CAPTURING TWO ELEMENTS IN UNAVOIDABLE MINORS OF 3-CONNECTED BINARY MATROIDS

DEBORAH CHUN AND JAMES OXLEY

Dedicated to Geoff Whittle, a mentor and a friend.

Abstract. Let M be a 3-connected binary matroid and let n be an integer exceeding two. Ding, Oporowski, Oxley, and Vertigan proved that there is an integer $f(n)$ so that if $|E(M)| > f(n)$, then M has a minor isomorphic to one of the rank-n wheel, the rank-n tipless binary spike, or the cycle or bond matroid of $K_{3,n}$. This result was recently extended by Chun, Oxley, and Whittle to show that there is an integer $g(n)$ so that if $|E(M)| > g(n)$ and $x \in E(M)$, then x is an element of a minor of M isomorphic to one of the rank-n wheel, the rank-n binary spike with a tip and a cotip, or the cycle or bond matroid of $K_{1,1,1,n}$. In this paper, we prove that, for each i in $\{2,3\}$, there is an integer $h_i(n)$ so that if $|E(M)| > h_i(n)$ and Z is an i-element rank-2 subset of M, then M has a minor from the last list whose ground set contains Z.

1. Introduction

In 1993, Oporowski, Oxley, and Thomas [8] showed that every sufficiently large 3-connected graph has a large wheel or a large $K_{3,n}$ as a minor. Ding, Oporowski, Oxley, and Vertigan generalized this graph result to find unavoidable minors of large 3-connected matroids, first in the binary case [5] and later in the general case [6]. Chun, Oxley, and Whittle [4] extended the latter result by proving that if x is an element of a sufficiently large 3-connected matroid M, then M has a large 3-connected minor that uses x and is from one of a small number of families of highly structured matroids. In this paper, we consider the problem of trying to capture two elements in a large highly structured 3-connected minor of M. Although we have been unable to solve this problem in the general case, we have solved it for binary matroids. Our solution is the main result of this paper. Because this result is a theorem for binary matroids, for the rest of the paper, we shall concentrate exclusively on such matroids.

The matroid terminology used here will follow Oxley [9]. In particular, we use $M(\mathcal{W}_k)$ to denote the cycle matroid of the k-spoked wheel, $[n]$ to mean the set $\{1, 2, \dots, n\}$, and J_n to denote the $n \times n$ matrix of all ones. The following is Ding, Oporowski, Oxley, and Vertigan's [5] unavoidable-minor result for large 3-connected binary matroids.

Theorem 1.1. *For every integer n exceeding 2, there is an integer $f(n)$ so that every 3-connected binary matroid with more than $f(n)$ elements contains a minor isomorphic to one of $M(\mathcal{W}_n)$, the vector matroid of the binary matrix $[I_n|J_n - I_n]$, or the cycle or bond matroid of $K_{3,n}$.*

The next theorem specializes Chun, Oxley, and Whittle's [4] main theorem to binary matroids. Let A_n be the binary matrix that is obtained from $J_n - I_n$ by replacing the 0 in the bottom right corner with a 1.

Date: September 8, 2012.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. binary matroid, 3-connected matroid, unavoidable minor.

The second author was supported by a grant from the National Security Agency.

Theorem 1.2. *For every integer n exceeding 2, there is an integer $g(n)$ so that if M is a 3-connected binary matroid with $|E(M)| \geq g(n)$ and $x \in E(M)$, then x is an element of a minor of M that is isomorphic to one of $M(\mathcal{W}_n)$, the vector matroid of the binary matrix $[I_n|A_n]$, or the cycle or bond matroid of $K_{1,1,1,n}$.*

If we want to find a large highly structured 3-connected minor of a matroid that captures not just a single element but some pair of elements, then, perhaps surprisingly, we do not need to alter the list of unavoidable minors. The following is the main result of the paper.

Theorem 1.3. *For every integer n exceeding 2, there is an integer $h(n)$ so that if M is a 3-connected binary matroid with $|E(M)| \geq h(n)$ and $\{x, y\} \subseteq E(M)$, then x and y are elements of a minor of M that is isomorphic to one of $M(\mathcal{W}_n)$, the vector matroid of the binary matrix $[I_n|A_n]$, or the cycle or bond matroid of $K_{1,1,1,n}$.*

The next corollary follows immediately by specializing the last theorem to graphic matroids.

Corollary 1.4. *For every integer n exceeding 2, there is an integer $j(n)$ so that if G is a simple 3-connected graph having at least $j(n)$ edges and $\{e, f\} \subseteq E(G)$, then e and f are edges of a minor of G that is isomorphic to $\mathcal{W}_n$ or $K_{1,1,1,n}$.*

This paper is structured as follows. The next section introduces some basic preliminaries. In Section 3, we modify a theorem of Bixby and Coullard stated in Section 2 into a form that we will use repeatedly in the proof of the main result. By Theorem 1.2, if x and y are elements of a large 3-connected binary matroid M, then M has a minor that contains x and is from one of four families of highly structured matroids. Sections 4–6 examine each of these four cases individually and show that M has a minor from one of the four special families that uses x and y. Section 7 completes the proof of the main theorem. Finally, in Section 8, we apply Theorem 1.3 to show that we can capture a triangle of the initial matroid in one of our special minors.

Corollary 1.5. *Let M be a 3-connected binary matroid, and let $\{x, y, z\}$ be a triangle of M. For every integer n exceeding 2, there is an integer $t(n)$ so that if $|E(M)| > t(n)$, then $\{x, y, z\}$ is a triangle of a minor N of M that is isomorphic to one of $M(\mathcal{W}_n)$, the vector matroid of the binary matrix $[I_n|A_n]$, or the cycle or bond matroid of $K_{1,1,1,n}$. Moreover, when $N \cong M(K_{1,1,1,n})$, the triangle $\{x, y, z\}$ can be chosen to be the one whose deletion from $K_{1,1,1,n}$ gives $K_{3,n}$.*

2. Preliminaries

In this section, we present some basic results that will be used throughout the paper. We begin by defining a fan. In a 3-connected matroid M, consider a sequence $(s_0, s_1, \ldots, s_n)$ of distinct elements of M with $n \geq 2$ so that, for all $i \geq 0$, every set $\{s_{2i}, s_{2i+1}, s_{2i+2}\}$ is a triangle of M and every set $\{s_{2i+1}, s_{2i+2}, s_{2i+3}\}$ is a triad of M. Here we call such a sequence a *fan*, noting that this specializes the terminology used in [9], where another related structure is also called a fan. In this paper, we will rely heavily on a modification of the next theorem, which is a result of Bixby and Coullard [2] (see also [9, p. 479]).

Theorem 2.1. *Let N be a 3-connected minor of a 3-connected matroid M. Suppose that $|E(N)| \geq 4$, that $x \in E(M) - E(N)$, and that M has no 3-connected proper minor that both contains x and has N as a minor. Then, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of the following holds:*

(i) $N_1 = M_1 \backslash x$.
(ii) $N_1 = M_1 \backslash x/e$, *and N_1 has an element t so that $\{e, x, t\}$ is a circuit of M_1.*
(iii) $N_1 = M_1 \backslash x, e/f$, *and N_1 has an element t so that (x, f, t, e) is a fan of M_1. Moreover, $M_1 \backslash x$ is 3-connected.*
(iv) $N_1 = M_1 \backslash x, e, f$, *and N_1 has two elements s and t so that (t, e, x, f, s) is a fan of M_1.*

(v) $N_1 = M_1\backslash x, e/f, g$, *and* N_1 *has an element* t *so that* (x, f, t, e, g) *is a fan of* M_1. *Moreover,* $M_1\backslash x$ *and* $M_1\backslash x/f$ *are* 3*-connected.*

The following basic connectivity result, which is known as Bixby's Lemma [1] (see also [9, p.333]), will be frequently used in the paper.

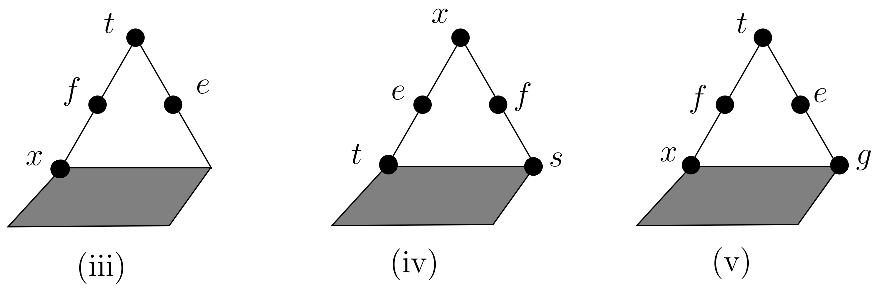

FIGURE 1. Cases (iii), (iv), and (v) of Theorem 2.1.

Lemma 2.2. *Let* M *be a* 3*-connected matroid and suppose* $e \in E(M)$. *Then either* $M\backslash e$ *or* M/e *has no non-minimal* 2*-separations, so either* $\mathrm{si}(M/e)$ *or* $\mathrm{si}(M^*/e)$ *is* 3*-connected.*

This paper will employ grafts, which are discussed in [9, Section 10.3]. A *graft* is a pair (G, γ) where G is a graph and γ is a subset of the vertex set of G. The *incidence matrix*, $A_{(G,\gamma)}$, of (G, γ) is the matrix that is obtained from the mod-2 vertex-edge incidence matrix of G by adjoining a new column e_γ corresponding to γ. Specifically, e_γ is the incidence vector of the set γ, that is, e_γ has a 1 in each row corresponding to a vertex of γ and a 0 in every other row. The matroid $M(G, \gamma)$ associated with the graft (G, γ) is the vector matroid $M[A_{(G,\gamma)}]$ where $A_{(G,\gamma)}$ is viewed as a matrix over $GF(2)$. Thus the *graft matroid* $M(G, \gamma)$ has ground set $E(G) \cup e_\gamma$. If the graft element e_γ is incident with an odd number of vertices, this element is a coloop in M. In this paper, we will require any graft element to be incident with an even number of vertices.

Let (G, γ) be a graft, and let $e \in E(G)$. To obtain the *deletion* $(G, \gamma)\backslash e$ and the *contraction* $(G, \gamma)/e$ of e from (G, γ), we delete or contract e from G leaving the set of vertices of γ unchanged except when e is contracted and has distinct ends u and v. In the exceptional case, $(G, \gamma)/e = (G/e, \gamma')$ where the vertex w that results from identifying u and v is in γ' if and only if exactly one of u and v is. Equivalently, $A_{(G/e,\gamma')}$ is obtained from $A_{(G,\gamma)}$ by deleting column e and replacing rows u and v with a single row equal to their sum modulo 2. Notice that if $|\gamma|$ is even, then so is $|\gamma'|$. The *minors* of (G, γ) are those grafts that can be produced by a sequence of single-edge deletions and contractions. For $e \in E(G)$, it is routine to check that $M((G, \gamma)\backslash e) = M(G, \gamma)\backslash e$ and $M((G, \gamma)/e) = M(G, \gamma)/e$.

The reader familiar with the matroid concept of roundedness may be reminded of it by the main theorem of this paper. Roundedness was introduced by Seymour [12] to encompass certain results that were concerned with relating particular minors of a matroid to specific elements of the matroid. The next lemma contains two examples of such results. The first part follows by combining results of Seymour [13] and Oxley and Reid [10] (see also [9, p.481]). The second part follows from the first.

Lemma 2.3. *Let* $t \in \{3, 4\}$ *and let* M *be a binary matroid with an* $M(\mathcal{W}_t)$*-minor.*

(i) *If* M *is* 3*-connected and* $e, f \in E(M)$, *then* M *has an* $M(\mathcal{W}_t)$*-minor using* $\{e, f\}$.
(ii) *If* M *is* 2*-connected and* $e \in E(M)$, *then* M *has an* $M(\mathcal{W}_t)$*-minor using* $\{e\}$.

3. A Modification of Bixby and Coullard's Theorem

By Theorem 2.1, if M is a 3-connected matroid with a 3-connected minor N and a fixed element x, then M has a 3-connected minor M' that uses x, has N as a minor, and has at most four more elements than N. As noted in [2], it is easy to see that M'', a smallest 3-connected minor of M that uses x and has a minor isomorphic to N, has at most $|E(N)|+1$ elements. In this section, we consider the case where M'' must also use a specified element of N. We will prove that, in this case, M'' has at most $|E(N)|+2$ elements.

