INFINITE MATROIDS

By JAMES G. OXLEY

[Received 12 August 1976—Revised 8 February 1977]

1. Introduction

This paper proves a number of results linking the operator and independent set approaches to infinite matroids (see Welsh [**12**, Chapter 20]).

Suppose that S is a fixed infinite set and Σ is the collection of independence spaces on S. Then a duality function Δ on Σ is an involution of Σ such that for all $\mathscr{I}$ in Σ and all finite subsets T of S, the restriction of $\Delta\mathscr{I}$ to T agrees with the finite dual of the contraction of $\mathscr{I}$ to T. In §2 we prove that there is no such duality function on Σ, while in §5 we show the existence of a distinguished class of preindependence spaces on S which includes all independence spaces on S, has well-defined operations of restriction and contraction, and is closed under a natural duality function.

In the operator approach of Klee [**8**] and Higgs [**5**], duality is introduced at the beginning. Both authors define several dual pairs of conditions on operators and characterize certain types of operators in terms of their collections of circuits or their collections of bases. In §4 we study the wIwE-operators of Klee [**8**]. These operators need not have circuits, bases, or hyperplanes. We determine the families of sets which can occur as collections of independent sets of such operators. In addition, we answer a question posed by Welsh (private communication, 1976) by proving that, whereas independence spaces fit naturally into the operator framework established by Klee, preindependence spaces do not.

In §5 we settle a question of Higgs [**6**] concerned with B-matroids [**5**, **6**, **7**]. Section 6 shows that B-matroids are a subclass of the class of inductive exchange systems of Brualdi and Scrimger [**4**].

2. Preindependence spaces, independence spaces, and duality

For the definitions of preindependence and independence spaces we follow Welsh [**12**, pp. 385, 387] (see also Mirsky [**11**, p. 90]). A maximal independent subset of a preindependence space is called a *base*. The following condition on a preindependence space $(S, \mathscr{I})$ holds for all independence spaces (see, for example, [**11**]).

(2.1.1) (Maximal condition). If $X \in \mathscr{I}$, then there is a base of $(S, \mathscr{I})$ containing X.

Again suppose that $(S, \mathscr{I})$ is a preindependence space and that $T \subseteq S$. Let

$$\mathscr{I}|T = \{X : X \subseteq T,\ X \in \mathscr{I}\}. \tag{2.1.2}$$

Then $(T, \mathscr{I}|T)$ is a preindependence space called the *restriction* of $\mathscr{I}$ to T. Now, if $(S, \mathscr{I})$ is an independence space and B is a base of $\mathscr{I}|(S \setminus T)$, then let

$$\mathscr{I}.T = \{X : X \subseteq T,\ X \cup B \in \mathscr{I}\}. \tag{2.1.3}$$

$\mathscr{I}.T$ does not depend on the choice of the base B (see, for example, [**3**]). Moreover, $(T, \mathscr{I}.T)$ is an independence space, the *contraction* of $\mathscr{I}$ to T. Brualdi and Scrimger [**4**] define an *exchange system* $(S, \mathscr{I})$ to be a preindependence space with the following additional property. If $T \subseteq S$, B_1 and B_2 are maximal members of $\mathscr{I}|T$, and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y$ and $(B_2 \setminus y) \cup x$ are maximal members of $\mathscr{I}|T$.

An *inductive exchange system* $(S, \mathscr{I})$ is an exchange system satisfying the maximal condition. Note that such systems satisfy the following strengthened exchange condition.

(2.1.4) If $T \subseteq S$, $A \in \mathscr{I}|T$, B is a maximal member of $\mathscr{I}|T$, and $x \in A \setminus B$, then there is an element y of $B \setminus A$ such that $(B \setminus y) \cup x$ is a maximal member of $\mathscr{I}|T$ and $(A \setminus x) \cup y \in \mathscr{I}|T$. If A is maximal in $\mathscr{I}|T$, then $(A \setminus x) \cup y$ is maximal in $\mathscr{I}|T$.

Inductive exchange systems will be looked at again in § 6.

