ON DENSITY-CRITICAL MATROIDS

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Abstract. For a matroid \( M \) having \( m \) rank-one flats, the density \( d(M) \) is \( \frac{m}{r(M)} \) unless \( m = 0 \), in which case \( d(M) = 0 \). A matroid is density-critical if all of its proper minors of non-zero rank have lower density. By a 1965 theorem of Edmonds, a matroid that is minor-minimal among simple matroids that cannot be covered by \( k \) independent sets is density-critical. It is straightforward to show that \( U_{1,k+1} \) is the only minor-minimal loopless matroid with no covering by \( k \) independent sets. We prove that there are exactly ten minor-minimal simple obstructions to a matroid being able to be covered by two independent sets. These ten matroids are precisely the density-critical matroids \( M \) such that \( d(M) > 2 \) but \( d(N) \leq 2 \) for all proper minors \( N \) of \( M \). All density-critical matroids of density less than 2 are series-parallel networks. For \( k \geq 2 \), although finding all density-critical matroids of density at most \( k \) does not seem straightforward, we do solve this problem for \( k = \frac{3}{4} \).

1. Introduction

Our notation and terminology follow Oxley [7]. For a positive integer \( k \), let \( \mathcal{M}_k \) be the class of matroids \( M \) for which \( E(M) \) is the union of \( k \) independent sets. We say such a matroid can be covered by \( k \) independent sets. Edmonds [3] gave the following characterization of the members of \( \mathcal{M}_k \).

Theorem 1.1. A matroid \( M \) has \( k \) independent sets whose union is \( E(M) \) if and only if, for every subset \( A \) of \( E(M) \),

\[
k \tau(A) \geq |A|.
\]

Clearly, \( \mathcal{M}_k \) is closed under deletion. However, \( \mathcal{M}_k \) is not closed under contraction. For example, the 6-element rank-3 uniform matroid \( U_{3,6} \) can be covered by two independent sets, yet contracting a point of this matroid gives \( U_{2,5} \), which cannot. For all \( k \), the loop is the unique minor-minimal matroid not in \( \mathcal{M}_k \). On that account, we limit the types of obstructions we consider. We first examine the minor-minimal loopless matroids that are not in \( \mathcal{M}_k \). We find the following result.

Proposition 1.2. The unique minor-minimal loopless matroid that cannot be covered by \( k \) independent sets is \( U_{1,k+1} \).
Restricting attention to minor-minimal simple matroids not in $\mathcal{M}_k$, we find much more structure. We have the following collection of ten matroids for the case when $k$ is two. In this result, $P(M_1, M_2)$ denotes the parallel connection of matroids $M_1$ and $M_2$, this matroid being unique when both $M_1$ and $M_2$ have transitive automorphism groups. Geometric representations of the nine of these ten matroids of rank at most four are shown in Figure 1. A diagram representing the tenth matroid, $P(M(K_4), M(K_4))$ is also given where we note that this matroid has rank five.
Theorem 1.3. The minor-minimal simple matroids that cannot be covered by two independent sets are $U_{2,5}$, $P(U_{2,4}, U_{2,4})$, $O_7$, $P_7$, $F_7^-$, $F_7^+$, $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, $M^*(K_{3,3})$, and $P(M(K_4), M(K_4))$.

The following consequence of Theorem 1.1 will be helpful.

Lemma 1.4. Let $M$ be a minor-minimal matroid that cannot be covered by $k$ independent sets. Then

$$kr(M) = |E(M)| - 1.$$ 

Moreover, $M$ has no coloops.

For a matroid $M$, we write $\varepsilon(M)$ for $|E(si(M))|$, the number of rank-one flats of $M$. The density $d(M)$ of $M$ is $\frac{\varepsilon(M)}{r(M)}$ unless $r(M) = 0$. In the exceptional case, $\varepsilon(M) = 0$ and we define $d(M) = 0$. We say that $M$ is density-critical when $d(N) < d(M)$ for all proper minors $N$ of $M$. Note that all density-critical matroids are simple. By Lemma 1.4 and Theorem 1.1, $M$ is a minor-minimal simple matroid that cannot be covered by $k$ independent sets if and only if $d(M) > k$ but $d(N) \leq k$ for all proper minors $N$ of $M$. Such matroids are strictly $k$-density-critical where, for $t \geq 0$, we say a matroid is strictly $t$-density-critical when its density is strictly greater than $t$ while all its proper minors have density at most $t$. Thus Theorem 1.3 explicitly determines all ten strictly 2-density-critical matroids.

We propose the following.

Conjecture 1.5. For all positive integers $k$, there are finitely many minor-minimal simple matroids that cannot be covered by $k$ independent sets.

More generally, we make the following conjectures. For $t > 0$, we say a matroid is $t$-density-critical when its density is at least $t$ while all of its proper minors have density strictly less than $t$.

Conjecture 1.6. For all $t \geq 0$, there are finitely many strictly $t$-density-critical matroids.

Conjecture 1.7. For all $t > 0$, there are finitely many $t$-density-critical matroids.

We also propose the following weakening of the last conjecture.

Conjecture 1.8. For all $t \geq 0$, there are finitely many density-critical matroids with density exactly $t$.

We note that these conjectures hold over any class of matroids that is well-quasi-ordered with respect to minors. In particular, by a result announced by Geelen, Gerards, and Whittle (see, for example, [4]), these conjectures hold within the class of matroids representable over a fixed finite field.

Because the two excluded minors for series-parallel networks, $U_{2,4}$ and $M(K_4)$, have density exactly two, for $k < 2$, all density-critical matroids of density at most $k$ are series-parallel networks. For $k > 2$, finding all
density-critical matroids of density at most \( k \) does not seem straightforward. However, we were able to solve this problem when \( k = \frac{9}{4} \). For all \( n \geq 2 \), we denote by \( P_n \) any matroid that can be constructed from \( n \) copies of \( M(K_3) \) via a sequence of \( n - 1 \) parallel connections. In particular, \( P_2 \cong M(K_4 \setminus e) \).

There are two choices for \( P_3 \) depending on which element of \( M(K_4 \setminus e) \) is used as the basepoint of the parallel connection with the third copy of \( M(K_3) \). We denote by \( M_{18} \) the 18-element matroid that is obtained by attaching, via parallel connection, a copy of \( M(K_4) \) at each element of an \( M(K_3) \).

**Theorem 1.9.** The following is a list of all pairs \((M, d)\) where \( M \) is a density-critical matroid of density \( d \) and \( d \leq \frac{9}{4} \):

\[
(U_{1,1}, 1), \; (U_{2,3}, \frac{3}{2}), \; (M(P_n), \frac{2n+1}{n+1}) \quad \text{for all} \quad n \geq 2, \; (U_{2,4}, 2), \; (M(K_4), 2), \; (P(M(K_4), M(K_4)), \frac{11}{4}), \; (P(U_{2,4}, M(K_4)), \frac{9}{4}), \; (M(K_5 \setminus e), \frac{9}{4}), \; (M^*(K_{3,3}), \frac{9}{4}), \; (M_{18}, \frac{9}{4}).
\]

2. Preliminaries

This section proves some preliminary results beginning with two that were stated in the introduction.