Theorem 3.1. *Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geq 4$. Let $x \in E(M) - E(N)$ and $y \in E(N)$. Suppose M has no 3-connected proper minor that uses $\{x,y\}$ and has N as a minor. Then either M has a minor that uses $\{x,y\}$ and is obtained from N by relabelling one element by x, or, for some (N_1, M_1) in $\{(N,M),(N^*,M^*)\}$, one of the following holds:*

(i) *$N_1 = M_1\backslash x$ and y is contained in N_1; or*
(ii) *$N_1 = M_1\backslash x/z$ and $\{x,z,y\}$ is a circuit of M_1.*

Proof. As M has no 3-connected proper minor that uses x and has N as a minor, for some (N_1, M_1) in $\{(N,M),(N^*,M^*)\}$, one of the five cases identified in Theorem 2.1 holds.

In case (v), $N_1 = M_1\backslash x, e/f, g$ where M_1 has (x,f,t,e,g) as a fan (see the diagram on the right in Figure 1). Then $M_1/f,g$ has t, e, and x in parallel. Thus $M_1/f,g\backslash t,e$ uses $\{x,y\}$ and is obtained from N_1 by relabelling t by x.

In case (iv), $N_1 = M_1\backslash x, e, f$ where M_1 has (t,e,x,f,s) as a fan (see the diagram in the middle of Figure 1). By symmetry, we may assume $t \neq y$. Since M_1 has $\{e,x,f\}$ as a triad, $N_1 = M_1/e\backslash f, x$. As $M_1/e\backslash f$ has $\{t,x\}$ as a circuit, $M_1/e\backslash f, t$ uses $\{x,y\}$ and is obtained from N_1 by relabelling t by x.

In case (iii), $N_1 = M_1\backslash x, e/f$ and (x,f,t,e) is a fan of M_1. Now $M_1\backslash e/f$ has $\{x,t\}$ as a circuit. Thus $M_1\backslash e/f\backslash t$ uses $\{x,y\}$ and is obtained from N_1 by relabelling t by x.

In case (ii), $N_1 = M_1\backslash x/f$ and $\{f,x,t\}$ is a circuit of M_1. As M_1/f has $\{x,t\}$ as a circuit, either $M_1/f\backslash t$ is obtained from N_1 by relabelling t by x; or $t = y$ and outcome (ii) of the theorem holds. □

4. A Large Wheel-Minor

In this section, we consider the case where a 3-connected matroid with two identified elements has a large wheel minor. We begin with two lemmas, the first of which relates to case (i) identified in Theorem 3.1.

Lemma 4.1. *Let M be a 3-connected binary matroid with distinct elements x and y. Suppose M has a minor $N \cong M(\mathcal{W}_k)$ for some integer k greater than 2 and that $|E(M) - E(N)| = 1$. Then there is an integer m with $m \geq \frac{k}{4}$ so that M has an $M(\mathcal{W}_m)$-minor that uses $\{x,y\}$.*

Proof. By Lemma 2.3, the theorem holds for $k \leq 16$, so we may assume that $k \geq 17$. The lemma clearly holds if $\{x,y\} \subseteq E(N)$. Hence we may assume that $x \in E(M) - E(N)$ and, by duality, that $M\backslash x = N$. Clearly M is the matroid of a graft (G, γ_x) with $G \cong \mathcal{W}_k$ where x corresponds to the graft element incident with the set $\gamma_x \subseteq V(G)$. Let the hub vertex of G be labelled by h.

This proof is divided into two main cases depending on whether or not h is in γ_x. We will operate on the matroid M by operating exclusively on the graft (G, γ_x) as described in the Section 2.

First, assume that $h \in \gamma_x$ and that y is a spoke of G. One endpoint of y is h, label the other v. As noted in Section 2, $|\gamma_x|$ is even. Since x is not parallel to any element of M, the set γ_x contains h and at least three other vertices. We now construct a new graft (G', γ'_x) on which we shall operate. When $v \notin \gamma_x$, we let $(G', \gamma'_x) = (G, \gamma_x)$. Now suppose $v \in \gamma_x$. Choose a vertex v'

of γ_x that is the shortest distance along the rim from v. Contract the edges of the shortest path from v to v' along the rim of G, noting that at most $\frac{k-1}{2}$ edges are removed this way. Label by v the composite vertex resulting from these contractions. Simplify the underlying graph without removing y to produce the graft (G', γ'_x) with $G' \cong \mathcal{W}_n$ for some $n \geq \frac{k+1}{2}$ and $\gamma'_x = \gamma_x - \{v, v'\}$. If $|\gamma'_x| = 2$, then $\gamma'_x = \{h, u\}$ for some $u \in V(G') - h$, and the graft element corresponds to an edge parallel to a spoke of G'. Then M has a $\mathcal{W}_n$-minor containing x and y, and the lemma holds. Hence we may assume that $|\gamma'_x| \geq 4$.

We have now constructed (G', γ'_x) both when v is and is not in γ_x. In each case, $G' \cong \mathcal{W}_n$ for some $n \geq \frac{k+1}{2}$. Let P be the shortest path along the rim of G' that contains v and has both endpoints in γ'_x. Label the end points of this path u and w. The edges of $G' - h$ not in $E(P)$ form a path from u to w. The vertices in $\gamma'_x - h$ partition the edges of this path into $|\gamma'_x - h|$ subpaths. Color each such subpath red or blue so that every vertex of $\gamma'_x - \{u, w\}$ meets a red edge and a blue edge. We may assume that there are at least as many blue edges as red edges. Contract the red edges and simplify the underlying graph without deleting y. The resulting graft, (G'', γ''_x), has $G'' \cong \mathcal{W}_m$ with $m \geq \frac{n}{2} \geq \frac{1}{2}(\frac{k+1}{2}) = \frac{k+1}{4}$. Moreover, γ''_x is $\{h, u\}$ or $\{h, w\}$. Thus the graft element is an edge parallel to a spoke f of G''. Recall that y is incident with h and v, and $v \notin \{u, w\}$. Therefore M has an $M(\mathcal{W}_m)$-minor that contains x and y, and the lemma holds.

Next, we assume that $h \in \gamma_x$ and that y is a rim element. Let P be the shortest path of $G - h$ that contains y and has both endpoints in γ_x. Label these endpoints u and w. As above, consider the path from u to w with edge set $E(G - h) - E(P)$. The vertices in $\gamma_x - h$ partition the edges of this path into $|\gamma_x - h|$ subpaths. Color each such subpath red or blue so that every vertex of $\gamma_x - \{u, w\}$ meets a red edge and a blue edge. Without loss of generality, there are no more than $\frac{k-|E(P)|}{2}$ red edges. Contract the red edges and simplify the underlying graph without deleting y. The resulting graft, (G', γ'_x), has $G' \cong \mathcal{W}_m$ with $m \geq k - \frac{k-|E(P)|}{2} \geq \frac{k+1}{2}$. Moreover, γ'_x is $\{h, u\}$ or $\{h, w\}$, the edge y lies on the rim of G'. Therefore, in (G', γ'_x), the graft element is parallel to a spoke edge of G'. It follows that M has an $M(\mathcal{W}_m)$-minor that contains x and y, and again the lemma holds.

We may now assume that $h \notin \gamma_x$. Partition the edges of $G - h$ into a red set and a blue set in the following way. Consider the $|\gamma_x|$ paths of $G - h$ with both endpoints in γ_x and with no two distinct paths having a common edge. As $|\gamma_x|$ is even, so is the number of such paths. Color each of these paths red or blue so that every vertex of γ_x meets a red edge and a blue edge. Without loss of generality, there are at most $\frac{k}{2}$ red edges.

Assume first that y is not a red edge, so either y is a spoke, or y is blue. Then contract all but one, say a, of the red edges. Simplify the underlying graph without deleting y to produce the graft (G', γ'_x) with $G' \cong \mathcal{W}_m$ with $m \geq k - (\frac{k}{2} - 1) \geq \frac{k}{2} + 1$. Then $\gamma'_x = \{u, w\}$, where u and w are endpoints of a. Thus $M(G', \gamma'_x)\backslash a$ is an $M(\mathcal{W}_m)$-minor of M using x and y, and the lemma holds.

It remains to consider the case when y is red. As $|\gamma_x| \geq 4$, there are at least two red paths and we can choose an edge z from a red path that does not contain y. Contract all the red edges other than y and z from (G, γ_x) and simplify the underlying graph to produce (G', γ'_x) where γ'_x is a 4-element set consisting of the endpoints of y and z. Choose a path of blue edges of G' that joins two distinct vertices of γ'_x and has at most half of the blue edges. Contract these edges and simplify the underlying graph to produce (G'', γ''_x) where G'' is a wheel in which y and z are adjacent rim edges and the graft element corresponds to a new edge x that completes a 3-cycle with y and z. Let H be the graph that is obtained from G'' by adding this new edge, and let e be the spoke of G'' that is adjacent to both y and z. We can simplify the graph H/e without deleting x or y to produce a graph isomorphic to $\mathcal{W}_m$ for some m. As at least half of the original blue rim edges of G remain and the number of blue edges was at least half of the original number of rim edges, we deduce that $m \geq \frac{k}{4}$ and the lemma follows. $\square$

We have dealt with the case where the removal of one element from a 3-connected binary matroid M results in a wheel. Lemma 4.3 considers the case where two elements need to be removed from M to produce a wheel. Before considering that, we require a technical lemma.

For an integer $k \geq 3$, let $[I_k|D_k]$ be the following binary matrix.

$$\begin{array}{c|ccccc} b_1 \quad b_2 \quad b_3 \quad \dots \quad b_k & a_1 & a_2 & a_3 & \dots & a_k \end{array}$$
$$\left[\begin{array}{c|ccccc} & 1 & 0 & 0 & \dots & 1 \\ & 1 & 1 & 0 & \dots & 0 \\ I_k & 0 & 1 & 1 & \dots & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & 0 & \dots & 1 \end{array}\right]$$

Then $M[I_k|D_k] \cong M(\mathcal{W}_k)$. The spoke and rim edges of $\mathcal{W}_k$ correspond to the column vectors labelled b_i and a_i, respectively, for $i \in [k]$. Let $V(k,2)$ be the k-dimensional vector space over GF(2) and view its elements as column vectors.

Lemma 4.2. *The set of vectors of $V(k,2)$ that are spanned by $\{a_1, a_2, \dots, a_k\}$ consists of precisely those vectors having an even number of ones.*

Proof. The set of vectors $(x_1, x_2, \dots, x_k)$ so that $\sum_{i=1}^{k} x_i \equiv_2 0$ forms a hyperplane H of $V(k,2)$. This hyperplane contains $\{a_1, a_2, \dots, a_k\}$. As the last set is a circuit of $M[I_k|D_k]$, it has rank k, and so spans H. $\square$

Lemma 4.3. *Let M be a 3-connected binary matroid with $M\backslash x/f = N \cong M(\mathcal{W}_k)$ for some integer k greater than 2. Suppose N has an element y so that $\{x, f, y\}$ is a circuit of M. Then there is an integer m with $m \geq \frac{k}{4}$ so that M has an $M(\mathcal{W}_m)$-minor that uses $\{x, y\}$.*

Proof. By Lemma 2.3, this theorem holds for $k \leq 16$. Hence we may assume that $k \geq 17$. We consider the following cases:

(I) y is a spoke element of N; and
(II) y is a rim element of N.

In M^*, the set $\{x, f, y\}$ is a cocircuit. Let H be the complementary hyperplane. As $M^*\backslash f/x = N^*$, the matroid $M^*|H = N^*\backslash y$. In the wheel N^*, the element y will be a rim element in case I and a spoke element in case II. The matroid M^* is represented in Figure 2. There is a unique

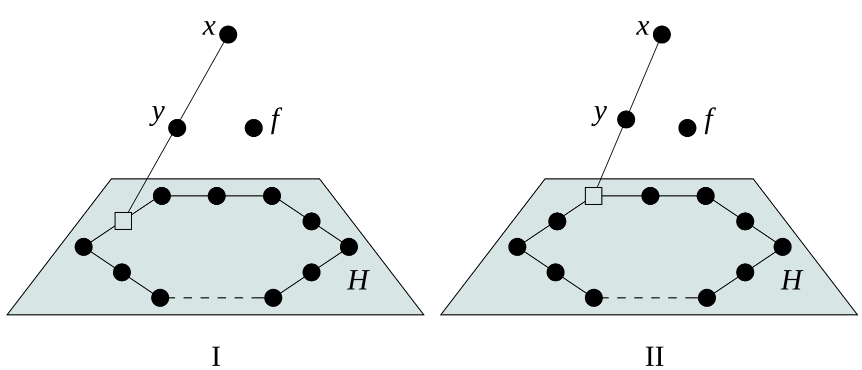

FIGURE 2. Geometric illustration of M^* for cases I and II.

binary matroid M_1 obtained by adding an element z to M^* to form a triangle with x and y. The matroid $M_1\backslash f/x$ has z parallel to y, and it is easy to see that $M_1|(H \cup z) \cong N^*$. Moreover, M_1/f

is an extension of $M_1|(H \cup z)$ by the elements x and y. Because we have added the element z, we will always be looking to delete it in our argument to ensure that we obtain a minor of M^*.

