

EXCLUDED MINORS ARE ALMOST FRAGILE

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ABSTRACT. Let M be an excluded minor for the class of \mathbb{P} -representable matroids for some partial field \mathbb{P} , and let N be a 3-connected strong \mathbb{P} -stabilizer such that N is neither a wheel nor a whirl. We prove that if M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor, then either M is bounded relative to N , or $M \setminus a, b$ is at most four elements from an N -fragile minor. Moreover, we prove that, up to replacing M by a $\Delta - Y$ -equivalent excluded-minor or its dual, if M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor, then we can choose a, b such that either M is bounded relative to N , or $M \setminus a, b$ is N -fragile.

1. INTRODUCTION

One of the longstanding goals of matroid theory is to find excluded-minor characterisations of classes of representable matroids. Indeed, to some extent, progress in matroid theory can be measured by success in problems of this type. Results to date include Tutte's excluded-minor characterisation of binary and regular matroids [9]; Bixby's, and independently Seymour's, excluded-minor characterisation of ternary matroids [9]; Geelen, Gerards and Kapoor's excluded-minor characterisation of $\text{GF}(4)$ -representable matroids [4]; and Hall, Mayhew and Van Zwam's excluded-minor characterisation of the near-regular matroids, that is, the matroids representable over all fields with at least three elements [6]. Recently Geelen, Gerards and Whittle announced a proof of Rota's Conjecture [5]. However, their techniques are extremal and give no insight into how one might find the exact list of excluded minors for such classes. Extending the range of known exact excluded-minor theorems for basic classes of matroids remains a problem of genuine interest and, indeed, a significant challenge that tests the state of the art of techniques in matroid theory.

At this stage we need to note that regular matroids and many other naturally arising classes of representable matroids such as near-regular, dyadic and $\sqrt[6]{1}$ -matroids [15] can be described as classes of matroids representable over an algebraic structure called a *partial field*. Of course, a field is an example of a partial field, and classes of matroids representable over partial fields enjoy many of the properties that hold for matroids representable over fields.

The immediate problem that looms large is that of finding the excluded minors for the class of $\text{GF}(5)$ -representable matroids. While this problem is beyond the range of current techniques, a road map for an attack is outlined

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in [12]. In essence, this road map reduces the problem to a finite sequence of problems of the following type. We have the class of \mathbb{P} -representable matroids for some fixed partial field \mathbb{P} . We have a 3-connected matroid N with the property that every \mathbb{P} -representation of N extends *uniquely* to a \mathbb{P} -representation of any 3-connected \mathbb{P} -representable matroid having N as a minor. Such a matroid N is called a *strong stabilizer* for the class of \mathbb{P} -representable matroids. With these ingredients, the goal is to bound the size of an excluded minor for the class of \mathbb{P} -representable matroids having the strong stabilizer N as a minor. This situation is a more general version of the one that arises in the proof of Rota's Conjecture for $\text{GF}(4)$. There, the partial field is $\text{GF}(4)$ and the strong stabilizer is $U_{2,4}$.

Ideally, we would develop techniques that would reduce problems of the above type to routine computation. But an annoying barrier arises. Let N be a matroid. A matroid M is *N -fragile* if, for all elements e of M , at most one of $M \setminus e$ or M/e has an N -minor. It seems that, for a strong stabilizer N for a partial field \mathbb{P} , to bound the size of an excluded minor for \mathbb{P} -representable matroids that contains N as a minor, we need to have some insight into the structure of \mathbb{P} -representable N -fragile matroids. The goal of this paper is to make progress towards demonstrating that this is, in essence, the fundamental problem. We prove, under a particular assumption, that if M is an excluded minor for the class of \mathbb{P} -representable matroids having an N -minor, then either the size of M is bounded relative to N , or M is at most four elements away from being an N -fragile matroid. More specifically, we have the following:

Theorem 1.1. *Let \mathbb{P} be a partial field, let M be an excluded minor for the class of \mathbb{P} -representable matroids, and let N be a strong stabilizer for the class of \mathbb{P} -representable matroids such that $|E(N)| \geq 4$ and N is neither a wheel nor a whirl. Assume that M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor. Then one of the following holds.*

- (a) $|E(M)| \leq |E(N)| + 16$.
- (b) $M \setminus a, b$ is N -fragile.
- (c) *There is some minor M' of $M \setminus a, b$ such that M' is N -fragile and $|E(M \setminus a, b) - E(M')| \leq 4$.*

Theorem 1.1 is a consequence of theorems in this paper that give more explicit information about the specific structure that arises, but these theorems are more technical and need some preparation. Unfortunately there is an elephant in the room and we had better front up to it. In Theorem 1.1 we make the assumption that there is a pair a, b of elements of our matroid M that has the property that $M \setminus a, b$ is 3-connected with an N -minor. To date, we have no guarantee that such a pair exists. However, progress towards finding such a pair is happening. In a recent doctoral dissertation Williams [17] proved the following theorem. We say that a pair a, b of elements of a 3-connected matroid M is *detachable* if either $M \setminus a, b$ or $M/a, b$ is 3-connected.

Theorem 1.2. *Let M be a 3-connected matroid with $|E(M)| \geq 12$. Then one of the following holds.*

- (a) M is a spike.

- (b) M contains a detachable pair.
- (c) There is a matroid obtained by performing a single Δ - Y -exchange on M that contains a detachable pair.

From our point of view a Δ - Y -exchange is not problematic as excluded minors for partial fields are preserved under this operation. We would like an analogue of Theorem 1.2 that preserves a copy of a fixed minor N . In fact, Williams's thesis contains significantly more information about detachable pairs than that given in Theorem 1.2. With the extra information from the thesis, along with additional work, it is to be hoped that such an analogue could be proved in the not-too-distant future.

All going well, this will mean that our excluded minor for \mathbb{P} -representable matroids will either have bounded size or will be very close to an N -fragile matroid. Current techniques for bounding the size of an excluded minor in the latter case rely on obtaining explicit information about the structure of N -fragile matroids and this needs to be done on a case-by-case basis. Even for quite simple matroids this can be a difficult problem. Here is an example. Recall the non-Fano matroid F_7^- . The barrier to finding the excluded minors for the class of dyadic matroids is that we do not understand the structure of dyadic F_7^- -fragile matroids and such an understanding seems some way off.

On the other hand, we do know the structure of $U_{2,5}$ - and $U_{3,5}$ -fragile matroids for two interesting partial fields [3]. The first is the partial field \mathbb{H}_5 which was introduced by Pendavingh and Van Zwam [12]. The class of matroids representable over this field is the class obtained by taking the 3-connected matroids that have exactly six inequivalent representations over $GF(5)$ and closing the class under minors. This class forms the bottom layer of Pendavingh and Van Zwam's hierarchy of $GF(5)$ -representable matroids. Finding excluded minors for this class would be a key first step towards finding the excluded minors for matroids representable over $GF(5)$.

The other partial field is the *2-regular* or *2-uniform* partial field, denoted \mathbb{U}_2 . This is a member of a family of partial fields. The matroids representable over \mathbb{U}_0 and \mathbb{U}_1 are the regular and near-regular matroids respectively. Regular matroids are the matroids representable over all fields, and near-regular matroids are the matroids representable over all fields with at least three elements. Let \mathcal{M}_4 denote the matroids representable over all fields of size at least four. It would certainly be interesting to have a characterisation of the class \mathcal{M}_4 . The class of \mathbb{U}_2 -representable matroids is contained in \mathcal{M}_4 , and it is known [14] that this class is a proper subclass of \mathcal{M}_4 . Nonetheless, knowing the excluded minors for \mathbb{U}_2 would be a key step towards characterising the class \mathcal{M}_4 . The interesting matroids to uncover are the excluded minors for \mathbb{U}_2 that belong to \mathcal{M}_4 . Attention could then be focussed on members of \mathcal{M}_4 having these matroids as minors. It is possible that these will form thin, highly structured classes.

With the results of this paper, further work on detachable pairs, and the characterisation of the \mathbb{U}_2 - and \mathbb{H}_5 -representable $U_{2,5}$ - and $U_{3,5}$ -fragile matroids, there is real hope that obtaining the full list of excluded minors for these classes is an achievable goal. Beyond these classes all bets are off. Experience with graph minors tells us that we must expect to hit a wall quite

soon — consider, for example, the excluded minors for the class of toroidal graphs or the class of Δ - Y -reducible graphs [18]. We know from [7] that there are at least 564 excluded minors for $\text{GF}(5)$ -representable matroids. It is possible that obtaining the full list will be forever beyond our reach. But the quest is surely a worthy one.

2. PRELIMINARIES AND THE MAIN THEOREMS

In this section we gather preliminaries on matroid connectivity and representation theory that are used throughout the paper. We will then be able to state the main results. Most of the results and terminology on matroid connectivity can either be found in Oxley [9] or in the recent literature on removing elements relative to a fixed basis [11, 16, 2]. The results and terminology on matroid representation theory can be found in [13, 12, 8]. Any undefined terminology or notation used in this paper will follow these sources.

2.1. Connectivity. We use the following result known as Bixby’s Lemma [1] (see also [9, Lemma 8.7.3]).

Lemma 2.1. *Let M be a 3-connected matroid, and let $e \in E(M)$. Then $\text{si}(M/e)$ or $\text{co}(M/e)$ is 3-connected.*

A 3-separation (X, Y) of M is a *vertical 3-separation* if $\min\{r(X), r(Y)\} \geq 3$. We say that a partition $(X, \{z\}, Y)$ is a *vertical 3-separation* of M when both $(X \cup \{z\}, Y)$ and $(X, Y \cup \{z\})$ are vertical 3-separations and $z \in \text{cl}(X) \cap \text{cl}(Y)$. We will write (X, z, Y) for $(X, \{z\}, Y)$.

Lemma 2.2. [11, Lemma 3.1] *Let M be a 3-connected matroid, and $e \in E(M)$. If $\text{si}(M/e)$ is not 3-connected, then M has a vertical 3-separation (X, e, Y) .*

We write “by orthogonality” to refer to the property that a circuit and a cocircuit cannot meet in one element. However, in the context of partitions of the form $(X, \{e\}, Y)$, we will also write “by orthogonality” to refer to an application of the next lemma.

Lemma 2.3. *Let e be an element of a matroid M , and let $(X, \{e\}, Y)$ be a partition of $E(M)$. Then $e \in \text{cl}(X)$ if and only if $e \notin \text{cl}^*(Y)$.*

The following result is an elementary consequence of orthogonality.

Lemma 2.4. *Let M be a 3-connected matroid. If X is a rank-2 subset and $|X| \geq 4$, then $M \setminus x$ is 3-connected for all $x \in X$.*

We will use the following two results on series classes. We omit the easy proof of the first lemma.

Lemma 2.5. *Let M be a 3-connected matroid, and let $u \in E(M)$ be an element such that $\text{co}(M \setminus u)$ is 3-connected. If S and S' are distinct series classes of $M \setminus u$, then either $S \cup S'$ is independent or $\text{co}(M \setminus u) \cong U_{1,3}$.*

Lemma 2.6. *Let M be a 3-connected matroid, and let $u \in E(M)$ be an element such that $\text{co}(M \setminus u)$ is 3-connected and $\text{co}(M \setminus u) \not\cong U_{1,3}$. Let S be a series class of $M \setminus u$ such that $|S| \geq 2$. If there is some element $s \in S$ such that $\text{si}(M/s)$ is not 3-connected, then*

- (i) $|S| = 2$;
- (ii) $M \setminus u$ has exactly two distinct non-trivial series classes; and
- (iii) $S - s$ contains an element s' such that $\text{si}(M/s')$ is 3-connected.

Proof. Since M/s is not 3-connected, it follows from the dual of Lemma 2.4 that $|S| = 2$, so (i) holds.

For (ii), let (A, s, B) be a vertical 3-separation of M , which must exist by Lemma 2.2. We may assume without loss of generality that $u \in A$. Then $(A - u, B)$ is a 2-separation of $M/s \setminus u$, and the matroid $M/s \setminus u$ is 3-connected up to series classes because $\text{co}(M \setminus u)$ is 3-connected. Hence $A - u$ is contained in a series class of $M \setminus u$. We claim that S and $A - u$ are disjoint. Since $S \cup u$ is a triad of M , it follows from the vertical 3-separation (A, s, B) and orthogonality that $S \cup u$ is not contained in $A \cup s$. Therefore $S \subseteq B \cup s$, so S and $A - u$ are distinct series classes. Since $s \in \text{cl}_M(A)$, there is a circuit C of M such that $s \in C \subseteq A \cup s$. Moreover, $u \in C$ by orthogonality between C and the triad $S \cup u$. Next we show that $A - u$ and S are the only series classes of $M \setminus u$. Suppose there is some series pair S' of $M \setminus u$ disjoint from $S \cup (A - u)$. Then $S' \cup u$ is a triad of M that meets the circuit C in the single element u ; a contradiction to orthogonality. Thus S and $A - u$ are the only series classes of $M \setminus u$, so (ii) holds.

Finally, let $s' \in S - s$. Suppose that $\text{si}(M/s')$ is not 3-connected. Then, by Lemma 2.2, there is some vertical 3-separation (A', s', B') of M . We may assume without loss of generality that $u \in A'$. By the previous paragraph, $A' - u$ is a series class of $M \setminus u$. By (ii), $A' - u = A - u$, and thus $(A', s', B') = (A, s', (B - s') \cup s)$. Then $s' \in \text{cl}_M(A)$, so there is some circuit C' of M such that $s' \in C' \subseteq A \cup s'$, and $u \in C'$ by orthogonality. But then we have distinct circuits $C \subseteq A \cup s$ and $C' \subseteq A \cup s'$ such that $u \in C \cap C'$. By circuit elimination, there is a circuit C'' of M such that $C'' \subseteq S \cup (A - u)$. Thus, by Lemma 2.5, $\text{co}(M \setminus u) \cong U_{1,3}$, a contradiction. \square

Let k be a positive integer, and let (P, Q) be a k -separation. We call the set $\text{cl}(P) \cap \text{cl}(Q)$ the *guts* of (P, Q) , and $\text{cl}^*(P) \cap \text{cl}^*(Q)$ the *coguts* of (P, Q) . The next three results state elementary properties of 3-separations that we shall use frequently. We use the notation $e \in \text{cl}^{(*)}(X)$ to mean $e \in \text{cl}(X)$ or $e \in \text{cl}^*(X)$.

Lemma 2.7. *Let X be an exactly 3-separating set in a 3-connected matroid, and suppose that $e \in E(M) - X$. Then $X \cup e$ is 3-separating if and only if $e \in \text{cl}^{(*)}(X)$.*

Lemma 2.8. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M . Suppose $|X| \geq 3$ and $x \in X$. Then*

- (i) $x \in \text{cl}^{(*)}(X - x)$; and
- (ii) $(X - x, Y \cup x)$ is exactly 3-separating if and only if x is in exactly one of $\text{cl}(X - x) \cap \text{cl}(Y)$ and $\text{cl}^*(X - x) \cap \text{cl}^*(Y)$.

Lemma 2.9. [2, Lemma 2.11] *Let (X, Y) be a 3-separation of a 3-connected matroid M . If $X \cap \text{cl}(Y) \neq \emptyset$ and $X \cap \text{cl}^*(Y) \neq \emptyset$, then $|X \cap \text{cl}(Y)| = 1$ and $|X \cap \text{cl}^*(Y)| = 1$.*

We write “by uncrossing” to refer to an application of the next result.

Lemma 2.10. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$. Then the following hold.*

- (i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating.*
- (ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

We employ the following results when we encounter fans.

Lemma 2.11. [2, Lemma 2.12] *Let M be a 3-connected matroid such that $|E(M)| \geq 7$. Suppose that M has a fan F of at least 4 elements, and let f be an end of F .*

- (i) *If f is a spoke element, then $\text{co}(M \setminus f)$ is 3-connected and $\text{si}(M/f)$ is not 3-connected.*
- (ii) *If f is a rim element, then $\text{si}(M/f)$ is 3-connected and $\text{co}(M \setminus f)$ is not 3-connected.*

Lemma 2.12. [2, Lemma 3.3] *Let M be a matroid with distinct elements f_1, f_2, f_3, f_4 . If the only triangle containing f_3 is $\{f_1, f_2, f_3\}$ and the only triad containing f_2 is $\{f_2, f_3, f_4\}$, then $\text{si}(M/f_3) \cong \text{co}(M \setminus f_2)$.*

Let M be a matroid and B be a basis of M . Let F be a 4-element fan of M with ordering (f_1, f_2, f_3, f_4) where $\{f_1, f_2, f_3\}$ is a triangle. We say that F is a *type-I fan relative to B* or a *type-II fan relative to B* if $B \cap F$ is $\{f_1, f_3\}$ or $\{f_1, f_3, f_4\}$, respectively.

Let M and N be matroids, and let x be an element of M . If $M \setminus x$ has an N -minor, then x is *N -deletable*. If M/x has an N -minor, then x is *N -contractible*. If neither $M \setminus x$ nor M/x has an N -minor, then x is *N -essential*. If x is both N -deletable and N -contractible, then we say that x is *N -flexible*. We say that the matroid M is *N -fragile* if no element of M is N -flexible. If M is N -fragile and has an N -minor, then we say that M is *strictly N -fragile*. In this paper, N -fragile will always mean strictly N -fragile.

In this paper, we will always be trying to keep some minor when removing elements. We will frequently have some fixed basis. Let M be a 3-connected matroid, let B be a basis of M , and let N be a 3-connected minor of M . An element e of M is *(N, B) -robust* if either

- (i) $e \in B$ and M/e has an N -minor; or
- (ii) $e \in E - B$ and $M \setminus e$ has an N -minor.

Note that an N -flexible element of M is clearly (N, B) -robust for any basis B of M .

An element e of M is *(N, B) -strong* if either

- (i) $e \in B$, and $\text{si}(M/e)$ is 3-connected and has an N -minor; or
- (ii) $e \in E - B$, and $\text{co}(M \setminus e)$ is 3-connected and has an N -minor.

The next three results give some useful conditions for when we can keep an N -minor when dealing with 2-separations.

Lemma 2.13. [2, Lemma 4.3] *Let N be a 3-connected matroid such that $|E(N)| \geq 4$. If M has an N -minor, then $\text{si}(M)$ has an N -minor.*

Lemma 2.14. [11, Lemma 2.6] *Let e and f be distinct elements of a 3-connected matroid M , and suppose that $\text{si}(M/e)$ is 3-connected. Then either $M/e \setminus f$ is connected or $\text{si}(M/e) \cong U_{2,3}$ and M has no triangle containing*

$\{e, f\}$. Moreover, if no non-trivial parallel class of M/e contains f , then $M/e/f$ is connected.

Lemma 2.15. [11, Lemma 2.7] *Let (X, Y) be a 2-separation of a connected matroid M and let N be a 3-connected minor of M . Then $\{X, Y\}$ has a member S such that $|E(N) \cap S| \leq 1$. Moreover, if $s \in S$, then*

- (i) M/s has an N -minor if M/s is connected; and
- (ii) $M \setminus s$ has an N -minor if $M \setminus s$ is connected.

Let $(X, \{z\}, Y)$ be a vertical 3-separation of a matroid M . Then (X, Y) has a non-minimal 2-separation of M/z . That is, (X, Y) is a 2-separation such that $|X| \geq 3$ and $|Y| \geq 3$. If M/z has a 3-connected minor N , then it follows from a well-known result [9, Proposition 8.3.5] that either $|E(N) \cap X| \leq 1$ or $|E(N) \cap Y| \leq 1$. We refer to X as the *non- N -side* and Y as the *N -side* of $(X, \{z\}, Y)$ when $|E(N) \cap X| \leq 1$.

The following result is an essential tool for dealing with elements that are (N, B) -robust but not (N, B) -strong. It is a routine upgrade of [2, Lemma 4.5] that also covers the case when the N -side of the vertical 3-separation is not closed.