**Proof of Proposition 1.2.** Clearly, \( U_{1,k+1} \) is a minor-minimal loopless matroid that cannot be covered by \( k \) independent sets. Conversely, suppose that \( M \) is a minor-minimal loopless matroid that cannot be covered by \( k \) independent sets. Certainly, \( M \) contains some element \( e \). Let \( P \cup \{e\} \) be the parallel class of \( M \) that contains \( e \) where \( P = \{e_1, e_2, \ldots, e_{\ell}\} \) and \( e \not\in P \).

Now \( M/e \setminus P \) is loopless, so, by minimality, \( M/e \setminus P \) can be covered by \( k \) independent sets \( \{A_1, A_2, \ldots, A_\ell\} \). Note that each \( A_j \cup \{e\} \) is independent in \( M \), so if \( |P| = \ell \leq k - 1 \), then \( \{A_1 \cup \{e_1\}, A_2 \cup \{e_2\}, \ldots, A_\ell \cup \{e_\ell\}, A_{\ell+1} \cup \{e\}, \ldots, A_k \cup \{e\}\} \) is a set of \( k \) independent sets that covers \( M \). Thus \( |P| \geq k \), and so \( M \cong U_{1,k+1} \).

Since \( U_{1,k+1} \) is a \((k+1)\)-element cocircuit, the matroids having no \( U_{1,k+1} \)-minor are precisely the matroids for which every cocircuit has at most \( k \) elements.

**Proof of Lemma 1.4.** Take \( x \) in \( E(M) \). Then \( M \setminus x \) can be covered by \( k \) independent sets. Thus, by Theorem 1.1,

\[
|E(M)| > kr(M) \geq kr(M \setminus x) \geq |E(M \setminus x)| = |E(M)| - 1.
\]

We deduce that \( kr(M) = |E(M)| - 1 \) and \( r(M) = r(M \setminus x) \) so \( M \) has no coloops.

**Lemma 2.1.** Let \( M \) be a density-critical matroid of rank at least two. For each subset \( S \) of \( E(M) \),

\[
|E(M)| - \varepsilon(M/S) > d(M)r(S).
\]

In particular, every element of \( M \) is in a triangle and is in at least two triangles when \( d(M) \geq 2 \).
Proof. Since $M$ is density-critical and therefore simple,

$$
\frac{\varepsilon(M/S)}{r(M/S)} < \frac{\varepsilon(M)}{r(M)} = \frac{|E(M)|}{r(M)}.
$$

Hence $r(M)\varepsilon(M/S) < |E(M)|(r(M) - r(S))$, so

$$
r(M)d(M)r(S) = |E(M)|r(S) < r(M)(|E(M)| - \varepsilon(M/S)).
$$

Thus $d(M)r(S) < |E(M)| - \varepsilon(M/S)$. In particular, $d(M) < |E(M)| - \varepsilon(M/e)$ for all $e$ in $E(M)$. Hence every such element $e$ is in at least one triangle, and $e$ is in at least two triangles when $d(M) \geq 2$. □

The next result will be useful in the proof of Theorem 1.3.

Lemma 2.2. In a 3-connected matroid $M$, let $F$ be a $2k$-element set $\{b_1, a_1, b_2, a_2, \ldots, b_k, a_k\}$. Suppose $\{b_1, b_2, \ldots, b_k\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a circuit for all $i$, where $b_{k+1} = b_1$. Then $M|F$ is a wheel of rank at least three or a whirl of rank at least two.

Proof. Since $M$ is 3-connected with at least four elements, it is simple. Now $M|F$ has $\{a_i, b_{i+1}, a_{i+1}\}$ as a triad, where $a_{k+1} = a_1$. By a result of Seymour [8] (see also [7, Lemma 8.8.5(ii)]), $M|F$ is a wheel or a whirl of rank $k$. □

3. The Matroids that Cannot Be Covered by Two Independent Sets

In this section, we prove Theorem 1.3 thereby specifying all of the minor-minimal simple matroids that cannot be covered by two independent sets.

Proof of Theorem 1.3. It is straightforward to check that each of the matroids listed is a minor-minimal simple matroid that cannot be covered by two independent sets. Now let $M$ be such a matroid. The next two assertions are immediate consequences of Lemmas 1.4, 2.1, and 1.1. However, we include proofs independent of Edmonds’s result for completeness.

3.1.1. Every element of $M$ is contained in at least two triangles.

Let $e$ be an element of $M$ and let $M' = si(M/e)$. By minimality, $M'$ has a partition into two independent sets $A$ and $B$. Suppose $e$ is not in a triangle. Then $E(M') = E(M) - \{e\}$ and we have $r_M(A \cup \{e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = |B| + 1$, so $M$ is covered by the independent sets $A \cup \{e\}$ and $B \cup \{e\}$, which is a contradiction.

Now suppose $e$ is in exactly one triangle $\{e, c, d\}$ of $M$. We may assume that $M' = M/e\setminus c$ and that $d \in A$. Then $r_M(A \cup \{c\}) = r_M(A \cup \{c, e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = r_{M'}(B) + 1 = |B| + 1$, so $M$ is covered by the independent sets $A \cup \{c\}$ and $B \cup \{e\}$. This contradiction implies that 3.1.1 holds.

3.1.2. $|E(M)| \leq 2r(M) + 1$ and $|A| \leq 2r(A)$ for every proper subset $A$ of $E(M)$. 

Suppose $A$ is a proper subset of $E(M)$. By the minimality of $M$, we can cover $M \setminus A$ by two independent sets, and so $|A| \leq 2r(A)$. It follows easily that $|E(M)| \leq 2r(M) + 1$. Thus 3.1.2 holds.

We construct a simple auxiliary graph $G$ from $M$, the vertices of which are the elements of $M$; two such vertices are adjacent exactly when they share a triangle in $M$. Next, we show the following.

**3.1.3.** Let $Z$ be the vertex set of a component of $G$. Then $M \setminus Z$ has a wheel or a whirl as a restriction.

We may assume that $M \setminus Z$ has no line with four or more points otherwise $M$ has a rank-2 whirl as a restriction. For $b_1$ in $Z$, by 3.1.1, we can construct a maximal sequence $b_1, a_1, b_2, a_2, \ldots, b_n$ of distinct elements such that $\{b_1, b_2, \ldots, b_n\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a triangle for all $i$ in $\{1, 2, \ldots, n-1\}$. Then $n \geq 3$.

Now $M$ has a triangles $\{b_n, a_n, b_{n+1}\}$ and $\{b_0, a_0, b_1\}$ that differ from $\{b_{n-1}, a_{n-1}, b_n\}$ and $\{b_1, a_1, b_2\}$, respectively. Let $A' = \{b_1, a_1, b_2, a_2, \ldots, b_{n-1}, a_{n-1}, b_n\}$. Assume that both $\{a_n, b_{n+1}\}$ and $\{a_0, b_0\}$ avoid $A'$. Then $|A' \cup \{a_n, b_{n+1}\}| = 2n + 1 = 2r(A' \cup \{a_n, b_{n+1}\}) + 1$. Thus, by 3.1.2, $A' \cup \{a_0, b_0\} = E(M)$. By symmetry, $A' \cup \{a_n, b_{n+1}\} = E(M)$. Hence $\{a_n, b_{n+1}\} = \{b_0, a_0\}$, so $\{b_n, a_n, b_{n+1}, b_1\}$ is a 4-point line, a contradiction.