First we consider case II. The following matrix represents M_1/f.

$$\begin{array}{c}
\begin{array}{ccccc|ccccccc} z & e_2 & e_3 & \dots & e_k & e_{k+1} & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y \end{array} \\
\left[\begin{array}{c|ccccccc}
 & 1 & 0 & 0 & \dots & 1 & a_1 & a_1+1 \\
 & 1 & 1 & 0 & \dots & 0 & a_2 & a_2 \\
I_k & 0 & 1 & 1 & \dots & 0 & a_3 & a_3 \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & \dots & 1 & a_k & a_k
\end{array}\right]
\end{array}$$

By possibly interchanging x and y, we may assume that $\sum_{i=1}^{k} a_i$ is even. Then, by Lemma 4.2, x is spanned by the set $C = \{e_{k+1}, e_{k+2}, \dots, e_{2k}\}$ and $y \notin \mathrm{cl}_{M_1/f}(C)$. A smallest circuit C_x that contains x and is contained in $C \cup x$ has at most $\frac{k}{2} + 1$ elements, otherwise a smaller such circuit can be found in the symmetric difference of C_x and C. We can certainly choose an element i of $[k]$ so that e_{k+i} is an element of $C_x - x$.

The matroid $(M_1/f)/(C_x - \{x, e_{k+i}\})$ has x parallel to e_{k+i}. Notice that y is not a loop of this matroid, as $y \notin \mathrm{cl}_{M_1/f}(C)$. Simplify $(M_1/f)/(C_x - \{x, e_{k+i}\})$ without deleting x or y to produce M_2 (see Figure 3). First suppose that $\{x, y, z\}$ is a triangle of a rank-$r(M_2)$ wheel restriction of

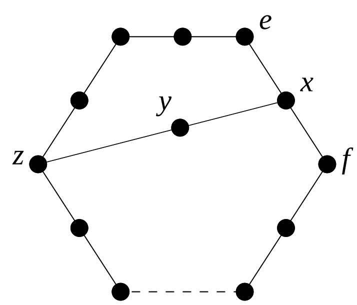

FIGURE 3. Geometric illustration of M_2.

M_2. In this case, contract one rim element other than x or y to make z parallel to another element and then delete z to produce a wheel minor of M^* that uses x and y and has rank at least $\frac{k}{2}$.

We may now suppose that $\{x, y, z\}$ is not a triangle of a rank-$r(M_2)$ wheel restriction of M_2. Then Figure 3 shows that $\{x, y, z\}$ is a triangle of two different wheel restrictions of M_2 that both have rank at least four. These wheels share the elements $\{e, f, x, y, z\}$ and contain x as a spoke. Restrict M_2 to one of these wheels of maximum rank. Contract one rim element other than y to make z parallel to an element of M^*. Then delete z to obtain a minor of M^* that uses x and y and is isomorphic to $M(\mathcal{W}_m)$ for some integer m with $m \geq \frac{r(M_2)+2}{2} - 1 \geq \frac{1}{2}(k - (|C_x| - 2)) \geq \frac{k+2}{4}$.

We may now assume that case I holds, that is, z is a rim element of the wheel $M_1/f\backslash x, y$. The following matrix represents M_1/f.

$$\begin{array}{c}
\begin{array}{ccccc|ccccccc} e_1 & e_2 & e_3 & \dots & e_k & z & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y \end{array} \\
\left[\begin{array}{c|ccccccc}
 & 1 & 0 & 0 & \dots & 1 & a_1 & a_1+1 \\
 & 1 & 1 & 0 & \dots & 0 & a_2 & a_2+1 \\
I_k & 0 & 1 & 1 & \dots & 0 & a_3 & a_3 \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & \dots & 1 & a_k & a_k
\end{array}\right]
\end{array}$$

First, assume that $\sum_{i=1}^{k} a_i$ is even. Let $I = \{e_{k+2}, e_{k+3}, \ldots, e_{2k}\}$. By Lemma 4.2, the vectors labelled by elements of I span the hyperplane of $V(k, 2)$ containing vectors with an even number of non-zero entries. Hence the independent set I spans x and y, and the sets $I \cup x$ and $I \cup y$ contain unique circuits, C_x and C_y, of M_1/f.

As M_1/f is binary, the symmetric difference $\{x, y, z\} \triangle (I \cup z)$, which equals $\{x, y\} \cup I$, is the union of disjoint circuits. The set I is independent, so these disjoint circuits are precisely C_x and C_y, and $C_x \dot{\cup} C_y = \{x, y\} \cup I$. Without loss of generality, $|C_x| \leq |C_y|$, so $|C_x| \leq \frac{|I \cup \{x,y\}|}{2} = \frac{(k-1)+2}{2}$.

Choose i in $[k] - \{1\}$ so that e_{k+i} is an element of $C_x - x$. Then $M_1/f/(C_x - \{x, e_{k+i}\})$ has x parallel to e_{k+i}. Notice that y is not a loop of this matroid, as $y \notin \mathrm{cl}_{M_1/f}(C_x)$. Simplify $(M_1/f)/(C_x - \{x, e_{k+i}\})$ without deleting x or y to produce M_3.

Suppose first that x is in a triangle with e_1 or e_2, as shown in Figure 4. In this case, as indicated

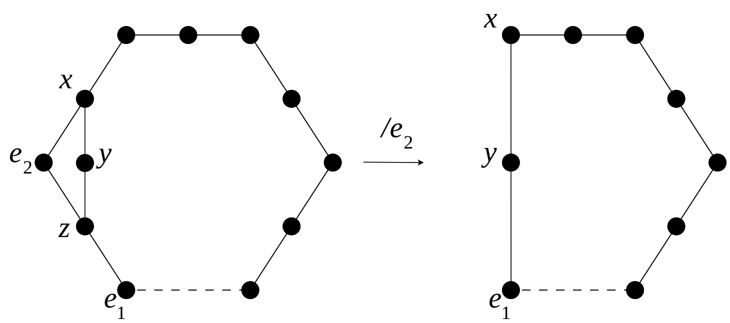

Figure 4. Geometric illustration of one possible configuration of M_3.

in that figure, we contract a spoke element and delete z and another spoke element to produce a wheel minor of M^* that uses x and y and has rank at least $\frac{k-1}{2}$.

We may now suppose that x is not in a triangle with e_1 or e_2 (see Figure 5). Then $\{x, y, z\}$

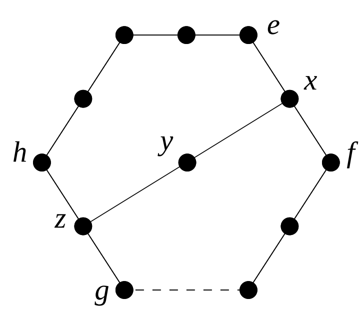

Figure 5. Geometric illustration of M_3.

is a triangle of two different wheel restrictions of M_3 that share the elements $\{e, f, g, h, x, y, z\}$ and together use all the elements of M_3. In each of these wheels, x and z are spokes and y is a rim element. Restrict to one of these wheels of maximum rank s. Then $s \geq \frac{r(M_2)}{2} + 1 \geq \frac{1}{2}(k - (|C_x| - 2)) + 1 \geq \frac{k+6}{4}$. Contract one rim element other than y to make z parallel to an element other than x or y. Then delete z to produce a minor of M^*. This minor has x and y and is isomorphic to $M(\mathcal{W}_m)$ for some integer m with $m \geq s - 1 \geq \frac{k+2}{4}$.

We may now assume, in case I, that $\sum_{i=1}^{k} a_i$ is odd. Recall that this case came from case I depicted in Figure 2. Because M^* is binary, there is a unique binary matroid, M_4, obtained by

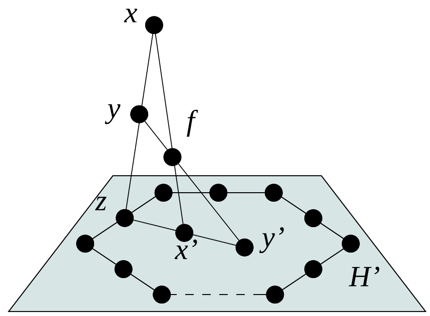

FIGURE 6. Geometric illustration of M^* with x', y', and z added to produce the matroid M_4. Here H' is the complement of triad $\{x, f, y\}$.

adding elements z, x' and y' so that $\{x, y, z\}$, $\{x, f, x'\}$, and $\{y, f, y'\}$ are triangles (see Figure 6). The following matrix represents M_4.

$$\begin{array}{c}
\begin{array}{cccccc|ccccccccc}
e_1 & e_2 & e_3 & \dots & e_k & f & z & e_{k+2} & e_{k+3} & \dots & e_{2k} & x & y & x' & y'
\end{array} \\
\left[\begin{array}{c|ccccccccc}
 & 1 & 0 & 0 & \dots & 1 & a_1 & a_1+1 & a_1 & a_1+1 \\
 & 1 & 1 & 0 & \dots & 0 & a_2 & a_2+1 & a_2 & a_2+1 \\
 & 0 & 1 & 1 & \dots & 0 & a_3 & a_3 & a_3 & a_3 \\
I_{k+1} & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & \dots & 1 & a_k & a_k & a_k & a_k \\
 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & 0
\end{array}\right]
\end{array}$$

Let H' be the hyperplane $E(M_4) - \{x, f, y\}$ of M_4. Since $\sum_{i=1}^{k} a_i$ is odd, by Lemma 4.2, neither x' nor y' is spanned by $\{e_{k+2}, e_{k+3}, \dots, e_{2k}\}$. The independent sets $I_x = \{x', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ and $I_y = \{y', e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ span H'. Now $I_x \cup I_y$ is the symmetric difference of the circuits $\{z, e_{k+2}, e_{k+3}, \dots, e_{2k}\}$ and $\{x', y', z\}$ of M_4, so $I_x \cup I_y$ is a union of disjoint circuits. Now each such circuit must contain $\{x', y'\}$ as both I_x and I_y are independent, so $I_x \cup I_y$ is a circuit.

Choose i in $[k] - \{1, 2\}$ and let B_i be the independent set $\{e_i, e_{k+2}, e_{k+3}, \dots, e_{2k}\}$. Then H' is spanned by B_i. Notice that $B_i \cup x' = I_x \cup e_i$ and this set contains a unique circuit C_x, which must contain $\{x', e_i\}$. Similarly, there is a unique circuit $C_y \subseteq B_i \cup y' = I_y \cup e_i$ and $\{y', e_i\} \subseteq C_y$. Now $C_x \triangle C_y$ is a disjoint union of circuits and is a non-empty subset of the circuit $I_x \cup I_y$. Hence $C_x \cap C_y = \{e_i\}$. Thus $C_x \triangle C_y = I_x \cup I_y$. Without loss of generality, $|C_x| \leq |C_y|$, so $|C_x| \leq \frac{|I_x \cup I_y|}{2} + 1 = \frac{k+3}{2}$.

Contract $C_x - \{x', e_i\}$ from M_4 to make x' parallel to e_i. Since y' is not contained in the closure of $C_x - \{x, e_i\}$, the element y' has not become a loop in this process. Simplify the matroid without deleting any element of $\{x, y, f, x', y', z\}$ to produce the matroid M_5 illustrated on the left in Figure 7. Clearly $M_5 \backslash \{x, y, f\}$ has two wheel restrictions that have x' and z as spokes and that together use all of the elements of $E(M_5) \backslash \{x, y, f\}$. Let R be the set of rim elements of one of these wheels of minimum rank. In M_5, contract $R - \{e_1, e_2, y'\}$ to make y' parallel to one of e_1 or e_2, thereby making x' parallel to the other (see Figure 7 right). Now delete the added elements, x', y', and z, and simplify to produce an $M(\mathcal{W}_m)$-minor of M^*, for some m with $m \geq \frac{r(M_5)}{2} + 1 \geq \frac{1}{2}(k - (|C_x| - 2)) + 1 \geq \frac{1}{2}(k - (\frac{k-1}{2})) + 1 = \frac{k+1}{4} + 1$. □

5. A Large Spike-Minor

In this section, we examine the case where a 3-connected binary matroid with two identified elements has a large spike-minor. The rank-n binary spike with no tip or cotip has $[I_n | J_n - I_n]$ as a representation and will be denoted by S_n. The rank-n binary spike with a tip and no cotip

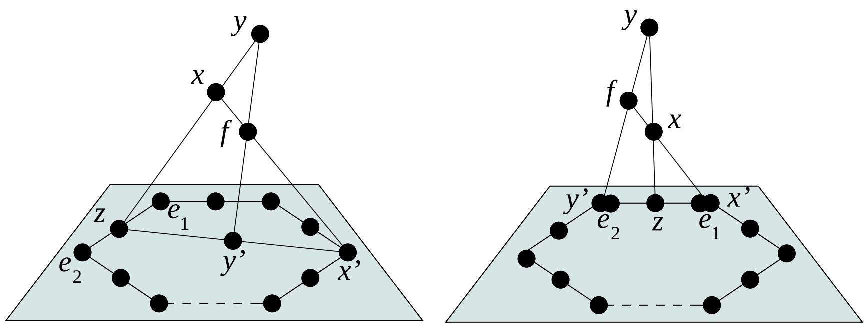

FIGURE 7. Geometric illustration of M_5 (left) and one of its minors (right).

has $[I_n|J_n - I_n|\mathbf{1}]$ as a representation where $\mathbf{1}$ is the column of n ones. This column represents the tip of the spike and, for all $i \in [n]$, the elements represented by the ith column and the $(i+n)$th column form a triangle with the tip. Delete any column of the last matrix other than $\mathbf{1}$ to produce a representation for T_n, the rank-n binary spike with a tip and a cotip. If the deleted element was in a triangle with c and the tip, then c is the cotip of this spike. Deleting the tip from T_n results in a rank-n binary spike with a cotip and no tip. Observe that $T_3 \cong M(\mathcal{W}_3)$.