Consider now the extension of duality for finite matroids to duality for independence spaces. If $(S, \mathscr{I})$ is a preindependence space then let

$$\mathscr{I}^* = \{X : S \setminus X \text{ contains a base of } \mathscr{I}\}. \tag{2.1.5}$$

If $(S, \mathscr{I})$ is an independence space, then $(S, \mathscr{I}^*)$ is a preindependence space with the maximal condition. Thus if S is finite, $(S, \mathscr{I}^*)$ is a matroid, the *dual* of the matroid $(S, \mathscr{I})$. In general however, $(S, \mathscr{I}^*)$ need not satisfy the finite character condition (see, for example, [**12**]).

Let S be an arbitrary infinite set and let Σ be the set of independence spaces on S. A *duality function* Δ on Σ is a mapping from Σ into Σ such that for all $\mathscr{I}$ in Σ

$$\Delta\Delta\mathscr{I} = \mathscr{I} \tag{2.2.1}$$

and

$$(\Delta\mathscr{I})|T = (\mathscr{I}.T)^* \quad \text{for all } T \subset\subset S. \tag{2.2.2}$$

The second of these conditions expresses agreement between Δ and duality for finite matroids. Note that the function defined by (2.1.5) fails as a duality function only because it may map an element of Σ outside Σ.

(2.3) THEOREM. *There is no duality function on the collection Σ of independence spaces on an infinite set S.*

Proof. Assume that there is a duality function Δ on Σ. Then for each non-negative integer k, let $\mathscr{I}_k = \{X: X \subseteq S, |X| \leqslant k\}$. Clearly $(S, \mathscr{I}_k)$ is an independence space.

Now for k an arbitrary but fixed non-negative integer, if $T \subset\subset S$, then $S \setminus T$ is infinite and so $\mathscr{I}_k | (S \setminus T)$ contains a base of $\mathscr{I}_k$. Thus $\mathscr{I}_k.T = \{\emptyset\}$, hence $(\mathscr{I}_k.T)^* = \{X: X \subseteq T\}$, and so, by (2.2.2), $T \in \Delta\mathscr{I}_k$. It follows that $\Delta\mathscr{I}_k$ contains all finite subsets of S and hence that $\Delta\mathscr{I}_k = \{X: X \subseteq S\}$. Therefore, for j and k distinct non-negative integers, we have, by (2.2.1), that $\mathscr{I}_j = \Delta(\Delta\mathscr{I}_j) = \Delta(\Delta\mathscr{I}_k) = \mathscr{I}_k$, a contradiction.

Las Vergnas [9] defines a function Δ on Σ which maps Σ into Σ and satisfies (2.2.2) as well as the following weakened form of (2.2.1):

$$\Delta\Delta\Delta\mathscr{I} = \Delta\mathscr{I} \quad \text{for all } \mathscr{I} \text{ in } \Sigma. \tag{2.2.3}$$

Alternatively, one can look for a more general class of infinite matroids on which there is a permutation satisfying (2.2.1) and (2.2.2) or some pair of corresponding conditions. Formally we seek, for an arbitrary infinite set S, a collection of conditions on independent sets which define on every non-empty subset U of S a distinguished class $\mathscr{D}_U$ of preindependence spaces so that:

(2.4.1) $\mathscr{D}_U$ includes all independence spaces on U;

(2.4.2) on $\mathscr{D}_U$, (2.1.2) and (2.1.3) give well-defined operations of restriction and contraction such that if $V \subseteq U$, the restriction or contraction of a member of $\mathscr{D}_U$ is in $\mathscr{D}_V$;

(2.4.3) the function defined by (2.1.5) is a permutation Δ of $\mathscr{D}_U$ satisfying (2.2.1) and (2.2.2), where the latter is to hold for all finite subsets T of U.

Assume that for all non-empty subsets U of S such a class $\mathscr{D}_U$ of preindependence spaces on U exists. We now determine some conditions on the members of $\mathscr{D}_U$. If $\mathscr{I} \in \mathscr{D}_U$ and $T \subseteq U$, then $\mathscr{I} | (U \setminus T)$ must have a base if $\mathscr{I}.T$ is to be well defined. Also, if $\mathscr{I}.T$ is to be independent of the choice of base for $\mathscr{I} | (U \setminus T)$, then the following must hold.