We may now assume that $b_{n+1}$ is a member $c_i$ of $\{b_1, a_i\}$ for some $i$ with $1 \leq i \leq n - 1$. Then $\{c_i, b_{i+1}, b_{i+2}, \ldots, b_n\}$ is an independent set in $M \setminus Z$ such that every two consecutive elements in the given cyclic order are in a triangle. Thus, by Lemma 2.2, $M \setminus Z$ has a wheel or whirl of rank $n - i + 1$ as a restriction. Hence 3.1.3 holds.

**3.1.4.** For some component of $G$ having vertex set $Z$, the matroid $M \setminus Z$ is not a wheel or a whirl.

Assume that this fails. Then, by 3.1.1, the only components of $G$ are rank-2 whirls or rank-3 wheels. Assume there are $s$ of the former and $t$ of the latter. Then $|E(M)| = 4s + 6t = 2(2s + 3t)$. Clearly $r(M) \leq 2s + 3t$. By 3.1.2, equality must hold here. Hence each component of $G$ corresponds to a wheel or whirl component of $M$. As each wheel and each whirl can be covered by two independent sets, so too can $M$, a contradiction. Thus 3.1.4 holds.

Now take a component of $G$ having vertex set $Z$ such that $M \setminus Z$ is not a wheel or a whirl. By 3.1.3, consider a wheel or whirl restriction of $M \setminus Z$ with basis $B = \{b_1, b_2, \ldots, b_n\}$ and ground set $W = \{b_1, a_1, b_2, a_2, \ldots, b_n, a_n\}$. Let $\{b_i, a_i, b_{i+1}\}$ be a triangle for all $i$ where $b_{n+1} = b_1$. As $W \neq Z$, there is a point $\beta_1$ in $W$ that is contained in a triangle $\{\beta_1, \alpha_1, \beta_2\}$ that is not a triangle of $M \setminus W$. If $M \setminus W$ is a rank-2 whirl or a rank-3 wheel, then, by symmetry, we may assume that $\beta_1 = a_1$. If, instead, $M \setminus W$ is neither a rank-2 whirl nor a rank-3 wheel, then 3.1.1 guarantees that such a triangle $\{\beta_1, \alpha_1, \beta_2\}$ exists with $\beta_1 = a_1$. By repeatedly using 3.1.1, we can construct a sequence
\( \beta_1, \alpha_1, \ldots, \beta_{m+1} \) where \( \{\beta_i, \alpha_i, \beta_{i+1}\} \) is a triangle for all \( i \) in \( \{1, 2, \ldots, m\} \) and \( B \cup \{\beta_2, \ldots, \beta_m\} \) is dependent but \( B \cup \{\beta_2, \ldots, \beta_m\} \) is independent. By potentially interchanging \( \alpha_m \) and \( \beta_{m+1} \), we may assume that \( \alpha_m \notin W \). Let \( Q = \{\beta_1, \alpha_1, \ldots, \beta_{m+1}\} \). Then
\[
\rho(W \cup Q) = \rho(W \cup (Q - \{\beta_{m+1}\})) = n + m - 1.
\]
(1)
As \( |W \cup (Q - \{\beta_{m+1}\})| = 2(n + m - 1) + 1 = 2\rho(W \cup (Q - \{\beta_{m+1}\})) + 1 \), we deduce, by 3.1.2, that
\[
W \cup (Q - \{\beta_{m+1}\}) = E(M).
\]
(2)
Hence
\[
\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\}).
\]
(3)
Assume that the theorem fails. We now show that

3.1.5. \( M|Z \) has no wheel-restriction of rank exceeding three and no whirl-restriction of rank exceeding two.

Assume that this fails. Then we may assume that \( M|W \) is a wheel of rank at least four or a whirl of rank at least three. Now \( \rho(W) = n \) and \( \rho(Q) \leq m + 1 \). By (1) and submodularity, \( \rho(\text{cl}(W) \cap \text{cl}(Q)) \leq 2 \). Assume \( W \) does not span \( M \). Then, by (1) and (2), we see that \( m > 1 \) and the only possible elements of \( W \) that can lie in triangles with elements of \( Q - W \) are \( \beta_1 \) and \( \beta_{m+1} \). But a wheel of rank at least four and a whirl of rank at least three have at least three elements that are in unique triangles. Hence one of these elements will violate 3.1.1.

We now know that \( W \) spans \( M \), so the unique element of \( Q - W \) is \( \alpha_3 \). Each of \( a_1, a_2, \ldots, a_n \) must be in a triangle with \( \alpha_1 \), the other element of which is in \( W \). Assume both \( \{a_1, \alpha_1, a_3\} \) and \( \{a_1, \alpha_1, a_{n-1}\} \) are triangles. Then \( n = 4 \). Suppose \( \{a_2, \alpha_1, a_4\} \) is also a triangle. Then, by Lemma 2.2, for each \( i \) in \( \{2, 4\} \), deleting \( a_i \) from \( M|(W \cup Q) \) gives a wheel or whirl of rank four. As \( \{b_1, b_4, \alpha_1, a_2\} \) and \( \{b_2, b_3, \alpha_1, a_4\} \) are circuits, both of these deletions are wheels. It follows that \( M|(W \cup Q) \cong M^*(K_{3,3}) \), so \( M \cong M^*(K_{3,3}) \), a contradiction. Thus, we may assume that \( \{a_2, \alpha_1, a_4\} \) is not a triangle. Since \( \alpha_1 \notin \text{cl}(\{b_1, b_2, b_3\}) \cup \text{cl}(\{b_2, b_3, b_4\}) \), there is no triangle containing \( \{a_2, \alpha_1\} \), a contradiction.

We may now assume that \( \{a_1, \alpha_1, a_3\} \) is not a triangle. Then, by 3.1.1, \( W \) has distinct elements \( x \) and \( y \) such that \( \{a_1, \alpha_1, x\} \) and \( \{a_3, \alpha_1, y\} \) are triangles. Thus \( \{a_1, a_3, x, y\} \) contains a circuit. Now \( \{a_1, a_3\} \) is not in a triangle of \( M|W \). Moreover, if \( \{a_1, x, y\} \) is a triangle, then \( \{x, y\} = \{b_1, b_2\} \). Using the triangles, \( \{a_1, \alpha_1, x\} \) and \( \{a_3, \alpha_1, y\} \), we deduce that \( a_3 \notin \text{cl}(\{b_1, b_2\}) \), a contradiction. It follows that \( \{a_1, a_3, x, y\} \) is a circuit of \( M \). Thus \( M|W \) is either a rank-3 whirl or a rank-4 wheel.

Suppose \( M|W \) is a rank-3 whirl. Then \( M \) is an extension of this matroid by \( \alpha_1 \) in which every element is in at least two triangles. If \( \{a_1, a_2, \alpha_1\} \) or \( \{a_2, a_3, \alpha_1\} \) is a triangle, then one easily checks that \( M \cong O_7 \) or \( M \cong P_7 \), a contradiction. Hence we may assume that none of \( \{a_1, a_2, \alpha_1\}, \{a_2, a_3, \alpha_1\}, \)
or \{a_3, a_1, \alpha_1\} is a triangle. Then, to avoid having \(U_{2,5}\) as a minor of \(M\), we must have \(\{a_1, b_3, \alpha_1\}, \{a_2, b_1, \alpha_1\}\), and \(\{a_3, b_2, \alpha_1\}\) as triangles, that is, \(M \cong F_7^-\), a contradiction.