First, we prove a technical lemma.

Lemma 5.1. *For some $n \geq 4$, let N be the rank-n binary spike with a tip t and no cotip. Let M be a 3-connected binary matroid so that $M \backslash x = N$. If T is the set of elements of a minimal set of triangles of N spanning x, then both $M|(T \cup x)$ and $M \backslash (T - t)$ are spikes with tip t and cotip x.*

Proof. There is a unique binary matroid M' that is obtained from M by adding an element z so that $\{t, x, z\}$ is a triangle. It is straightforward to show using a binary matrix representation for M' that both $M'|(T \cup x \cup z)$ and $M' \backslash (T - t)$ are binary spikes with tip t. The lemma follows immediately from this. $\square$

We use this lemma when considering a 3-connected binary matroid M that is a single-element extension of T_n.

Lemma 5.2. *Let $N \cong T_n$ for some integer n greater than 2. Let M be a 3-connected binary matroid with elements x and y so that $M \backslash x = N$. Then there is an integer m with $m \geq \frac{n}{2}$ so that M has a T_m-minor that uses $\{x, y\}$.*

Proof. By Lemma 2.3, as T_3 is isomorphic to $M(\mathcal{W}_3)$, the theorem holds for $n \leq 6$. Thus we may assume $n \geq 7$. The matroid N has n copunctual lines, $L_1, L_2, \dots, L_{n-1}, L'_n$ where $L_i = \{t, e_i, f_i\}$ for each i in $[n-1]$ and $L'_n = \{t, e_n\}$. Let M_1 be the unique binary matroid obtained by adding z to M so that $\{t, e_n, z\}$ is a triangle. Let $L_n = \{t, e_n, z\}$. The following is a representation of M_1.

$$\begin{array}{c|cccccc|ccccccccc|}
 & e_1 & e_2 & e_3 & \dots & e_{n-1} & e_n & f_1 & f_2 & f_3 & \dots & f_{n-1} & z & t & x \\
1 & & & & & & & 0 & 1 & 1 & \dots & 1 & 1 & 1 & x_1 \\
2 & & & & & & & 1 & 0 & 1 & \dots & 1 & 1 & 1 & x_2 \\
3 & & & & I_n & & & 1 & 1 & 0 & \dots & 1 & 1 & 1 & x_3 \\
\vdots & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
n-1 & & & & & & & 1 & 1 & 1 & \dots & 0 & 1 & 1 & x_{n-1} \\
n & & & & & & & 1 & 1 & 1 & \dots & 1 & 0 & 1 & x_n
\end{array}$$

If x is spanned by some L_i in M_1, then, as M is simple, x is parallel to z in M_1. Thus, deleting z and any element other than x, y, or t from M_1 gives a T_n-minor of M containing x and y.

We may now assume that x is not spanned by any L_i. Then M_1 is 3-connected. Let A be the set of elements of a minimal subset of $\{L_1, L_2, \ldots, L_n\}$ whose closure contains x. Let k be the number of lines L_i that are subsets of A. Let $B = E(M_1) - (A - t)$. By Lemma 5.1, $x \in \mathrm{cl}_{M_1}(B)$, and $M_1|(A \cup x)$ and $M_1|(B \cup x)$ are spikes with tip t and cotip x. Note that $k \leq n - k$. We may assume that $A = \{t, e_1, f_1, e_2, f_2, \ldots, e_k, f_k\}$ or $A = \{t, e_1, f_1, e_2, f_2, \ldots, e_{k-1}, f_{k-1}, e_n, z\}$. Thus, for some c in $\{0, 1\}$, either

$$x_i = \begin{cases} c & \text{if } 1 \leq i \leq k, \\ c-1 & \text{otherwise;} \end{cases} \qquad \text{or} \qquad x_i = \begin{cases} c-1 & \text{if } k \leq i \leq n-1, \\ c & \text{otherwise.} \end{cases}$$

The element y may be contained in A. By the symmetry of the matroid M_1, we may assume that, if it is, $y \in \{t, e_1, f_1, e_n\}$. Let $M_2 = M_1/\{e_2, e_3, \ldots, e_{k-1}\} \backslash \{f_2, f_3, \ldots, f_{k-1}\}$. The matroid M_2 has the following representation.

$$\begin{array}{c} \\ 1 \\ k \\ k+1 \\ \vdots \\ n-1 \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccc|cccccccc} e_1 & e_k & e_{k+1} & \cdots & e_{n-1} & e_n & f_1 & f_k & f_{k+1} & \cdots & f_{n-1} & z & t & x \end{array} \\
\left[\begin{array}{c|cccccccc}
 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & c \\
 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & x_k \\
 & 1 & 1 & 0 & \cdots & 1 & 1 & 1 & c-1 \\
I_{n-k+2} & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & c-1 \\
 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & x_k - 1
\end{array}\right]
\end{array}$$

Since $c \in \{0, 1\}$ and $x_k \in \{0, 1\}$, one of four cases holds. First, if $c = x_k = 0$, then both $\{e_1, f_k, x\}$ and $\{e_k, f_1, x\}$ are triangles of M_2. Secondly, if $c = x_k = 1$, then $\{e_1, e_k, x\}$ and $\{f_1, f_k, x\}$ are triangles of M_2. In either of these cases, contract e_1 provided $e_1 \neq y$, otherwise contract f_1. In the resulting matroid, x is parallel to an element in $\{e_k, f_k\}$, and either $\{t, f_1\}$ or $\{t, e_1\}$ is a circuit. Now delete z and simplify without deleting x or y. The result is a T_m-minor of M using x and y where $m = n - (k-2) - 1 \geq \frac{n}{2} + 1$.

In the third case, $c = 1$ and $x_k = 0$, so $\{e_1, e_n, x\}$ and $\{f_1, z, x\}$ are triangles of M_2. Finally, if $c = 0$ and $x_k = 1$, then $\{e_1, z, x\}$ and $\{e_n, f_1, x\}$ are triangles of M_2. In these last two cases, if the triangle containing $\{x, e_1\}$ avoids y, contract e_1, otherwise contract f_1. In both cases, x is parallel to an element of M_2 other than y. Delete z and simplify without deleting x or y to produce a spike-minor of M that uses $\{x, y\}$ and has a tip but possibly no cotip. If this minor has no cotip, delete an element other than t, x, or y to produce a T_m-minor for some m with $m \geq \frac{n}{2} + 1$. □

We now consider the case where two elements must be removed from M to form a T_n-minor.

Lemma 5.3. *Let M be a 3-connected binary matroid with $M \backslash x/f = N \cong T_n$ for some integer n with $n \geq 4$. Suppose N has an element y so that $\{x, f, y\}$ is a circuit of M. Then there is an integer m with $m \geq \frac{n-1}{2}$ so that M has a T_m-minor that uses $\{x, y\}$.*

Proof. By Lemma 2.3, since $T_3 \cong M(\mathcal{W}_3)$, the theorem holds for $n \leq 7$. Thus we may assume $n \geq 8$. In M^*, the set $\{x, f, y\}$ is a cocircuit. Let M_0 be the unique binary matroid obtained by adding z to M^* so that $\{x, y, z\}$ is a triangle of M_0. In $M_0 \backslash \{x, f, y\}$, which is isomorphic to T_n, the element z is either (1) the tip, (2) the cotip, or (3) neither the tip nor the cotip. In the first case, M_0/f has the following representation.

$$
\begin{array}{cc}
\begin{array}{cccccc} e_1 & e_2 & e_3 & \dots & e_{n-1} & e_n \end{array} & \begin{array}{cccccccc} f_1 & f_2 & f_3 & \dots & f_{n-1} & z & x & y \end{array} \\
\left[\begin{array}{c} \\ \\ I_n \\ \\ \\ \end{array}\right. & \left.\begin{array}{cccccccc}
0 & 1 & 1 & \dots & 1 & 1 & x_1 & x_1+1 \\
1 & 0 & 1 & \dots & 1 & 1 & x_2 & x_2+1 \\
1 & 1 & 0 & \dots & 1 & 1 & x_3 & x_3+1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & x_{n-1} & x_{n-1}+1 \\
1 & 1 & 1 & \dots & 1 & 1 & x_n & x_n+1
\end{array}\right]
\end{array}
$$

Let k be the number of non-zero members of $\{x_1, x_2, \dots, x_n\}$. By switching x and y if necessary, we may assume that $k \le \frac{n}{2}$, so that $n - k \ge \frac{n}{2}$. Suppose first that $k = 1$. If $x_n = 1$, then delete f_{n-1} and e_n from M_0/f to produce a T_n-minor using x and y. If $x_j = 1$ for some $j \ne n$, then delete e_j and f_j to produce a T_n-minor using x and y. In either case, we produce a T_{n-1}-minor of M^* by contracting some remaining f_i and then deleting z.

We may now assume that $k > 1$. Without loss of generality, $x_1 = x_2 = \dots = x_{k-1} = 1$ and either (i) $x_k = 1$ or (ii) $x_n = 1$. In case (i), contract $\{e_2, e_3, \dots, e_k\}$ and delete $\{f_3, f_4, \dots, f_k\}$. Then deleting $\{e_1, f_1, z\}$ gives a T_{n-k+1}-minor of M^* using x and y and having tip f_2 and cotip e_n. In case (ii), first contract $\{e_2, e_3, e_4, \dots, e_{k-1}, e_n\}$ and delete $\{f_3, f_4, \dots, f_{k-1}\}$. Then deleting $\{e_1, f_1, z, f_{n-1}\}$ gives a T_{n-k+1}-minor of M^* using x and y and having tip f_2 and cotip e_{n-1}.

In case (2), M_0/f has the following representation.

$$
\begin{array}{cc}
\begin{array}{cccccc} e_1 & e_2 & e_3 & \dots & e_{n-1} & z \end{array} & \begin{array}{cccccccc} f_1 & f_2 & f_3 & \dots & f_{n-1} & t & x & y \end{array} \\
\left[\begin{array}{c} \\ \\ I_n \\ \\ \\ \end{array}\right. & \left.\begin{array}{cccccccc}
0 & 1 & 1 & \dots & 1 & 1 & x_1 & x_1 \\
1 & 0 & 1 & \dots & 1 & 1 & x_2 & x_2 \\
1 & 1 & 0 & \dots & 1 & 1 & x_3 & x_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & x_{n-1} & x_{n-1} \\
1 & 1 & 1 & \dots & 1 & 1 & x_n & x_n+1
\end{array}\right]
\end{array}
$$

By switching x and y if necessary, we may assume that $x_n = 0$. Let k be the number of non-zero members of $\{x_1, x_2, \dots, x_{n-1}\}$.

Assume first that $k \ge \frac{n-1}{2}$. Without loss of generality, we may suppose that $x_1 = x_2 = \dots = x_{n-k-1} = 0$. Contract $\{e_1, e_2, \dots, e_{n-k-1}\}$ and delete $\{f_1, f_2, f_3, \dots, f_{n-k-1}\}$ to produce a matroid in which t and y are parallel. Deleting z and t from this matroid gives a T_{k+1}-minor with tip y and cotip x. As $k + 1 \ge \frac{n+1}{2}$, the result holds.

We may now assume that $k \le \frac{n-2}{2}$. As x is not a loop, $x_j = 1$ for some $j \ne n$. Without loss of generality, we may assume that $x_1 = x_2 = \dots = x_k = 1$. Contract $\{e_2, e_3, \dots, e_k\}$ and delete $\{f_2, f_3, \dots, f_k\}$ to produce a matroid in which x is parallel to e_1 and $\{t, x, f_1\}$ and $\{x, y, z\}$ are circuits. Thus $\{t, y, z, f_1\}$ is also a circuit. Now contracting f_1 and deleting e_1, z, and t gives a minor of M^* isomorphic to T_m with tip x and cotip y and with $m = n - k \ge n - \frac{n-2}{2} \ge \frac{n}{2} + 1$.