(2.5) If B_1 and B_2 are bases of $\mathscr{I} | (U \setminus T)$ and $X \subseteq T$, then $B_1 \cup X \in \mathscr{I}$ if and only if $B_2 \cup X \in \mathscr{I}$.

By (2.1.5), if $\mathscr{I} \in \mathscr{D}_U$, every element of $\mathscr{I}^*$ is contained in a base of $\mathscr{I}^*$. That is, $(U, \mathscr{I}^*)$ has the maximal condition, and so, by (2.4.3), (2.2.1) and (2.4.2), a restriction of an element of $\mathscr{D}_U$ has the maximal condition.

(2.6) LEMMA. *Let $\mathscr{D}_S$ be a class of preindependence spaces on S satisfying* (2.4.1)–(2.4.3). *Suppose that $\mathscr{I} \in \mathscr{D}_S$ and $U \subseteq S$. Let B_1 and B_2 be bases of $\mathscr{I} | U$ and x be in $B_1 \setminus B_2$. Then there is an element y of $B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y$ is a base of $\mathscr{I} | U$.*

Proof. Clearly B_1 and B_2 are bases of $\mathscr{I} | (B_1 \cup B_2)$ and B_2 is a base of $\mathscr{I} | [(B_1 \cup B_2) \setminus x]$. If $B_1 \setminus x$ is a base of $\mathscr{I} | [(B_1 \cup B_2) \setminus x]$, then by (2.5), since $(B_1 \setminus x) \cup x \in \mathscr{I}$, $B_2 \cup x \in \mathscr{I}$, a contradiction. Therefore, $B_1 \setminus x$ is not a base of $\mathscr{I} | [(B_1 \cup B_2) \setminus x]$. However, $\mathscr{I} | [(B_1 \cup B_2) \setminus x]$ has the maximal condition and hence has a base B which properly contains $B_1 \setminus x$. This implies that there is an element y of $B_2 \setminus B_1$ such that

$$(B_1 \setminus x) \cup y \in \mathscr{I} | (B_1 \cup B_2).$$

Thus $(B_1 \setminus x) \cup y \in \mathscr{I} | U$. Now consider the contraction

$$(\mathscr{I} | U).[U \setminus (B_1 \setminus x)]$$

of $\mathscr{I} | U$. By (2.4.2), $\{x\}$ is a base of this preindependence space and $\{y\}$ is independent. Hence $\{y\}$ is a base of $(\mathscr{I} | U).[U \setminus (B_1 \setminus x)]$ and so $(B_1 \setminus x) \cup y$ is a base of $\mathscr{I} | U$, as required.

To summarize, the following are necessary conditions for $\mathscr{I}$ to belong to $\mathscr{D}_S$.

(2.7.1) *$(S, \mathscr{I})$ is a preindependence space.*

(2.7.2) *For all $T \subseteq S$, $\mathscr{I} | T$ has the maximal condition.*

(2.7.3) *For all $T \subseteq S$, if B_1 and B_2 are bases of $\mathscr{I} | T$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y$ is a base of $\mathscr{I} | T$.*

Section 5 shows that these three conditions are sufficient to define a class of infinite matroids for which (2.4.1)–(2.4.3) are satisfied.

3. Closure operators and infinite matroids

For the basic definitions and notation associated with the operator approach to infinite matroids, we shall essentially follow Klee [8] (see also Welsh [12]). In particular, Klee uses five dual pairs of conditions on an operator f on a set S. Listing one from each pair, these are:

(vwI) if X is finite, Y is independent, and $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$;

(wI) if $x \in f(Y)$, then $f(x \cup Y) = f(Y)$;

(I) if $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$;

(C_F) if $p \in f(Y)$, then there is a finite subset U of Y such that $p \in f(U)$;

(C) if $p \in f(Y)$, then there is a minimal subset U of Y such that $p \in f(U)$ and U is independent.