We are left with the possibility that \(M|W\) is a rank-4 wheel. Since it has \(\{a_1, a_3, x, y\}\) as a circuit, it follows that \(\{x, y\} = \{a_2, a_4\}\). Then \(M\) has either \(\{a_1, a_2, \alpha_1\}\) and \(\{a_3, a_4, \alpha_1\}\) as triangles or \(\{a_1, a_4, \alpha_1\}\) and \(\{a_2, a_3, \alpha_1\}\) as triangles. By symmetry, we may assume that we are in the second case. Then, by submodularity using the sets \(\{b_1, b_2, a_1, a_4, b_4, \alpha_1\}\) and \(\{b_2, b_3, a_2, a_3, b_4, \alpha_1\}\), we deduce that \(r(\{b_1, b_4, \alpha_1\}) = 2\). It follows that \(M \cong M(K_5\setminus e)\), a contradiction. We conclude that 3.1.5 holds.

Now suppose that \(W\) spans \(Z\). If \(M|W\) is a rank-2 whirl, then \(M|Z \cong U_{2,5}\), a contradiction. If \(M|W\) is a rank-3 wheel, then one easily checks that \(M|Z\) is isomorphic to one of \(O_7\), \(F_7^-\), or \(F_7\), a contradiction.

We may now assume that \(W\) does not span \(Z\). Then \(m > 1\). By (3), \(\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\})\). We will first suppose that \(\beta_{m+1} = \beta_i\) for some \(i\) in \(\{1, 2, \ldots, m\}\). Then \(\{\beta_i, \beta_{i+1}, \ldots, \beta_m\}\) is an independent set and \(\{\beta_j, \alpha_j, \beta_{j+1}\}\) is a triangle for all \(j\) in \(\{i, i+1, \ldots, m\}\). By 3.1.5 and Lemma 2.2, for \(R = \{\beta_i, \alpha_i, \beta_{i+1}, \alpha_{i+1}, \ldots, \beta_m, \alpha_m\}\), the matroid \(M|R\) is a rank-3 wheel or a rank-2 whirl. Then the matroid obtained from \(M|Z\) by contracting \(\{\alpha_2, \alpha_3, \ldots, \alpha_{i-1}\}\) and simplifying is the parallel connection of \(M|W\) and \(M|R\), that is, \(M|Z\) has as a minor one of \(P(U_{2,4}, U_{2,4}), P(U_{2,4}, M(K_4))\), and \(P(M(K_4), M(K_4))\), a contradiction.

Finally, suppose that \(\beta_{m+1} \in W\). Consider the first case and take \(\alpha_{m-1} = \beta_1\). Then, by 3.1.5 and Lemma 2.2, with \(R = \{\beta_{i+1}, \alpha_{i+1}, \ldots, \beta_{m+1}, \alpha_{m+1}\}\), we have that \(M|R\) is a rank-3 wheel or a rank-2 whirl. Contracting \(\{\alpha_2, \alpha_3, \ldots, \alpha_{i-1}\}\) from \(M|Z\) and simplifying, we obtain one of \(P(U_{2,4}, U_{2,4})\), \(P(U_{2,4}, M(K_4))\), and \(P(M(K_4), M(K_4))\), a contradiction. In the second case, when \(\beta_{m+1} \in W\), we recall that \(\beta_1 = \alpha_1\). Suppose that \(\{\beta_1, \beta_{m+1}\}\) is not in a triangle of \(M|W\). Then \(M|W \cong M(K_4)\) and \(\beta_{m+1} = b_3\). By assumption, \(\{b_1, b_2, b_3\} \cup \{\beta_2, \ldots, \beta_m\}\) is independent. By Lemma 2.2, the triangles \(\{b_1, b_2, a_1\}, \{a_1, \alpha_1, \beta_2\}, \ldots, \{\beta_m, \alpha_m, b_3\}, \{b_3, a_3, b_1\}\) imply that \(M|Z\) has a wheel or whirl of rank at least four as a restriction, a contradiction. We deduce that \(\{\beta_1, \beta_{m+1}\}\) is in a triangle of \(M|W\). Then, by symmetry, we may assume that \(\beta_{m+1} = b_1\). We let \(\alpha_{m+1} = b_2\). Then, for \(R = \{\beta_1, \alpha_1, \ldots, \beta_{m+1}, \alpha_{m+1}\}\), we have that \(M|R\) is a rank-3 wheel or a rank-2 whirl. But \(\alpha_1 \notin cl(W)\), so \(M|R\) is a rank-3 wheel. If \(M|W\) is a rank-2 whirl, then \(O_7\) is a restriction of \(M|Z\), a contradiction. If \(M|W\) is a rank-3 wheel, then \(M|(W \cup R)\) has rank four and consists of two copies of \(M(K_4)\) sharing a triangle. This matroid is \(M(K_5\setminus e)\), a contradiction. □

4. The density-critical matroids of small density

In this section, we prove Theorem 1.9. The following result [6] (see also [7, Lemma 4.3.10]) will be used repeatedly in this proof.
Lemma 4.1. Let $M$ be a connected matroid having at least two elements and let \{$e_1, e_2, \ldots, e_m$\} be a cocircuit of $M$ such that $M/e_i$ is disconnected for all $i$ in \{1, 2, \ldots, m-1\}. Then \{e_1, e_2, \ldots, e_{m-1}\} contains a 2-circuit of $M$.

We shall make repeated use of the following consequence of this lemma.

Corollary 4.2. Let $M$ be a simple connected matroid and $Z$ be a non-empty non-spanning subset of $E(M)$. Then $M$ has a simple connected minor $N$ such that $N|Z = M|Z$ and $r(N) = r_M(Z)$.

Proof. Let $C^*$ be a cocircuit of $M$ that is disjoint from $\text{cl}(Z)$. As $M$ is simple, it follows by Lemma 4.1 that there is an element $e$ of $C^*$ such that $M/e$ is connected. Since $e \notin \text{cl}(Z)$, we see that $(M/e)|Z = M|Z$. Clearly we can label $\text{si}(M/e)$ so that its ground set contains $Z$. If $r(M) - r(Z) = 1$, then we take $N = \text{si}(M/e)$. Otherwise we repeat the above process using $\text{si}(M/e)$ in place of $M$. After $r(M) - r(Z)$ applications of this process, we obtain the desired minor $N$. □

The next result, which was proved by Dirac [2], follows easily by induction after recalling that a connected matroid with no minor isomorphic to $U_{2,4}$ or $M(K_4)$ is isomorphic to the cycle matroid of a series-parallel network.

Lemma 4.3. Let $M$ be a simple matroid having no minor isomorphic to $U_{2,4}$ or $M(K_4)$. Then

$$|E(M)| \leq 2r(M) - 1.$$

We omit the elementary proof of the next result a consequence of which is that every density-critical matroid is connected.

Lemma 4.4. Let $M_1$ and $M_2$ be matroids of rank at least one. Then

$$d(M_1 \oplus M_2) \leq \max\{d(M_1), d(M_2)\}.$$

Moreover, equality holds here if and only if $d(M_1) = d(M_2)$.