Lemma 2.16. *Let N be a 3-connected minor of a 3-connected matroid M . Let $(X, \{z\}, Y)$ be a vertical 3-separation of M such that M/z has an N -minor, where $|X \cap E(N)| \leq 1$.*

- (i) *If $Y \cup \{z\}$ is closed, then every element of X is N -contractible and there is at most one element x of X that is not N -deletable. Moreover, if such an element x exists, then $x \in \text{cl}^*(Y)$ and $z \in \text{cl}(X - \{x\})$.*
- (ii) *If $Y \cup \{z\}$ is not closed, then every element of $X - \text{cl}_M(Y)$ is N -contractible, and at most one element of X is not N -deletable. Moreover, if such an element x exists, then $x \in \text{cl}_M^*(\text{cl}_M(Y)) - \text{cl}_M(Y)$ and $z \in \text{cl}_M(X - (\text{cl}_M(Y) \cup x))$.*

Proof. Part (i) is immediate from [2, Lemma 4.5]. We may therefore assume that $Y \cup \{z\}$ is not closed. Let $s \in X \cap \text{cl}_M(Y)$. We first show that s is N -deletable. Since (X, Y) is a 2-separation of the connected matroid M/z and $s \in \text{cl}_{M/z}(X) \cap \text{cl}_{M/z}(Y)$, it follows that $M/z \setminus s$ is connected. Then, by Lemma 2.15, $M/z \setminus s$ has an N -minor, so s is N -deletable in M . Thus any element of X that is not N -deletable belongs to $X - \text{cl}_M(Y)$. The remaining properties will follow from the next claim.

2.16.1. *The partition $((X - s), z, Y \cup s)$ is a vertical 3-separation of M .*

Subproof. We must show that $((X - s) \cup z, Y \cup s)$ and $(X - s, Y \cup \{s, z\})$ are vertical 3-separations of M . By Lemma 2.8 (i), $s \in \text{cl}_M^{(*)}(X - s)$. Since $s \in \text{cl}_M(Y)$ it follows from orthogonality that $s \notin \text{cl}_M^*(X - s)$. Therefore $s \in \text{cl}_M(X - s)$ and, by Lemma 2.8 (ii), $(X - \{s\}, Y \cup \{s, z\})$ is exactly 3-separating. By a similar argument, we see that $((X - s) \cup z, Y \cup s)$ is exactly 3-separating. As $s \in \text{cl}_M(X - s)$, we see that $r(X - s) = r(X) \geq 3$. Therefore $(X - s, Y \cup \{s, z\})$ and $((X - s) \cup z, Y \cup s)$ are vertical 3-separations of M . Hence the partition $(X - s, z, Y \cup s)$ is a vertical 3-separation of M . \square

By repeatedly applying 2.16.1, we see that $(X - \text{cl}_M(Y), z, \text{cl}_M(Y) - z)$ is a vertical 3-separation of M satisfying the hypotheses of (i), so the remaining properties of (ii) follow by applying (i) to this partition. \square

2.2. Representation Theory. A *partial field* is a pair (R, G) , where R is a commutative ring with unity, and G is a subgroup of the group of units of R such that $-1 \in G$. If $\mathbb{P} = (R, G)$ is a partial field, then we write $p \in \mathbb{P}$ whenever $p \in G \cup \{0\}$.

Let \mathbb{P} be a partial field, and let A be an $X \times Y$ matrix with entries from \mathbb{P} . Then A is a \mathbb{P} -matrix if every subdeterminant of A is contained in \mathbb{P} . If $X' \subseteq X$ and $Y' \subseteq Y$, then we write $A[X', Y']$ to denote the submatrix of A induced by X' and Y' . If $Z \subseteq X \cup Y$, then we denote by $A[Z]$ the submatrix induced by $X \cap Z$ and $Y \cap Z$, and we denote by $A - Z$ the submatrix induced by $X - Z$ and $Y - Z$.

Theorem 2.17. [12, Theorem 2.8] *Let \mathbb{P} be a partial field, and let the $X \times Y$ matrix A be a \mathbb{P} -matrix. Let*

$$(1) \quad \mathcal{B} = \{X\} \cup \{X \Delta Z \mid |X \cap Z| = |Y \cap Z|, \det(A[Z]) \neq 0\}.$$

Then \mathcal{B} is the set of bases of a matroid on $X \cup Y$.

We say that the matroid in Theorem 2.17 is \mathbb{P} -representable, and that A is a \mathbb{P} -representation of M . We write $M = M[I|A]$ if A is a \mathbb{P} -matrix, and M is the matroid whose bases are described in Theorem 2.17.

Let A be an $X \times Y$ \mathbb{P} -matrix, and let $x \in X$ and $y \in Y$ such that $A_{xy} \neq 0$. Then we define A^{xy} to be the $(X \Delta \{x, y\}) \times (Y \Delta \{x, y\})$ \mathbb{P} -matrix given by

$$(A^{xy})_{uv} = \begin{cases} A_{xy}^{-1} & \text{if } uv = yx \\ A_{xy}^{-1} A_{xv} & \text{if } u = y, v \neq x \\ -A_{xy}^{-1} A_{uy} & \text{if } v = x, u \neq y \\ A_{uv} - A_{xy}^{-1} A_{uy} A_{xv} & \text{otherwise.} \end{cases}$$

We say that A^{xy} is obtained from A by *pivoting* on xy .

Two \mathbb{P} -matrices are *scaling equivalent* if one can be obtained from the other by repeatedly scaling rows and columns by non-zero elements of \mathbb{P} . Two \mathbb{P} -matrices are *geometrically equivalent* if one can be obtained from the other by a sequence of the following operations: scaling rows and columns by non-zero entries of \mathbb{P} , permuting rows, permuting columns, and pivoting.

Let \mathbb{P} be a partial field, and let M and N be 3-connected \mathbb{P} -representable matroids such that N is a minor of M . Suppose the ground set of N is $X' \cup Y'$, where X' is a basis of N . We say that N is a \mathbb{P} -stabilizer for M if, whenever A_1 and A_2 are $X \times Y$ \mathbb{P} -matrices (where $X' \subseteq X$ and $Y' \subseteq Y$) such that

- (i) $M = M[I|A_1] = M[I|A_2]$;
- (ii) $A_1[X', Y']$ is scaling equivalent to $A_2[X', Y']$; and
- (iii) $N = M[I|A_1[X', Y']] = M[I|A_2[X', Y']]$,

then A_1 is scaling equivalent to A_2 .

Let \mathcal{M} be a class of matroids. We say that N is a \mathbb{P} -stabilizer for \mathcal{M} if N is a \mathbb{P} -stabilizer for every 3-connected \mathbb{P} -representable matroid $M \in \mathcal{M}$ with an N -minor. We say that N is a *strong \mathbb{P} -stabilizer* for \mathcal{M} if N is a \mathbb{P} -stabilizer for \mathcal{M} and, for every 3-connected \mathbb{P} -representable matroid $M \in \mathcal{M}$

with an N -minor, every \mathbb{P} -representation of N extends to a \mathbb{P} -representation of M . When we say that N is a “strong \mathbb{P} -stabilizer” omitting reference to a class of matroids, the omitted class is the class of \mathbb{P} -representable matroids.

Let B be a basis for a matroid M , and let Z be a subset of $E(M)$. We write $M_B[Z]$ to denote the minor $M/(B - Z)\setminus(B^* - Z)$.

Theorem 2.18. [8, Theorem 5.5] *Let D be an $X_N \times Y_N$ \mathbb{P} -matrix such that $N = M[I|D]$. Choose $B, E_N \subseteq E$ such that B is a basis of $M \setminus \{a, b\}$, $E_N \subseteq E - \{a, b\}$ is such that $M_B[E_N] = N$, and $X_N \subseteq B$. Suppose $M \setminus a$ and $M \setminus b$ are \mathbb{P} -representable. Then there exists an $B \times (E - B)$ matrix A with entries in \mathbb{P} such that*

- (i) $A - a$ and $A - b$ are \mathbb{P} -matrices;
- (ii) $M[I|A - a] = M \setminus a$ and $M[I|A - b] = M \setminus b$; and
- (iii) $A[E_N]$ is scaling equivalent to D .

Moreover, the matrix A is unique up to row and column scaling.

We call the matrix A of Theorem 2.18 a *companion matrix* for M .

A companion matrix for an excluded minor contains a certificate of non-representability over \mathbb{P} . Let B be a basis of M , and let A be a $B \times (E(M) - B)$ matrix with entries in \mathbb{P} . A subset Z of $E(M)$ *incriminates* the pair (M, A) if $A[Z]$ is square and one of the following holds:

- (i) $\det(A[Z]) \notin \mathbb{P}$;
- (ii) $\det(A[Z]) = 0$ but $B \triangle Z$ is a basis of M ;
- (iii) $\det(A[Z]) \neq 0$ but $B \triangle Z$ is dependent in M .

The next result follows immediately from the definition.

Lemma 2.19. *Let A be an $X \times Y$ \mathbb{P} -matrix, where X and Y are disjoint, and $X \cup Y = E$. Exactly one of the following statements is true:*

- (i) A is a \mathbb{P} -matrix and $M = M[I|A]$.
- (ii) Some $Z \subseteq X \cup Y$ incriminates (M, A) .

The next theorem shows that there is some companion matrix A for M that has a 4-element incriminating set.

Theorem 2.20. [8, Theorem 5.8] *Suppose $A - u$, $A - v$ are \mathbb{P} -matrices, and $M \setminus u = M[I|(A - u)]$, $M \setminus v = M[I|(A - v)]$. Suppose $Z \subseteq X \cup Y$ incriminates (M, A) . Then there is some $X' \times Y'$ matrix A' , and $a, b \in X'$, such that:*

- (i) $u, v \in Y'$;
- (ii) $A - u$ is geometrically equivalent to $A' - u$;
- (iii) $A - v$ is geometrically equivalent to $A' - v$; and
- (iv) $\{a, b, u, v\}$ incriminates (M, A') .

Let N be a 3-connected non-binary matroid. A matroid M with an N -minor is N -stable if, whenever (X, Y) is a 2-separation of M with $|X \cap E(N)| \leq 1$, then the matroid M_X corresponding to X in the 1- or 2-sum decomposition of M induced by (X, Y) is binary.

The following result is proved by Hall, Mayhew, and Van Zwam in [6] (See Proposition 3.1 and 3.2).

Lemma 2.21. *Let M be a \mathbb{P} -representable matroid, and let N be a 3-connected strong \mathbb{P} -stabilizer minor of M . If M is N -stable, then M is strongly \mathbb{P} -stabilized by N .*

The next lemma can be proved by a straightforward modification of the proof of Mayhew, Whittle, and Van Zwam in [8, Theorem 5.12]. The conditions (iv) and (v) of [8, Theorem 5.12] are changed from “ $M_B[Z_1]$ and $M_B[Z_2]$ are 3-connected up to series-parallel classes” to “ $M_B[Z_1]$ and $M_B[Z_2]$ are N -stable” using Lemma 2.21.

Lemma 2.22. *Let N be a strong stabilizer for the class of \mathbb{P} -representable matroids, and suppose that $C \subseteq E(M)$ is such that $M_B[C]$ is (strictly) N -fragile. If there exist subsets $Z, Z_1, Z_2 \subseteq E(M)$ such that:*

- (i) $a \in Z_1 - Z_2$ and $b \in Z_2 - Z_1$;
- (ii) $C \cup \{x, y\} \subseteq Z \subseteq Z_1 \cap Z_2$;
- (iii) $M_B[Z]$ is connected;
- (iv) $M_B[Z_1]$ is N -stable;
- (v) $M_B[Z_2]$ is N -stable;
- (vi) $\{a, b, x, y\}$ incriminates $(M_B[Z_1 \cup Z_2], A[Z_1 \cup Z_2])$;

then $M_B[Z_1 \cup Z_2]$ is not strongly \mathbb{P} -stabilized by N .

We write “by an allowable pivot” to refer to an application of either of the next two results.

Lemma 2.23. [8, Lemma 5.10] *Let A be a $B \times B^*$ companion matrix for M . Suppose that $\{a, b, x, y\}$ incriminates (M, A) , where $\{x, y\} \subseteq B$ and $\{a, b\} \subseteq B^*$. If $p \in \{x, y\}$, $q \in B^* - \{a, b\}$, and $A_{pq} \neq 0$, then $\{x, y, a, b\} \triangle \{p, q\}$ incriminates (M, A^{pq}) .*

Lemma 2.24. [8, Lemma 5.11] *Let A be a $B \times B^*$ companion matrix for M . Suppose that $\{a, b, x, y\}$ incriminates (M, A) , where $\{x, y\} \subseteq B$ and $\{a, b\} \subseteq B^*$. If $p \in B - \{x, y\}$, $q \in B^* - \{a, b\}$ are such that $A_{pq} \neq 0$, and either $A_{pa} = A_{pb} = 0$ or $A_{xq} = A_{yq} = 0$, then $\{a, b, x, y\}$ incriminates (M, A^{pq}) .*

The elements of a set $\{a, b, x, y\}$ that incriminates (M, A) label a 2×2 submatrix $A[\{a, b, x, y\}]$ of A . We will refer to the next result by saying “the bad submatrix has no zero entries.”

Lemma 2.25. *Let A be a $B \times B^*$ companion matrix for M . Suppose that $\{a, b, x, y\}$ incriminates (M, A) , where $\{x, y\} \subseteq B$ and $\{a, b\} \subseteq B^*$. If $\{a, b, x, y\}$ incriminates the pair (M, A) , then $A_{ij} \neq 0$ for $i \in \{x, y\}$ and $j \in \{a, b\}$.*

Proof. Suppose that $A_{ij} = 0$ for some $i \in \{x, y\}$ and $j \in \{a, b\}$. We can assume without loss of generality that $A_{xb} = 0$. Then $\det(A[\{a, b, x, y\}]) \in \mathbb{P}$. Since $\{a, b, x, y\}$ incriminates the pair (M, A) , it follows that either

- (i) $B \triangle \{a, b, x, y\}$ is a basis of M but $\det(A[\{a, b, x, y\}]) = 0$; or
- (ii) $B \triangle \{a, b, x, y\}$ is dependent in M but $\det(A[\{a, b, x, y\}]) \neq 0$.

Assume that (i) holds. As $B \triangle \{a, b, x, y\}$ is a basis of M , but $\det(A[\{a, b, x, y\}]) = A_{xa} \cdot A_{yb} = 0$ and non-zero elements of \mathbb{P} are units, it follows that $A_{xa} = 0$ or $A_{yb} = 0$. Suppose that $A_{xa} = 0$. Let

$B' = B \Delta \{a, b, x, y\}$. Now B and B' are bases of M and $x \in B - B'$, so, by basis exchange, there is some $z \in B' - B = \{a, b\}$ such that $(B - x) \cup z$ is a basis of M . This is a contradiction because $M \setminus b = M[I|A - b]$, $M \setminus a = M[I|A - a]$ and $A_{xa} = A_{xb} = 0$, so both $(B - x) \cup a$ and $(B - x) \cup b$ are dependent in M . Thus $A_{xa} \neq 0$. Similarly, since $a \in B' - B$, it follows that $(B' - a) \cup x$ or $(B' - a) \cup y$ is a basis of $M \setminus a = M[I|A - a]$. Thus $A_{yb} \neq 0$; a contradiction.

Therefore (ii) holds, so $B' = B \Delta \{a, b, x, y\}$ is dependent in M , but B' is a basis of $M[I|A]$. Since $\det(A[\{a, b, x, y\}]) = A_{xa} \cdot A_{yb} \neq 0$, it follows that $A_{xa} \neq 0$ and $A_{yb} \neq 0$. Now $M \setminus b = M[I|A - b]$ and $A_{xa} \neq 0$, so $(B - x) \cup a$ is a basis of M . Similarly, $M \setminus a = M[I|A - a]$ and $A_{yb} \neq 0$, so $(B - y) \cup b$ is also a basis of M . Let $B_1 = (B - x) \cup a$ and $B_2 = (B - y) \cup b$. Then $x \in B_2 - B_1$, so, by basis exchange, there is some $z \in B_1 - B_2$ such that $(B_2 - x) \cup z$ is a basis of M . But $B_1 - B_2 = \{a, y\}$, so then $(B_2 - x) \cup z$ is either $(B - x) \cup b$ or B' . This is a contradiction because B' is dependent by assumption while, since $A_{xb} = 0$, it follows that $(B - x) \cup b$ is dependent in $M \setminus a = M[I|A - a]$ and hence in M . \square

Our setup is as follows.

- (i) N is a 3-connected strong \mathbb{P} -stabilizer for the class of \mathbb{P} -representable matroids such that $|E(N)| \geq 4$ and N is neither a wheel nor a whirl.
- (ii) M is an excluded minor for the class of \mathbb{P} -representable matroids, and M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor.
- (iii) M has a $B \times B^*$ companion matrix A and $\{a, b, x, y\}$ incriminates (M, A) , where $\{x, y\} \subseteq B$ and $\{a, b\} \subseteq B^*$.

2.3. The main theorems. We will prove that either $M \setminus a, b$ has some local structure containing $\{x, y\}$ that implies a bound on $|E(M)|$ relative to $|E(N)|$, or that M has a basis B with at most one (N, B) -strong element outside of $\{x, y\}$. In the case that some basis B has an (N, B) -strong element z outside of $\{x, y\}$, we will show that B can be chosen such that $\{x, y, z\}$ is a triad of $M \setminus a, b$. We say that a basis B is a *robust basis* for M if, for any basis B' of M subject to the setup, the number of (N, B') -robust elements of $M \setminus a, b$ outside of $\{x', y'\}$ is at most the number of (N, B) -robust elements of $M \setminus a, b$ outside of $\{x, y\}$. When M has some basis B that has an (N, B) -strong element outside of $\{x, y\}$, the maximum number of (N, B) -robust elements outside of $\{x, y\}$ in a robust basis is subject to the constraint that B displays a triad of the form $\{x, y, z\}$, where z is an (N, B) -strong element of $M \setminus a, b$.

We have the following theorem, which gives the structure of $M \setminus a, b$ for any pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor.

Theorem 2.26. *Let M be an excluded minor for the class of \mathbb{P} -representable matroids, and let N be a strong \mathbb{P} -stabilizer. If M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor, then at least one of the following holds:*

- (a) *Either $|E(M)| \leq |E(N)| + 16$; or*
- (b) *there is some robust basis B for M such that either:*
 - (i) *$M \setminus a, b$ is N -fragile, and $M \setminus a, b$ has at most one (N, B) -robust element $z \in B^* - \{a, b\}$ outside of $\{x, y\}$. Moreover, if $z \in$*

- $B^* - \{a, b\}$ is (N, B) -robust, then z is an (N, B) -strong element of $M \setminus a, b$, and $\{x, y, z\}$ is a triad of $M \setminus a, b$; or
- (ii) $M \setminus a, b$ is not N -fragile, but the only N -flexible elements of $M \setminus a, b$ are contained in a triad $\{x, y, z\}$ of $M \setminus a, b$ for some (N, B) -strong element $z \in B^*$. Moreover, the only (N, B) -robust elements of $M \setminus a, b$ are in $\{x, y, z\}$, and M has a cocircuit $\{a, b, x, y, z\}$ that contains a triangle $\{p, x, y\}$ for some $p \in \{a, b\}$.
 - (iii) $M \setminus a, b$ is not N -fragile, but the only N -flexible elements of $M \setminus a, b$ are contained in $\{x, y, z_1, z_2\}$, where z_1 is (N, B) -strong and (z_2, z_1, x, y) a type-II fan relative to B . Moreover, the fan (z_2, z_1, x, y) is maximal, the only (N, B) -robust elements of $M \setminus a, b$ are in $\{x, y, z_1, z_2\}$, and M has a cocircuit $\{a, b, x, y, z\}$ that contains a triangle $\{p, x, y\}$ for some $p \in \{a, b\}$.

