, q)$ is said to *distinguish* A if for all e in A, $f_i(e) \neq 0$ for some i in $\{1, 2, \ldots, k\}$. Let M be a rank n matroid on a set S and suppose that M is representable over $GF(q)$.

(3.1) THEOREM. (Crapo and Rota [**5**, p. 16.4]). *If $k \in \mathbf{Z}^+$ and φ is a representation of M in $V(n, q)$, then the number of k-tuples of linear functionals on $V(n, q)$ which distinguish $\varphi(S)$ equals $P(M; q^k)$.*

Thus for a matroid M representable over $GF(q)$

(3.2) $$P(M; q^k) \geq 0 \text{ for all } k \text{ in } \mathbf{Z}^+.$$

The *critical exponent* $c(M, q)$ of M is defined by

(3.3) $$c(M, q) = \begin{cases} \infty, \text{ if } M \text{ has a loop;} \\ \min\{j \in \mathbf{Z}^+ : P(M; q^j) > 0\}, \text{ otherwise.} \end{cases}$$

Thus if M has no loops,

(3.4) $$P(M; q^k) > 0 \text{ for } k = c(M, q), c(M, q) + 1, \ldots .$$

The critical exponent has the following alternative interpretation. If M is a rank n, loopless matroid representable over $GF(q)$ and φ is a representation of M in $V(n, q)$, then $c(M, q)$ is the least number k of hyperplanes $H_1, H_2, \ldots, H_k$ of $V(n, q)$ such that $\left(\bigcap_{i=1}^{k} H_i \right) \cap \varphi(S) = \varnothing$.

(3.5) THEOREM. *If M is representable over $GF(q)$ and M has no loops, then*

$$c(M, q) \leqslant \left\{\log_q \left(1 + \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\}.$$

Proof. By induction on $|S|$. The result is true for $|S| = 1$. Assume it true for all matroids on sets of fewer than n elements and let M be a matroid on a set of size n. If M is not simple, then, since $c(M, q) = c(\bar{M}, q)$ and $\mathcal{R}(M) = \mathcal{R}(\bar{M})$, the result follows by the induction assumption. Thus suppose that M is simple and let $\{x_1, x_2, \ldots, x_k\}$ be a cocircuit of M of minimal size. Then by Lemma 2.7, (3.3) and (3.4), $c(M; q) \leqslant \max\{\{\log_q (k+1)\}, c(M \backslash x_1, x_2, \ldots, x_k, q)\}$. But $k = \min_{C^* \in \mathcal{C}^*(M)} |C^*|$, thus $k \leqslant \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)$. Moreover, by the induction assumption,

$$\begin{aligned} c(M \backslash x_1, \ldots, x_k, q) &\leqslant \left\{\log_q \left(1 + \max_{N \in \mathcal{R}(M \backslash x_1, \ldots, x_k)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\} \\ &\leqslant \left\{\log_q \left(1 + \max_{N \in \mathcal{R}(M)} \left(\min_{C^* \in \mathcal{C}^*(N)} |C^*|\right)\right)\right\}. \end{aligned}$$

The required result now follows by induction.

(3.6) COROLLARY. *Suppose that M is a matroid representable over $GF(q)$. If there is a covering of S with cocircuits each of size less than q^k, then $c(M, q) \leqslant k$.*

Proof. Since there is a covering of S with cocircuits each of size less than q^k, M has no loops and $\left\{\log_q \left(1 + \max_{p \in S} \left(\min_{p \in C^* \in \mathcal{C}^*(M)} |C^*|\right)\right)\right\} \leqslant \{\log_q (1 + (q^k - 1))\} = k$. The rest of the proof resembles the proof of Corollary 2.11.

Lindström [**10**, Theorem 18] proved the preceding result for the case $q = 2$.

Now suppose that M is a rank n matroid representable over $GF(q)$. If M has no loops, then $c(M, q) = c(\bar{M}, q)$. As $c(M, q)$ does not depend on the representation chosen for M in $V(n, q)$, it will be convenient in the next two proofs to identify $\bar{M}$ with a restriction of $V(n, q)$ to which it is isomorphic. The closure operator of $V(n, q)$ will be denoted by σ.

(3.7) PROPOSITION. *If T is a non-empty subset of S, then*

$$c(M \mid T, q) \leqslant c(M.T, q).$$

Proof. If $M.T$ has a loop, then the result is immediate. Thus suppose that $M.T$ has no loops. We may also assume that M is simple. Let

$M' = V(n, q)/\sigma(S \setminus T)$. As $M.T$ is loopless, $\sigma(S \setminus T) \cap T = \emptyset$ and, using [**14**, 3.334], we get that $M' \mid T \cong M.T$. Suppose that $\{H_1, H_2, \ldots, H_k\}$ is a minimal set of hyperplanes of M' such that $\left(\bigcap_{i=1}^{k} H_i\right) \cap T = \emptyset$. Then, as $\overline{M'} \cong \overline{V(n', q)}$, where $M.T$ has rank n', it follows that $k = c(M.T, q)$. But, for all $1 \leq i \leq k$, $H_i \cup \sigma(S \setminus T)$ is a hyperplane of $V(n, q)$. Therefore, since $\sigma(S \setminus T) \cap T = \emptyset$, we have $c(M \mid T, q) \leq k = c(M.T, q)$, as required.