The next result will be useful in identifying the density-critical matroids of density at most two.

Lemma 4.5. Let $M$ be a density-critical matroid with $d(M) \leq 2$. If $(X_1, X_2)$ is a 2-separation of $M$, then there is an element $p$ in $\text{cl}(X_1) \cap \text{cl}(X_2)$, and $M = P(M|(X_1 \cup \{p\}), M|(X_2 \cup \{p\}))$.

Proof. As $(X_1, X_2)$ is a 2-separation of $M$, for some element $q$ not in $E(M)$, we can write $M$ as $M_1 \oplus_M M_2$ where each $M_i$ has ground set $X_i \cup \{q\}$. Let $|E(M_i)| = n_i$ and $r(M_i) = r_i$. Assume that both $M_1$ and $M_2$ are simple. Then $\frac{|E(M)|}{r(M)} > \frac{|E(M_i)|}{r(M_i)}$, so

$$\frac{n_1 + n_2 - 2}{r_1 + r_2 - 1} > \frac{n_1}{r_1}.$$

Hence

$$r_1n_2 - 2r_1 > r_2n_1 - n_1.$$
By symmetry,
\[ r_2 n_1 - 2r_2 > r_1 n_2 - n_2. \]
Adding the last two inequalities gives \( n_1 + n_2 > 2(r_1 + r_2) \), so \( n_i > 2r_i \)
for some \( i \). Thus \( d(M_i) > 2 \). Since \( M \) is density-critical with density at
most two, this is a contradiction. We conclude that \( M_1 \) or \( M_2 \), say \( M_1 \), is
non-simple. Thus it has an element \( p \) in parallel with the basepoint \( q \) of the
2-sum. Hence \( M = P(M(X_1 \cup \{p\}), M(X_2 \cup \{p\})) \).

\[ \square \]

**Lemma 4.6.** Let \( N \) be a simple connected matroid in which all but at most
one element is in at least two triangles. Then \( N \) has no 2-cocircuits. Moreover, if \( N \) has \( \{a, b, c\} \) as a triad, then either

(i) \( \{a, b, c\} \) is contained in a 4-point line and \( N = P(U_2, A, N \setminus \{a, b, c\}) \);

or

(ii) \( N \) has a triangle \( \{x, y, z\} \) such that \( N\{a, b, c, x, y, z\} \cong M(K_4) \)

and \( N \) is the generalized parallel connection of \( N\{a, b, c, x, y, z\} \) and \( N \setminus \{a, b, c\} \) across the triangle \( \{x, y, z\} \).

**Proof.** As \( N \) has at most one element that is not in at least two triangles, \( N \) has no 2-cocircuits. Suppose \( \{a, b, c\} \) is a triad of \( N \). If \( \{a, b, c\} \) is also a
triangle, then \( \{a, b, c\} \) is 2-separating in \( N \). Moreover, \( \{a, b, c\} \) is contained in
a 4-point line \( \{a, b, c, d\} \) and (i) holds.

We may now assume that \( \{a, b, c\} \) is not a triad of \( N \). Then, because at
least two of \( a, b, \) and \( c \) are in at least two triangles, the hyperplane \( E(N) - \{a, b, c\} \) of \( N \) contains distinct elements \( x, y, \) and \( z \) such that \( \{a, b, z\}, \{a, y, c\}, \) and \( \{x, b, c\} \) are triangles. Now

\[ r(\{x, y, z\}) \leq r(E(N) - \{a, b, c\}) + r(cl(\{a, b, c\})) - r(N) = r(N) - 1 + 3 - r(N) = 2. \]

Thus \( \{x, y, z\} \) is a triangle of \( N \) and \( N\{a, b, c, x, y, z\} \cong M(K_4) \). It follows
by a result of Brylawski [1] (see also [7, Proposition 11.4.15]) that (ii) holds.

\[ \square \]

**Corollary 4.7.** Let \( N \) be a simple connected matroid in which all but at most
one element is in at least two triangles and \( d(N) \leq \frac{9}{4} \). If \( r(N) = 2 \),
then \( N \cong U_{2,4} \). If \( r(N) = 3 \), then \( N \cong M(K_4) \). If \( r(N) = 4 \), then \( N \cong P(U_{2,4}, M(K_4)), M(K_5 \setminus e), \) or \( M^*(K_{3,3}) \).

**Proof.** We omit the straightforward proof for the case when \( r(N) \in \{2, 3\} \).
Assume \( r(N) = 4 \). By Lemma 4.6, \( N \) has no 2-cocircuits. Now suppose \( N \) has \( \{a, b, c\} \) as a triad. If (i) of Lemma 4.6 holds, then \( N = P(U_{2,4}, N \setminus \{a, b, c\}) \). By the result in the rank-3 case, \( N \setminus \{a, b, c\} \cong M(K_4) \),
so \( N \cong P(U_{2,4}, M(K_4)) \). If, instead, (ii) of Lemma 4.6 holds, then \( N \) is the
generalized parallel connection across a triangle \( \{x, y, z\} \) of \( M(K_4) \) and \( N \setminus \{a, b, c\} \). In the latter, \( E(N \setminus \{a, b, c, x, y, z\}) \) must be a triad of \( N \), so \( N \setminus \{a, b, c\} \cong M(K_4) \). Hence \( N \) is the generalized parallel connection across
a triangle of two copies of \( M(K_4) \), so \( N \cong M(K_5 \setminus e) \).
We may now assume that \( N \) has no triads. Then every cocircuit of \( N \) has at least four elements. As \( N \) certainly has a plane that contains two intersecting triangles, \( \{ x, f_1, g_1 \} \) and \( \{ x, f_2, g_2 \} \), we deduce that \( |E(N)| \geq 9 \), so \( |E(N)| = 9 \). Let \( \{ a, b, c, d \} \) be the cocircuit \( E(N) - \{ x, f_1, f_2, g_1, g_2 \} \). Because \( N \) has no plane with more than five points and has all but at most one element in two triangles, we may assume that \( \{ a, b, g_1 \} \) and \( \{ a, c, g_2 \} \) are triangles of \( N \). Then \( N \setminus d \) has \( \{ x, f_1, g_1 \}, \{ g_1, b, a \}, \{ a, c, g_2 \}, \{ g_2, f_2, x \} \) as triangles. By Lemma 2.2, \( N \setminus d \) is a rank-4 wheel or whirl. In this matroid, \( f_1, b, c, \) and \( f_2 \) are in unique triangles. It follows that \( N \) must have \( \{ d, f_1, c \} \) and \( \{ d, b, f_2 \} \) as triangles. Thus \( N \setminus d \) is a rank-4 wheel. Likewise, \( N \setminus f_1 \) and \( N \setminus c \) are also rank-4 wheels, so \( N \cong M^*(K_{3,3}) \).

**Lemma 4.8.** Let \( N \) be a simple matroid of rank at least three in which every element is in at least two triangles. Suppose \( e \in E(N) \). Then

(i) \( e \) is in a plane of \( N \) having at least seven points; or
(ii) every element of \( \text{si}(N/e) \) is in at least two triangles; or
(iii) \( N \) has a \( U_{2,4} \)- or \( M(K_4) \)-restriction using \( e \).