If the excluded minor M is sufficiently larger than the stabilizer N , then up to performing at most one Δ - Y -exchange, we can eliminate cases (b)(ii) and (b)(iii) of Theorem 2.26 by choosing a different deletion pair. We note that Oxley, Semple, and Vertigan [10, Theorem 1.1] proved that excluded minors for the class of \mathbb{P} -representable matroids are closed under $\Delta - Y$ -exchange.

Theorem 2.27. *Let M be an excluded minor for the class of \mathbb{P} -representable matroids, and let N be a strong \mathbb{P} -stabilizer. Suppose that M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor. Then, for some (M_0, N_0) in $\{(M, N), (\nabla(M^*), N^*)\}$, where $\nabla(M^*)$ is a matroid obtained from M^* by a single Y - Δ -exchange, M_0 is an excluded minor with an N_0 -minor having a pair of elements a', b' such that $M_0 \setminus a', b'$ is 3-connected with an N_0 -minor and at least one of the following holds:*

- (a) $|E(M_0)| \leq |E(N_0)| + 16$; or
- (b) $r(M_0) \leq r(N_0) + 8$; or
- (c) *there is some robust basis B_0 for M_0 such that $M_0 \setminus a', b'$ is N_0 -fragile, and $M_0 \setminus a', b'$ has at most one (N_0, B_0) -robust element $z \in (B_0)^* - \{a', b'\}$ outside of $\{x', y'\}$. Moreover, if $z \in (B_0)^* - \{a', b'\}$ is an (N_0, B_0) -robust element, then z is an (N_0, B_0) -strong element of $M_0 \setminus a', b'$, and $\{x', y', z\}$ is a triad of $M_0 \setminus a', b'$.*

The remainder of the paper is structured as follows. In Section 3, we establish a link between (N, B) -strong elements and confining sets. In Section 4, we bound $|E(M)|$ relative to $|E(N)|$ in the case when $M \setminus a, b$ has a confining set. In Section 5 we show that elements of $M \setminus a, b$ that are (N, B) -robust but not (N, B) -strong give rise to a path of 3-separations of $M \setminus a, b$. Finally, in Section 6, we use the structure given by the path of 3-separations in Section 5 to bound the number of (N, B) -robust elements and prove Theorems 2.26 and 2.27.

3. STRONG ELEMENTS

In this section, we prove results about the (N, B) -strong elements of $M' = M \setminus a, b$ outside of the elements x and y of the set $\{a, b, x, y\}$ that incriminates

(M, A) . The main result here is a proof that M' has at most two (N, B) -strong elements outside of x and y , and that any such elements are in $B^* - \{a, b\}$.

Lemma 3.1. *If u is an (N, B) -strong element of M' such that $u \notin \{x, y\}$, then $u \notin B$.*

Proof. Suppose that u is an (N, B) -strong element of M' such that $u \notin \{x, y\}$ and that $u \in B$. Let $Z = E(M) - \{a, b, u\}$, $Z_1 = E(M) - \{b, u\}$, $Z_2 = E(M) - \{a, u\}$, and $C = E(N)$. Then, by Lemma 2.22, the matroid M/u is not strongly \mathbb{P} -stabilized by N . But u is (N, B) -strong, so $M \setminus a, b/u$, and hence M/u , is 3-connected up to parallel classes. Then it follows from Lemma 2.21 that M/u is strongly \mathbb{P} -stabilized by N ; a contradiction. \square

A subset S of $E(M)$ is a *segment* if every 3-element subset of S is a triangle. A *cosegment* is a segment of M^* .

Lemma 3.2. *Let u be an (N, B) -strong element of M' such that $u \notin \{x, y\}$. If C is a cosegment of M' such that $u \in C$ and $|C| \geq 4$, then $|C| = 4$ and $B \cap C = \{x, y\}$.*

Proof. Since C is a corank-2 subset, it follows that $|C \cap B^*| \leq 2$. Hence $|C \cap B| \geq |C| - 2$. Since u is (N, B) -strong, it follows from Lemma 3.1 that $u \in B^*$. Then $M' \setminus u$ has an N -minor, and hence the elements of the series class $C - u$ of $M' \setminus u$ are N -contractible. Suppose that there is some $c \in C$ that is a member of $B - \{x, y\}$. Then c is N -contractible, and M'/c is 3-connected by the dual of Lemma 2.4, so c is an (N, B) -strong element; a contradiction to Lemma 3.1. We deduce that $|C| = 4$ and that $B \cap C = \{x, y\}$. \square

We say that a matroid Q is *3-connected up to series pairs* if $\text{co}(Q)$ is 3-connected and non-trivial series classes of Q have exactly 2 elements.

Lemma 3.3. *If $u \in B^* - \{a, b\}$ is an (N, B) -strong element of M' such that $M' \setminus u$ is 3-connected up to series pairs, then $M \setminus a, u$ or $M \setminus b, u$ is not N -stable.*

Proof. Suppose that both $M \setminus a, u$ and $M \setminus b, u$ are N -stable. Then either $M \setminus a, u$ is 3-connected up to series pairs, or b is in the guts of some 2-separation (S, T) of $M \setminus a, u$ where S or T is a series pair of $M \setminus a, u$. By symmetry, either $M \setminus b, u$ is 3-connected up to series pairs, or a is in the guts of some 2-separation (S, T) of $M \setminus b, u$ where S or T is a series pair of $M \setminus b, u$. Hence $M \setminus u$ is N -stable. But, by Lemma 2.22 with $Z = E(M) - \{a, b, u\}$, $Z_1 = E(M) - \{b, u\}$, $Z_2 = E(M) - \{a, u\}$, and $C = E(N)$, the matroid $M \setminus u$ is not strongly \mathbb{P} -stabilized by N . Thus $M \setminus u$ is not N -stable by Lemma 2.21; a contradiction. \square

We omit the routine proof of the next lemma.

Lemma 3.4. *Let $Q \setminus e$ be a matroid with an N -minor. Suppose that $Q \setminus e$ is 3-connected up to series pairs, and that Q is not N -stable. Then, for some matroid R , we have $Q = R \oplus_2 U_{2,4}$, where $E(U_{2,4})$ contains e and the elements of some non-trivial series class of $Q \setminus e$.*

We call a non-trivial series class that is contained in a $U_{2,4}$ -minor an *unstable series pair*. Thus, if M' has an (N, B) -strong element $u \notin \{x, y\}$ where $M' \setminus u$ is 3-connected up to series pairs, then it follows from Lemma 3.3 that $M' \setminus u, a$ or $M' \setminus u, b$ has an unstable series pair.

We now show that unstable series pairs must meet $\{x, y\}$.

Lemma 3.5. *Let u be an (N, B) -strong element of M' such that $u \notin \{x, y\}$, and let S be an unstable series pair of $M' \setminus u$. Then $S \cap B \subseteq \{x, y\}$.*

Proof. Let $S = \{s_1, s_2\}$. Note that, since S is a series pair of $M' \setminus u$, both s_1 and s_2 are N -contractible elements of M' . We also note that $S \cap B$ is non-empty because S is codependent. Suppose that $s_1 \in B - \{x, y\}$. Then s_1 is not (N, B) -strong by Lemma 3.1, so it follows that $\text{si}(M'/s_1)$ is not 3-connected. Hence $\text{si}(M'/s_2)$ is 3-connected by Lemma 2.6, so it follows from Lemma 3.1 that either $s_2 \in \{x, y\}$ or $s_2 \in B^* - \{a, b\}$.

3.5.1. *Up to an allowable pivot, we can assume that $s_2 \in \{x, y\}$.*

Subproof. Suppose that $s_2 \in B^* - \{a, b\}$. Then $A_{s_1 s_2} \neq 0$ because $\{u, s_1, s_2\}$ is a triad of M' . If $A_{x s_2} = A_{y s_2} = 0$, then a pivot on $A_{s_1 s_2}$ is allowable, and s_2 is an $(N, B \Delta \{s_1, s_2\})$ -strong element with $s_2 \in (B \Delta \{s_1, s_2\}) - \{x, y\}$, which contradicts Lemma 3.1. Thus we shall assume that $A_{x s_2} \neq 0$. Then a pivot on $A_{x s_2}$ is an allowable pivot, and s_2 takes the place of x as a member of the set $\{a, b, s_2, y\}$ that incriminates $(M, A^{x s_2})$. \square

By 3.5.1 we may assume that $s_2 = x$. Since $\{s_1, s_2\}$ is an unstable series pair, we have $a \in \text{cl}_{M \setminus b}(\{s_1, s_2\})$ where $\{s_1, s_2\} \subseteq B$. Hence $A_{a j} \neq 0$ if and only if $j \in \{s_1, s_2\}$. But then $A_{a y} = 0$; a contradiction because the bad submatrix has no zero entries. This contradiction arose from the assumption that some member of $S \cap B$ was outside of $\{x, y\}$. Therefore $S \cap B \subseteq \{x, y\}$. \square

Lemma 3.6. *Let u and v be distinct (N, B) -strong elements of M' outside of $\{x, y\}$ such that both $M' \setminus u$ and $M' \setminus v$ are 3-connected up to series pairs. If $M \setminus a, u$ is not N -stable, then $M \setminus a, v$ is N -stable.*

Proof. Suppose that both $M \setminus a, u$ and $M \setminus a, v$ are not N -stable, and let S_u and S_v be unstable series pairs for u and v respectively. First suppose $S_u \cap S_v = \emptyset$. Then $S_v \subseteq E(M) - S_u$, so $b \in \text{cl}_M(E(M) - S_u)$; a contradiction because $b \notin \text{cl}(E(M) - S_u)$. Next suppose that $|S_u \cap S_v| = 1$. Then, by Lemma 3.4, $S_u \cup b$ and $S_v \cup b$ are triangles of M , so it follows that $S_u \cup S_v$ is a triangle of M' , and so $\{u, v\} \cup S_u \cup S_v$ is a 5-element fan of M' with rim ends u, v . But then $\text{co}(M' \setminus v)$ is not 3-connected; a contradiction because v is an (N, B) -strong element of M' . Therefore $S_u = S_v$. But then $\{u, v\} \cup S_u$ is a 4-point cosegment of M' ; a contradiction because $M' \setminus u$ is 3-connected up to series pairs. Therefore $M \setminus a, v$ is N -stable. \square

Lemma 3.7. *If M' has a 4-element cosegment C such that $C \cap B = \{x, y\}$, then M' has no (N, B) -strong elements outside of C .*

Proof. Suppose M' has an (N, B) -strong element v outside of C . Then it follows from Lemma 3.2 that $M' \setminus v$ is 3-connected up to series pairs. Hence $M' \setminus v$ has an unstable series pair S' by Lemma 3.3, and S' meets $\{x, y\}$ by

Lemma 3.5. Then the subset $C \cup S' \cup v$ has corank at most 3 in M' , so $S' = \{x, y\}$ by Lemma 3.5. But then $C \cup v$ has corank 2; a contradiction because $|(C \cup v) \cap B^*| = 3$. \square

These results are enough to bound the number of (N, B) -strong elements outside of $\{x, y\}$. The bound on the number of (N, B) -strong elements is a key ingredient in many subsequent arguments.

Lemma 3.8. *M' has at most two (N, B) -strong elements outside of $\{x, y\}$.*

Proof. Assume that M' has an (N, B) -strong element u outside of $\{x, y\}$ such that $M' \setminus u$ has a series class S with $|S| \geq 3$. Then M' has a 4-point cosegment C such that $\{u, x, y\} \subseteq C$ and $C \cap B = \{x, y\}$ by Lemma 3.2. Thus, by Lemma 3.7, M' has at most two (N, B) -strong elements outside of $\{x, y\}$.

We may now assume that $M' \setminus u$ is 3-connected up to series pairs for each (N, B) -strong element u of M' outside of $\{x, y\}$. Then it follows from Lemma 3.6 and Lemma 3.3 that M' has at most two (N, B) -strong elements outside of $\{x, y\}$. \square

We say that a subset G of M' is a *confining set* if $G \cap B = \{x, y\}$, and there are triads T and T' of M' such that

- (i) $G = T \cup T'$;
- (ii) $|T \cap T'| \in \{1, 2\}$;
- (iii) $G \subseteq \text{cl}^*(G \cap B^*)$; and
- (iv) if $|T \cap T'| = 1$, then $G \cap B^*$ has at least one (N, B) -strong element.

Note that a confining set with $|T \cap T'| = 2$ is a 4-element cosegment. A confining set with $|T \cap T'| = 1$ has corank 3 in M' . We will see later that a confining set enables us to bound the number of rows and columns of the companion matrix that are obstacles to allowable pivots, and hence enables us to bound $|E(M)|$ relative to $|E(N)|$.

Now we seek to prove that we see either a confining set of M' or at most one (N, B) -strong element outside $\{x, y\}$.

Lemma 3.9. *If M' does not have a confining set, then either*

- (i) *there is some basis B' of M' and some set $\{a, b, x', y'\}$ that incriminates (M, A') such that M' has exactly one (N, B') -strong element u outside of $\{x', y'\}$, and $\{u, x', y'\}$ is a triad of M' ; or*
- (ii) *there is no basis B' of M' and set $\{a, b, x', y'\}$ that incriminates (M, A') such that M' has an (N, B') -strong element outside of $\{x', y'\}$.*

Proof. Suppose that M' does not have a confining set. Then we can assume, by Lemma 3.2, that $M' \setminus u$ is 3-connected up to series pairs for each (N, B) -strong element of M' outside of $\{x, y\}$. We can also assume the basis B is chosen such that M' has at least one (N, B) -strong element u outside of $\{x, y\}$ or else (ii) holds and we are done. Then by Lemma 3.3 we may assume that $(M \setminus u) \setminus a$ is not N -stable, and that S_u is an unstable series pair of $M' \setminus u$.

Suppose that u is the only (N, B) -strong element of M' outside of $\{x, y\}$. If $S_u = \{x, y\}$, then (i) holds. Assume that $S_u \neq \{x, y\}$. Then it follows from

Lemma 3.3 that $S_u = \{x, s\}$ for some $s \in B^* - \{a, b, u\}$. Now b is spanned by S_u and $A_{yb} \neq 0$ because the bad submatrix has no zero entries, so it follows that $A_{ys} \neq 0$. Hence a pivot on A_{ys} is allowable. Let $B' = B \Delta \{s, y\}$. If y is not (N, B') -strong, then (i) holds. We may now assume that the basis B is chosen for M' such that M' has two (N, B) -strong elements, u and v , outside of $\{x, y\}$.

We may assume that $(M \setminus u) \setminus a$ and $(M \setminus v) \setminus b$ are not N -stable, but that $(M \setminus u) \setminus b$ and $(M \setminus v) \setminus a$ are N -stable by Lemma 3.6 and Lemma 3.3. Let S_u and S_v be unstable series pairs for u and v .

Suppose that the triads $S_u \cup u$ and $S_v \cup v$ are disjoint. Thus, by Lemma 3.5, we may assume that $S_u = \{s, x\}$ and $S_v = \{t, y\}$ for some $s, t \in B^* - \{u, v\}$. Then $A_{ys} = 0$ because $S_v \cup v$ is a triad of M' . But then s is spanned by $B - \{y\}$, and hence b is spanned by $B - \{y\}$. Then $A_{by} = 0$; a contradiction because the bad submatrix has no zero entries.

Suppose that $|(S_u \cup u) \cap (S_v \cup v)| = 1$. Then $S_u \cup S_v \cup \{u, v\}$ has corank 3 in M' , so it follows from Lemma 3.5 that $(S_u \cup S_v) \cap B = \{x, y\}$. But then $S_u \cup S_v \cup \{u, v\}$ is a confining set of M' ; a contradiction.

We shall therefore assume that $|(S_u \cup u) \cap (S_v \cup v)| = 2$. Then $S_u \cup S_v \cup \{u, v\}$ is a corank-2 subset of M' . Then we can assume by Lemma 3.5 that $S_u \cup S_v \cup \{u, v\} = \{u, v, x\}$. Since b is spanned by x and v , and $A_{yb} \neq 0$ because the bad submatrix has no zero entries, it follows that $A_{yv} \neq 0$. Hence a pivot on A_{yv} is allowable by Lemma 3.5. Let $B' = B \Delta \{v, y\}$, $x' = x$, and $y' = v$. Then u is an (N, B') -strong element outside of $\{x', y'\}$ and $\{u, x', y'\}$ is a triad. Suppose that y is an (N, B') -strong element of M' . If y is cospanned by $\{u, x', y'\}$, then M' has a 4-point cosegment $\{u, y, x', y'\}$; a contradiction because M' has no confining set. Thus y is not cospanned by $\{u, x', y'\}$. Let S_y be an unstable series class for y . Then $S_y \cup y$ meets $\{u, x', y'\}$ in a single element by Lemma 3.5, so M' has a confining set by the above argument; a contradiction. Thus y is not (N, B') -strong, and (i) holds. \square

4. CONFINING SETS

In this section we suppose that M' has a confining set G , and we prove that $|E(M)| \leq |E(N)| + 16$. We begin with the following restriction on the (N, B) -strong elements of M' .

Lemma 4.1. *If M' has a confining set G , then M' has no (N, B) -strong elements outside of G .*

Proof. Suppose that G is a 4-element cosegment. Then it follows from Lemma 3.7 that M' has no (N, B) -strong elements outside of G .

Assume that $G = \{u, v, w, x, y\}$, where $\{u, v, w\} \subseteq B^*$. Suppose t is an (N, B) -strong element outside of $\{x, y\}$. Then $t \in B^*$ by Lemma 3.1.

We first show that $M' \setminus t$ is 3-connected up to series pairs. Suppose that M' has a cosegment C containing t with $|C| \geq 4$. Then $C = \{s, t, x, y\}$ for some $s \in B^*$ by Lemma 3.2. If $s \notin \{u, v, w\}$, then there is some (N, B) -strong element of M' outside of the confining set C ; a contradiction. Thus we shall assume that $C = \{u, t, x, y\}$. Then $G \cup C$ is a triad that meets a 4-element cosegment, so it has corank at most 3; a contradiction because

$G \cup C$ has a four-element subset $\{t, u, v, w\}$ contained in B^* . Therefore $M' \setminus t$ is 3-connected up to series pairs, and $M' \setminus t$ has a series pair that meets $\{x, y\}$ by Lemma 3.5.

Let $T = \{t, x, z\}$ be the triad of M' containing t and meeting $\{x, y\}$. If $z \in G$, then $G \cup T$ has corank at most 3 but contains a 4-element subset $\{t, u, v, w\}$ of B^* ; a contradiction. Thus $z \notin G$, so $z \in B^*$ by Lemma 3.5. Then $G \cup T$ has corank 4 but contains a 5-element subset $\{t, u, v, w, z\}$ of B^* ; a contradiction. \square

The following results consider allowable pivots of M' . The routine proof of the first is omitted.

Lemma 4.2. *Let $z \in B - \{x, y\}$. Then $A_{zw} \neq 0$ for some $w \in B^* - G$ if and only if $z \notin \text{cl}^*(G \cap B^*)$.*

Lemma 4.3. *If G is a confining set, then $A_{xz} = A_{yz} = 0$ for all $z \in B^* - G$.*

Proof. Since $\{x, y\} \subseteq \text{cl}^*(G \cap B^*)$, it follows that $A_{xz} = A_{yz} = 0$ for all $z \in B^* - G$. \square

Lemma 4.4. *If $A_{wz} \neq 0$ for some $w \in B - \{x, y\}$ and $z \in B^* - G$, then a pivot on A_{wz} is allowable. Moreover, G is a confining set relative to the basis $B \Delta \{w, z\}$.*

Proof. Suppose that $A_{wz} \neq 0$ for some $w \in B - \{x, y\}$ and $z \in B^* - G$. Then the pivot on A_{wz} is allowable by Lemma 4.3. Since $G \cap B = G \cap (B \Delta \{w, z\})$ and $G \cap B^* = G \cap (B^* \Delta \{w, z\})$, the pivot preserves the property that G is a confining set. \square

Next we show that a confining set, together with allowable pivots, imposes the following restrictions on the elements of $E(M') - G$.

