WHEN EXCLUDING ONE MATROID PREVENTS INFINITE ANTICHAINS

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Abstract. Geelen, Gerards, and Whittle have announced that there are no infinite sets of binary matroids none of which is isomorphic to a minor of another. In this paper, we use this result to determine precisely when a minor-closed class of matroids with a single excluded minor does not contain such an infinite antichain.

1. Introduction

The matroid terminology used here will follow Oxley [9]. For a matroid N, let $EX(N)$ denote the class of matroids having no minor isomorphic to N. Tutte [12] proved that $EX(U_{2,4})$ is the class of binary matroids. Robertson and Seymour [11] proved a conjecture of Wagner that there are no infinite antichains of graphs. They also conjectured, though apparently not in print [4, 5], that, for all prime powers q, this theorem can be extended to the class of matroids representable over $GF(q)$. Geelen, Gerards, and Whittle [6] have announced that they have proved this conjecture for $q = 2$; that is, under the minor ordering, $EX(U_{2,4})$ does not contain an infinite antichain. This theorem prompts the question as to precisely when $EX(N)$ does not contain an infinite antichain. The purpose of this note is to answer this question. The following theorem is our main result.

Theorem 1.1. *Under the minor ordering, $EX(N)$ does not contain an infinite antichain if and only if N is a minor of $U_{2,4} \oplus_2 U_{1,3}$ or $U_{2,4} \oplus_2 U_{2,3}$.*

2. Infinite Antichains

The proof that certain classes $EX(N)$ contain infinite antichains will use three examples of such antichains.

1991 *Mathematics Subject Classification.* 05B35, 05C83, 06A07.
Key words and phrases. infinite antichain, minor-closed class, excluded minor.
The second author was supported by the National Security Agency.

Example 2.1. For all $n \geq 3$, let P_n be the rank-3 matroid consisting of a ring of n three-point lines, that is, P_n has ground set $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ and its only non-spanning circuits are $\{x_1, y_1, x_2\}, \{x_2, y_2, x_3\}, \dots, \{x_n, y_n, x_1\}$. The set $\{P_n : n \geq 3\}$ is an infinite antichain [2, p. 155].

Example 2.2. For all $k \geq 2$, let T_k be the matroid that is obtained by taking the direct sum of two k-element circuits and truncating this to rank k. Oxley, Prendergast, and Row [10] proved that the set $\{T_k : k \geq 2\}$ is an infinite antichain.

Example 2.3. For all $r \geq 2$, let N_r be the tipless binary spike of rank $2r$, that is, the vector matroid of the matrix $[I_{2r}|J_{2r} - I_{2r}]$ over $GF(2)$ where J_{2r} is the $2r \times 2r$ matrix of all ones. Let M_r be a matroid obtained from N_r by relaxing a pair of complementary circuit-hyperplanes. Kahn (in [9, p. 471]) proved that the set $\{M_r : r \geq 2\}$ is an infinite antichain no member of which has a $U_{2,5}$- or $U_{3,5}$-minor.

A binary relation $\leq$ on a set Q is a *quasi-order* if it is reflexive and transitive. A *well-quasi-order* is a quasi-order such that, for every infinite sequence $q_1, q_2, \dots$ of members of Q, there are indices i and j such that $i < j$ and $q_i \leq q_j$. For example, the set $\mathbb{N}$ of natural numbers under the usual ordering is a well-quasi-order. If $\mathcal{M}$ is a class of matroids that is closed under isomorphism and minors, then $\mathcal{M}$ is a quasi-order under the minor relation $\leq_m$. It is well-known, and we shall give examples below to show this, that, when $\mathcal{M}$ is the class of all matroids, $(\mathcal{M}, \leq_m)$ is not a well-quasi-order. This paper determines precisely when $(EX(N), \leq_m)$ is a well-quasi-order.

For a quasi-order $(Q, \leq)$, let $Q^{<w}$ be the set of all finite sequences of members of Q. For $(p_1, p_2, \dots, p_m)$ and $(q_1, q_2, \dots, q_n)$ in $Q^{<w}$, define $(p_1, p_2, \dots, p_m) \leq^{<w} (q_1, q_2, \dots q_n)$ if there are indices $i_1, i_2, \dots i_m$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $p_j \leq q_{i_j}$ for all j in $\{1, 2, \dots, m\}$. Higman [7] proved the following fundamental result.

Lemma 2.4. *If $(Q, \leq)$ is a well-quasi-order, then $(Q^{<w}, \leq^{<w})$ is a well-quasi-order.*

Let $(Q_1, \leq_1), (Q_2, \leq_2), \dots (Q_k, \leq_k)$ be quasi-orders. For $(p_1, p_2, \dots p_k)$ and $(q_1, q_2, \dots q_k)$ in $Q_1 \times Q_2 \times \cdots \times Q_k$, define $(p_1, p_2, \dots, p_k) \leq (q_1, q_2, \dots, q_k)$ if $p_j \leq_j q_j$ for all j in $\{1, 2, \dots, k\}$. As noted, for example, in [3], the following is a well-known consequence of Lemma 2.4.