**Proof.** Assume that neither (i) nor (iii) holds. We show that every element of \( \text{si}(N/e) \) is in at least two triangles. First consider a triangle \( \{ e, c_1, c_2 \} \) of \( N \) containing \( e \). Let \( \{ c_1, d_1, f_1 \} \) and \( \{ c_2, d_2, f_2 \} \) be triangles of \( N \) where neither contains \( e \). If \( r(\{ e, c_1, d_1, f_1, c_2, d_2, f_2 \}) = 4 \), then, in \( \text{si}(N/e) \), the element \( c \) corresponding to \( c_1 \) and \( c_2 \) is in at least two triangles. Now suppose \( r(\{ e, c_1, d_1, f_1, c_2, d_2, f_2 \}) = 3 \). Since \( N \) has no plane with more than six points, we may assume that \( f_1 = f_2 \). Rename this element \( f \). If \( \{ e, d_1, d_2 \} \) is not a triangle, then \( \text{si}(N/e) \) has a 4-point line containing \( c \), so \( c \) is in at least two triangles of this matroid. If \( \{ e, d_1, d_2 \} \) is a triangle of \( N \), then \( N \setminus \{ e, c_1, c_2, d_1, d_2, f \} \) \( \cong M(K_4) \), a contradiction.

Now let \( f \) be an element of \( N \) that is not in a triangle with \( e \). Let \( \{ f, g_1, h_1 \} \) and \( \{ f, g_2, h_2 \} \) be triangles of \( N \). Then \( \text{si}(N/e) \) has at least two triangles containing \( f \) otherwise \( N \setminus \{ e, f, g_1, h_1, g_2, h_2 \} \) \( \cong M(K_4) \), a contradiction.

Recall that \( M_{18} \) is the 18-element matroid that is obtained by attaching, via parallel connection, a copy of \( M(K_4) \) at each element of an \( M(K_3) \).

**Lemma 4.9.** Let \( N \) be a simple connected non-empty matroid in which every element is in a \( U_{2,4} \)- or \( M(K_4) \)-restriction. Assume that \( d(N) \leq \frac{9}{4} \) but \( d(N') < \frac{9}{4} \) for all proper minors \( N' \) of \( N \). Then \( N \) is isomorphic to \( U_{2,4}, M(K_4), P(U_{2,4}, M(K_4)), P(M(K_4), M(K_4)), M(K_5 \setminus e) \), or \( M_{18} \).

**Proof.** Since \( d(N') \leq \frac{9}{4} \) for all minors \( N' \) of \( N \), we see that, in any such \( N' \), no line has more than four points and no plane has more than six points. Next we show the following.

**4.9.1.** If \( N \) has a 4-point line, then \( N \) is isomorphic to \( U_{2,4} \) or \( P(U_{2,4}, M(K_4)) \).
This is immediate if \( r(N) = 2 \). Because \( N \) has no plane with more than six points, \( r(N) \neq 3 \). Let \( L \) be a 4-point line of \( N \) and let \( Z \) be a subset of \( E(N) \) not containing \( L \) such that \( N|Z \) is isomorphic to \( U_{2,4} \) or \( M(K_4) \). If \( L \cap Z \neq \emptyset \), then again, since \( N \) has no plane with more than six points, we deduce that \( N \cong P(U_{2,4}, M(K_4)) \). We may now assume that \( L \cap Z = \emptyset \). If \( r(L \cup Z) \leq r(Z) + 1 \), then \( N \) has a rank-3 or rank-4 restriction of density exceeding \( \frac{9}{4} \), a contradiction. We deduce that \( r(L \cup Z) = r(Z) + 2 \).

By Corollary \( 4.2 \), \( N \) has a simple connected minor \( N' \) such that \( N'|(L \cup Z) = N|(L \cup Z) \) and \( r(N') = r(Z) + 2 \). As \( N' \) is connected, it has an element \( x' \) that is not in the closure of \( L \) or of \( Z \). Then \( N'/x' \) has \( N|L \) and \( N|Z \) as restrictions and has rank \( r(Z) + 1 \). Thus \( \sigma(N'/x') \) has either a plane with more than six points or has \( P(U_{2,4}, M(K_4)) \) as a restriction. Each possibility yields a contradiction, so \( 4.9.1 \) holds.

We may now assume that every element of \( N \) is in an \( M(K_4) \)-restriction. We may also assume that \( N \) is not isomorphic to \( M(K_4) \) or \( P(M(K_4), M(K_4)) \). Next we show the following.

4.9.2. Let \( X \) and \( Y \) be distinct subsets of \( E(N) \) such that both \( N|X \) and \( N|Y \) are isomorphic to \( M(K_4) \). If \( |X \cap Y| \geq 2 \), then \( N \cong M(K_5 \setminus e) \).

Since \( N \) has no plane with more than six points, \( r(X \cup Y) \geq 3 \). As \( |X \cap Y| \geq 2 \), it follows by submodularity that \( r(X \cup Y) = 4 \) and \( r(X \cap Y) = 2 \).

As \( d(N|(X \cup Y)) \leq \frac{9}{4} \), we deduce that \( |X \cup Y| = 9 \), so \( |X \cap Y| = 3 \) and \( N = N|(X \cup Y) \). Moreover, \( N|X \) and \( N|Y \) meet in a triangle \( \Delta \). By Lemma \( 4.6 \), \( N \) is the generalized parallel connection of \( N|X \) and \( N|Y \) across \( \Delta \). Thus \( N \cong M(K_5 \setminus e) \) as each of \( N|X \) and \( N|Y \) is isomorphic to \( M(K_4) \), so \( 4.9.2 \) holds.

We may now assume that \( E(N) \) has at least three distinct subsets \( X \) with \( N|X \cong M(K_4) \) and that no two such subsets meet in more than one element.

4.9.3. \( N \) does not have \( P(M(K_4), M(K_4)) \) as a restriction.

Assume that \( N|X \cong P(M(K_4), M(K_4)) \) and \( N|Y \cong M(K_4) \) where \( Y \not\subseteq X \). Suppose \( |X \cap Y| = k \) where \( k \in \{1, 2\} \). Then \( r(X \cup Y) \leq 8 - k \) and \( |X \cup Y| = 17 - k \), so

\[
\frac{9}{4} \geq d(N|(X \cup Y)) \geq \frac{17 - k}{8 - k}.
\]

Simplifying we obtain the contradiction that \( 4 \geq 5k \geq 5 \). We deduce using 4.9.2 that \( |X \cap Y| = 0 \). Then \( r(X \cup Y) = 8 \) otherwise \( d(N|(X \cup Y)) > \frac{9}{4} \).

By Corollary \( 4.2 \), \( N \) has a simple connected minor \( N' \) such that \( N'|(X \cup Y) = N|(X \cup Y) \) and \( r(N') = 8 \). As \( N|(X \cup Y) \) is disconnected, \( N' \) must contain an element that is not in \( X \cup Y \). Hence \( |E(N')| \geq 18 \), so \( d(N') \geq \frac{9}{4} \). Thus \( N' = N \) and \( |E(N)| = 18 \), so \( N \) has a single element \( z \) that is not in \( X \cup Y \). The \( M(K_4) \)-restriction of \( N \) that contains \( z \) is forced to have more than one element in common with \( Y \) or one of the \( M(K_4) \)-restrictions of \( N|X \). This contradiction to 4.9.2 completes the proof of 4.9.3.
We now know that any two $M(K_4)$-restrictions of $N$ have disjoint ground sets. Let $X, Y,$ and $Z$ be distinct subsets of $E(N)$ such that each of $N|X, N|Y,$ and $N|Z$ is isomorphic to $M(K_4)$. Next we show the following.