Lemma 4.5. *There is no element $z \in E(M') - G$ such that either:*

- (i) *z is N -deletable, $\text{co}(M' \setminus z)$ is 3-connected, and $z \notin \text{cl}^*(G \cap B^*)$; or*
- (ii) *z is N -contractible and $\text{si}(M'/z)$ is 3-connected.*

Proof. Suppose that $E(M') - G$ contains an element z satisfying the properties described in (i). Then $z \in B - \{x, y\}$ by Lemma 4.1, and it follows from Lemma 4.2 that there is some $w \in B^* - G$ such that $A_{zw} \neq 0$. Hence a pivot on A_{zw} is allowable by Lemma 4.4 and preserves the confining set G . Let $B' = B \Delta \{w, z\}$. Then M' has a confining set G and an (N, B') -strong element z outside of G ; a contradiction of Lemma 4.1.

Now suppose that $E(M') - G$ contains an element z satisfying the properties described in (ii). Since $z \notin G$ it follows from Lemma 4.1 that $z \in B^* - G$. Then $A_{xz} = A_{yz} = 0$ by Lemma 4.3, so there is some $w \in B - \{x, y\}$ such that $A_{wz} \neq 0$ because M' has no loops. Then a pivot on A_{wz} is allowable by Lemma 4.4. Let $B' = B \Delta \{w, z\}$. Then M' has an (N, B') -strong element z in $B' - \{x, y\}$; a contradiction of Lemma 3.1. \square

We employ the following consequence of the Splitter Theorem. A *splitter sequence* for N in M' is a pair $((C, D), (x_1, \dots, x_n))$, where C and D are disjoint subsets of $E(M')$, and (x_1, \dots, x_n) is an ordering of the elements of $C \cup D$ such that:

- (i) $M'/C \setminus D \cong N$; and

- (ii) $M'/(C \cap \{x_1, \dots, x_i\}) \setminus (D \cap \{x_1, \dots, x_i\})$ is 3-connected for all $i \in \{1, \dots, n\}$.

Given a splitter sequence for N in M' , the problem of bounding $|E(M)|$ by some function of $|E(N)|$ is reduced to bounding the size of $|C \cup D|$.

The next result is a tool for dealing with elements of a splitter sequence that are outside of G . If such an element z is (N, B) -robust, then it cannot be (N, B) -strong by Lemma 4.1, so there is a vertical 3-separation associated with z . We now determine various properties of such a vertical 3-separation.

Lemma 4.6. *Let R be a 3-connected minor of a 3-connected matroid Q , and let $Z = (z_1, \dots, z_n)$ be a splitter sequence ordering for $E(Q) - E(R)$. Let z_i be a contractible element for some $i \in \{1, \dots, n\}$. If $\text{si}(Q/z_i)$ is not 3-connected, then there is a vertical 3-separation (X, z_i, Y) such that $|Y - \{z_1, \dots, z_{i-1}\}| \leq 1$, and there is at most one element of Y that is not N -flexible in Q . Moreover, if $s \in Y$ is not N -flexible in Q , then s is N -contractible in Q and $\text{si}(Q/s)$ is 3-connected.*

Proof. Since $\text{si}(Q/z_i)$ is not 3-connected, Q has a vertical 3-separation (X, z_i, Y) . We assume that $|E(R) \cap Y| \leq 1$ and that $X \cup z_i$ is closed in Q . But $(X - \{z_1, \dots, z_{i-1}\}, Y - \{z_1, \dots, z_{i-1}\})$ is not a 2-separation of the 3-connected matroid $Q/z_i/C \cap \{z_1, \dots, z_{i-1}\} \setminus D \cap \{z_1, \dots, z_{i-1}\}$, so $|Y - \{z_1, \dots, z_{i-1}\}| \leq 1$. The remaining properties follow immediately from Lemma 2.16 (i). \square

Let $C \cup D$ be the elements of the splitter sequence (z_1, \dots, z_n) for N in M' , where $N \cong M'/C \setminus D$. A key to bounding the size of $|C \cup D|$ is to bound the number of elements in $\text{cl}^*(G) - G$. We will bound $|D \cap (\text{cl}^*(G) - G)|$, but we first need the following lemma for corank-3 confining sets.

Lemma 4.7. *Let G have corank 3 in M' . If $z', z'' \in C \cup D$ are in $\text{cl}^*(G) - G$, then there is no partition (X, Y) of $G \cup \{z', z''\}$ such that both $r^*(X)$ and $r^*(Y)$ are at most two.*

Proof. Suppose that (X, Y) is a partition of $G \cup \{z', z''\}$ such that $\max\{r^*(X), r^*(Y)\} \leq 2$. We claim that either z' or z'' is an element that contradicts Lemma 4.5 (ii). Since $|G \cup \{z', z''\}| = 7$, we may assume that $|X| \geq 4$. Then X is a cosegment with at least four elements that contains at least one element $z \in \{z', z''\}$, so $\text{si}(M/z)$ is 3-connected by the dual of Lemma 2.4. Hence z is not N -contractible by Lemma 4.5 (ii), so $z \in D$.

First suppose that $z', z'' \in X$. Then $z', z'' \in D$, but z' is in a series class $X \cup z'$ of $M' \setminus z''$, so z' is N -contractible in $M' \setminus z''$ and hence in M' ; a contradiction of Lemma 4.5 (ii).

We may now assume that $z' \in X$ and $z'' \in Y$, so X and $\text{cl}^*(Y)$ are both 4-element cosegments. Hence both $\text{si}(M'/z')$ and $\text{si}(M'/z'')$ are 3-connected by the dual of Lemma 2.4. By the definition of a confining set, there is some element $u \in G - \{x, y\}$ that is (N, B) -strong in M' , and u belongs to either X or Y . Hence, in $M' \setminus u$, either z' or z'' is in a non-trivial series class, so at least one of z' and z'' is N -contractible in M' ; a contradiction of Lemma 4.5 (ii). \square

Lemma 4.8. *There are at most two elements of D that belong to $\text{cl}^*(G) - G$.*

Proof. Suppose that there are at least three elements $z, z', z'' \in (\text{cl}^*(G) - G) \cap D$. Then $z, z', z'' \in B - \{x, y\}$. We shall assume that z comes after z' and z'' in the splitter-sequence ordering.

Now if G is a 4-element cosegment of M' , then z and z' are elements of the cosegment. Since z is in a non-trivial series class of $M' \setminus z'$, the element z is N -contractible in M' . By the dual of Lemma 2.4, M'/z is 3-connected, so z' is an (N, B) -strong element of $B - \{x, y\}$; a contradiction of Lemma 3.1.

Assume G is not a 4-point cosegment. Then $\text{cl}^*(G)$ has corank 3 in M' . We first show that z is N -contractible in M' . This is certainly true if $\{z, z', z''\}$ is a triad of M' , for, in that case, z is N -contractible since it is in a series pair of $M' \setminus z'$, and z' is N -deletable in M' . We may therefore assume that $\{z, z', z''\}$ is not a triad of M' . Then $\{z, z', z''\}$ is a cobasis for $\text{cl}^*(G)$. As $M' \setminus \{z, z', z''\}$ has an N -minor, it follows that the minor obtained from M' by deleting any cobasis for $\text{cl}^*(G)$ and contracting the remaining elements of $\text{cl}^*(G)$ has an N -minor. In particular, the element z is N -contractible in the matroid $M' \setminus (G \cap B^*)$, so it follows that z is N -contractible in M' .

Now z is an N -contractible element of M' , so it follows from Lemma 4.5 (ii) that $\text{si}(M'/z)$ is not 3-connected. Hence there is a vertical 3-separation (X, z, Y) of M' for some X and Y . But then either X or Y must cospan $\text{cl}^*(G)$ by Lemma 4.7. Assume X spans $\text{cl}^*(G)$. Then $z \in \text{cl}^*(X)$, and by the definition of a vertical 3-separation, $z \in \text{cl}(Y)$; a contradiction to orthogonality. \square

Now we work towards a proof that $D \subseteq \text{cl}^*(G)$. We first need the following easy observation on the elements that are N -flexible in M' .

Lemma 4.9. *If c is N -flexible, then $c \in \text{cl}^*(G)$.*

Proof. Suppose that $c \notin \text{cl}^*(G)$. Then $\text{si}(M'/c)$ is not 3-connected by Lemma 4.5 (ii), so $\text{co}(M' \setminus c)$ is 3-connected by Bixby's Lemma. Then it follows from Lemma 4.5 (i) that $c \in \text{cl}^*(G)$. \square

Lemma 4.10. *$D \subseteq \text{cl}^*(G)$.*

Proof. Suppose that there is some $z \in D$ outside of $\text{cl}^*(G)$. Since z is N -deletable and $z \notin \text{cl}^*(G)$, it follows from Lemma 4.5 (i) that $\text{co}(M' \setminus z)$ is not 3-connected. Thus, by Lemma 4.6, there is a vertical 3-separation (X, z, Y) of $(M')^*$ such that at most one element of Y is not N -flexible. We claim that every element of Y is in $\text{cl}^*(G)$. The claim follows immediately from Lemma 4.9 unless $s \in Y$ is the single element of Y that is not N -flexible. By Lemma 4.6, the element s is N -deletable and $\text{co}(M' \setminus s)$ is 3-connected, so, by Lemma 4.5(i), $s \in \text{cl}^*(G)$. Thus every element of Y is in $\text{cl}^*(G)$. Now it follows from the definition of a vertical 3-separation in $(M')^*$ that $r^*(Y) \geq 3$. Thus Y spans $\text{cl}^*(G)$, and so $\text{cl}^*(Y) = \text{cl}^*(G)$. But $z \in \text{cl}^*(Y)$ because (X, z, Y) is a vertical 3-separation in $(M')^*$, so $z \in \text{cl}^*(G)$; a contradiction. \square

We can now show that $|\text{cl}^*(G)| \leq 7$.

Lemma 4.11. *There are at most two elements of $C \cup D$ in $\text{cl}^*(G) - G$.*

Proof. Suppose that there are three elements $p, q, s \in C \cup D$ such that $p, q, s \in \text{cl}^*(G) - G$. We may assume, by Lemma 4.8, that $p \in C$. Hence

$\text{si}(M'/p)$ is not 3-connected by Lemma 4.5 (ii). Let (X, p, Y) be a vertical 3-separation of M' . Then by Lemma 4.7 we may assume that X cospans $G \cup \{q, s\}$, and hence p . But then $p \in \text{cl}^*(X)$ and $p \in \text{cl}(Y)$; a contradiction to orthogonality. \square

It remains to bound the number of elements of C outside of $\text{cl}^*(G)$.

Lemma 4.12. *There are at most seven elements of C that are not in $\text{cl}^*(G)$.*

Proof. Suppose that there are at least eight elements of $C - \text{cl}^*(G)$, and let p_1, \dots, p_8 be the first eight such elements in the splitter-sequence ordering. It follows from Lemma 4.5 (ii) that there is some vertical 3-separation (X_i, p_i, Y_i) of M' for each $i \in \{1, \dots, 8\}$. Assume that $|Y_i \cap E(N)| \leq 1$ for all $i \in \{1, \dots, 8\}$.

4.12.1. *There is some pair $i, j \in \{1, \dots, 8\}$ such that $|Y_i \cap Y_j| \geq 2$.*

Subproof. It follows from Lemma 4.5, Lemma 4.6 and Lemma 4.9 that $Y_i \subseteq \text{cl}^*(G)$ for all $i \in \{1, \dots, 8\}$. By Lemma 4.11 $|\text{cl}^*(G)| \leq 7$. We know that $|Y_i| \geq 3$ because (X_i, p_i, Y_i) is a vertical 3-separation of M' . Moreover, if $|Y_i| = 3$, then Y_i is a triad of M' .

Suppose that $|Y_1| \geq 5$. If $|Y_1 \cap Y_2| \geq 2$, then Y_1, Y_2 is the desired pair. Otherwise $|Y_1 \cap Y_2| = 1$ and $Y_1 \cup Y_2$ covers $\text{cl}^*(G)$. But then either $|Y_1 \cap Y_3| \geq 2$ or $|Y_2 \cap Y_3| \geq 2$. By symmetry, it follows that we may assume that $|Y_i| \leq 4$ for each $i \in \{1, \dots, 8\}$.

Now suppose that $|Y_1| = |Y_2| = 4$. Then either $|Y_1 \cap Y_2| \geq 2$, or $|Y_1 \cap Y_2| = 1$ and $Y_1 \cup Y_2$ covers $\text{cl}^*(G)$. But then either $|Y_1 \cap Y_3| \geq 2$ or $|Y_2 \cap Y_3| \geq 2$. It follows that we may assume that $|Y_i| \geq 4$ for at most one $i \in \{1, \dots, 8\}$.

Assume that labels are chosen such that Y_1, \dots, Y_7 are triads. Suppose that $|Y_i \cap Y_j| \leq 1$ for all $i, j \in \{1, \dots, 7\}$. We first consider the case that there are $i, j \in \{1, \dots, 8\}$ such that $Y_i \cap Y_j = \emptyset$. We shall assume that labels are chosen such that $Y_1 \cap Y_2 = \emptyset$. Then it follows that $Y_3 \cap Y_4 \cap Y_5 \neq \emptyset$. Let $y \in Y_3 \cap Y_4 \cap Y_5$. Now consider Y_6 . It follows that $y \in Y_6$ or else $|Y_i \cap Y_6| \geq 2$ for some $i \in \{1, 2\}$. But now $Y_6 - \{y\}$ has at least two elements of $Y_1 \cup Y_2$, and every element in $Y_1 \cup Y_2$ is in $Y_3 \cup Y_4 \cup Y_5$. Thus $|Y_i \cap Y_6| \geq 2$ for some $i \in \{3, 4, 5\}$, which is a contradiction. We shall therefore assume that $|Y_i \cap Y_j| = 1$ for all $i, j \in \{1, \dots, 7\}$. But then $|Y_i \cap Y_8| \geq 2$ for some $i \in \{1, \dots, 7\}$, and we have the desired pair. \square

Now let Y_i and Y_j be a pair such that $|Y_i \cap Y_j| \geq 2$. By uncrossing, the sets $Y_i \cup Y_j$, $Y_i \cup Y_j \cup p_i$, $Y_i \cup Y_j \cup p_j$, $Y_i \cup Y_j \cup \{p_i, p_j\}$ are all 3-separating. Moreover, since p_1, \dots, p_8 are not N -flexible by Lemma 4.9, it follows that $|\{p_1, \dots, p_8\} \cap X_k| = 7$ for $k \in \{i, j\}$, so in particular $|X_i \cap X_j| \geq 2$. Hence $Y_i \cup Y_j$, $Y_i \cup Y_j \cup p_i$, $Y_i \cup Y_j \cup p_j$, $Y_i \cup Y_j \cup \{p_i, p_j\}$ are sides of exact 3-separations of M' and p_i, p_j are guts elements. Then $(X_i \cap X_j, p_i, Y_i \cup Y_j \cup p_j)$ is a vertical 3-separation of M' unless $r(X_i \cap X_j) \leq 2$. But if $r(X_i \cap X_j) \leq 2$, then $(X_i \cap X_j) \cup \{p_i, p_j\}$ is a segment of M' with at least four elements, so, in particular, p_i belongs to a non-trivial parallel class of M'/p_j . Then p_i is N -deletable in M'/p_j and hence in M' , so $p_i \in \text{cl}^*(G)$ by Lemma 4.9; a contradiction. Thus $(X_i \cap X_j, p_i, Y_i \cup Y_j \cup p_j)$ is a vertical 3-separation of M' , and either $|(X_i \cap X_j) \cap E(N)| \leq 1$ or $|(Y_i \cup Y_j \cup p_j) \cap E(N)| \leq 1$. Then, in either case, there is an element p_k in the non- N -side of $(X_i \cap X_j, p_i, Y_i \cup Y_j \cup p_j)$.

Since $\text{si}(M'/p_k)$ is not 3-connected, it follows from Lemma 4.6 that p_k is N -flexible in M' . Hence $p_k \in \text{cl}^*(G)$ by Lemma 4.9; a contradiction. \square

Finally, we are in position to prove the main result of this section.

Lemma 4.13. $|E(M)| \leq |E(N)| + 16$.

Proof. Recall that $D \subseteq \text{cl}^*(G)$ by Lemma 4.10, and $|\text{cl}^*(G)| \leq 7$ by Lemma 4.11. Moreover, $|C - \text{cl}^*(G)| \leq 7$ by Lemma 4.12. Therefore $|E(M)| \leq |E(N)| + |\text{cl}^*(G)| + |C - \text{cl}^*(G)| + |\{a, b\}| \leq |E(N)| + 16$. \square

5. ROBUST ELEMENTS

In this section, we consider the structure of M' that arises from elements that are (N, B) -robust but not (N, B) -strong. A *path of 3-separations* of M' is a partition (P_1, \dots, P_n) of $E(M')$ such that $(P_1 \cup \dots \cup P_i, P_{i+1} \cup \dots \cup P_n)$ is a 3-separation of M' for each $i \in \{1, \dots, n-1\}$. The main result of this section will show that elements that are (N, B) -robust but not (N, B) -strong give rise to a certain path of 3-separations of M' .

We assume that M' has no confining set. By Lemma 3.9 we may assume that the basis B of M' is chosen such that either

- (RB1) There is some element $u \in B^* - \{a, b\}$ such that u is an (N, B) -strong element and $\{u, x, y\}$ is a triad of M' ; or
- (RB2) there are no (N, B) -strong elements of M' outside of $\{x, y\}$.

If B satisfies (RB2), then we can assume there is no basis B' that satisfies (RB1). Furthermore, we assume that our basis B is chosen to have the maximum number of (N, B) -robust elements outside of $\{x, y\}$. Thus if we change the basis of M' from B to B' by way of allowable pivots on elements of $E(M') - \{u, x, y\}$, then the number of (N, B') -robust elements of M' cannot be greater than the number of (N, B) -robust elements of M' .

If M' has an (N, B) -strong element u outside of $\{x, y\}$, then we let $S' = \{u, x, y\}$. Otherwise we let $S' = \{x, y\}$. Thus S' is the set of possible (N, B) -strong elements of M' .

Let (X, z, Y) be a vertical 3-separation of M' . We say that Y is *z -closed* if $Y = \text{cl}^*(Y)$ and $Y = \text{cl}(Y) - \{z\}$. We use z -closure to ensure that the (N, B) -strong elements of M' are contained in the non- N -side of a vertical 3-separation of M' . A set is *fully closed* if it is both closed and coclosed. Given a subset A of $E(M')$, we use $\text{fcl}_{M'}(A)$ to denote the smallest fully closed set that contains A . Thus, the set Y is z -closed if $\text{fcl}_{M'/z}(Y) = Y$.

Lemma 5.1. *If $z \in B$ and z is (N, B) -robust but not (N, B) -strong, then there is some vertical 3-separation (X, z, Y) of M' such that Y is z -closed and $|Y \cap E(N)| \leq 1$.*

Proof. By Lemma 2.2, M' has a vertical 3-separation (X, z, Y) , and we may assume that $|Y \cap E(N)| \leq 1$. The elements of $\text{fcl}_{M'/z}(Y) - Y$ can be ordered (y_1, \dots, y_m) such that $Y \cup \{y_1, \dots, y_i\}$ is 2-separating for all $i \in \{1, \dots, m\}$. Let $Y_i = Y \cup \{y_1, \dots, y_i\}$ and $X_i = X - \{y_1, \dots, y_i\}$ for each $i \in \{1, 2, \dots, m\}$. We also let $(X_0, Y_0) = (X, Y)$. Suppose that $|Y_j \cap E(N)| \geq 2$ for some $j \in \{1, 2, \dots, m\}$. We shall assume that j is the smallest index such that $|Y_j \cap E(N)| \geq 2$. Then, since j is the smallest index, $|Y_{j-1} \cap E(N)| \leq 1$, so $|X_{j-1} \cap E(N)| \geq 3$ because $|E(N)| \geq 4$. Hence $|X_j \cap E(N)| \geq 2$.