It remains to consider case (3), that is, z forms a triangle with t and some element e. Without loss of generality, M_0/f has the following matrix representation.

$$
\begin{array}{cc}
\begin{array}{cccccc} e & e_2 & e_3 & \dots & e_{n-1} & e_n \end{array} & \begin{array}{cccccccc} z & f_2 & f_3 & \dots & f_{n-1} & t & x & y \end{array} \\
\left[\begin{array}{c} \\ \\ I_n \\ \\ \\ \end{array}\right. & \left.\begin{array}{cccccccc}
0 & 1 & 1 & \dots & 1 & 1 & x_1 & x_1 \\
1 & 0 & 1 & \dots & 1 & 1 & x_2 & x_2+1 \\
1 & 1 & 0 & \dots & 1 & 1 & x_3 & x_3+1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \dots & 0 & 1 & x_{n-1} & x_{n-1}+1 \\
1 & 1 & 1 & \dots & 1 & 1 & x_n & x_n+1
\end{array}\right]
\end{array}
$$

As long as e is not parallel to x or y in M_0/f, we can contract e, delete z, and relabel t as z to give this matrix the same form as the matrix representing M_0/f in case (1). In this case, we reduce case (3) to case (1) and find a T_m-minor of M^* using x and y for some integer $m \geq n-1-\frac{n-1}{2} = \frac{n-1}{2}$. Now suppose e is parallel to x or y in M_0/f. Then M_0 has $\{f, e, x\}$ or $\{f, e, y\}$ as a triangle. Thus $M_0|\{z, t, e, x, f, y\}$ is isomorphic to $M(\mathcal{W}_3)$. It is straightforward to check that $M_0/f\backslash t, e$ has $\{x, y, z\}$ as a triangle and is isomorphic to T_n, where x or y is the tip. Contracting the cotip from this copy of T_n and then deleting z gives a T_{n-1}-minor of M^* that uses x and y. Hence the required result holds. $\square$

6. A Large Minor Isomorphic to the Cycle or Bond Matroid of $K_{1,1,1,n}$

In this section, we examine the case when M has a minor isomorphic to the cycle or bond matroid of $K_{1,1,1,n}$. We will refer to Figure 8, which shows the graph of $K_{1,1,1,n}$ and illustrates the geometry of this rank-$(n+2)$ matroid. First, we consider the case where the deletion of one element of M results in an $M(K_{1,1,1,n})$-minor.

Lemma 6.1. *Let M be a 3-connected binary matroid so that $M\backslash x = N \cong M(K_{1,1,1,n})$ for some positive integer n. Suppose $y \in E(N)$. Then there is an integer m with $m \geq \frac{n-1}{2}$ so that x and y are elements of a minor of M isomorphic to T_m or $M(K_{1,1,1,m})$.*

Proof. By Lemma 2.3, as $T_3 \cong M(\mathcal{W}_3)$, we may assume that $n \geq 8$. Clearly $M = M(G, \gamma_x)$ where $G \cong K_{1,1,1,n}$. Label G as in Figure 8. By symmetry, we may assume that y is a_1a_2 or a_1b_1. If $|\gamma_x| = 2$, then, as M is simple, we may assume that $\gamma_x = \{b_2, b_i\}$ for some i in $\{1, 3\}$. Then $M/a_3b_2\backslash\{a_1b_2, a_2b_2, a_3b_i\}$ is an $M(K_{1,1,1,n-1})$-minor of M that uses x and y.

We may now assume that $|\gamma_x| \geq 4$. Let A_x and B_x be the sets $\{a_1, a_2, a_3\} \cap \gamma_x$ and $\{b_1, b_2, \ldots, b_n\} \cap \gamma_x$, respectively.

First, let $|B_x| \leq \frac{n}{2}+1$. Assume a_1 or a_2 is not in γ_x. Then $|B_x| \geq 2$ so, without loss of generality, $b_2 \in B_x$. Contract the edges from vertices of $B_x - b_2$ to a_3 and label the resulting composite vertex a_3. Simplify the underlying graph without deleting y. The resulting graft (G', γ'_x) has $G' \cong K_{1,1,1,m}$ for some m with $m = n - |B_x - b_2| \geq \frac{n}{2}$. In G', the edge y has a_1 as one endpoint, and the other endpoint is in $\{a_2, a_3, b_1\}$. Moreover, γ'_x consists of b_2 and some subset of $\{a_1, a_2, a_3\}$. Since $|\gamma'_x|$ is even, and either a_1 or a_2 is not in γ'_x, the set $\gamma'_x = \{a_i, b_2\}$ for some $i \in [3]$. In $M(G', \gamma'_x)$, then, x is parallel to the element a_ib_2 and, since y is not incident with b_2 in G', the matroid $M(G'\backslash a_ib_2, \gamma'_x)$ is an $M(K_{1,1,1,m})$-minor of M using x and y.

Now assume that both a_1 and a_2 are in γ_x. Since $|\gamma_x| \geq 4$, there is a vertex b_k in B_x. If $b_1 \in B_x$, let $k = 1$, otherwise, let b_k be any vertex of B_x. Contract a_2b_k from the graft, labelling the resulting vertex a_2. Simplify the underlying graph without deleting y to produce the graft (G', γ'_x), with $G' \cong K_{1,1,1,n-1}$ and $\gamma'_x = \gamma_x - \{a_2, b_k\}$. If $|\gamma'_x| = 2$, then $\gamma'_x = \{a_1, a_3\}$ or $\gamma'_x = \{a_1, b_i\}$ for some

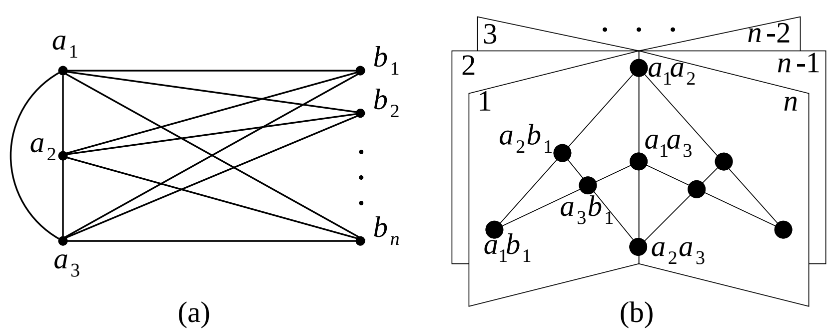

Figure 8. (a) $K_{1,1,1,n}$ and (b) a geometric illustration of $M(K_{1,1,1,n})$

$i \neq 1$. In either case, $M(G', \gamma'_x)$ has x parallel to some element other than y, so we may simplify to produce an $M(K_{1,1,1,n-1})$-minor containing x and y. Thus we may assume that $|\gamma_x - \{a_2, b_k\}| \geq 4$. Since $a_2 \notin \gamma'_x$, this case is reduced to the case considered in the previous paragraph, and M has an $M(K_{1,1,1,m})$-minor using x and y with $m \geq \frac{n-1}{2}$.

Finally, we may assume $|B_x| \geq \frac{n+1}{2} + 1$. Then $|B_x| \geq 5$, so $|B_x - b_1| \geq 4$. Without loss of generality, $\{b_2, b_3, b_4\} \subseteq B_x$. For every $a_i \in A_x$, contract the edge $a_i b_{i+1}$ and label the resulting vertex a_i. Also contract the set of edges from $\{b_1, b_2, \ldots, b_n\} - B_x$ to a_3 and label the composite vertex a_3. The resulting graft has graft element $\gamma'_x = B_x - \{b_{i+1} : a_i \in A_x\}$ and has the vertex set $\{a_1, a_2, a_3\} \cup \gamma'_x$. Simplify the underlying graph without deleting y to produce the graft (G', γ'_x) with $G' \cong K_{1,1,1,m}$ for some integer m with $m = |B_x| - |A_x| \geq \frac{n-1}{2} - 1$.

At this point, $y \in \{a_1a_2, a_1b_1, a_1a_3\}$. Without loss of generality, $y \neq a_1a_3$. Delete the vertex a_3 from G' to produce a graft (G'', γ'_x) where $\gamma'_x = V(G'') - \{a_1, a_2\}$. Clearly $M(G'')$ can be obtained from T_{m+1} by deleting the cotip. As $\gamma'_x = V(G'') - \{a_1, a_2\}$, it follows easily that $M(G'', \gamma'_x)$ is isomorphic to T_{m+1} and uses $\{x, y\}$. Since $m + 1 \geq \frac{n-1}{2}$, the lemma follows. $\square$

We now consider the matroid $M^*(K_{1,1,1,n})$. While $M(K_{1,1,1,n})$ is depicted in Figure 8, it will still be useful to develop a geometric illustration for $M^*(K_{1,1,1,n})$ itself. In $K_{3,n+1}$, let the vertex classes be labelled $\{a_1, a_2, a_3\}$ and $\{b_0, b_1, \ldots, b_n\}$. Perform a $Y - \Delta$ exchange on the triad $\{b_0a_1, b_0a_2, b_0a_3\}$. The resulting triangle is $\{a_1a_2, a_2a_3, a_1a_3\}$ and the resulting graph is $K_{1,1,1,n}$. Thus, in $M^*(K_{1,1,1,n})$, if we perform a $Y - \Delta$ exchange on the triad $\{a_1a_2, a_2a_3, a_1a_3\}$, we get $M^*(K_{3,n+1})$. Geometrically, $M^*(K_{3,n+1})$ can be formed as follows. Take the direct sum of n triangles $Z_i = \{a_1b_i, a_2b_i, a_3b_i\}$ for all $i \in [n]$. There is a unique binary matroid M_0 that can be obtained by adding elements z_1, z_2, and z_3 so that $\{a_jb_1, a_jb_2, \ldots, a_jb_n, z_j\}$ is a circuit of M_0 for each $j \in [3]$. By taking the symmetric difference of these three $(n+1)$-element circuits and the n triangles Z_i, we find that $\{z_1, z_2, z_3\}$ is a triangle of M_0. From above, we see that performing a $\triangle$-Y exchange on the triangle $\{z_1, z_2, z_3\}$ gives the triad $\{a_1a_2, a_2a_3, a_1a_3\}$ in the matroid $M^*(K_{1,1,1,n})$ where $A_1 = \{a_1a_2, a_1a_3, a_1b_1, a_1b_2, \ldots, a_1b_n\}$, $A_2 = \{a_1a_2, a_2a_3, a_2b_1, a_2b_2, \ldots, a_2b_n\}$, and $A_3 = \{a_1a_3, a_2a_3, a_3b_1, a_3b_2, \ldots, a_3b_n\}$ are circuits. While $M^*(K_{1,1,1,n})$ has rank $2n+1$, an illustration is useful. Figure 9 shows triad $\{a_1a_2, a_1a_3, a_2a_3\}$ complementing a hyperplane labelled H. The white squares indicate the position of triangle $\{z_1, z_2, z_3\}$ which was removed. The other triangles are shown as vertical, 3-point lines and each circuit A_i is indicated by a horizontal line that bends at a white square so that each such line includes $n+2$ points.

We now extend the remarks above to make some observations that will be helpful in the proofs of the next result and Corollary 1.5. Let $Z_{n+1} = \{z_1, z_2, z_3\}$ and fix k in $\{2, 3, \ldots, n-1\}$. Then $(Z_1 \cup Z_2 \cup \cdots \cup Z_k, Z_{k+1} \cup Z_{k+2} \cup \cdots \cup Z_{n+1})$ is an exact 3-separation of $M^*(K_{3,n+1})$. There is a unique binary matroid M'_0 that is obtained from M_0 by adding elements z'_1, z'_2, and z'_3 so that $\{a_jb_1, a_jb_2, \ldots, a_jb_k, z'_j\}$ is a circuit of M'_0 for each $j \in [3]$. Let $Z' = \{z'_1, z'_2, z'_3\}$. Then Z' is a circuit of M'_0 as is $\{a_jb_{k+1}, a_jb_{k+2}, \ldots, a_jb_n, z_j, z'_j\}$ for each $j \in [3]$. Moreover, M_0 is the 3-sum of $M'_0|(Z_1 \cup Z_2 \cup \cdots \cup Z_k \cup Z')$ and $M'_0|(Z_{k+1} \cup Z_{k+2} \cup \cdots \cup Z_{n+1} \cup Z')$ across Z'. We observe that the last two matroids are isomorphic to $M^*(K_{3,k+1})$ and $M^*(K_{3,n-k+2})$. By performing a $\triangle - Y$ exchange on $\{z_1, z_2, z_3\}$, we see that $M^*(K_{1,1,1,n})$ is the 3-sum of $M^*(K_{3,k+1})$ and $M^*(K_{1,1,1,n-k+1})$.