Corollary 2.5. *If $(Q_i, \leq_i)$ is a well-quasi-order for all i in $\{1, 2, \dots, k\}$, then $(Q_1 \times Q_2 \times \dots Q_k, \leq_1 \times \leq_2 \times \cdots \times \leq_k)$ is a well-quasi-order.*

Let M be a uniform matroid with ground set $\{x_1, x_2, \dots, x_n\}$. Replace each element x_i by k_i parallel elements for some $k_i \geq 1$ where if $r(M) = 0$, each $k_i = 1$. We call the resulting matroid a *parallel extension of a uniform matroid*. Its dual is a *series extension of a uniform matroid*. Note that this terminology differs from Oxley [9] where parallel and series extensions require the addition of a single element.

Lemma 2.6. *There are no infinite antichains of series extensions of uniform matroids.*

Proof. Associate the pair $(r, s-r)$ and the s-tuple $(k_1, k_2, \dots, k_s)$ with $k_1 \leq k_2 \leq \cdots \leq k_s$ to each series extension of a non-empty uniform matroid $U_{r,s}$. From above, $\mathbb{N}^2 \times \mathbb{N}^{<w}$ is a well-quasi-order. Thus the class of series extensions of uniform matroids is a well-quasi-order. □

3. $EX(N)$

In the next lemma, $\mathcal{W}^3$ denotes the rank-3 whirl, while Q_6 and P_6 are obtained from $\mathcal{W}^3$ by relaxing one and two circuit-hyperplanes, respectively.

Lemma 3.1. *The class $EX(U_{2,4} \oplus_2 U_{1,3})$ consists of direct sums of binary matroids and series extensions of uniform matroids.*

Proof. Let $M \in EX(U_{2,4} \oplus_2 U_{1,3})$. Assume M is 3-connected. Observe that $M \in EX(\mathcal{W}^3, Q_6, P_6)$. Thus, by [8, Theorem 1.5], M is binary or uniform. Now assume M is connected, but not 3-connected. Then $M = M_1 \oplus_2 M_2$ for some connected matroids M_1 and M_2. Suppose M is non-binary. Then, without loss of generality, M_1 is non-binary. Hence, M_1 has a $U_{2,4}$-minor. Furthermore, Bixby [1] proved that every element of M_1, so, in particular, the basepoint p of the 2-sum, is in a $U_{2,4}$-minor of M_1. Thus, no cocircuit of M_2 containing p has more than two elements. Hence, M_2 is a circuit. Thus, every 2-sum decomposition of M has a circuit as one part. It follows without difficulty that M is a series extension of a uniform matroid, and it is straightforward to complete the proof of the lemma. □

Corollary 3.2. *The classes $EX(U_{2,4} \oplus_2 U_{1,3})$ and $EX(U_{2,4} \oplus_2 U_{2,3})$ do not contain infinite antichains.*

Proof. By duality, it suffices to prove the result for $EX(U_{2,4} \oplus_2 U_{1,3})$. If $M \in EX(U_{2,4} \oplus_2 U_{1,3})$, then, by the previous lemma, we can write M as $M_0 \oplus M_1 \oplus \cdots \oplus M_k$ for some $k \geq 0$ where M_0 is binary and every other M_i is a series extension of a uniform matroid. Note that we shall allow M_0 to be $U_{0,0}$. Let Q_B denote the class of binary matroids and let Q_S denote the class of series extensions of uniform matroids. By [6] and Lemma 2.6, neither Q_B nor Q_S contains any infinite antichains. By Lemma 2.4 and Corollary 2.5, $Q_B \times Q_S^{<w}$ is a well-quasi-order. Thus $EX(U_{2,4} \oplus_2 U_{1,3})$ is a well-quasi-order. □

We now prove the main theorem.

Proof of Theorem 1.1. Assume $EX(N)$ contains an infinite antichain. Then, by Corollary 3.2, N is not a minor of $U_{2,4} \oplus_2 U_{1,3}$ or $U_{2,4} \oplus_2 U_{2,3}$.

Assume N is not a minor of $U_{2,4} \oplus_2 U_{1,3}$ or $U_{2,4} \oplus_2 U_{2,3}$, so $|E(N)| \geq 3$. If $r(N) \geq 4$ or $r(N^*) \geq 4$, then $EX(N)$ contains $\{P_n : n \geq 3\}$ or $\{P_n^* : n \geq 3\}$, respectively. Hence, $r(N) \leq 3$ and $r(N^*) \leq 3$. Thus $|E(N)| \leq 6$. Observe that $EX(U_{0,2} \oplus U_{1,1})$ and $EX(U_{2,2} \oplus U_{0,1})$ contain $\{P_n : n \geq 3\}$ and $\{P_n^* : n \geq 3\}$, respectively; both $EX(U_{1,2} \oplus U_{1,2})$ and $EX(U_{2,4} \oplus_2 U_{2,4})$ contain $\{T_k : k \geq 4\}$; and $EX(U_{3,5})$ and $EX(U_{2,5})$ contain $\{M_r : r \geq 2\}$ and $\{M_r^* : r \geq 2\}$, respectively. Hence we may assume that N has no minor isomorphic to $U_{0,2} \oplus U_{1,1}$, $U_{2,2} \oplus U_{0,1}$, $U_{1,2} \oplus U_{1,2}$, $U_{2,5}$, $U_{3,5}$, or $U_{2,4} \oplus_2 U_{2,4}$. It is not difficult to check that this leaves no remaining choices for N. □

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