But then (X_j, Y_j) is a 2-separation of M'/z such that $|X_j \cap E(N)| \geq 2$ and $|Y_j \cap E(N)| \geq 2$; a contradiction. Hence $|Y_i \cap E(N)| \leq 1$ for all $i \in \{1, \dots, m\}$. Thus (X_i, Y_i) is a 2-separation such that X_i is always the N -side for each i , so $|X_i| \geq 3$ for all i . In particular, (X_m, Y_m) is a 2-separation of M'/z such that Y_m is fully closed. Since M' is 3-connected, it follows that $z \in \text{cl}_{M'}(X_m) \cap \text{cl}_{M'}(Y_m)$. Finally X_m is not a parallel class of M'/z because Y_m is fully closed, so $r_{M'}(X_m) \geq 3$. Thus (X_m, z, Y_m) is the desired z -closed vertical 3-separation of M' . \square

We need the following results in order to handle fans.

Lemma 5.2. [2, Lemma 3.4] *Let M be a 3-connected matroid and let B be a basis of M . Suppose there is an element $b \in B$ such that $\text{si}(M/b)$ is not 3-connected, and let (X, b, Y) be a vertical 3-separation of M . Then one of the following holds:*

- (i) *there are distinct elements $s_1, s_2 \in X$ that are removable with respect to B ; or*
- (ii) *there are distinct elements $s_1 \in X$ and $s_2 \in \text{cl}^*(X) \cap B$, and a vertical 3-separation (X', b', Y') of M such that $X' \cup s_2$ is a 4-element cosegment containing s_1 , the element $b' \in B$ is not removable with respect to B , and $X' \cup b' \subseteq X \cup b$; or*
- (iii) *there are distinct elements $s_1 \in X$ and $s_2, s_3 \in \text{cl}(X) \cap (E(M) - B)$ that are removable with respect to B ; or*
- (iv) *M has a type I or type II fan relative to B contained in $X \cup b$.*

Lemma 5.3. [2, Lemma 4.8] *Let M be a 3-connected matroid, let N be a 3-connected minor of M such that $|E(N)| \geq 4$, and let B be a basis of M . Suppose that $b \in B$ is an element that is (N, B) -robust but not (N, B) -strong, and let (X, b, Y) be a vertical 3-separation of M such that $|X \cap E(N)| \leq 1$. Then one of the following holds:*

- (i) *there are distinct (N, B) -strong elements $s_1, s_2 \in X$; or*
- (ii) *there are distinct (N, B) -strong elements $s_1 \in X$ and $s_2 \in \text{cl}^*(X) \cap B$;*
or
- (iii) *there are distinct (N, B) -strong elements $s_1 \in X$ and $s_2, s_3 \in \text{cl}(X) \cap (E(M) - B)$; or*
- (iv) *M has a type I or type II fan relative to B contained in $X \cup b$.*

The next lemma is a consequence of Lemma 5.3 and Lemma 2.16.

Lemma 5.4. *Let $z \in B - \{x, y\}$ be an element of M' that is (N, B) -robust but not (N, B) -strong, and let (X, z, Y) be a vertical 3-separation of M' such that Y is z -closed and $|Y \cap E(N)| \leq 1$. If there is at most one (N, B) -strong element of M' contained in Y , then there is a type-I or type-II fan F relative to B that is contained in $Y \cup z$ with ordering $(\alpha, \beta, \gamma, \delta)$ such that β, γ, δ are N -contractible and α, β, γ are N -deletable.*

Proof. Since Y is z -closed, it follows from Lemma 5.3 that M' has a type-I or type-II fan F relative to B with ordering $(\alpha, \beta, \gamma, \delta)$ such that $F \subseteq Y \cup z$. Since $\{\beta, \gamma, \delta\}$ is a triad of M' contained in Y , it follows from orthogonality that $\beta, \gamma, \delta \notin \text{cl}_{M'}(X)$. Hence β, γ , and δ are N -contractible by Lemma 2.16 (ii). Since $\{\alpha, \beta, \gamma\}$ is a triangle of M' , it follows that α, β , and γ are also N -deletable. \square

We assume for the remainder of the paper that $|E(M)| \geq 9$ to use Lemma 2.11. Note that, for an excluded minor M having $|E(M)| < 9$, it is clear that $|E(M)| \leq |E(N)| + 16$, so outcome (i) of Theorem 2.26 and 2.26 holds.

Recall that S' is the set of possible (N, B) -strong elements of M' . We now prove that $S' \subseteq Y$.

Lemma 5.5. *Let $z \in B - \{x, y\}$ be an element of M' that is (N, B) -robust but not (N, B) -strong, and let (X, z, Y) be a vertical 3-separation of M' such that Y is z -closed and $|Y \cap E(N)| \leq 1$. Then $S' \subseteq Y$.*

Proof. Suppose that there are at least two distinct (N, B) -strong elements in Y . The (N, B) -strong elements of M' contained in Y must belong to S' by the definition of S' . If $|S'| = 2$, then it follows immediately that $S' \subseteq Y$. If $|S'| = 3$, then S' is a triad, so $S' \subseteq Y$ because Y is coclosed.

We may therefore assume that there is at most one (N, B) -strong element of M' contained in Y . Then it follows from Lemma 5.4 that there is a type-I or type-II fan F relative to B contained in $Y \cup z$ with ordering $(\alpha, \beta, \gamma, \delta)$ such that β, γ, δ are N -contractible and α, β, γ are N -deletable.

We need the following subproofs.

5.5.1. $\{x, y\} \cap \{\alpha, \gamma\} \neq \emptyset$.

Subproof. Assume that $\{x, y\} \cap \{\alpha, \gamma\} = \emptyset$. Suppose β is an (N, B) -strong element of M' . Then, since $\beta \notin B$, it follows that $S' = \{\beta, x, y\}$ is a triad of M' . Since $\{\alpha, \beta, \gamma\}$ is a triangle that meets $\{\beta, x, y\}$, it follows from orthogonality that x or y is in $\{\alpha, \gamma\}$; a contradiction because $\{x, y\} \cap \{\alpha, \gamma\} = \emptyset$. Thus β is not an (N, B) -strong element of M' . Since β is N -flexible, $\text{co}(M' \setminus \beta)$ is not 3-connected. Thus, by Bixby's Lemma, $\text{si}(M' / \beta)$ is 3-connected. Since $\{\alpha, \beta, \gamma\}$ is a triangle of M' the cobasis element β is spanned by the basis elements α and γ , so $A_{i\beta} \neq 0$ if and only if $i \in \{\alpha, \gamma\}$. In particular, since $\{x, y\} \cap \{\alpha, \gamma\} \neq \emptyset$, this means that $A_{\alpha\beta} \neq 0$ and $A_{x\beta} = A_{y\beta} = 0$. Thus a pivot on $A_{\alpha\beta}$ is an allowable pivot. But then β is an $(N, B \Delta \{\alpha, \beta\})$ -strong element outside of $\{x, y\}$ such that $\beta \in B \Delta \{\alpha, \beta\}$; a contradiction of Lemma 3.1. \square

We now know that α or γ is a member of $\{x, y\}$. Since δ is N -contractible and $\text{si}(M / \delta)$ is 3-connected by Lemma 2.11, it follows from Lemma 3.1 that if $\delta \in B$, then $\delta \in \{x, y\}$. Hence $\{x, y\} \subseteq F$, and $S' \subseteq \text{cl}^*(F) \subseteq Y$. Thus we may assume that $\delta \in B^*$.

We first handle the case when $\alpha \in \{x, y\}$.

5.5.2. *If $\alpha \in \{x, y\}$, then $S' \subseteq Y$.*

Subproof. Assume that $\alpha = x$. Suppose that β is an (N, B) -strong element of M' . Then $\{\beta, x, y\}$ is a triad of M' , so $S' \subseteq \text{cl}^*(F) \subseteq Y$, as required. Now suppose that β is not an (N, B) -strong element of M' . Consider the entry $A_{\alpha\beta}$. Since $\{\alpha, \beta, \gamma\}$ is a triangle of M' it follows that $A_{\alpha\beta} \neq 0$, so a pivot on $A_{\alpha\beta}$ is an allowable pivot. We then have a basis $B' = B \Delta \{\alpha, \beta\}$ of M' and α is an (N, B') -strong element outside of $\{\beta, y\}$. By the choice of basis B , there is some element $u \in B^*$ such that u is (N, B) -strong and $\{u, x, y\}$ is a triad. Since β and δ are not (N, B) -strong, it follows that $u \in E(M') - F$. But then $y \in \{\beta, \gamma\}$ by orthogonality, so $y = \gamma$. Therefore $S' \subseteq \text{cl}^*(F) \subseteq Y$. \square

We may now assume that $\alpha \notin \{x, y\}$. Suppose that $\gamma = x$. If β is (N, B) -strong, then $\{\beta, x, y\}$ is a triad. But then $\{\beta, \delta, x, y\}$ is a 4-point cosegment; a contradiction because M' has no confining set. We deduce that β is not (N, B) -strong. Suppose that $\text{co}(M' \setminus x)$ is 3-connected. Now $A_{x\beta} \neq 0$ since $\{\alpha, \beta, x\}$ is a triangle of M' , so a pivot on $A_{x\beta}$ is an allowable pivot. We then have a basis $B' = B \Delta \{\gamma, \beta\}$ such that x is (N, B) -strong element outside of $\{\beta, y\}$. By the choice of B , there is some (N, B) -strong element $u \in B^*$ such that $\{u, x, y\}$ is a triad. Since $u \notin F$, it follows from orthogonality that $\alpha = y$. Thus $S' \subseteq \text{cl}^*(F) \subseteq Y$.

We may now assume that $\text{co}(M' \setminus x)$ is not 3-connected. Then $\text{si}(M'/x)$ is 3-connected by Bixby's Lemma. Since β is not (N, B) -strong, there is a vertical 3-separation (X, β, Y) of $(M')^*$. By orthogonality we may assume that $x \in X$ and $\alpha \in Y$. Consider $(X - x, x, Y \cup \beta)$. It cannot be a vertical 3-separation of M' since $\text{si}(M'/x)$ is 3-connected. Thus $r(X - x) \leq 2$ and so X contains a triangle. By orthogonality, X is a triangle and $X = \{x, \delta, \mu\}$ for some $\mu \in E(M')$. Moreover, $\mu \in \text{cl}(Y)$ or else $\{\beta, x, \delta, \mu\}$ is a 4-point cosegment; a contradiction to orthogonality. Thus M' has a 5-point fan with ordering $(\alpha, \beta, x, \delta, \mu)$. Now $\text{co}(M' \setminus \mu)$ is 3-connected by Lemma 2.11 and μ is N -deletable since μ is in a non-trivial parallel class in M'/x . Now, if $\mu \in B^* - \{a, b\}$, then μ is (N, B) -strong and outside of $\{x, y\}$. Then $\{\mu, x, y\}$ is a triad. Then $\alpha = y$ by orthogonality; a contradiction to the assumption that $\alpha \notin \{x, y\}$. We deduce that $\mu \in B$.

Now $\text{co}(M' \setminus x)$ is not 3-connected. Then there is a vertical 3-separation (X, x, Y) of $(M')^*$. By orthogonality, we may assume that $\alpha \in X$ and $\beta \in Y$. Consider the partition $(X \cup x, \beta, Y - \beta)$. Since $\text{si}(M'/\beta)$ is 3-connected by Bixby's Lemma, it follows that $(X \cup x, \beta, Y - \beta)$ cannot be a vertical 3-separation of M' . Thus $r(Y - \beta) = 2$, so Y is a triangle of M' . By orthogonality, $\mu \in X$ and $\delta \in Y$. Moreover, $\varepsilon \in \text{cl}(X)$ or else $\{\varepsilon, x, \beta, \delta\}$ is a 4-point cosegment; a contradiction to orthogonality. Thus there is some element $\varepsilon \in E(M')$ such that $\{\alpha, \mu, \varepsilon\}$ is a triangle. But $\alpha, \mu \in B$ and $\{\alpha, \mu, \varepsilon\}$ is a triangle, so it follows that $\varepsilon \in B^*$. We claim that ε is an (N, B) -strong element of M' . That ε is (N, B) -robust follows from the fact that β is N -contractible and $\{\delta, \varepsilon\}$ is a parallel pair in M'/β . The fact that $\text{co}(M' \setminus \varepsilon)$ is 3-connected follows from Bixby's Lemma, since $(F, \varepsilon, E(M') - F)$ is a vertical 3-separation of M' so $\text{si}(M'/\varepsilon)$ is not 3-connected. Thus ε is an (N, B) -strong element of M' outside of $\{x, y\}$. Thus $\{\varepsilon, x, y\}$ is a triad of M' . But $\{x, y\}$ meets the triangle $\{\beta, \delta, \varepsilon\}$ in a single element; a contradiction to orthogonality. \square

Next we handle (N, B) -robust elements of B^* .

Lemma 5.6. *Let $z \in B^*$ be an element of M' that is (N, B) -robust but not (N, B) -strong, and let (X, z, Y) be a vertical 3-separation of $(M')^*$ such that Y is z -closed in $(M')^*$ and $|Y \cap E(N)| \leq 1$. Then $S' \subseteq Y$.*

Proof. Suppose that there are at least two distinct (N, B) -strong elements in Y . The (N, B) -strong elements of M' contained in Y must belong to S' by the definition of S' . If $|S'| = 2$, then it follows immediately that $S' \subseteq Y$. If $|S'| = 3$, then S' is a triad, so $S' \subseteq Y$ because Y is coclosed.

We may therefore assume that there is at most one (N, B) -strong element of M' contained in Y . Then it follows from the dual of Lemma 5.4 that there is a type-I or type-II fan F relative to B^* that is contained in $Y \cup z$ with ordering $(\alpha, \beta, \gamma, \delta)$ such that β, γ, δ are N^* -contractible and α, β, γ are N^* -deletable in $(M')^*$.

Suppose that F is a type-II fan relative to B^* . Then δ is an (N^*, B^*) -strong element of $(M')^*$ by Lemma 2.11. Hence $(M')^*$ has a triangle $\{\delta, x, y\}$. By orthogonality, $\beta \in \{x, y\}$ or $\gamma \in \{x, y\}$, so $S' \subseteq Y$ because Y is z -closed.

We may now assume that F is a type-I fan relative to B^* . If γ is an (N^*, B^*) -strong element of $(M')^*$, then $(M')^*$ has a triangle $\{\gamma, x, y\}$. By orthogonality, $\beta \in \{x, y\}$ or $\gamma \in \{x, y\}$, so $S' \subseteq Y$ because Y is z -closed. Therefore we may also assume that $\text{si}((M')^*/\gamma)$ is not 3-connected.

5.6.1. $\{x, y\} \cap \{\beta, \delta\} \neq \emptyset$.

Subproof. Suppose that $\{x, y\} \cap \{\beta, \delta\} = \emptyset$. Then, since $\{\beta, \gamma, \delta\}$ is a triad of $(M')^*$, it follows that $A_{x\gamma} = A_{y\gamma} = 0$ and $A_{\beta\gamma} \neq 0$. Hence a pivot on $A_{\beta\gamma}$ is allowable, and $\gamma \in B' = B\Delta\{\beta, \gamma\}$. But then M' has γ is an (N, B') -strong element in $B' - \{x, y\}$; a contradiction of Lemma 3.1. \square

Suppose $\delta \in \{x, y\}$. Then there is some z such that $A_{\delta z} \neq 0$, and a pivot on $A_{\delta z}$ is allowable. Hence M' has a basis $B' = B\Delta\{\delta, z\}$ with an (N, B') -strong element δ in $(B')^*$. By the choice of basis, B must have an (N, B) -strong element $u \in B^*$ such that $S' = \{u, x, y\}$ is a triangle of $(M')^*$. By orthogonality, either $\beta \in S'$ or $\gamma \in S'$. Hence $S' \subseteq Y$ because Y is z -closed. A similar argument holds if $\beta \in \{x, y\}$ and $\text{si}((M')^*/\beta)$ is 3-connected.

We may now assume that $\beta \in \{x, y\}$ and that $\text{si}((M')^*/\beta)$ is not 3-connected. Let (P, β, Q) be a vertical 3-separation of $(M')^*$. Since β is in a triad of $(M')^*$, we may assume that $\gamma \in P$ and $\delta \in Q$. The partition $(P - \gamma, \gamma, Q \cup \beta)$ cannot be a vertical 3-separation of M' by Bixby's Lemma, so it follows that P is a triad of $(M')^*$. By orthogonality, $\alpha \in P$. Thus $P = \{\alpha, \gamma, p\}$ for some p .

Let (R, γ, S) be a vertical 3-separation of $(M')^*$. Since γ is in a triad of $(M')^*$, we may assume that $\beta \in R$ and $\delta \in S$. The partition $(R - \beta, \beta, S \cup \gamma)$ cannot be a vertical 3-separation of M' by Bixby's Lemma, so it follows that R is a triad of $(M')^*$. By orthogonality, $\alpha \in R$. Therefore $R = \{\alpha, \beta, t\}$ for some t .

Then $\{\alpha, \beta, \gamma, \delta, p, t\}$ is a corank-3 subset of $(M')^*$. Hence at least one of p and t are in B^* . We may assume that $t \in B^*$. It follows that t is an (N^*, B^*) -strong element of $(M')^*$. But then $S' = \{t, x, y\}$ is a triangle of $(M')^*$, so it follows by z -closure that $S' \subseteq Y$. \square

We now work towards a proof that all of the (N, B) -robust elements of M' are contained in the non- N -side of some vertical 3-separation.

Lemma 5.7. *Let Q be a 3-connected matroid and (A, Z, B) a partition of $E(Q)$ with $|A|, |B| \geq 2$. If, for all $z \in Z$, there is a path $(A', \{z\}, B')$ of 3-separations such that $A \subseteq A'$ and $B \subseteq B'$, then there is an ordering (z_1, \dots, z_n) of the elements of Z such that (A, z_1, \dots, z_n, B) is a path of 3-separations of Q .*

Proof. We argue by induction on $|Z| = n$. If $n = 1$, then the lemma holds. Suppose that $n \geq 2$ and that the lemma holds for all $m \leq n - 1$. Let $z \in Z$. Then Q has a path of 3-separations (A_z, z, B_z) such that $A \subseteq A_z$ and $B \subseteq B_z$.

5.7.1. $(A_1, Z_1, B_1) = (A, A_z - A, B_z \cup z)$ and $(A_2, Z_2, B_2) = (A_z \cup z, B_z - B, B)$ are paths of 3-separations of Q that satisfy the inductive hypotheses.

Subproof. It is clear that (A_1, Z_1, B_1) is a partition of $E(M)$ and that $|Z_1| < |Z|$. It remains to prove that, for each $x \in Z_1$, there is a path of 3-separations of the form (A'', x, B'') such that $A_1 \subseteq A''$ and $B_1 \subseteq B''$. Since $x \in Z$ there is a path of 3-separations (A_x, x, B_x) such that $A \subseteq A_x$ and $B \subseteq B_x$. Consider the partition $(A_x \cap A_z, x, B_x \cup B_z \cup z)$ of Q . It follows from uncrossing B_x and $B_z \cup z$ that $B_x \cup B_z \cup z$ is 3-separating. Also, by uncrossing A_x and A_z , we see that $A_x \cap A_z$ is 3-separating. Thus $(A_x \cap A_z, x, B_x \cup B_z \cup z)$ is a path of 3-separations of Q such that $A_1 \subseteq A_x \cap A_z$ and $B_1 \subseteq B_x \cup B_z \cup z$. Therefore the inductive hypotheses hold for (A_1, Z_1, B_1) . The same argument shows they also hold for (A_2, Z_2, B_2) . \square

It follows from 5.7.1 and the induction assumption that there are paths of 3-separations $(A_1, x_1, \dots, x_p, B_1)$ and $(A_2, y_1, \dots, y_q, B_2)$ of Q . Then combining these paths, we see that $(A, x_1, \dots, x_p, z, y_1, \dots, y_q, B)$ is also a path of 3-separations of Q . \square

We show in the next lemma that elements on the non- N -side of a vertical 3-separation that are not N -flexible are not (N, B) -robust.