4.9.4. $r(X \cup Y) = 6$. Moreover, $r(X \cup Y \cup Z) = 9$ unless $N \cong M_{18}$.

As $|X \cup Y| = 12$ and $d(N|(X \cup Y)) < \frac{9}{4}$, we deduce that $r(X \cup Y) = 6$. The density constraint also means that $r(X \cup Y \cup Z) \geq 8$. Suppose $r(X \cup Y \cup Z) = 8$. Then $d(N|(X \cup Y \cup Z)) = \frac{9}{4}$, so $N = N|(X \cup Y \cup Z)$. Now $r(N/Z) = 5$. As $\frac{12}{5} > \frac{9}{4}$, we must have some parallel elements in $N/Z$. As $Z$ is skew to each of $X$ and $Y$, we know that $(N/Z)|X = N|X$ and $(N/Z)|Y = N|Y$. Thus there must be elements $x$ of $X$ and $y$ of $Y$ that are parallel in $N/Z$. If there is a second such parallel pair, then $r(N/Z) \leq 4$, a contradiction. In $N$, we see that $r(Z \cup \{x, y\}) = 4$. Hence, in $N/x$, we obtain a 7-point plane $Z \cup y$ unless $\{x, y, z\}$ is a triangle of $N$ for some $z$ in $Z$. Observe that each of $N/x$, $N/y$, and $N/z$ is disconnected, so $N$ is obtained from $M(K_3)$ by attaching a copy of $M(K_4)$ via parallel connection at each element. Thus $N \cong M_{18}$ and 4.9.4 holds.

By Corollary 4.2, $N$ has a simple connected minor $N'$ of rank 9 such that $N'|(X \cup Y \cup Z) = N|(X \cup Y \cup Z)$. As $N'$ is connected, there is an element $g$ of $E(N') - (X \cup Y \cup Z)$. Since $N'$ has no plane with more than six points, $g$ is not in the closure of any of $X, Y, \text{ or } Z$ in $N'$. As $N'/g$ has rank 8 but has density less than $\frac{9}{4}$, the eighteen elements of $X \cup Y \cup Z$ cannot all be in distinct parallel classes of $N'/g$. Thus $N'$ has a triangle $\{x, y, g\}$ where we may assume that $x \in X$ and $y \in Y$. Since $N'|(X \cup Y \cup Z \cup g)$ has $Z$ as a component, there is an element $h$ of $E(N')$ that is in neither $\text{cl}_{N'}(X \cup Y)$ nor $\text{cl}_{N'}(Z)$. As above, $N'$ has a triangle $\{h, z, t\}$ where $t \in X \cup Y$ and $z \in Z$. Contracting $g$ and $h$ from $N'|(X \cup Y \cup Z \cup \{g, h\})$ and simplifying, we get a rank-7 matroid with 16 elements. As $\frac{16}{7} > \frac{9}{4}$, we have a contradiction that completes the proof of Lemma 4.9.

Lemma 4.10. Let $N$ be a simple connected matroid having an element $z$ such that each of $N$ and $\text{si}(N/z)$ has every element in at least two triangles. If $d(N) \leq \frac{9}{4}$ and $d(N') < \frac{9}{4}$ for all proper minors $N'$ of $N$, then $N$ is isomorphic to $P(U_{2,4}, M(K_4)), M(K_5 \setminus e), \text{or } M^*(K_{3,3})$.

Proof. We argue by induction on $r(N)$, which must be at least three. Suppose it is exactly three. Since $\text{si}(N/z)$ has density less than $\frac{9}{4}$, it is isomorphic to $U_{2,4}$. As $d(N) \leq \frac{9}{4}$, we see that $|E(N)| \leq 6$. By Lemma 4.6, $N$ has no 2-cocircuits. Thus $N$ has a triangle whose complement is a triad. By Lemma 4.6 again, $N \cong M(K_4)$ and we get a contradiction. Hence $r(N) \geq 4$. If $r(N) = 4$, then, by Corollary 4.7, $N$ is isomorphic to $P(U_{2,4}, M(K_4)), M(K_5 \setminus e), \text{or } M^*(K_{3,3})$.

Now assume the result holds for $r(N) < k$ and let $r(N) = k \geq 5$. Let $N_1 = \text{si}(N/z)$. Every element of $N_1$ is in at least two triangles. Let $N_2$ be a component of $N_1$. By Lemma 4.8, either every element of $N_2$ is in a $U_{2,4}$- or $M(K_4)$-restriction, or $N_2$ has an element $z_2$ such that every element
of \( \text{si}(N_2/z_2) \) is in at least two triangles. If the latter occurs, then, by the induction assumption, \( N_2 \) is isomorphic to \( P(U_{2,4}, M(K_4)), M(K_5 \setminus e) \), or \( M^*(K_{3,3}) \). Each of these matroids has density \( \frac{9}{4} \), a contradiction. Thus every element of \( N_2 \) is in a \( U_{2,4} \)- or \( M(K_4) \)-restriction. As \( d(N_2) < \frac{9}{4} \), Lemma 4.9 implies that \( N_2 \), and hence each component of \( N_1 \), is isomorphic to one of \( U_{2,4}, M(K_4), or P(M(K_4), M(K_4)) \).

Suppose that \( N_2 = N_1 \). Then, as \( r(N) \geq 5 \), we deduce that \( N_1 \cong P(M(K_4), M(K_4)) \). As \( N_1 = \text{si}(N/z) \), we see that \( r(N) = 6 \). Because \( d(N) \leq \frac{9}{4} \), it follows that \( |E(N)| \leq 13 \). Since \( z \) is in at least two triangles of \( N \), we deduce that \( |E(N)| \geq |E(N_1)| + 3 = 14 \), a contradiction.

We may now assume that \( N_1 \) has more than one component. Hence, for some \( k \geq 2 \), there is a collection \( N^1, N^2, \ldots, N^k \) of connected matroids such that \( E(N^i) \cap E(N^j) = \{z\} \) for all \( i \neq j \), the matroid \( N^i/z \) is connected for all \( i \), and \( N \) is the parallel connection of \( N^1, N^2, \ldots, N^k \) across the common basepoint \( z \). As noted above, each \( \text{si}(N^i/z) \) is isomorphic to one of \( U_{2,4}, M(K_4), or P(M(K_4), M(K_4)) \). As every element of \( N \) is in at least two triangles, every element of each \( N^i \) except possibly \( z \) is in at least two triangles of \( N \). Thus, by Corollary 4.7, \( N^i \cong M(K_4) \); or \( r(N^i) = 4 \) and \( |E(N^i)| = 9 \); or \( r(N^i) > 4 \). In the first case, \( \text{si}(N^i/z) \not\cong U_{2,4} \); in the second case, \( d(N^i) = \frac{9}{4} \). Both of these possibilities give contradictions, so \( \text{si}(N^i/z) \cong P(M(K_4), M(K_4)) \) for each \( i \). As \( z \) is in at least two triangles of \( N \), we may assume the elements of two such triangles lie in \( E(N^1) \cup E(N^2) \). As \( |E(\text{si}(N^i/z))| = 11 \) and \( r(N^i/z) = 5 \), we see that \( |E(N^1) \cup E(N^2)| \geq 25 \) and \( r(E(N^1) \cup E(N^2)) = 11 \). But \( \frac{25}{11} > \frac{9}{4} \), a contradiction. \( \square \)

