BOUNDING THE NUMBER OF BASES OF A MATROID

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Abstract. Thomassen proved in 2010 that the number of spanning trees of a graph with vertex degrees \(d_1, d_2, \ldots, d_n\) is at most \(d_1d_2\ldots d_{n-1}\). This note generalizes this result to show that if \(A\) is a matrix representing a rank-\(r\) matroid \(M\) over a field and \(S_1, S_2, \ldots, S_r\) are the supports of the rows of \(A\), then the number of bases of \(M\) is at most \(|S_1||S_2|\ldots|S_r|\). More generally, it is shown that if \(C^*_1, C^*_2, \ldots, C^*_r\) are cocircuits of a rank-\(r\) matroid \(N\) such that the deletion of any \(k\) of these cocircuits from \(N\) drops the rank by at least \(k\), then the number of bases of \(N\) is at most \(|C^*_1||C^*_2|\ldots|C^*_r|\).

1. Introduction

A number of authors including Alon [1] have given bounds on the number \(\tau(G)\) of spanning trees in a graph \(G\) in terms of the degree sequence of the graph. In particular, Kostochka [4] proved that, when \(G\) is simple having \(n\) vertices of degrees \(d_1, d_2, \ldots, d_n\), we have
\[
\tau(G) \leq \frac{1}{n-1}d_1d_2\ldots d_n.
\]
Thomassen [7] proved a similar result for an arbitrary graph \(G\) showing that
\[
\tau(G) \leq d_1d_2\ldots d_{n-1}.
\]
Recently, Klee, Naranyan, and Sauermann [3] have proved that, when \(G\) is simple,
\[
\tau(G) \leq \frac{1}{n^2}(d_1 + 1)(d_2 + 1)\ldots(d_n + 1).
\]
The purpose of this note is to prove a bound on the number \(b(M)\) of bases of a matroid \(M\) that generalizes Thomassen’s bound. The terminology used here for graphs and matroids will follow [6]. When \(G\) is a loopless 2-connected graph, the set of edges meeting a fixed vertex of \(G\) forms a cocircuit of its cycle matroid \(M(G)\).

Since \(r(M(G)) = |V(G)| - 1\), the following consequence of a result of Bondy and Welsh [2, Lemma 3.2] is a matroid analogue of Thomassen’s result.

Theorem 1.1. Let \(M\) be a rank-\(r\) matroid. Let \(C^*_1, C^*_2, \ldots, C^*_r\) be a collection of cocircuits of \(M\) such that no \(C^*_j\) is contained in \(\cup_{i \neq j} C^*_i\). Then
\[
b(M) \leq |C^*_1||C^*_2|\ldots|C^*_r|.
\]

Bondy and Welsh [2, Lemma 3.1] proved that the condition on cocircuits in the last theorem is equivalent to the assertion that \(\{C^*_1, C^*_2, \ldots, C^*_r\}\) is the set of fundamental cocircuits with respect to some cobasis of \(M\). Thus, such a collection of cocircuits has the property that the deletion of any \(k\) of them drops the rank by at least \(k\). With this condition alone, we can get the same conclusion as in the last theorem. This is our main result.
Theorem 1.2. Let $M$ be a rank-$r$ matroid. Let $C_1^r, C_2^r, \ldots, C_r^r$ be a collection of cocircuits of $M$ such that, for all $k$ in $\{1, 2, \ldots, r\}$, the deletion of the union of $k$ of these cocircuits from $M$ has rank at most $r - k$. Then $b(M) \leq |C_1^r||C_2^r| \ldots |C_r^r|$. Moreover, equality holds if and only if $M$ is the direct sum of $r$ rank-one uniform matroids with ground sets $C_1^r, C_2^r, \ldots, C_r^r$ and a rank-zero matroid.

2. The proof and some consequences

Let $(T_1, T_2, \ldots, T_r)$ be a collection of subsets of the ground set of a rank-$r$ matroid $M$ such that $r(M \setminus (T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k})) \leq r - k$ for all subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, r\}$. Then each $T_i$ must contain a cocircuit of $M$. We call $(T_1, T_2, \ldots, T_r)$ a full codependent family of $M$. For instance, if $(S_1, S_2, \ldots, S_r)$ is a presentation of a rank-$r$ transversal matroid $N$, then $(S_1, S_2, \ldots, S_r)$ is a full codependent family of $N$. Theorem 1.2 follows immediately from the next result, which also implies the subsequent corollary.

Proposition 2.1. Let $M$ be a rank-$r$ matroid. Let $(T_1, T_2, \ldots, T_r)$ be a full codependent family of $M$. Then $b(M) \leq |T_1||T_2| \ldots |T_r|$. Moreover, equality holds if and only if $M$ is the direct sum of $r$ rank-one uniform matroids with ground sets $T_1, T_2, \ldots, T_r$ and a rank-zero matroid.