Lemma 5.8. *Let $z \in B - \{x, y\}$ be an element of M' that is (N, B) -robust but not (N, B) -strong, and let $(X, \{z\}, Y)$ be a vertical 3-separation of M' such that $|E(N) \cap Y| \leq 1$ and Y is z -closed. If $\mu \in Y - S'$ and μ is not N -flexible, then μ is not (N, B) -robust. Moreover, if such a μ exists, then it is unique, and $(X \cup \mu, z, Y - \mu)$ is a vertical 3-separation of M' .*

Proof. Clearly either μ is not N -contractible or not N -deletable. Assume that μ is not N -contractible. Then $\mu \in \text{cl}(X)$ and μ is N -deletable by Lemma 2.16. Suppose that μ is (N, B) -robust. Then $\mu \in B^* - \{a, b\}$. But $(X, \mu, (Y - \mu) \cup z)$ is a vertical 3-separation of M' , so it follows by Bixby's Lemma that $\text{co}(M' \setminus \mu)$ is 3-connected. Thus μ is an (N, B) -strong element, which is a contradiction because $\mu \notin S'$. Therefore μ is not (N, B) -robust, so $\mu \in B$. Thus there can be at most one such element μ because $\{z, \mu\}$ spans the guts of the 3-separation $(X, Y \cup z)$.

We may now assume that μ is not N -deletable. Then it follows from Lemma 2.16 that μ is in the coguts of $(X, Y \cup z)$, and that μ is the only element of Y that is not N -deletable. Moreover, μ is the only element of Y that is not N -flexible. Seeking a contradiction, suppose that μ is an (N, B) -robust element. Then $\mu \in B - \{x, y\}$. But $(X, \mu, (Y - \mu) \cup z)$ is a vertical 3-separation of $(M')^*$, so $\text{si}(M'/\mu)$ is 3-connected by Bixby's Lemma. Thus μ is an (N, B) -strong element of M' ; a contradiction to Lemma 3.1. Therefore μ is not (N, B) -robust.

For the second statement, note that M/z is the two sum of M_X and M_Y with basepoint z' say, where $M_X \setminus z' = (M/z)|_X$ and $M_Y \setminus z' = (M/z)|_Y$. If there is some μ that is not N -deletable, then $\{z', \mu\}$ is a cocircuit in M_Y .

Hence there cannot be any element μ' of Y in the guts of $(X, Y \cup z)$ or else $\{z, \mu'\}$ is a circuit of M_Y ; a contradiction to orthogonality. Therefore the element μ is unique. Now consider the partition $(X \cup \mu, z, Y - \mu)$. If μ is a guts element, then it is clear that $(X \cup \mu, z, Y - \mu)$ is a vertical 3-separation of M' . Assume that μ is a coguts element. Suppose that $(X \cup \mu, z, Y - \mu)$ is not a vertical 3-separation of M' . Then $r(Y) \leq 2$. But $Y - \mu$ spans z , and $\{x, y\} \subseteq Y - \mu$, so $\{x, y, z\}$ is a triangle of M' and $\{x, y, z\} \subseteq B$; a contradiction. \square

Note that a similar lemma holds for elements of $B^* - \{a, b\}$ that are (N, B) -robust but not (N, B) -strong.

Lemma 5.9. *Let $z \in B^* - \{a, b\}$ be an element of M' that is (N, B) -robust but not (N, B) -strong, and let $(X, \{z\}, Y)$ be a vertical 3-separation of $(M')^*$ such that $|E(N) \cap Y| \leq 1$ and $S' \subseteq Y$. If $\mu \in Y - S'$ and μ is not N -flexible, then μ is not (N, B) -robust. Moreover, if such a μ exists, then it is unique, and $(X \cup \mu, z, Y - \mu)$ is a vertical 3-separation of $(M')^*$.*

Let (X', z, Y') be a vertical 3-separation such that Y' is z -closed. Let Y be the 3-separating set obtained from Y' by removing any element of $Y' - S'$ that is not (N, B) -robust, and let (X, z, Y) be the corresponding vertical 3-separation. We say that (X, z, Y) is a *good separation for z* . The idea here is that, for each (N, B) -robust element of M' , there is a good separation. All of the elements on the non- N -side of the good separation are N -flexible, and therefore (N, B) -robust. We now show that a good separation induces a path of 3-separations in M' .

Lemma 5.10. *Let (X, z, Y) be a good separation for z , and let $Z = Y - S'$. Then there is an ordering (z_1, \dots, z_n) of Z such that $(X, z, z_n, \dots, z_1, S')$ is a path of 3-separations of M' . Moreover, the elements of $\{z_n, \dots, z_1\} \cup S'$ are N -flexible, and $|E(N) \cap (\{z_i, \dots, z_1\} \cup S')| \leq 1$ for all $i \in \{1, \dots, n\}$.*

Proof. Consider the partition $(X \cup z, Z, S')$ of $E(M')$. We claim that, for each $z_i \in Z$, there is a path of 3-separations of the form (X_i, z_i, Y_i) such that $X \cup z \subseteq X_i$ and $S' \subseteq Y_i$.

Fix an N -minor of M'/z on $A = \{a_1, \dots, a_k\}$ say. Since $z_i \in Y$, removing $z_i \in Z$ keeps an N -minor on A' , where either $A' = \{a_1, \dots, a_k\}$ or $A' = (\{a_1, \dots, a_k\} - z_i) \cup y_i$ for some $y_i \in Y$. Since z_i is (N, B) -robust but not (N, B) -strong, there is a vertical 3-separation of M' or $(M')^*$ of the form (X_i, z_i, Y_i) where $|A' \cap Y_i| \leq 1$. By Lemma 5.1 or its dual, we may assume that Y_i is z_i -closed. Hence $S' \subseteq Y_i$ by Lemma 5.5 or Lemma 5.6.

Since $|E(N)| \geq 4$, it follows that $|X \cap X_i| \geq |E(N)| - 2 \geq 2$. Therefore, by uncrossing $X \cup z$ and X_i , it follows that $X \cup X_i \cup z$ is 3-separating. Similarly, by uncrossing $X \cup z$ and $X_i \cup z_i$, we deduce that $X \cup X_i \cup z \cup z_i$ is 3-separating. Therefore the partition $(X \cup X_i \cup z, z_i, Y \cap Y_i)$ is a path of 3-separations of M' .

It follows from Lemma 5.7 that there is an ordering (z_1, \dots, z_n) of Z such that $(X, z, z_n, \dots, z_1, S')$ is a path of 3-separations of M' . The second statement follows from Lemma 5.8 and Lemma 5.9. \square

6. THE MAIN RESULTS

Suppose that M' has an element z that is (N, B) -robust but not (N, B) -strong. Then M' has a path of 3-separations of the form $(X, z, z_n, \dots, z_1, S')$ by Lemma 5.10. Moreover, the elements of $\{z_n, \dots, z_1\} \cup S'$ are N -flexible. In this section, we study these paths of 3-separations, and use them to prove the main results.

Recall that $|S'| \in \{2, 3\}$. In the case where $|S'| = 3$, we shall let z_1 label the (N, B) -strong element outside of $\{x, y\}$ and we relabel the elements of $Y - S'$ accordingly, so that we can always write the path of 3-separations as $(X, z, z_n, \dots, z_1, \{x, y\})$.

We say that an element $z_i \in Z$ is a *guts* or *coguts* element according to whether z_i is in the guts or coguts of the 3-separation $(X \cup \{z_n, \dots, z_i\}, \{z_{i-1}, \dots, z_1\} \cup \{x, y\})$.

Lemma 6.1. *If $|E(M)| \geq 10$, then $\{x, y, z_1\}$ is not a triangle.*

Proof. If $z_1 \in S'$, then $S' = \{x, y, z_1\}$ and S' is a triad, so $\{x, y, z_1\}$ is not a triangle since M' is 3-connected. We may therefore assume that $z_1 \notin S'$, so B is a basis of M' satisfying (RB2). Suppose that $\{x, y, z_1\}$ is a triangle. Now, if $z_1 \in B$, then the triangle $\{x, y, z_1\}$ is contained in the basis B ; a contradiction. Thus $z_1 \in B^*$, and $\text{co}(M' \setminus z_1)$ is not 3-connected because z_1 is (N, B) -robust but not (N, B) -strong.

Let (P, z_1, Q) be a vertical 3-separation of $(M')^*$. Since $\{x, y, z_1\}$ is a triangle of M' , it follows from orthogonality that $|P \cap \{x, y\}| = |Q \cap \{x, y\}| = 1$. We shall therefore assume that $x \in P$ and $y \in Q$. By Lemma 2.16, either x or y is N -flexible in M' . Since $\{x, y, z_1\}$ is a triangle of M' , it follows that both x and y are N -deletable in M' . Hence $\text{co}(M' \setminus x)$ and $\text{co}(M' \setminus y)$ are also not 3-connected because M' does not have a robust basis with an (N, B) -strong element outside of $\{x, y\}$. Now consider the paths of 3-separations $(P \cup z_1, y, Q - y)$ and $(P - x, x, Q \cup z_1)$ of M' . If $r(P \cup z_1) \geq 3$ and $r(Q - y) \geq 3$, then $(P \cup z_1, y, Q - y)$ is a vertical 3-separation of M' , so $\text{si}(M'/y)$ is not 3-connected, which is a contradiction to Bixby's Lemma. Therefore $r(P \cup z_1) \leq 2$ or $r(Q - y) \leq 2$. But if $r(P \cup z_1) \leq 2$, then $M' \setminus z_1$ is 3-connected since $|P \cup z_1| \geq 4$; a contradiction to the assumption that z_1 is not (N, B) -strong. Thus $r(Q - y) \leq 2$, and hence $r(Q) \leq 2$. Similarly, it follows that $r(P - x) \leq 2$, and hence $r(P) \leq 2$. Then since $x \in P$ and $y \in Q$, and $\text{co}(M' \setminus x)$ and $\text{co}(M' \setminus y)$ are not 3-connected, it follows from Lemma 2.4 that $|P| \leq 3$ and $|Q| \leq 3$. Therefore $|E(M)| \leq |P| + |Q| + |\{a, b, z_1\}| \leq 9$, a contradiction. \square

Lemma 6.2. $z_1 \in B^*$.

Proof. The statement is clearly true if z_1 is an (N, B) -strong element of M' . Suppose that M' has no (N, B) -strong elements outside of $\{x, y\}$, and suppose that $z_1 \in B$. Then $\text{si}(M'/z_1)$ is not 3-connected because M' has no (N, B) -strong elements in $B - \{x, y\}$.

Now M' has an (N, B) -robust element $z \in Z$ that is either in the closure or coclosure of $\{z_1, x, y\}$. If z is in the coclosure of $\{z_1, x, y\}$, then $\{z, z_1, x, y\}$ is a 4-element cosegment of M' , which is a contradiction to the fact that $\text{si}(M'/z_1)$ is not 3-connected. Thus z is in the closure of $\{z_1, x, y\}$.

Then $(E(M') - \{z, z_1, x, y\}, z, \{z_1, x, y\})$ is a vertical 3-separation of M' , so $\text{si}(M'/z)$ is not 3-connected. Hence $\text{co}(M'\setminus z)$ is 3-connected by Bixby's Lemma. But, since $\{z, z_1, x, y\}$ contains a circuit of M' , it follows that $z \in B^*$. Thus z is an (N, B) -strong element outside of $\{x, y\}$; a contradiction. Therefore $z_1 \in B^*$. \square

Lemma 6.3. *Let $z \in Z$. Then z is a guts element if and only if $z \in B$.*

Proof. Suppose that z is a guts element. Then, by Lemma 6.1 and Lemma 6.2, we may assume that $z = z_i$ for some $i \geq 2$. If z is not N -deletable, then $z \in B$ because z is an (N, B) -robust element. Thus we may assume that z is N -deletable. Since z is in the guts of a vertical 3-separation, it follows that $\text{co}(M'\setminus z)$ is 3-connected by Bixby's Lemma, so $z \in B$ because z is not an (N, B) -strong element outside of S' .

Conversely, suppose that z is a coguts element. By Lemma 6.2 we shall assume that $z = z_i$ for some $i \geq 2$. If z is not N -contractible, then $z \in B^*$ because z is an (N, B) -robust element. Thus we shall assume that z is N -contractible. Now z is in the guts of a vertical 3-separation of $(M')^*$, so it follows from Bixby's Lemma that $\text{si}(M'/z)$ is 3-connected. Thus $z \in B^*$ because M' has no (N, B) -strong elements in $B - \{x, y\}$. \square

We need the following result of Whittle and Williams [16].

Lemma 6.4. [16, Lemma 2.13] *Let M be a 3-connected matroid with a triad $\{a, b, c\}$ and circuit $\{a, b, c, d\}$. Then at least one of the following holds.*

- (i) *Either $\text{co}(M\setminus a)$ or $\text{co}(M\setminus c)$ is 3-connected.*
- (ii) *There exist $a', c' \in E(M)$ such that both $\{a, a', b\}$ and $\{b, c, c'\}$ are triangles.*
- (iii) *There exists $f \in E(M)$ such that $\{a, b, c, f\}$ is a cosegment.*

We can now handle the case when M' has an element outside of S' that is (N, B) -robust but not (N, B) -strong.

Lemma 6.5. *If M' has an element z outside of S' that is (N, B) -robust but not (N, B) -strong, then either*

- (i) *$F = (z, z_1, x, y)$ is a maximal type-II fan relative to B and z is unique; or*
- (ii) *$|E(M)| \leq |E(N)| + 7$.*

Proof. If $|E(M)| \leq 10$, then (ii) holds and we are done. We may therefore assume that $|E(M)| \geq 10$. Hence $\{x, y, z_1\}$ is a triad of M' and $z_1 \in B^*$ by Lemma 6.1 and Lemma 6.2.

Now consider the element $z_2 \in Z$.

6.5.1. $z_2 \in B$.

Subproof. Suppose $z_2 \notin B$. Then z_2 is a coguts element by Lemma 6.3, so $\{x, y, z_1, z_2\}$ is a 4-element cosegment; a contradiction because M' has no confining set. Thus $z_2 \in B$. \square

By Lemma 6.3, it follows that z_2 is spanned by the triad $\{x, y, z_1\}$. Hence $\{x, y, z_1, z_2\}$ is either a 4-element fan or a 4-element circuit of M' .

6.5.2. $\text{cl}(\{x, y, z_1\}) - \{x, y, z_1\} = \{z_2\}$.

Subproof. Suppose there is some $z \notin \{x, y, z_1, z_2\}$ such that $z \in \text{cl}(\{x, y, z_1\})$. If $z \notin \text{cl}(E(M') - \{x, y, z_1\})$, then $z \in \text{cl}^*(\{x, y, z_1\})$, and so $\{x, y, z_1, z\}$ must be a 4-point cosegment of M' ; a contradiction because M' has no confining set. Thus $(E(M') - \{x, y, z_1\}, z, \{x, y, z_1\})$ is a vertical 3-separation of M' , so $\text{si}(M'/z)$ is not 3-connected. Thus $\text{co}(M' \setminus z)$ is 3-connected by Bixby's Lemma. Since z is also a guts element of the vertical 3-separation $(E(M') - \{x, y, z_1\}, z_2, \{x, y, z_1\})$ of M' , it follows from Lemma 2.16 that the element z is N -deletable. Now, if $z \in B^*$, then z is an (N, B) -strong element outside of S' ; a contradiction. Thus $z \in B$. But then the subset $\{x, y, z, z_2\} \subseteq B$ is independent and contained in the rank-3 subset $\text{cl}(\{x, y, z_1\})$; a contradiction. \square

The following claim is the key step for enabling allowable pivots.

6.5.3. *Either (i) holds or $A_{za} = A_{zb} = 0$ for all $z \in B - \{x, y, z_2\}$.*

Subproof. Suppose $\{x, y, z_1, z_2\}$ is a 4-element circuit. Now, since M' has no confining set and $\{x, y, z_1, z_2\}$ is closed by 6.5.2, it follows from Lemma 6.4 that either $\text{co}(M' \setminus x)$ or $\text{co}(M' \setminus y)$ is 3-connected. We may assume without loss of generality that $\text{co}(M' \setminus x)$ is 3-connected. Now $A_{xz_1} \neq 0$ because $\{x, y, z_1, z_2\}$ is a circuit, so a pivot on A_{xz_1} is allowable. Let $B' = B \Delta \{x, z_1\}$. Then x is an (N, B') -strong element outside of $\{z_1, y\}$. Thus x has an unstable pair that meets $\{z_1, y\}$ by Lemma 3.3, so $\{z_1, y\}$ is the unstable series pair for x and we may assume that $b \in \text{cl}(\{y, z_1\})$. By the choice of B , it follows that z_1 is an (N, B) -strong element outside of $\{x, y\}$ and that $a \in \text{cl}(\{x, y\})$. Since $z_1 \in \text{cl}(\{x, y, z_2\})$, it follows that $b \in \text{cl}(\{x, y, z_2\})$. Now $a, b \in \text{cl}(\{x, y, z_2\})$, so $A_{za} = A_{zb} = 0$ for all $z \in B - \{x, y, z_2\}$.

We may therefore assume that $\{x, y, z_1, z_2\}$ is a 4-element fan with ordering (z_2, z_1, x, y) , where $\{z_2, z_1, x\}$ is a triangle. Suppose that $\text{co}(M' \setminus x)$ is 3-connected. Then $A_{xz_1} \neq 0$ because z_1 is a triangle with x , so a pivot on A_{xz_1} is allowable. Let $B' = B \Delta \{x, z_1\}$. Then x is an (N, B') -strong element outside of $\{z_1, y\}$. Thus x has an unstable series pair that meets $\{z_1, y\}$ by Lemma 3.3, so $\{z_1, y\}$ is the unstable series pair for x and we may assume that $b \in \text{cl}(\{y, z_1\})$. By the choice of B , it follows that z_1 is an (N, B) -strong element outside of $\{x, y\}$ and that $a \in \text{cl}(\{x, y\})$. Since $z_1 \in \text{cl}(\{x, y, z_2\})$, we deduce that $b \in \text{cl}(\{x, y, z_2\})$. Now $a, b \in \text{cl}(\{x, y, z_2\})$, so $A_{za} = A_{zb} = 0$ for all $z \in B - \{x, y, z_2\}$.

Suppose that $\text{co}(M' \setminus x)$ is not 3-connected. Then the fan (z_2, z_1, x, y) is maximal. We claim z_2 is the only element outside of S' that is (N, B) -robust but not (N, B) -strong. If there is another such element z' , then it must be in the coguts of the fan (z_2, z_1, x, y) . Since (z_2, z_1, x, y) is maximal, there is a 4-element cocircuit $\{z', z_2, z_1, x\}$ of M' . Now $\{z_2, z_1, x\}$ cannot be contained in a 4-element segment by orthogonality, and $\{x, z_2\}$ cannot be contained in a triad because (z_2, z_1, x, y) is maximal. Therefore it follows from the dual of Lemma 6.4 that either $\text{si}(M'/z_2)$ or $\text{si}(M'/z_1)$ is 3-connected. But $\text{si}(M'/z_2)$ is not 3-connected because z_2 is not (N, B) -strong, and $\text{si}(M'/z_1)$ is not 3-connected because $\text{si}(M'/z_1) \cong \text{co}(M' \setminus x)$ by Lemma 2.12; a contradiction. \square

Assume that (i) does not hold, and suppose that $|E(M)| \geq |E(N)| + 8$. Suppose that $q \in B^* - z_1$ is N -flexible. Then since q is not (N, B) -strong,

$\text{co}(M \setminus q)$ is not 3-connected. Hence $\text{si}(M'/q)$ is 3-connected by Bixby's Lemma. Since $q \notin \text{cl}(z_1, x, y)$ by 6.5.2 it follows that $A_{pq} \neq 0$ for some $p \in B - \{x, y, z_2\}$. Since $A_{pa} = A_{pb} = 0$ by 6.5.3, a pivot on A_{pq} is allowable. But then $B' = B \triangle \{p, q\}$ has an (N, B') -strong element in $B' - \{x, y\}$; a contradiction to Lemma 3.1. Thus M' has no N -flexible elements in $B^* - z_1$.