We conclude the paper by proving Theorem 1.9. In this proof, we will make extensive use of the Cunningham-Edmonds canonical tree decomposition of a connected matroid. The definition and properties of this decomposition may be found in [7, Section 8.3]. In brief, associated with each connected matroid \( M \), there is a tree \( T \) that is unique up to the labelling of its edges. Each vertex of \( T \) is labelled by a circuit, a cocircuit, or a 3-connected matroid with at least four elements. Moreover, no two adjacent vertices of \( T \) are labelled by circuits and no two adjacent vertices are labelled by cocircuits. For an edge \( e \) of \( T \) whose endpoints are labelled by matroids \( M_1 \) and \( M_2 \), the ground sets of these two matroids meet in \( \{e\} \). When we contract \( e \) from \( T \), the composite vertex that results by identifying the endpoints of \( e \) is labelled by the 2-sum of \( M_1 \) and \( M_2 \). By repeating this process, contracting all of the remaining edges of \( T \) one by one, we eventually obtain a single-vertex tree. Its vertex is labelled by \( M \).

Each edge \( f \) of \( T \) induces a partition of \( E(M) \). This partition is a 2-separation of \( M \) displayed by \( f \). The remaining 2-separations of \( M \) coincide with those that are displayed by those vertices of \( T \) that are labelled by circuits or cocircuits. For such a vertex \( v \) having label \( N \), there is a partition \( \{X_1, X_2, \ldots, X_k\} \) of \( E(M) - E(N) \) induced by the components of \( T - v \). A partition \( (X, Y) \) of \( E(M) \) is displayed by the vertex \( v \) if each \( X_i \) is contained...
in \(X\) or \(Y\). Every such partition of \(E(M)\) with both \(X\) and \(Y\) having at least two elements is a 2-separation of \(M\) and these 2-separations along with those displayed by the edges of \(T\) are all of the 2-separations of \(M\). Recall that, for all \(n \geq 2\), we denote by \(P_n\) any matroid that can be constructed from \(n\) copies of \(M(K_3)\) via a sequence of parallel connections.

**Proof of Theorem 1.9.** Let \(M\) be a density-critical matroid with \(d(M) \leq \frac{9}{4}\). Suppose \(d(M) \geq 2\). By Lemma 2.1, every element of \(M\) is in at least two triangles. By Corollary 4.7, if \(r(M) \in \{2, 3\}\), then \(M\) is \(U_{2,4}\) or \(M(K_4)\). We may now assume that \(r(M) \geq 4\). By Lemma 4.8, either every element of \(M\) is in a \(U_{2,4}\)- or \(M(K_4)\)-restriction, or, for some element \(z\) of \(M\), every element of \(\text{si}(M/z)\) is in at least two triangles. In the first case, by Lemma 4.9, \(M\) is isomorphic to \(P(U_{2,4}, M(K_4))\), \(P(M(K_4), M(K_4))\), \(M(K_5 \setminus e)\), or \(M_{18}\). In the second case, by Lemma 4.10, \(M\) is isomorphic to \(P(U_{2,4}, M(K_4))\), \(M(K_5 \setminus e)\), or \(M^*(K_{3,3})\). Thus the theorem identifies all possible density-critical matroids with density in \([2, \frac{9}{4}]\) and one easily checks that each of the matroids identified is indeed density-critical.

Now suppose that \(d(M) < 2\). By Lemma 4.4, \(M\) is connected. Clearly, if \(r(M) = 1\) or \(2\), then \(M\) is isomorphic to \(U_{1,1}\) or \(U_{2,3}\). As \(U_{2,4}\) and \(M(K_4)\) both have density 2, \(M\) is a series-parallel network (see, for example, [7, Corollary 12.2.14]). Thus, in the Cunningham-Edmonds canonical tree decomposition \(T\) of \(M\), every vertex is labelled by a circuit or a cocircuit. Since \(M\) is simple, for every vertex of \(T\) that is labelled by a cocircuit \(C^*\), at most one element of \(C^*\) is in \(E(M)\). Let \(e\) be an edge of \(T\) that meets the vertex labelled by \(C^*\). Then, for the 2-separation \((X, Y)\) of \(M\) that is displayed by \(e\), Lemma 4.5 implies that \(M\) has an element \(p\) in \(\text{cl}(X) \cap \text{cl}(Y)\). Thus \(p \in C^*\), so \(C^*\) contains exactly one element of \(M\).

Now take a vertex of \(T\) that is labelled by a circuit \(C\) where \(C = \{e_1, e_2, \ldots, e_k\}\) and suppose that \(k \geq 4\). Suppose \(e_1 \in E(M)\). Then \(M/e_1\) is simple having rank \(r(M) - 1\). As \(\frac{|E(M)| - 1}{r(M) - 1} < \frac{|E(M)|}{r(M)}\), we obtain the contradiction that \(|E(M)| < r(M)|. We deduce that \(C \cap E(M) = \emptyset\). Now \(T \setminus e_1, e_2\) has exactly three components. Let \(T'\) be the one containing \(e_3\) and let \(X\) be the subset of \(E(M)\) corresponding to \(T'\). Then \((X, E(M) - X)\) is a 2-separation of \(M\). By Lemma 4.5, there is an element \(p\) of \(M\) that is in \(\text{cl}(X) \cap \text{cl}(E(M) - X)\). But the tree decomposition implies that there is no such element. We deduce that \(C\) has exactly three elements. Thus every vertex of \(T\) that is labelled by a circuit is labelled by a triangle. Since every vertex of \(T\) that is labelled by a cocircuit has exactly one element of \(E(M)\) in that cocircuit, a straightforward induction argument establishes that, for some \(n \geq 2\), the matroid \(M\) is obtained from \(n\) copies of \(M(K_3)\) by a sequence of \(n - 1\) parallel connections. Thus \(M \cong P_n\).

Finally, we show by induction that \(P_n\) is density-critical. This is true for \(n = 1\). Assume it true for \(n < m\) and let \(n = m \geq 2\). Take \(x\) in \(P_n\). Assume first that \(x\) is in exactly one triangle \(\{x, y, z\}\). Then \(\text{si}(P_n/x) \cong P_n/x \cup z\). As the last matroid is easily seen to be isomorphic to the density-critical matroid
We deduce that every minor of $P_n/x$ has density less that $d(P_n)$. Now assume that $x$ is in at least two triangles of $P_n$. Then $si(P_n/x)$ is easily seen to be the direct sum of a collection of matroids each of which is isomorphic to some $P_k$ with $k < n$. By Lemma 4.4 and the induction assumption, every minor of $P_n/x$ has density less that $d(P_n)$. We conclude that $P_n$ is density-critical, so the theorem is proved.

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