Proof. Let $N$ be the transversal matroid having $(T_1, T_2, \ldots, T_r)$ as a presentation and having ground set $E(M)$. Clearly, $b(N) \leq |T_1||T_2| \ldots |T_r|$. Let $B$ be a basis of $M$. We shall show that $B$ is a basis of $N$. Consider the family $(B \cap T_1, B \cap T_2, \ldots, B \cap T_r)$. Suppose that $|(T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k}) \cap B| < k$ for some collection of $k$ of the sets $T_i$. By assumption, $r(M \setminus (T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k})) \leq r - k$, so $B$ is not a basis of $M$, a contradiction. Thus $|(T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k}) \cap B| \geq k$. Hence, by Hall’s Marriage Theorem, the family $(B \cap T_1, B \cap T_2, \ldots, B \cap T_r)$ has a transversal, so $B$ is a basis of $N$, as desired. Therefore $\mathcal{R}(M) \subseteq \mathcal{R}(N)$, so $b(M) \leq b(N) \leq |T_1||T_2| \ldots |T_r|$. Clearly $b(M) = |T_1||T_2| \ldots |T_r|$ when $M$ is the direct sum of $r$ rank-one uniform matroids with ground sets $T_1, T_2, \ldots, T_r$ and a rank-zero matroid. Now suppose that $b(M) = |T_1||T_2| \ldots |T_r|$. Then $\mathcal{R}(M) = \mathcal{R}(N)$, and $b(N) = |T_1||T_2| \ldots |T_r|$, so the sets $T_1, T_2, \ldots, T_r$ in the presentation of the transversal matroid $N$ are pairwise disjoint. Thus $N$ is the direct sum of $r$ rank-one uniform matroids with ground sets $T_1, T_2, \ldots, T_r$ and a rank-zero matroid with ground set $E(M) - (T_1 \cup T_2 \cup \cdots \cup T_r)$. As $\mathcal{R}(M) = \mathcal{R}(N)$, we deduce that $M = N$, so $M$ is the specified direct sum. \hfill \square

Corollary 2.2. Let $S_1, S_2, \ldots, S_r$ be the supports of the rows of a matrix $A$ that represents a rank-$r$ matroid $M$. Then $b(M) \leq |S_1||S_2| \ldots |S_r|$

We remark that Theorem 1.2 has Thomassen’s result as a consequence while Theorem 1.1 does not. For example, the set consisting of all but the largest vertex bond in a 2-connected bipartite graph $G$ does not have the property that no cocircuit in the collection is contained in the union of the others.

For matroids of rank two, we can obtain tight bounds on $b(M)$ that can be leveraged to improve on the bound on $b(M)$ for representable matroids.

Proposition 2.3. Let $M$ be a rank-2 matroid and let $C_1^2$ and $C_2^2$ be distinct cocircuits of $M$. Then

$$b(M) \leq |C_1^2||C_2^2| - \left(\frac{|C_1^2 \cap C_2^2| + 1}{2}\right)$$
and equality holds if and only if no element of $C^*_1 \cap C^*_2$ is in a 2-circuit. Furthermore, if $M$ is binary, then $b(M) = |C^*_1||C^*_2| - |C^*_1 \cap C^*_2|^2$.

Proof. Let $(A_1, A_2, A_3) = (C^*_1 - C^*_2, C^*_2 - C^*_1, C^*_1 \cap C^*_2)$. Then any basis of $M$ is a member of exactly one of the following sets:

$\mathcal{B}_1 = \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_2\}$; $\mathcal{B}_2 = \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_3\}$; $\mathcal{B}_3 = \{(x_1, x_2) : x_1 \in A_2, x_2 \in A_3\}$; $\mathcal{B}_4 = \{(x_1, x_2) : x_1, x_2 \in A_3, x_1 \neq x_2\}$.

Therefore, $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \subseteq \mathcal{B}(M) \subseteq \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. It follows that $|\mathcal{B}(M)| \leq |A_1||A_2| + |A_1||A_3| + |A_2||A_3| + {\binom{|A_2|}{2}} = |C^*_1||C^*_2| - \left(\frac{|C^*_1 \cap C^*_2|}{2}\right) + 1$. Moreover, equality holds in the last bound if and only if no two elements of $A_3$ are in parallel.

When $M$ is binary, the set $\mathcal{B}_4$ contains no bases of $M$, so $|\mathcal{B}(M)| = |A_1||A_2| + |A_1||A_3| + |A_2||A_3| = |C^*_1||C^*_2| - |C^*_1 \cap C^*_2|^2$. □

Lemma 2.4. Let $A$ be a matrix over a field such that $A$ has $r$ rows and rank $r$. For some $k$ with $1 \leq k \leq r - 1$, let $C$ and $D$ be the submatrices of $A$ consisting of the first $k$ and the last $r - k$ rows, respectively. Then $b(M[A]) \leq b(M[C])b(M[D])$.

Proof. Let $B$ be a basis of $M[A]$. Then the submatrix $A'$ of $A$ whose columns are labelled by the elements of $B$ has non-zero determinant. Using a Laplace expansion of $\det A'$ (see, for example, [5, p. 180]), we see that this determinant is a sum of terms each of which is plus or minus the product of the determinants of a $k \times k$ submatrix $C'$ of $C$ and an $(r - k) \times (r - k)$ submatrix $D'$ of $D$ such that every column label of $A'$ is a column label of exactly one of $C'$ and $D'$. Because $\det A'$ is non-zero, there must be such a pair $(C', D')$ for which both $\det C'$ and $\det D'$ are non-zero. Hence $B$ can be written as the disjoint union of a basis of $M[C]$ and a basis of $M[D]$. Thus $b(M[A]) \leq b(M[C])b(M[D])$. □

As an example of how we can combine the last two results, we have the following result, whose straightforward proof we omit.

Proposition 2.5. Let $A$ be an $r \times n$ binary matrix representing a rank-$r$ binary matroid $M$ where $r$ is even. Let $S_1, S_2, \ldots, S_r$ be the supports of the rows of $A$. Then $b(M)$ is at most

$$\left(1 - \frac{|S_1\cap S_2|^2}{|S_1||S_2|}\right)\left(1 - \frac{|S_3\cap S_4|^2}{|S_3||S_4|}\right) \cdots \left(1 - \frac{|S_{r/2}\cap S_{r/2}|^2}{|S_{r/2}|^2}\right).$$

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References