Suppose that M' has an (N, B) -robust element $p \in B - \{x, y, z_2\}$. Then M' has a path of 3-separations of the form $(X', p, z'_n, \dots, z'_3, z_2, z_1, x, y)$, and, by 6.5.2 and Lemma 6.3, it follows that M' must have an N -flexible element in $B^* - z_1$; a contradiction.

Therefore no element of $B - \{x, y, z_2\}$ is N -contractible. By 6.5.2 and 6.5.3, for each element $q \in B^* - z_1$, there is some $p \in B - \{x, y, z_2\}$ such that a pivot on A_{pq} is allowable. Since B is a robust basis, M' cannot have more $(N, B \triangle \{p, q\})$ -robust elements than (N, B) -robust elements, so it follows that there are also no N -contractible elements in $B^* - z_1$.

Now any (N, B) -robust elements of M' outside of $\{x, y, z_1, z_2\}$ must be coguts elements, and there can be at most one of them else there is an N -flexible element of $B^* - z_1$. Call such an element z_3 . Then the set of (N, B) -robust element of M is $R = \{a, b, x, y, z_1, z_2, z_3\}$. Since $|E(M)| \geq |E(N)| + 8$, there is an element q outside of R that is either N -deletable or N -contractible in M' , but is not (N, B) -robust in M' . Suppose that q is N -contractible, so $q \in B^* - R$. Then $q \notin \text{cl}(\{x, y, z_2\})$, so there is some $p \in B - \{x, y, z_2\}$ such that $A_{pq} \neq 0$. Since $A_{pa} = A_{pb} = 0$ by 6.5.3, a pivot on A_{pq} is allowable. But with $B' = B \triangle \{p, q\}$, there are more (N, B') -robust elements than there were (N, B) -robust elements; a contradiction to the choice of B . Suppose $q \in B - R$ and q is N -deletable. Since z_1 is in a circuit of M' contained in $\{x, y, z_1, z_2\}$, it follows from orthogonality that $q \notin \text{cl}^*(R \cap B^*)$. Thus there is some $p \in B^* - R$ such that $A_{qp} \neq 0$. Since $A_{qa} = A_{qb} = 0$ by 6.5.3, a pivot on A_{qp} is allowable. Again letting $B' = B \triangle \{p, q\}$, we see there are more (N, B') -robust elements than there were (N, B) -robust elements; a contradiction to the choice of B . Therefore any N -deletable or N -contractible element of M is (N, B) -robust, so $|E(M)| \leq |E(N)| + 7$. \square

By Lemma 6.5, if Theorem 2.26(a) does not hold for M , that is, if $|E(M)| \geq |E(N)| + 17$, then M has at most one (N, B) -robust element outside of S' . The case when M' has no (N, B) -robust elements outside of $\{x, y\}$ is handled next.

Lemma 6.6. *If M' has no (N, B) -robust elements outside of $\{x, y\}$, then $M \setminus a, b$ is N -fragile.*

Proof. The elements outside of $\{x, y\}$ are not (N, B) -robust. Thus the elements of $B - \{x, y\}$ are not N -contractible, and the elements of $B^* - \{a, b\}$ are not N -deletable. Thus it remains to show that $M' \setminus x$ and $M' \setminus y$ have no N -minor. Suppose that x is N -deletable. There is some $x' \in B^* - \{a, b\}$ such that $A_{xx'} \neq 0$ because x is not a coloop of M' . Then perform an allowable pivot on $A_{xx'}$. Let $B' = B \triangle \{x, x'\}$. Now $x \in (B')^* - \{a, b\}$, so x is an (N, B') -robust element outside of $\{x', y\}$; a contradiction because B was chosen to have the most (N, B) -robust elements outside of $\{x, y\}$. Thus x is not N -deletable. A similar argument shows that y is not N -deletable. \square

We can now prove Theorem 2.26, which we restate here for ease of reference.

Theorem 6.7. *Let M be an excluded minor for the class of \mathbb{P} -representable matroids, and let N be a strong \mathbb{P} -stabilizer. If M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor, then at least one of the following holds:*

- (a) *Either $|E(M)| \leq |E(N)| + 16$; or*
- (b) *there is some robust basis B for M such that either:*
 - (i) *$M \setminus a, b$ is N -fragile, and $M \setminus a, b$ has at most one (N, B) -robust element $z \in B^* - \{a, b\}$ outside of $\{x, y\}$. Moreover, if $z \in B^* - \{a, b\}$ is (N, B) -robust, then z is an (N, B) -strong element of $M \setminus a, b$, and $\{x, y, z\}$ is a triad of $M \setminus a, b$; or*
 - (ii) *$M \setminus a, b$ is not N -fragile, but the only N -flexible elements of $M \setminus a, b$ are contained in a triad $\{x, y, z\}$ of $M \setminus a, b$ for some (N, B) -strong z of B^* . Moreover, the only (N, B) -robust elements of $M \setminus a, b$ are in $\{x, y, z\}$, and M has a cocircuit $\{a, b, x, y, z\}$ that contains a triangle $\{p, x, y\}$ for some $p \in \{a, b\}$.*
 - (iii) *$M \setminus a, b$ is not N -fragile, but the only N -flexible elements of $M \setminus a, b$ are contained in $\{x, y, z_1, z_2\}$, where z_1 is (N, B) -strong and (z_2, z_1, x, y) a type-II fan relative to B . Moreover, the fan (z_2, z_1, x, y) is maximal, the only (N, B) -robust elements of $M \setminus a, b$ are in $\{x, y, z_1, z_2\}$, and M has a cocircuit $\{a, b, x, y, z\}$ that contains a triangle $\{p, x, y\}$ for some $p \in \{a, b\}$.*

Proof. It follows from Lemma 3.9 that M' has either a confining set or a robust basis B . If M' has a confining set, then (a) holds by Lemma 4.13. Assume that M' has a robust basis B and that (a) does not hold. If M' has no (N, B) -robust elements outside of $\{x, y\}$, then (b)(i) holds by Lemma 6.6. We shall therefore assume M' has an (N, B) -robust element outside of $\{x, y\}$.

Suppose first that all (N, B) -robust elements of M' outside of $\{x, y\}$ are (N, B) -strong. Then M' has exactly one (N, B) -strong element z , and the set $\{x, y, z\}$ is a triad of M' by Lemma 3.9. If M' is N -fragile, then (b)(i) holds. Suppose then that M' is not N -fragile. Since N -flexible elements are (N, B) -robust, it follows that the only N -flexible elements of M' are in $\{x, y, z\}$. To show that (b)(ii) holds, it remains to prove that $\{a, b, x, y, z\}$ is a cocircuit of M' , and that $\{p, x, y\}$ is a triangle of M for some $p \in \{a, b\}$. It follows from Lemma 3.3 that $\{a, x, y\}$ or $\{b, x, y\}$ is a triangle of M . Assume that $\{b, x, y\}$ is a triangle of M . Then either $\{b, x, y, z\}$ or $\{a, b, x, y, z\}$ is a cocircuit of M that contains the triangle $\{b, x, y\}$. If $A_{xp} = A_{yp} = 0$ for all $p \in B^* - \{a, b, z\}$, then there is an allowable pivot to a basis B' where M' has more (N, B') -robust elements than (N, B) -robust elements, a contradiction because B is a robust basis. We shall assume that $A_{yp} \neq 0$ for some $p \in B^* - \{a, b, z\}$. If $\{b, x, y, z\}$ is a cocircuit of M , then pivoting on a nonzero entry A_{yp} for $p \in B^* - \{a, b, z\}$ gives a companion matrix A^{yp} with $A_{xa}^{yp} = 0$ because $\{b, y, z\}$ cospans x ; a contradiction because the bad submatrix $A^{yp}[\{x, p, a, b\}]$ has no zero entries. Therefore $\{a, b, x, y, z\}$ is a cocircuit of M that contains the triangle $\{b, x, y\}$, so (b)(ii) holds.

Finally, suppose that some (N, B) -robust element of M' outside of $\{x, y\}$ is not (N, B) -strong. Then M' has a maximal fan (x, y, z_1, z_2) that is a type-II fan relative to B , and $\{x, y, z_1, z_2\}$ contains all of the (N, B) -robust elements of M' by Lemma 6.5. Hence M' is not N -fragile, and $\{x, y, z_1, z_2\}$ contains all of the N -flexible elements of M' . That $\{a, b, x, y, z\}$ is a cocircuit of M' , and that $\{p, x, y\}$ is a triangle of M for some $p \in \{a, b\}$ follow by the same argument in the preceding paragraph, so (b)(iii) holds. \square

We handle a few special cases next before we prove the second main result.

Lemma 6.8. *If (b)(ii) or (b)(iii) of Theorem 6.7 holds, and $\{a, b\} \subseteq \text{cl}_M(\{x, y\})$, then $|E(M)| \leq |E(N)| + 6$.*

Proof. Suppose that $\{a, b\} \subseteq \text{cl}_M(\{x, y\})$ but that $|E(M)| \geq |E(N)| + 7$. Then there is at least one element p in $E(M) - \{a, b, x, y\}$ that is either N -deletable or N -contractible in M' but not (N, B) -robust. Suppose that p is N -deletable. Then $p \in B - \{x, y\}$. Now $A_{pa} = A_{pb} = 0$ because $\{a, b\} \subseteq \text{cl}_M(\{x, y\})$. Moreover, there is some $q \in B^* - \{a, b, z\}$ such that $A_{pq} \neq 0$ because p is not a coloop nor is p in a series pair $\{p, z\}$ of M' . By Lemma 6.5, q is not (N, B) -robust either. Then a pivot on A_{pq} is allowable, and $B' = B \Delta \{p, q\}$ is a basis for M' and there are more (N, B') -robust elements than there are (N, B) -robust elements; a contradiction to the choice of B .

We may therefore assume that q is N -contractible in M' . Suppose that $q \in \text{cl}_M(\{x, y\})$. Then, since x is N -contractible in M' and $\{q, y\}$ is a parallel pair in M'/x , it follows that q is also N -deletable in M' . Hence q is an (N, B) -robust element; a contradiction. Thus $q \notin \text{cl}_M(\{x, y\})$, so it follows that $A_{pq} \neq 0$ for some non-robust $p \in B - \{x, y\}$. Then a pivot on A_{pq} is allowable, and $B' = B \Delta \{p, q\}$ is a basis for M' such that there are more (N, B') -robust elements than there are (N, B) -robust elements; a contradiction to the choice of B . \square

Lemma 6.9. *If (b)(ii) or (b)(iii) of Theorem 6.7 holds, and there is some $p \in B - \{x, y\}$ such that $p \in \text{cl}_M(\{x, y, z\})$ and $\{a, b\} \subseteq \text{cl}_M(\{p, x, y\})$, then $|E(M)| \leq |E(N)| + 7$.*

Proof. Let $R = \{a, b, x, y, z_1, p\}$. Note that if $p \in \text{cl}_{M'}(\{x, y, z\})$ and M' has a type-II fan relative to B , then p must be the end spoke element of the fan. Suppose that $|E(M)| \geq |E(N)| + 8$. Then M has at least two elements outside of R , each of which is either N -deletable or N -contractible. Since R contains all of the (N, B) -robust elements of M' , it follows that these elements are not (N, B) -robust.

Suppose q is N -deletable element such that $q \in B - \{x, y, p\}$. Then there is some $p \in B^* - \{a, b, z_1\}$ such that $A_{qp} \neq 0$ because M' is 3-connected. Since $\{a, b\} \subseteq \text{cl}_M(\{p, x, y\})$, it follows that $A_{qa} = A_{qb} = 0$, so a pivot on A_{qp} is allowable. But, with $B' = B \Delta \{p, q\}$, there are more (N, B') -robust elements than there are (N, B) -robust elements; a contradiction to the choice of B .

We now know that M' has at least two N -contractible elements in $B^* - \{a, b, z_1\}$. We claim that at least one of these elements is not in $\text{cl}(\{x, y, p\})$. The claim is clear if M' has a type-II fan relative to B . Otherwise M' has a

triad $\{x, y, z_1\}$ and at least two N -contractible elements $p, q \in \text{cl}(\{x, y, p\})$. But then q is in the guts of the vertical 3-separation $(\{x, y, z_1\}, p, E(M') - \{x, y, z_1\})$ of M' , so q is also N -deletable by Lemma 2.16; a contradiction because q is not (N, B) -robust. Thus we may assume that $q \notin \text{cl}(\{x, y, p\})$, so there is some $s \in B - \{x, y, p\}$ such that $A_{sq} \neq 0$. Then $A_{sa} = A_{sb} = 0$, so a pivot on A_{sq} is allowable. But with $B' = B \triangle \{s, q\}$, there are more (N, B') -robust elements than there are (N, B) -robust elements; a contradiction to the choice of B . \square

We now prove our second main result, Theorem 2.27, first restating it.

Theorem 6.10. *Let M be an excluded minor for the class of \mathbb{P} -representable matroids, and let N be a strong \mathbb{P} -stabilizer. Suppose that M has a pair of elements a, b such that $M \setminus a, b$ is 3-connected with an N -minor. Then, for some (M_0, N_0) in $\{(M, N), (\nabla(M^*), N^*)\}$, where $\nabla(M^*)$ is a matroid obtained from M^* by a single Y - Δ -exchange, M_0 is an excluded minor with an N_0 -minor having a pair of elements a', b' such that $M_0 \setminus a', b'$ is 3-connected with an N_0 -minor and at least one of the following holds:*

- (a) $|E(M_0)| \leq |E(N_0)| + 16$; or
- (b) $r(M_0) \leq r(N_0) + 8$; or
- (c) *there is some robust basis B_0 for M_0 such that $M_0 \setminus a', b'$ is N_0 -fragile, and $M_0 \setminus a', b'$ has at most one (N_0, B_0) -robust element $z \in (B_0)^* - \{a', b'\}$ outside of $\{x', y'\}$. Moreover, if $z \in (B_0)^* - \{a', b'\}$ is an (N_0, B_0) -robust element, then z is an (N_0, B_0) -strong element of $M_0 \setminus a', b'$, and $\{x', y', z\}$ is a triad of $M_0 \setminus a', b'$.*

Proof. We shall assume that neither (a) nor (b) holds for the excluded minor M and the deletion pair $\{a, b\}$. If (b)(i) of Theorem 6.7 holds for M and the pair $\{a, b\}$, then (c) holds. We may therefore assume that (b)(ii) or (b)(iii) of Theorem 6.7 holds for M and the pair $\{a, b\}$. Thus, we may assume that M has $\{b, x, y\}$ as a triangle. Then $M \setminus a, b$ has an (N, B) -strong element $z \in B^* - \{a, b\}$, and the only (N, B) -robust elements of M' are contained in the set R , where $R \in \{\{x, y, z\}, \{x, y, z_1, z_2\}\}$ is either a triad or a type-II fan relative to B , respectively. We make the following observation about the series pairs of $M' \setminus z$.

6.10.1. $\{x, y\}$ is the only series pair for $M' \setminus z$.

Subproof. If R is a triad, this follows because if $\{p, q\}$ were another series pair, then p and q are N -contractible in M' , and at least one of p, q is in $B - \{x, y\}$, so M' has an (N, B) -robust element outside of R ; a contradiction. If R is a type-II fan relative to B , this follows because the fan R is maximal. \square

Since (a) does not hold, it follows from Lemma 6.8 that a is not in the span of $\{b, x, y\}$. Next, we show that M , or a familiar modification of M , has a delete pair that is contained in a triangle. This triangle will provide additional leverage in later orthogonality arguments.

6.10.2. *Up to duality and performing at most one Δ - Y operation, there is a deletion pair $\{p, q\}$ for M such that $M \setminus p, q$ is 3-connected with an N -minor, and p and q are in a triangle of M .*

Subproof. We consider the following cases:

- (I) $\{a, x, z\}$ or $\{a, y, z\}$ is a triangle of M ; or
- (II) neither $\{a, x, z\}$ nor $\{a, y, z\}$ is a triangle of M .

We may first assume that (I) holds, and let $\{a, x, z\}$ be a triangle. We note that the proof when $\{a, y, z\}$ is a triangle of M is identical to the following argument after interchanging x and y . We claim that $\{b, x\}$ is a deletion pair with the desired properties. Clearly $M \setminus b$ is 3-connected and has an N -minor. Since z is N -contractable in $M \setminus a, b$, it follows that z is also N -contractible in $M \setminus b$. But x and a are in parallel in $M \setminus b/z$, so $M \setminus b, x/z$ has an N -minor. Thus $M \setminus x, b$ has an N -minor. It remains to show that $M \setminus x, b$ is 3-connected.

Suppose that $\text{co}(M \setminus b, x)$ is not 3-connected. Then there is some vertical 3-separation (P, x, Q) of $(M \setminus b)^*$. Since $\{a, x, z\}$ is a triangle, we may assume, by orthogonality, that $z \in P$ and $a \in Q$. Assume that $y \in P$. Then $(P \cup x \cup a, Q - a)$ is an exact 3-separation of $M \setminus b$ by Lemma 2.8, so $a \in \text{cl}^{(*)}(Q - a)$. But $a \in \text{cl}(P \cup x)$ because $\{a, x, z\}$ is a triangle of $M \setminus b$, and $a \in \text{cl}^*(P \cup x)$ because $\{a, x, y, z\}$ is a cocircuit of $M \setminus b$; a contradiction to orthogonality. Thus $\text{co}(M \setminus b, x)$ is 3-connected. Thus, if $M \setminus b, x$ is not 3-connected, then there is a triad C of $M \setminus b$ that contains x . By orthogonality with the triangle $\{a, x, z\}$, the triad C meets $\{a, z\}$. But $a \notin C$ because $M \setminus a, b$ is 3-connected. Thus $M \setminus b$ has a triad C that meets $\{x, z\}$. But then C is a triad of $M \setminus a, b$ too, so M' has a 4-point cosegment $C \cup \{x, y, z\}$; a contradiction because (i) does not hold, so M' has no confining set. Thus $M \setminus b, x$ is 3-connected with an N -minor, and $\{b, x\}$ is contained in a triangle of M .

We may now assume that (II) holds. Consider the excluded minor $\Delta_T(M)$, where $T = \{b, x, y\}$. We claim that either $\{b, x\}$ or $\{b, y\}$ is a pair of elements such that $\Delta_T(M)/b, x$ or $\Delta_T(M)/b, y$ is 3-connected with an N -minor. Hence either $\{b, x\}$ or $\{b, y\}$ is a deletion pair of $\nabla_T(M^*)$ with the desired properties. We see that $\Delta_T(M)/b \cong M \setminus b$, so $\Delta_T(M)/b$ is 3-connected and has an N -minor. Now $\Delta_T(M)/b, x \cong M \setminus b/x$ and $\Delta_T(M)/b, y \cong M \setminus b/y$. Now x and y are N -contractible in M' , so $M \setminus b/x$ and $M \setminus b/y$ have an N -minor. Thus $\Delta_T(M)/b, x$ and $\Delta_T(M)/b, y$ have N -minors.