(3.8) THEOREM. *If S is a disjoint union of cocircuits, then $c(M, q) = 1$.*

Proof. By (1.4) we may assume that M is simple. Let S be a disjoint union of the cocircuits $C_1^*, C_2^*, \ldots, C_k^*$ of M. If $k = 1$, then $M \cong U_{1,1}$ and the result is immediate. Assume therefore that $k > 1$. For all $1 \leq i \leq k$, $S \setminus C_i^*$ is a hyperplane of M, hence $S \setminus C_i^* = S \cap H_i$ where H_i is a hyperplane of $V(n, q)$. Let $\bigcap_{i=1}^{k} H_i = F$. Then F is a flat of $V(n, q)$ of rank $n-k$ and $F \cap S = \emptyset$. Moreover $\bigcup_{i=1}^{k}\left(\bigcap_{\substack{j \neq i \\ j=1}}^{k} H_j\right) \supseteq S$, hence $\left(\bigcup_{i=1}^{k}\left(\bigcap_{\substack{j \neq i \\ j=1}}^{k} H_j\right)\right) \setminus F \supseteq S$.

Now let $M' = V(n, q)/F$. Then M' has rank k and $\left\{\left(\bigcap_{\substack{j \neq i \\ j=1}}^{k} H_j\right) \setminus F\right\}_{1 \leq i \leq k}$ is a subset of the set of rank one flats of M'. It is a routine exercise in linear algebra to show that there is a hyperplane of $V(k, q)$ avoiding k linearly independent vectors. Therefore, as $\overline{M'} \cong \overline{V(k, q)}$, there is a hyperplane H' of M' such that $H' \cap \left(\left(\bigcup_{i=1}^{k}\left(\bigcap_{\substack{j \neq i \\ j=1}}^{k} H_j\right)\right) \setminus F\right) = \emptyset$. Therefore $(H' \cup F) \cap \left(\left(\bigcup_{i=1}^{k}\left(\bigcap_{\substack{j \neq i \\ j=1}}^{k} H_j\right)\right) \setminus F\right) = \emptyset$ and so $(H' \cup F) \cap S = \emptyset$. But $H' \cup F$ is a hyperplane of $V(n, q)$ and so $c(M, q) = 1$, as required.

Brylawski [**3**, Theorem 10.3] and Heron [**8**, p. 102] proved the preceding result and its converse for the case $q = 2$. To see that the converse does not hold for $q > 2$, consider the affine planes $AG(2, q)$.

The following result is a straightforward corollary of a result of Lindström [**10**, Theorem 15].

(3.9) PROPOSITION. *Let M be a loopless binary matroid on a set S and suppose that k is the least positive integer j such that $S = \bigcup_{i=1}^{j} S_i$ and $M.S_i$ is a disjoint union of cocircuits for all $1 \leq i \leq j$. Then $k = c(M, 2)$.*

We use this to get a result linking α, β, χ and π for binary matroids.

(3.10) PROPOSITION. *Let M be a loopless binary matroid on a set S. Then*
(i) $\{\log_2 \chi(M)\} \leq \alpha(M) \leq \beta(M)\{\log_2 \pi(M)\}$; *and*
(ii) $\dfrac{|S|}{\Delta(M)\{\log_2 \pi(M)\}} \leq \beta(M)$.

Proof. The left-hand inequality in (i) follows from the obvious inequalities $\{\log_2 \chi(M)\} \leq c(M, 2) \leq \alpha(M)$. Now suppose that $k = \{\log_2 \pi(M)\}$. Then $\pi(M) \leq 2^k$ and so, by Proposition 3.9, $S = \bigcup_{j=1}^{k} B_j$ where $M.B_j$ is a disjoint union of cocircuits for all $1 \leq j \leq k$. If B_j is a union of t_j disjoint cocircuits, then $t_j \leq \beta(M)$ and furthermore, $\sum_{j=1}^{k} t_j \geq \alpha(M)$ and $\Delta(M) \sum_{j=1}^{k} t_j \geq |S|$. Thus $k\beta(M) \geq \alpha(M)$ and $k\beta(M)\Delta(M) \geq |S|$. The right-hand inequality in (i) and inequality (ii) follow immediately.

Since $\beta(M(K_n)) = 1$ and $\chi(M(K_n)) = \pi(M(K_n)) = n$, a consequence of (i) above is that $\alpha(M(K_n)) = \{\log_2 n\}$.

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Note added in proof. The author has proved conjecture (1.3) for binary matroids. The proof will appear in a paper entitled 'Cocircuit coverings and packings for binary matroids'.