Now we consider the case where the deletion of one element of M produces an $M^*(K_{1,1,1,n})$-minor.

Lemma 6.2. *Let M be a 3-connected binary matroid so that $M \backslash x = N \cong M^*(K_{1,1,1,n})$ for a positive integer n. Suppose $y \in E(N)$. Then there is an integer m with $m \geq \frac{n}{4} - 2$ so that M has an $M^*(K_{1,1,1,m})$-minor that uses $\{x, y\}$.*

Proof. As $M^*(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 2.3, the theorem holds for $n \leq 12$. Thus we may assume that $n \geq 13$. We will also assume N is labelled as in Figure 9, with triangles $Z_i = \{a_1b_i, a_2b_i, a_3b_i\}$ for all $i \in [n]$ and a triad $Z_0^* = \{a_1a_2, a_1a_3, a_2a_3\}$.

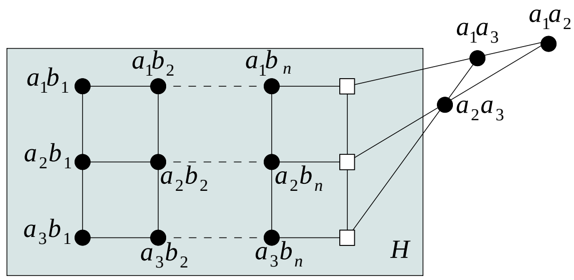

FIGURE 9. A geometric illustration of $M^*(K_{1,1,1,n})$.

Let C_x be a circuit of M containing x meeting a minimum-sized subset $\mathcal{Z}$ of $\{Z_1, Z_2, \dots, Z_n\}$. Subject to this, choose C_x so that $|C_x - \{a_1a_2, a_1a_3, a_2a_3\}|$ is minimized. Then $|C_x \cap Z_i| \leq 1$ for all $i \in [n]$; otherwise, for some i, a circuit contained in $C_x \triangle Z_i$ containing x contradicts the choice of C_x. Let $k = |\mathcal{Z}|$. Without loss of generality, $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_k\}$ and $y \in \{a_1a_2, a_1b_1, a_1b_{k+1}\}$.

First, we assume $k > \frac{3}{4}n$. By the pigeonhole principle, for some $j \in [3]$, say $j = 1$, the set C_x meets $\{a_jb_1, a_jb_2, \dots, a_jb_n\}$ in at least $\frac{1}{3}|\mathcal{Z}|$ elements. Thus $C_x \triangle \{a_1b_1, a_1b_2, \dots, a_1b_n,\ a_1a_2, a_1a_3\}$ contains a circuit C'_x containing x and avoiding at least $\frac{|\mathcal{Z}|}{3}$ triangles of N. Then C'_x meets at most $n - \frac{|\mathcal{Z}|}{3}$ triangles of N. But $n - \frac{|\mathcal{Z}|}{3} < \frac{3}{4}n$ so we have contradicted the choice of C_x. Thus $k \leq \frac{3}{4}n$.

Next suppose $k = 0$. Then $x \in \mathrm{cl}(Z_0^*)$. As M is binary, $M\backslash x$ is illustrated in Figure 10 with the four possible locations for x in M represented by squares. If x is not in $\mathrm{cl}(H)$, then delete

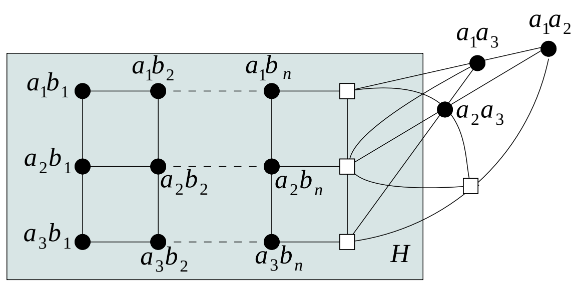

FIGURE 10. $M\backslash x$ with boxes representing the four possible locations for x.

a_1a_3 to produce an $M^*(K_{1,1,1,n})$-minor using x and y. Thus we may assume $x \in \mathrm{cl}(H)$. If x is not in a triangle with a_1a_3 and a_2a_3, then we can contract one of these elements to produce an $M^*(K_{3,n+1})$-minor using x and y. In this case, we can easily find an $M^*(K_{1,1,1,n-1})$-minor using x and y. Thus we may assume $\{x, a_1a_3, a_2a_3\}$ is a triangle (see Figure 10). If $y \neq a_1a_2$, then $M/a_1a_2 \cong M^*(K_{3,n+1})$ and we can easily find an $M^*(K_{1,1,1,n-1})$-minor using x and y. Therefore we may assume $y = a_1a_2$ and M is the vector matroid of the following binary matrix.

$$
\begin{array}{c}
\begin{array}{ccccccc|cccc|ccc}
a_1b_1 & \cdots & a_1b_n & a_2b_1 & \cdots & a_2b_n & a_1a_2 & a_3b_1 & a_3b_2 & \cdots & a_3b_n & a_1a_3 & a_2a_3 & x
\end{array}\\
\left[\begin{array}{c|cccc|ccc}
 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 \\
 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & \cdots & 1 & 1 & 0 & 1 \\
I_{2n+1} & 1 & 0 & \cdots & 0 & 0 & 1 & 1 \\
 & 0 & 1 & \cdots & 0 & 0 & 1 & 1 \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & \cdots & 1 & 0 & 1 & 1 \\
 & 0 & 0 & \cdots & 0 & 1 & 1 & 0
\end{array}\right]
\end{array}
$$

We now construct a representation for M^*. From the matrix $[I_{2n+1}|D]$ representing M, first construct $[D^T|I_{n+3}]$. In the resulting matrix, we add rows $n+1$ and $n+2$ to row $n+3$. Finally, we adjoin a new row that is the sum of all the current rows to get the following matrix.

$$
\begin{array}{c}
\begin{array}{ccc|ccc|c|ccc|ccc}
a_1b_1 & \cdots & a_1b_n & a_2b_1 & \cdots & a_2b_n & a_1a_2 & a_3b_1 & \cdots & a_3b_n & a_1a_3 & a_2a_3 & x
\end{array}\\
\left[\begin{array}{ccc|ccc|c|ccc|ccc}
 & & & & & & 0 & & & & 0 & 0 & 0 \\
 & I_n & & & I_n & & \vdots & & I_n & & \vdots & \vdots & \vdots \\
 & & & & & & 0 & & & & 0 & 0 & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

Therefore M is cographic with its dual represented by the graph G shown in Figure 11. It is

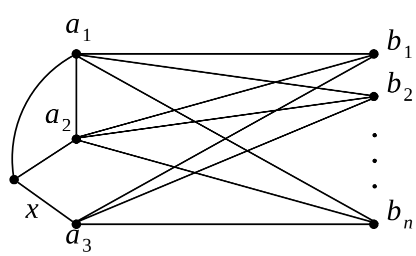

FIGURE 11. A graph G representing M^*.

easy to see that $G/a_3b_n \cong K_{1,1,1,n}$, and this graph contains x and y. Therefore $M\backslash a_3b_n$ is an $M^*(K_{1,1,1,n})$-minor of M that uses $\{x,y\}$.

We may now assume that $k \geq 1$. Just as we may delete a triad from $K_{1,1,1,n}$ to produce $K_{1,1,1,n-1}$, we may contract a triangle of $M^*(K_{1,1,1,n})$ to produce $M^*(K_{1,1,1,n-1})$. Contract the triangles $Z_k, Z_{k-1}, \ldots, Z_2$ one-by-one in order until one of the following holds:

(1) x is in $\mathrm{cl}(Z_j)$ for some $j \in [n]$, or
(2) x is in $\mathrm{cl}(Z_j \cup Z_0^*)$ for some $j \in [n]$.

The resulting matroid, M_1, is a single-element extension of $M^*(K_{1,1,1,m})$ for some $m \geq n-k \geq \frac{n}{4}$.

In case (1), M_1 has $x \in \mathrm{cl}(Z_j)$ for some $j \in [n]$. By the minimality of $|\mathcal{Z}|$, it follows that $j=1$ and $k \geq 2$. If x is not parallel to y, we may simplify M_1 to obtain an $M^*(K_{1,1,1,m})$-minor of M using $\{x,y\}$, so assume x and y are parallel in M_1. Recall that $y \in \{a_1b_1, a_1a_2, a_1b_{k+1}\}$. In this case, $y = a_1b_1$. Let M_0 be the matroid obtained from M by contracting the triangles of $\mathcal{Z}$ other than Z_1 and Z_2. By the minimality of $|\mathcal{Z}|$, contracting Z_2 from M_0 creates the parallel class $\{x, y\}$. Hence $\{x, a_1b_1, a_ib_2\}$ is a circuit for some $i \in [3]$. Since $M_0\backslash x$ has $Z_1 \cup Z_2$ as a 3-separating set, and $x \in \mathrm{cl}(Z_1 \cup Z_2)$, the matroid M_0 can be represented as a 3-sum of the type shown in Figure 12 (see [9, Proposition 9.3.4]).

If $i \neq 1$, then, without loss of generality, $i = 2$. Then contracting $\{a_1b_2, a_2b_1\}$ and deleting $\{a_3b_1, a_3b_2\}$ gives an $M^*(K_{1,1,1,n-k+1})$-minor using $\{x, y\}$. If $i = 1$, then x is parallel to a gray element in Figure 12, and x and y are elements of $M_0/\{a_2b_2, a_3b_1\}\backslash\{a_2b_1, a_3b_2\}$, which is isomorphic to $M^*(K_{1,1,1,n-k+1})$. As $k \geq \frac{3}{4}n$, we have that $n - k + 1 \geq \frac{n}{4} + 1$, so the result holds in case (1).

Now consider case (2). In M_1, the element x is in $\mathrm{cl}(Z_0^* \cup Z_j)$. By the minimality of $|\mathcal{Z}|$, it follows that $j = 1$. Since $Z_1 \cup Z_0^*$ is a 3-separating set in $M_1\backslash x$ and $x \in \mathrm{cl}_{M_1}(Z_1 \cup Z_0^*)$, we can view M_1 as the 3-sum shown in Figure 13. Recall that $y \in \{a_1a_2, a_1b_1, a_1b_{k+1}\}$.

The set $\{x, a_1a_2, a_1a_3, a_2a_3, a_1b_1, a_2b_1, a_3b_1\}$ contains a minimum-sized subset C'_x that is a circuit of M_1 containing x. As M_1 is binary, $|C'_x \cap \{x, a_1a_2, a_1a_3, a_2a_3\}|$ in even. As $k \geq 1$, the circuit C'_x meets Z_1. We may assume $C'_x \cap Z_1 = \{a_ib_1\}$, otherwise $C'_x \triangle Z_1$ contains a circuit containing x that contradicts the minimality of C'_x. Therefore either $\{x, a_1a_2, a_1a_3, a_2a_3, a_ib_1\}$ or $\{x, a_ja_k, a_ib_1\}$ is a circuit for some $i \in [3]$ and some $a_ja_k \in \{a_1a_2, a_1a_3, a_2a_3\}$. By choosing the basis $\{a_1a_2, a_1a_3, a_2a_3, a_1b_1, a_2b_1\}$, we obtain the following binary representation for the left side, M_2, of the 3-sum displayed in Figure 13.

$$\begin{array}{c}
\begin{array}{ccccc|ccccc} a_1a_2 & a_1a_3 & a_2a_3 & a_1b_1 & a_2b_1 & a_3b_1 & e & f & g & x \end{array}\\
\left[\begin{array}{c|ccccc}
 & 0 & 1 & 1 & 0 & x_1 \\
 & 0 & 1 & 0 & 1 & x_2 \\
I_5 & 0 & 0 & 1 & 1 & x_3 \\
 & 1 & 1 & 0 & 1 & x_4 \\
 & 1 & 0 & 1 & 1 & x_5
\end{array}\right]
\end{array}$$

Assume $a_1a_2 \in C'_x$. Then C'_x is either $\{x, a_1a_2, a_1a_3, a_2a_3, a_ib_1\}$ or $\{x, a_1a_2, a_ib_1\}$ for some $i \in [3]$, so $M_2/\{a_1a_3, a_2a_3\}$ has the following representation for some (x_4, x_5) in $\{(1, 0), (0, 1), (1, 1)\}$.

$$\begin{array}{c}
\begin{array}{ccc|ccccc} a_1a_2 & a_1b_1 & a_2b_1 & a_3b_1 & e & f & g & x \end{array}\\
\left[\begin{array}{c|ccccc}
 & 0 & 1 & 1 & 0 & 1 \\
I_3 & 1 & 1 & 0 & 1 & x_4 \\
 & 1 & 0 & 1 & 1 & x_5
\end{array}\right]
\end{array}$$

If (x_4, x_5) is $(1, 0)$ or $(1, 1)$, then contracting a_2b_1 from this matroid produces a rank-2 matroid with every gray element parallel to another element and with x not parallel to y. Thus we may simplify $M_1/\{a_1a_3, a_2a_3, a_2b_1\}$ to find an $M^*(K_{3,n-k+1})$-minor using x and y. If, instead, $(x_4, x_5) = (0, 1)$, then contract one element of $\{a_1a_2, a_1b_1\} - y$ from $M_1/\{a_1a_3, a_2a_3\}$ to find an $M^*(K_{3,n-k+1})$-minor using x and y. In either case, we can easily find an $M^*(K_{1,1,1,n-k-1})$-minor of M using $\{x, y\}$. As $n - k - 1 \geq \frac{n}{4} - 1$, the lemma follows.