Suppose that $\text{si}(M \setminus b/x)$ is not 3-connected. Then there is a vertical 3-separation (P, x, Q) of $M \setminus b$. But $M \setminus a, b, z/x$ is 3-connected, so it follows that $Q = \{a, z, q\}$ for some q . Hence the set Q is a triad of $M \setminus b$ because Q is a 3-separating set and $r(Q) \geq 3$. But then $M \setminus a, b$ is not 3-connected because $a \in Q$ and Q is a triad of $M \setminus b$; a contradiction. Thus $M \setminus b/x$, and hence $\Delta_T(M)/b, x$, is 3-connected up to parallel pairs. The same argument shows that $\Delta_T(M)/b, y$ is 3-connected up to parallel pairs.

We claim that M has no triangle of the form $\{p, x, y\}$ where $p \in E(M) - \{a, b\}$, and that at most one of $\{y, z\}$ and $\{x, z\}$ is contained in a triangle. We shall first show that M has no triangle of the form $\{p, x, y\}$ with $p \in E(M) - \{a, b\}$. Suppose that $\{p, x, y\}$ is a triangle for some $p \in E(M) - \{a, b\}$. Then $p \notin R$. Since x is N -contractible in M' , and $\{p, y\}$ is a parallel pair of M'/x , it follows that p is N -deletable in M' . Moreover, since $\{x, y\} \subseteq B$ and $\{p, x, y\}$ is a triangle of M' , it follows that $p \in B^*$. Therefore p is an

(N, B) -robust element of M' ; a contradiction because M' has no (N, B) -robust elements outside of R . Next suppose that $M \setminus b$ has triangles $\{p, x, z\}$ and $\{q, y, z\}$ for some $p, q \in E(M) - \{b, x, y, z\}$. Suppose that $a \notin \{p, q\}$. Since z is N -contractible, and there are parallel pairs $\{p, x\}$ and $\{q, y\}$ in M'/z , it follows that the elements p and q are N -deletable in M' . Thus $p, q \in B - \{x, y\}$ because there are no (N, B) -robust elements in $B^* - \{z\}$; a contradiction because $\{p, q, x, y, z\}$ is a rank-3 set so the 4-element subset $\{p, q, x, y\}$ is dependent. We shall therefore assume that $a \in \{p, q\}$, so M has triangles $\{a, x, z\}$ and $\{p, y, z\}$. So then $p \in B - \{x, y\}$ as before, so then $\{a, b\} \subseteq \text{cl}_M(\{p, x, y\})$. Thus (a) holds by Lemma 6.9. We may now assume that, in $M \setminus b$, at most one of $\{y, z\}$ and $\{x, z\}$ is contained in a triangle.

Suppose that $M \setminus b$ has a triangle T' that contains x , and a triangle T'' that contains y . Then, by orthogonality with the cocircuit $\{a, x, y, z\}$, T' and T'' meet $\{a, y, z\}$. By (II), and since at most one of $\{y, z\}$ and $\{x, z\}$ is contained in a triangle, we may assume that the triangle T' has the form $\{x, a, q\}$ for some $q \in E(M) - \{a, b, x, y, z\}$. Then q is N -deletable because x is N -contractible in $M \setminus b$ and $\{a, q\}$ is a parallel pair of $M \setminus b/x$. Thus $q \in B - \{x, y\}$ because M' has no (N, B) -robust elements outside of $\{x, y, z\}$ or the type-II fan relative to B . Thus $\{a, b\} \subseteq \text{cl}_M(\{q, x, y\})$, and so (i) holds by Lemma 6.9, a contradiction. We may now assume that $M \setminus b$ has no triangle containing x . Since x is not in a triangle of $M \setminus b$, it follows that $M \setminus b/x$ is 3-connected. Since $\{b, x\}$ is contained in a triad of $\Delta_T(M)$, its dual $\nabla_T(M^*)$ has a delete pair $\{b, x\}$ that is contained in a triangle of $\nabla_T(M^*)$. \square

Up to replacing M by a matroid that is obtained by performing at most one Y - Δ operation on M^* , and up to replacing $\{a, b\}$ by another deletion pair, we shall assume that $\{a, b\}$ is contained in a triangle of M . Again we assume that neither (a) nor (b) holds, and that either (b)(ii) or (b)(iii) of Theorem 6.7 holds for M and $\{a, b\}$. Then there is a 5-element cocircuit $C = \{a, b, x, y, z\}$ of M . Since $\{a, b\}$ is contained in a triangle, either $\{a, b, z\}$ is a triangle of M , or $\{a, b, p\}$ is a triangle for some $p \notin \{x, y, z\}$. Let $C^+ = C$ in the former case and let $C^+ = C \cup \{p\}$ in the latter case.

6.10.3. *There are elements q' and q'' of M such that the following hold:*

- (i) *for each $q \in \{q', q''\}$, the matroid $M \setminus b, q$ is 3-connected with an N -minor, where $q \notin \text{cl}_M^*(R \cup C^+)$ and q is not in a triangle of the form $\{q, x, z\}$ or $\{q, y, z\}$; and*
- (ii) *for each $q \in \{q', q''\}$, either q is N -deletable in $M \setminus a, b$, or $M \setminus b$ has a triangle $T' = \{a, q, t\}$ for some $t \in \{x, y\}$, and T' is contained in a 4-element cocircuit of $M \setminus b$; and*
- (iii) *$q'' \notin \text{cl}_M^*(R \cup C^+ \cup \{q'\})$.*

Subproof. We shall first show that there is some N -deletable element q of $M \setminus a, b$ such that $q \notin \text{cl}_M^*(R \cup C^+)$ and q is not in a triangle of the form $\{q, x, z\}$ or $\{q, y, z\}$. Since (b) does not hold for M and $\{a, b\}$, we know that $r^*(M) \geq r^*(N) + 9$. Hence $r^*(M \setminus a, b) \geq r^*(N) + 7$. We observe that $(R \cup C^+) - \{a, b\}$ has corank at most 4 in $M \setminus a, b$, so $M \setminus a, b$ has at least three N -deletable elements outside of $\text{cl}_{M \setminus a, b}^*((R \cup C^+) - \{a, b\})$. Hence M has at least three elements outside of $\text{cl}_M^*(R \cup C^+)$. Since these N -deletable

elements are not (N, B) -robust elements, they belong to $B - \{x, y\}$. At least two of these elements are not in a triangle with either $\{x, z\}$ or $\{y, z\}$ because $M \setminus a, b$ cannot have triangles $\{s, x, z\}$ and $\{t, y, z\}$ spanning a rank-3 set $\{s, t, x, y, z\}$ with $\{s, t, x, y\} \subseteq B$. Thus such a q exists. We show that there is an element q with these properties such that $M \setminus b, q$ is 3-connected with an N -minor.

Suppose that $\text{co}(M \setminus b, q)$ is 3-connected, but that $M \setminus b, q$ is not 3-connected. Then $M \setminus b$ has a triad $\{q, s, t\}$. Hence either $\{q, s, t\}$ or $\{b, q, s, t\}$ is a cocircuit of M . Since $M \setminus a, b$ is 3-connected, it follows that $a \notin \{q, s, t\}$ and that $\{q, s, t\}$ is a triad of $M \setminus a, b$. Since q is N deletable in $M \setminus a, b$, it follows that s and t are N -contractible in $M \setminus a, b$. Note that $(R - \{x, y\}) \cap \{s, t\} = \emptyset$. To see this, observe that it follows when $R - \{x, y\} = \{z\}$ because $\{x, y\}$ is the only series pair of $M \setminus a, b, z$; while, for $(R - \{x, y\}) = \{z_1, z_2\}$, it follows because the type-II fan relative to B is a maximal fan. Thus either $\{s, t\} \subseteq B^* - \{a, b, z\}$, or $\{s, t\}$ meets $\{x, y\}$. Suppose that $\{s, t\} \subseteq B^* - \{a, b, z\}$. If $\{q, s, t\}$ is a triad of M , then $A_{qa} = A_{qb} = 0$, so there is an allowable pivot on A_{qs} or A_{qt} that gives a basis for $M \setminus a, b$ with more robust elements; a contradiction because B is a robust basis. If $\{b, q, s, t\}$ is a cocircuit of M , then it meets the triangle $\{b, x, y\}$ in one element, a contradiction to orthogonality. Therefore $\{s, t\}$ meets $\{x, y\}$. If $\{s, t\} = \{x, y\}$, then $M \setminus a, b$ has a 4-point cosegment $\{q, x, y, z\}$, a contradiction because M has no confining set. Thus we shall assume that $t = x$ and that $s \neq y$. Then $\{x, y, z\}$ and $\{q, s, t\}$ are triads that meet in x and $\{q, s, z\} \subseteq B^*$, so $\{x, y, z\} \cup \{q, s, t\}$ is a confining set; a contradiction. Therefore $M \setminus b, q$ is 3-connected.

We may now assume that $\text{co}(M \setminus b, q)$ is not 3-connected. We first show $\text{co}(M \setminus a, b, q)$ is 3-connected. Suppose not. Then there is a vertical 3-separation (X, q, Y) of $(M \setminus a, b)^*$ such that $|X \cap E(N)| \leq 1$ and $Y \cup q$ is closed in $(M \setminus a, b)^*$. Then it follows from Lemma 2.16 (i) that at most one element of X is not N -flexible, so at most one element of X is not (N, B) -robust. Therefore X or $X - s$ is contained in R for some $s \in \text{cl}_{M \setminus a, b}^*(X - \{s\})$ because the only (N, B) -robust elements of M' are in R . Hence $q \in \text{cl}^*(R \cup C^+)$, a contradiction. Thus $M \setminus a, b, q$ is 3-connected up to series pairs.

Since $\text{co}(M \setminus a, b, q)$ is 3-connected, but $\text{co}(M \setminus b, q)$ is not, it follows that there is a vertical 3-separation (P, q, Q) of $(M \setminus b)^*$. We may assume that $a \in Q$. Then $(P, Q - a)$ is not a vertical 2-separation of $(M \setminus a, b, q)^*$, so $Q - a$ must be a series class of $M \setminus a, b, q$. Hence $(Q - a) \cup q$ is a cosegment of $M \setminus a, b$. Suppose that $|Q - a| \geq 3$. Then $Q - a$ meets B , and since the elements of $Q - a$ in B are (N, B) -strong, it follows that $(Q - a) \cap B \subseteq \{x, y\}$. If $(Q - a) \cap B = \{x, y\}$, then q is cospanned by $\{x, y\}$, a contradiction. Thus $(Q - a) \cap B$ contains exactly one element of $\{x, y\}$. But then $M \setminus a, b$ has a confining set $\{x, y, z\} \cup ((Q - a) - \{x, y\})$, a contradiction. Therefore $|Q - a| = 2$, so $Q = \{a, s, t\}$ is a triangle of $M \setminus b$ because Q is 3-separating.

In $M \setminus b$, either $Q \cup q$ is a 4-element fan or a 4-element cocircuit. Consider the first case. Then $\{b, q\} \cup (Q - a)$ is a 4-element cocircuit of M because an excluded minor cannot have a 4-element fan. By orthogonality the triangles $\{b, x, y\}$ and $\{a, b, p\}$ of M must meet $Q - a$. But then $q \in \text{cl}_M^*((Q - a) \cup \{b\}) \subseteq \text{cl}^*(R \cup C^+)$, a contradiction. Assume that $Q \cup q$ is a 4-element

cocircuit of $M \setminus b$. By orthogonality with the triangle Q and the cocircuit $\{a, x, y, z\}$ of $M \setminus b$, it follows that $\{x, y, z\}$ meets $Q - a$. Moreover, we see that $z \notin Q - a$ because z has only one series pair $\{x, y\}$ in $M \setminus a, b$. Thus $\{x, y\}$ meets $Q - a$. We may now assume that $x \in Q - a$. We shall prove that $t \in Q - \{a, x\}$ is the required element. Since x is contractible in $M \setminus b$ and $\{a, t\}$ is a parallel pair of $M \setminus b/x$, it follows that $M \setminus b, t$ has an N -minor. Suppose that $t \in \text{cl}_M^*(C^+)$. Then $Q \cup b \subseteq C^+$, so $q \in \text{cl}_M^*(C^+)$, a contradiction. Thus $t \notin \text{cl}_M^*(C^+)$. We claim that M has no triangle of the form $\{t, x, z\}$ or $\{t, y, z\}$. If M has a triangle of the form $\{t, x, z\}$, then $\{a, t, x, z\}$ is a 4-segment of M ; a contradiction of the property that $\text{co}(M \setminus a, b, q)$ is 3-connected. The fact that M has no triangle of the form $\{t, y, z\}$ follows from orthogonality with the cocircuit $Q \cup q$ of $M \setminus b$.

It remains to prove that $M \setminus b, t$ is 3-connected. Suppose that (X, t, Y) is a vertical 3-separation of $(M \setminus b)^*$. By orthogonality we may assume that $a \in X$ and $x \in Y$. We may also assume without loss of generality that $q \in Y$. Now $X - a$ and X are exactly 3-separating in $M \setminus b$, so $a \in \text{cl}^{(*)}(X - a)$. But $a \in \text{cl}(Y \cup \{a, t\})$ and $a \in \text{cl}^*(Y \cup \{a, t\})$; a contradiction to orthogonality. Thus if $M \setminus b, t$ is not 3-connected, then there is some triad T' of $M \setminus b$ such that $t \in T'$. By orthogonality with the triangle $\{a, t, x\}$, the triad T' meets $\{a, x\}$. But $a \notin T'$ because $M \setminus a, b$ is 3-connected. Thus $t, x \in T'$. Now T' or $T' \cup b$ is a cocircuit of M , and $|T' - C^+| \geq 2$ because $t \notin \text{cl}_M^*(C^+)$. But $\{b, x, y\}$ is a triangle of M that meets T' in a single element, and $\{a, b, p\}$ is a triangle of M that meets $T' \cup b$ in a single element, a contradiction to orthogonality.

We now know that there is a q' satisfying properties (i) and (ii). By replacing $R \cup C^+$ with $R \cup C^+ \cup \{q'\}$ and repeating the above argument, we obtain q'' satisfying the properties (i) through (iii). \square

Let q' and q'' be elements as in 6.10.3. Assume that none of (a) through (c) holds for M and the pairs $\{b, q'\}$ and $\{b, q''\}$. Then, by (ii) or (iii) of Theorem 6.7, there is a 5-element cocircuit of M of the form $C' = \{x', y', z', b, q'\}$, where $\{x', y', z'\}$ is a triad of $M \setminus b, q'$ with an (N, B) -strong element z' outside of $\{x', y'\}$, and either $\{b, x', y'\}$ or $\{q', x', y'\}$ is a triangle of M .

Suppose that $\{b, x', y'\}$ is a triangle of M . Since M has a cocircuit C' , and M has triangles $\{b, x, y\}$ and $\{a, b, p\}$ for some $p \in C^+$, it follows from orthogonality and the fact that $q' \notin C$ that $\{x, y\}$ and $\{a, p\}$ meet $\{x', y', z'\}$. Therefore at least one of $\{x, y\}$ and $\{a, p\}$ meets $\{x', y'\}$. If either $\{x, y\}$ or $\{a, p\}$ meets $\{x', y'\}$ in a single element, then $\{x', y'\}$ spans a 4-element segment. Hence $\{x', y'\}$ spans a triangle in $M \setminus b, q'$. But then $\text{co}(M \setminus b, q', z')$ is not 3-connected by Lemma 2.11; a contradiction because z' is an (N, B') -strong element of $M \setminus b, q'$. Therefore $\{x', y', z'\} \subseteq \{a, p, x, y\} \subseteq C^+$. Hence $C' - \{q'\} \subseteq C^+$, so $q' \in \text{cl}_M^*(C^+)$; a contradiction because $q' \notin \text{cl}_M^*(R \cup C^+)$.

We may now assume that $\{q', x', y'\}$ is a triangle of M . The triangles $\{b, x, y\}$ and $\{a, b, s\}$ meet the cocircuit C' , so at least two elements of $\{b, x, y\}$ and two elements of $\{a, b, p\}$ are in C' by orthogonality. Therefore $\{x, y\}$ and $\{a, p\}$ meet $\{x', y', z'\}$.

First suppose that $C \cap \{x', y'\} = \emptyset$. Then $C^+ = C \cup \{p\}$ where $p \in \{x', y'\}$ and $z' \in \{x, y\}$. Suppose that a is N -deletable in $M \setminus b, q'$. Then $a \in B'$ since a is N -deletable in $M \setminus b, q'$, and $M \setminus b, q'$ has no (N, B') -robust elements

in $(B')^* - \{z'\}$. But then, the triangle $\{a, b, p\}$ implies that $A'_{vb} = 0$ for $v \in \{x', y'\} - \{p\}$; a contradiction because the entries of the bad submatrix $A[\{b, q', x', y'\}]$ are non-zero. We may assume that a is not N -deletable in $M \setminus b, q'$. Then q' is not N -deletable in $M \setminus a, b$, so it follows from 6.10.3 (b) that $M \setminus b$ has a triangle $T' = \{a, q', r\}$ for some $r \in \{x, y\}$, and T' is contained in a 4-element cocircuit C^* of $M \setminus b$. By orthogonality with C and C' , it follows that $T' = \{a, q', x\}$. Then, by orthogonality with the triangle $\{q', x', y'\}$, the cocircuit C^* of $M \setminus b$ meets $\{x', y'\}$. Hence $C^* \cup \{b\}$ must be a cocircuit of M by orthogonality with the triangle $\{b, x, y\}$ of M . Then $M \setminus b, q'$ has a 4-element cosegment $\{a, x', y', z'\}$, so (i) holds for M by Lemma 4.13, a contradiction.

We now know that $C \cap \{x', y'\} \neq \emptyset$. Then it follows from orthogonality with the triangle $\{q', x', y'\}$ that $\{x', y'\} \subset C$. By orthogonality with the cocircuit C' and triangles $\{b, x, y\}$ and $\{a, b, s\}$, one of $\{x', y'\}$ must be in $\{x, y\}$ and the other in $\{a, s\} \cap C$. We may therefore assume that $\{x', y'\}$ is either $\{x, a\}$ or $\{x, z\}$. But q' was chosen so that there is no triangle in M of the form $\{q', x, z\}$, so $\{q', x, a\}$ is a triangle of M . If q' is N -deletable but not (N, B) -robust in $M \setminus a, b$, it follows that $q' \in B - \{x, y\}$. Then, since $q', x \in B$ and $\{q', x, a\}$ is a triangle of M , it follows that $A_{ya} = 0$, a contradiction because the entries of the bad submatrix are non-zero.

We may therefore assume that q' is not N -deletable in $M \setminus a, b$. Then, by 6.10.3 (ii), $M \setminus b$ has a triangle $T' = \{a, q', s\}$ for some $s \in \{x, y\}$, and T' is contained in a 4-element cocircuit C^* of $M \setminus b$. Thus $T' = \{a, q', x\}$ and $C^* \cup \{b\}$ must be a cocircuit of M by orthogonality with the triangles $\{b, x, y\}$ and $\{a, b, p\}$. The same must hold for the element q'' . By 6.10.3 (ii), $M \setminus b$ has a triangle $T'' = \{a, q'', s\}$ for some $s \in \{x, y\}$. If $T'' = \{a, q'', x\}$, then we contradict orthogonality with C and the triangle $\{q', q'', x\}$. Therefore $T'' = \{a, q'', y\}$. By orthogonality with $C^* \cup \{b\}$ and T'' , it follows that either $q'' \in C^* \cup \{b\}$ or $y \in C^* \cup \{b\}$. But if $q'' \in C^* \cup \{b\}$, then q'' fails 6.10.3 (iii), a contradiction. Therefore $y \in C^* \cup \{b\}$. But then $C^* \cup \{b\} = \{a, b, q', x, y\}$, so $q' \in \text{cl}_M^*(R \cup C^+)$, a contradiction of 6.10.3 (i). Therefore at least one of (a) through (c) holds for M and the pair $\{b, q\}$ for some $q \in \{q', q''\}$. \square

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