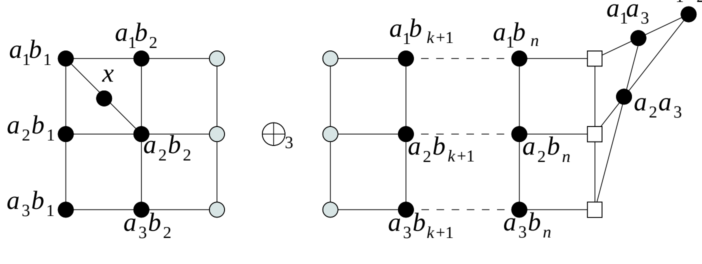

FIGURE 12. M_0 shown as a 3-sum across the gray triangle when $i = 2$.

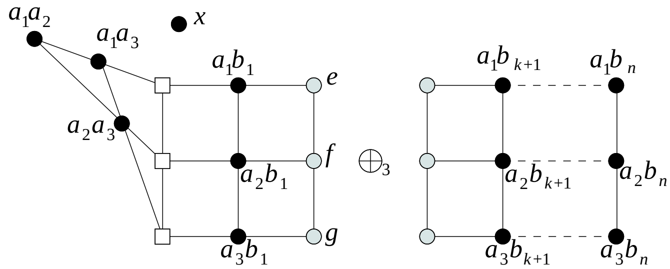

FIGURE 13. The matroid M_1 with cocircuit $\{x, a_1a_2, a_1a_3, a_2a_3\}$ illustrated as a 3-sum.

We may now assume that $a_1a_2 \notin C'_x$. Thus $C'_x = \{x, a_ja_k, a_ib_1\}$ for some $i \in [3]$ and some $a_ja_k \in \{a_1a_3, a_2a_3\}$. Thus, in the 5×10 matrix above representing M_2, we have $x_1 = 0$ and $(x_2, x_3) \in \{(1, 0), (0, 1)\}$, while $(x_4, x_5) \in \{(1, 0), (0, 1), (1, 1)\}$. By symmetry, we may assume $(x_2, x_3) = (1, 0)$. If $y \neq a_1b_1$, then $M_2/\{a_1b_1, a_2b_1\}\backslash a_3b_1$ has x parallel to a_1a_3. In this case, $M_1/\{a_1b_1, a_2b_1\} \backslash\{a_3b_1, a_1a_3\} \cong M^*(K_{1,1,1,n-k})$, and this matroid contains x and y. Thus we may assume that $y = a_1b_1$.

If $(x_4, x_5) \neq (1, 0)$, then $M_2/\{a_1a_2, a_1a_3, a_2a_3\}$ has y, a_2b_1, and a_3b_1 parallel to e, f, and g, respectively. Moreover, x is parallel to f or g. Therefore we may simplify $M_1/\{a_1a_2, a_1a_3, a_2a_3\}$ to find an $M^*(K_{3,n-k+1})$-matroid containing x and y. From this matroid, we can easily find an $M^*(K_{1,1,1,n-k-1})$-minor using x and y. Instead, we assume that $(x_4, x_5) = (1, 0)$, so (x_2, x_3, x_4, x_5) is $(1, 0, 1, 0)$. Then $M^*(K_{3,n-k+1}) \cong M_1/\{a_1a_2, a_2a_3, a_3b_1\}\backslash a_2b_1$. This minor contains $\{x, y\}$, so M_1 has an $M^*(K_{1,1,1,n-k-1})$-minor using $\{x, y\}$. As $n - k - 1 \geq \frac{n}{4} - 1$, the lemma follows. □

Next we consider the case where removing two elements of M produces an $M(K_{1,1,1,n})$-minor.

Lemma 6.3. *Let M be a 3-connected binary matroid so that $M\backslash x/f = N \cong M(K_{1,1,1,n})$ with $n \geq 1$. Let N have an element y so that $\{x, f, y\}$ is a circuit of M. Then there is an integer m with $m \geq \frac{n}{16} - 5$ so that M has an $M(K_{1,1,1,m})$-minor that uses $\{x, y\}$.*

Proof. As $M(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 2.3, the theorem holds for $n \geq 96$, so we may assume $n \geq 97$. In M^*, the set $\{x, f, y\}$ is a triad, and $M^*/x\backslash f \cong M^*(K_{1,1,1,n})$. There is a unique binary matroid, M_0, obtained from M^* by adding an element z so that $\{x, y, z\}$ is a circuit of M_0. Moreover, M_0 is 3-connected. Let H be the hyperplane of M^* that is the complement of $\{x, f, y\}$.

Now $z \in \mathrm{cl}_{M_0}(H)$ and M_0/x has the parallel pair $\{y, z\}$. Thus $M_0|(H \cup z) = M_0/x\backslash\{f, y\}$ $\cong M_0/x\backslash\{f, z\} = M^*/x\backslash f \cong M^*(K_{1,1,1,n})$. Hence M_0 contains z in an $M^*(K_{1,1,1,n})$-restriction. We will assume this restriction is labelled as in Figure 9. Without loss of generality, $z \in \{a_1b_1, a_1a_2\}$.

Consider M_0/f. Since $M_0/f\backslash\{x, y\} \cong M^*(K_{1,1,1,n})$, the matroids $M_0/f\backslash y$ and $M_0/f\backslash x$ are single-element extensions of $M^*(K_{1,1,1,n})$. If one of these is 3-connected, then without loss of generality, $M_0/f\backslash y$ is 3-connected. By Lemma 6.2, for some $k \geq \frac{n}{4} - 2$, the matroid $M_0/f\backslash y$ has an $M^*(K_{1,1,1,k})$-minor $(M_0/f\backslash y)/C\backslash D$ using $\{x, z\}$. Now $M_0/(C \cup f)\backslash D$ is the single-element extension of $M^*(K_{1,1,1,k})$ by an element y added so that $\{x, y, z\}$ is a circuit.

Suppose $M_0/(C \cup f)\backslash D$ is not 3-connected. Then y is parallel to an element c. In this case, $M_0/(C \cup f)\backslash(D \cup c) \cong M^*(K_{1,1,1,k})$, and $M_0/(C \cup f)\backslash(D \cup c)$ has $\{x, y, z\}$ as a triangle. Then, without loss of generality, $x = a_1b_1$, $y = a_2b_1$, and $z = a_3b_1$ (see Figure 9 taking n equal to k in that figure). Contract the cocircuit $\{a_1a_2, a_1a_3, a_2a_3\}$ from this matroid to produce an $M^*(K_{3,k})$-minor. Delete $\{z, a_2b_2\}$ and contract a_3b_2 to produce an $M^*(K_{1,1,1,k-2})$-minor using x and y. As we have deleted z, this minor is also a minor of M^*.

We may now assume that $M_0/(C \cup f)\backslash D$ is a 3-connected single-element extension of $M^*(K_{1,1,1,k})$ that uses $\{x, y\}$. By Lemma 6.2, this matroid has x and y in a minor, N_1, which is isomorphic to $M^*(K_{1,1,1,j})$ for some $j \geq \frac{k}{4} - 2 \geq \frac{n}{16} - 3$. Since x and y are not parallel in N_1, the element z has not been contracted to produce N_1. Therefore either z has been deleted to produce N_1 so N_1 is a minor of M^*, or z is an element of the triangle $\{x, y, z\}$ in N_1. In the latter case, using the argument above, we can delete z and identify an $M^*(K_{1,1,1,j-2})$-minor of M^* that contains x and y. Since $j - 2 \geq \frac{n}{16} - 5$, the lemma holds in this case.

It remains to consider the case when neither $M_0/f\backslash y$ nor $M_0/f\backslash x$ is 3-connected. As both M_0 and $M_0/f\backslash\{x, y\}$ are 3-connected, x and y are parallel to some elements, say e and d, in M_0/f. Thus $M_0/f\backslash\{e, d\} \cong M^*(K_{1,1,1,n})$, and $\{x, y, z\}$ is a triangle of this matroid. Again, by the argument above, we may delete z to get an $M^*(K_{1,1,1,n-2})$ minor of M^* using $\{x, y\}$. □

Finally, we consider the case where the removal of two elements from M produces an $M^*(K_{1,1,1,n})$-minor. One outcome in this case involves getting a spike-minor but does not mention x or y.

Lemma 6.4. *Let M be a 3-connected binary matroid so that $M\backslash x/f = N \cong M^*(K_{1,1,1,n})$ for some positive integer n. Let N have an element y so that $\{x, f, y\}$ is a circuit of M. Then there is an integer m with $m \geq \frac{n}{4} - 3$ so that either M has a minor isomorphic to T_m, or M has a minor that uses $\{x, y\}$ and is isomorphic to $M^*(K_{1,1,1,m})$.*

Proof. As $M^*(K_{1,1,1,1}) \cong M(\mathcal{W}_3)$, by Lemma 2.3, the theorem holds for $n \leq 16$, so we may assume $n \geq 17$. In addition, we may assume that M has no T_m-minor for any $m \geq \frac{n}{4} - 3$. In M^*, the set $\{x, f, y\}$ is a triad complementing a hyperplane H. The matroid $M^*/x\backslash f \cong M(K_{1,1,1,n})$. Let M_0 be the unique binary matroid obtained from M^* by adding an element z so that $\{x, y, z\}$ is a triangle. Then M_0 is 3-connected.

Now $M_0|(H \cup z) = M_0/x\backslash\{f, y\} \cong M_0/x\backslash\{f, z\} = M^*/x\backslash f \cong M(K_{1,1,1,n})$. Hence M_0 contains z in an $M(K_{1,1,1,n})$-restriction. We will assume this restriction is labelled as in Figure 8. Without loss of generality, $z \in \{a_1b_1, a_1a_2\}$. Since $M_0/f\backslash\{x, y\} \cong M(K_{1,1,1,n})$, both $M_0/f\backslash y$ and $M_0/f\backslash x$ are single-element extensions of $M(K_{1,1,1,n})$. If one of these matroids is 3-connected, then, without loss of generality, $M_0/f\backslash y$ is 3-connected. By Lemma 6.1, $M_0/f\backslash y$ has x and z in a minor, $(M_0/f\backslash y)/C\backslash D$, that is isomorphic to T_k or $M(K_{1,1,1,k})$ for some $k \geq \frac{n-1}{2}$. If $(M_0/f\backslash y)/C\backslash D \cong T_k$, then $(M_0/f)/C\backslash D$ is a spike T_k with an extra element, y, added in the closure of two elements. It is routine to check that $((M_0/f)/C\backslash D)/y\backslash z$ or $((M_0/f)/C\backslash D)/x\backslash z$ contains a T_{k-1}-minor. Since z has been deleted, T_{k-1} is also a minor of M^*, a contradiction. Therefore $(M_0/f\backslash y)/C\backslash D \cong M(K_{1,1,1,k})$. Now $M_0/(C \cup f)\backslash D$ is obtained from $M(K_{1,1,1,k})$ by adding y added so that $\{x, y, z\}$ is a circuit.

Assume $M_0/(C \cup f)\backslash D$ is not 3-connected. Then y is parallel to some element c. In this case, let $M_1 = M_0/(C \cup f)\backslash(D \cup c)$. Then $M_1 \cong M(K_{1,1,1,k})$, and M_1 has $\{x, y, z\}$ as a triangle. If $\{x, y, z\}$ is $\{a_1a_2, a_1a_3, a_2a_3\}$, then $M_1\backslash z$ has an $M(K_{1,1,1,k-1})$-minor using x and y. Otherwise, without loss of generality, $\{x, y, z\} = \{a_1a_2, a_1b_1, a_2b_1\}$ (see Figure 8(b) taking n equal to k in that figure). In M_1/a_3b_1, each of a_1a_3 and a_2a_3 is parallel to an element of $\{x, y, z\}$. Delete z and any elements parallel to x and y to produce a minor isomorphic to $M(K_{1,1,1,k-1})$ or $M(K_{1,2,k-1})$. In the latter case, we can easily find a minor isomorphic to $M(K_{1,1,1,k-2})$ that contains x and y. In either case, since we have deleted z, this minor is also a minor of M^*.

We may now assume that $M_0/(C \cup f)\backslash D$ is a 3-connected, single-element extension of $M(K_{1,1,1,k})$ that uses $\{x, y\}$. By Lemma 6.1, this matroid has x and y in a minor, N_1, that is isomorphic to $M(K_{1,1,1,j})$ or T_j for some $j \geq \frac{k-1}{2} \geq \frac{n-3}{4}$. Since x and y are not parallel in N_1, the element z has not been contracted to produce N_1. Therefore either z has been deleted to produce N_1 so N_1 is a minor of M^* and the lemma holds; or z is an element of the triangle $\{x, y, z\}$ in N_1. In the latter case, suppose first that $N_1 \cong T_j$. The spike T_j is not a minor of M^* by assumption. Therefore $z \in E(N_1)$ and this T_j-minor has triangle $\{x, y, z\}$. As the only triangles of T_j are those including the tip and, without loss of generality, x is not the tip of N_1, it is routine to check that $N_1/x\backslash z \cong T_{j-1}$. Since z has been deleted, the last matroid is a minor of M^*, a contradiction.

We may now assume that $N_1 \cong M(K_{1,1,1,j})$ and $\{x, y, z\}$ is a triangle of N_1. Then, using the argument in the second-last paragraph, we can find an $M(K_{1,1,1,j-2})$-minor of M_0 that uses $\{x, y\}$ and avoids z. Thus this matroid is also a minor of M^*. As $j - 2 \geq \frac{n-3}{4} - 2$, the lemma follows.

Finally, suppose that neither $M_0/f\backslash y$ nor $M_0/f\backslash x$ is 3-connected. As M_0 and $M_0/f\backslash\{x, y\}$ are both 3-connected, x and y are parallel to some elements, say e and d, in M_0/f. Thus $M_0/f\backslash\{e, d\} \cong M(K_{1,1,1,n})$, and $\{x, y, z\}$ is a triangle of this matroid. Again, by the argument above, we can delete z and produce an $M(K_{1,1,1,n-2})$ minor of M_0 using x and y that is also a minor of M^*. □

7. The Proof of the Main Result

The following theorem is a consequence of Theorem 4.2 of Chun, Oxley, and Whittle [4].

Theorem 7.1. *Let M be a connected matroid with an element x so that $M\backslash x$ is isomorphic to T_n for some $n \geq 6$. Then x is an element of a minor of M that is isomorphic to T_m for some $m \geq \frac{n}{2}$.*

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.3. By Theorem 1.2, there is a function g so that if $|E(M)| \geq g(100n)$, then M has a minor N that uses y and is isomorphic to $M(\mathcal{W}_{100n})$, T_{100n}, $M(K_{1,1,1,100n})$, or $M^*(K_{1,1,1,100n})$. If $x \in E(N)$, then the theorem holds, so we assume $x \in E(M) - E(N)$. Let M' be a minimum-sized 3-connected minor of M so that $\{x, y\} \subseteq E(M')$ and M' has an N-minor. By Theorem 3.1, for some (N_1, M_1) such that either $N_1 \cong N$ and $M_1 \cong M'$, or $N_1 \cong N^*$ and $M_1 \cong (M')^*$, one of the following holds:

(i) $N_1 = M_1\backslash x$ and y is contained in this minor; or
(ii) $N_1 = M_1\backslash x/z$ and $\{x, z, y\}$ is a circuit of M_1.

As $\{M(\mathcal{W}_{100n}), T_{100n}, M(K_{1,1,1,100n}), M^*(K_{1,1,1,100n})\}$ is closed under duality, we may assume that $N_1 \in \{M(\mathcal{W}_{100n}), T_{100n}, M(K_{1,1,1,100n}), M^*(K_{1,1,1,100n})\}$.

First, assume that $N_1 \cong M(\mathcal{W}_{100n})$. In cases (i) and (ii), by Lemmas 4.1 and 4.3, M_1 has an $M(\mathcal{W}_m)$-minor that uses $\{x, y\}$ for some $m \geq 25n$. Next assume that N_1 is isomorphic to $M(K_{1,1,1,100n})$ or $M^*(K_{1,1,1,100n})$. In case (i), by Lemmas 6.1 and 6.2, either M_1 has a T_k-minor, or x and y are elements of a minor of M_1 isomorphic to $M(K_{1,1,1,k})$ or $M^*(K_{1,1,1,k})$ for some $k \geq 25n - 2$. In case (ii), by Lemmas 6.3 and 6.4, either M_1 has a T_k-minor for some $k \geq 25n - 3$, or x and y are elements of a minor of M_1 isomorphic to $M(K_{1,1,1,m})$ or $M^*(K_{1,1,1,m})$ for some $m \geq \frac{25n}{4} - 5 \geq 4n$.

We may now assume M_1 has a T_k-minor for some $k \geq 25n - 3$. By Theorem 7.1, x is an element of a T_j-minor of M_1 for some $j \geq \frac{25n-3}{2}$. Let M'' be a minimum-sized 3-connected minor of M_1 that uses $\{x, y\}$ and has a T_j-minor. By Theorem 3.1, for some M_2 in $\{M'', (M'')^*\}$, one of the following holds:

(i) $T_j \cong M_2$ and $\{x, y\}$ is contained in this minor; or
(ii) $T_j \cong M_2\backslash x$, and y is contained in this minor; or
(iii) $T_j \cong M_2\backslash x/z$ and M_2 has $\{x, z, y\}$ as a triangle.

In cases (ii) and (iii), by Lemmas 5.2 and 5.3 respectively, x and y are elements of a minor of M_2 that is isomorphic to T_i for some $i \geq \frac{j-1}{2} \geq 6n - 2$. We conclude that the theorem holds. □

8. Capturing a triangle

In this section, we prove Corollary 1.5, showing that we can capture, in a large, highly structured minor, a triangle of the original 3-connected matroid. The proof will use the following result, which is a straightforward consequence of an extension of Tutte's Linking Theorem by Geelen, Gerards, and Whittle [7], see also [9, p. 323]. We omit the proof. A *doubled triangle* is the matroid that is obtained from a triangle by adding a new element in parallel to each existing element.

Lemma 8.1. *Let M be a connected matroid so that* $\mathrm{si}(M)$ *is 3-connected. Let T_1 and T_2 be disjoint triangles in M. Then M has as a minor a doubled triangle that has ground set $T_1 \cup T_2$ and has T_1 and T_2 as triangles.*

Proof of Corollary 1.5. Let $t(n) = h(2n)$ where h is the function whose existence is established in Theorem 1.3. By that theorem, M has a minor N_1 that uses x and y and is isomorphic to one of $M(\mathcal{W}_{2n})$, $M[I_{2n}|A_{2n}]$, $M(K_{1,1,1,2n})$, or $M^*(K_{1,1,1,2n})$. Thus there are subsets C and D of $E(M)$ so that $M/C\backslash D = N_1$. If $z \notin C \cup D$, then the result follows easily. If $z \in C$, then x and y are parallel in N_1, a contradiction. Thus we may assume that $z \in D$. Let $N_2 = M/C\backslash(D - z)$, so N_2 is a single-element extension of N_1. We may assume that N_2 is simple otherwise z is parallel to some element w, and interchanging w and z gives the result. Thus N_2 is 3-connected.

For each of the possibilities for N_1, we will identify an exactly 3-separating set A in N_1 such that A contains $\{x, y\}$ while each of A and $E(N_1) - A$ has at least four elements. Then $A \cup z$ is exactly 3-separating in N_2. Thus, by [9, Proposition 9.3.4], there is a unique binary extension N_3 of N_2 by a triangle T that is disjoint from $E(N_2)$ such that N_2 is the 3-sum of N_A and N_B across T, where $N_A = N_3|(A \cup z \cup T)$ and $N_B = N_3|((E(N_1) - A) \cup T)$. We show next that

8.1.1. *N_2 has a minor isomorphic to* $\mathrm{si}(N_B)$ *that can be labelled so that it uses $\{x, y, z\}$.*

Clearly $\mathrm{si}(N_3)$ is 3-connected and is obtained from N_3 by deleting those elements of T that are parallel to elements of N_2. Moreover, it is straightforward to check that each of $\mathrm{si}(N_A)$ and $\mathrm{si}(N_B)$ is 3-connected and can be obtained by deleting those elements of T that are parallel to elements of N_A and N_B, respectively. Now N_A is connected and has T and $\{x, y, z\}$ as disjoint triangles. Thus, by Lemma 8.1, N_A has a doubled triangle as a minor in which both T and $\{x, y, z\}$ are triangles. From this, 8.1.1 follows immediately.

When $N_1 \cong M(\mathcal{W}_{2n})$, let the spokes of the wheel, in cyclic order, be $(s_1, s_2, \dots, s_{2n})$. Clearly, we may assume that $\{x, y\} \subseteq \mathrm{cl}_{N_1}(\{s_1, s_2, \dots, s_{n+1}\})$. In this case, we let $A = \mathrm{cl}_{N_1}(\{s_1, s_2, \dots, s_{n+1}\})$. Then one easily checks that $N_B \cong M(\mathcal{W}_n)$, and the result follows.

Next suppose $N_1 \cong M[I_{2n}|A_{2n}]$. Then N_1 is a spike with tip t and cotip c, so it consists of $2n$ lines, $L_1, L_2, \dots, L_{2n}$, all passing through the tip t, where $L_1 = \{t, c\}$ and all other L_i have three points. In this case, we may assume that $\{x, y\} \subseteq L_1 \cup L_2 \cup L_3$. Letting A be $L_1 \cup L_2 \cup L_3$, we see that A is exactly 3-separating in N_1. Now, as is easily checked, N_B is a rank-$(2n-2)$ spike with a tip but no cotip, so N_2 has a minor isomorphic to such a spike that uses $\{x, y, z\}$. Deleting some element from this matroid not in $\{x, y, z\}$ gives a rank-$(2n-2)$ spike with a tip and cotip that uses $\{x, y, z\}$, and the result follows easily.

Finally, suppose N_1 is isomorphic to $M(K_{1,1,1,2n})$ or its dual. Then $E(N_1)$ is the union of $2n+1$ disjoint 3-element sets, $T_0, T_1, \dots, T_{2n}$, where, when $N_1 \cong M(K_{1,1,1,2n})$, the set T_0 is a triangle and every other T_i is a triad that spans T_0. We may assume that $\{x, y\}$ is contained in the 3-separating set $T_0 \cup T_1 \cup T_2$, letting this last set be A. When $N_1 \cong M(K_{1,1,1,2n})$, it is clear that $\mathrm{si}(N_B) \cong M(K_{1,1,1,2n-2})$ where the base triangle T of the 3-sum is spanned by each triad in N_B, and the required result follows. When $N_1 \cong M^*(K_{1,1,1,2n})$, we get, using the remarks preceding Lemma 6.2, that $N_B \cong M^*(K_{3,2n-1})$. From this minor, it is easy to find an $M^*(K_{1,1,1,2n-3})$-minor preserving the triangle $\{x, y, z\}$. Since $2n - 3 \geq n$, the required result follows. □

References

[1] R. E. Bixby, A simple theorem on 3-connectivity, Linear Algebra Appl. 45 (1982) 123–126.
[2] R. E. Bixby, C. R. Coullard, Finding a smallest 3-connected minor maintaining a fixed minor and a fixed element, Discrete Appl. Math. 15 (1987) 155–166.
[3] T. H. Brylawski, A decomposition for combinatorial geometries, Trans. Amer. Math. Soc. 171 (1972) 235–282.
[4] D. Chun, J. Oxley, G. Whittle, Capturing matroid elements in unavoidable 3-connected minors, European J. Combin. 33 (2012) 1100–1112.

[5] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected binary matroids, J. Combin. Theory Ser. B 66 (1996) 334–360.
[6] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected matroids, J. Combin. Theory Ser. B 71 (1997) 244–293.
[7] J. Geelen, B. Gerards, G. Whittle, Excluding a planar graph from GF(q)-representable matroids, J. Combin. Theory Ser. B 97 (2007) 971–998.
[8] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, J. Combin. Theory Ser. B 57 (1993) 239–257.
[9] J. Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.
[10] J. Oxley, T. J. Reid, The smallest rounded sets of binary matroids, European J. Combin. 11 (1990) 47–56.
[11] P. D. Seymour, A note on the production of matroid minors, J. Combin. Theory Ser. B 22 (1977) 289–295.
[12] P. D. Seymour, On minors of non-binary matroids, Combinatorica 1 (1981) 387–394.
[13] P. D. Seymour, Minors of 3-connected matroids, European J. Combin. 6 (1985) 375–382.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
Current address: Mathematics Department, West Virginia University Institute of Technology, Montgomery, West Virginia
E-mail address: deborah.chun@mail.wvu.edu

Louisiana State University, Baton Rouge, Louisiana
E-mail address: oxley@math.lsu.edu