The duals of these conditions are called respectively (vwE), (wE), (E), (H_F), and (H). Several relations between these conditions are proved in [8, p. 140]. An operator satisfying (wI), (wE), and (C) will be called simply a wIwEC-operator. Similar abbreviations will be used for other types of operators.

If f is an operator on the set S and $T \subseteq S$, define the *restriction*, f_T, of f to T by

$$f_T(X) = f(X) \cap T \quad \text{for all } X \subseteq T.$$

The *contraction*, f^T, of f to T is defined, as for finite matroids, by

$$f^T(X) = f(X \cup (S \setminus T)) \cap T \quad \text{for all } X \subseteq T.$$

Clearly both f_T and f^T are operators on T. If f^* denotes the dual operator of f, then the following familiar relationships hold (see [7, p. 246]):

(3.1.1) $$(f^*)^* = f;$$

(3.1.2) $$(f^*)_T = (f^T)^*.$$

Note that (3.1.1) and (3.1.2) are the operator forms of (2.2.1) and (2.2.2) respectively.

An operator f on a finite set S satisfies (vwI) and (vwE) if and only if f is the closure operator of a (finite) matroid on S. However, while the dual of a vwIvwE-operator on an arbitrary set is also a vwIvwE-operator, a restriction or contraction need not be: if f is an operator on S and $T \subseteq S$, then the set of f_T-independent sets is just the set of f-independent sets contained in T. Thus (vwI) is preserved under restriction. But, as the following example shows, (vwE) need not be preserved under restriction even in the presence of (I).

(3.2) Example. Let $S = \{1, 2, 3, 4, \ldots\}$ and define $f\colon 2^S \to 2^S$ by

$$f(X) = \begin{cases} X, & \text{for } X \text{ finite and } \{2,3\} \nsubseteq X, \\ X \cup \{1\}, & \text{for } X \text{ finite and } \{2,3\} \subseteq X, \\ S, & \text{for } X \text{ infinite.} \end{cases}$$

Clearly f is an operator and, since the f-spanning sets are precisely all infinite subsets of S, f satisfies (vwE). Furthermore, f satisfies (I) and hence (vwI).

Now let $T = \{1,2,3\}$ and consider f_T. Put $Y = \{2,3\}$, $p = 1$, and $X = \{3\}$. Then $p \cup Y$ is f_T-spanning, $p \in f_T(Y)$, and $p \notin f_T(Y \setminus X)$. Yet $x \notin f(p \cup (Y \setminus x))$ for the only element x of X. That is, f_T does not satisfy (vwE).

It may be deduced from [**7**, p. 247] that (wI), (wE), (I), (E), (C_F), and (H_F) are preserved under both restriction and contraction.

There is an important link between independence spaces and wIwEC_F-operators (see, for example, [**12**, p. 398]). Using this and a result of Mason [**10**], we can easily characterize all wIwE$C_F H_F$-operators.

A *B-matroid* (S,f) is an I-operator f on a non-empty set S such that, for all $T \subseteq S$, if X is an independent subset of T, then there is a ~~maximal independent subset~~ basis of T containing X.

4. Operators and preindependence spaces

In this section those families of sets which can occur as the collection of independent sets of a wIwE-operator are characterized. It is also shown that, unlike independence spaces, preindependence spaces cannot be described in the operator framework of Klee.

Welsh [**12**] notes that a wIwE-operator is not uniquely determined by its collection of independent sets. In fact, it is not difficult to show that an IE-operator is not uniquely determined by the pair consisting of its collection of independent sets and its collection of spanning sets.

Let $\mathscr{I}$ be the collection of independent sets of a vwIwE-operator f on a set S. Then

(4.1.1) *$(S, \mathscr{I})$ is a preindependence space.*

This follows from [**12**, p. 398], since if $T \subset\subset S$, then f_T is a wIwEC_F-operator on T.

The next two lemmas generalize familiar finite results and are not difficult to prove.

(4.2) LEMMA. *Suppose that f is a wE-operator on a set S, I is an f-independent subset of S and $x \in S \setminus I$. Then $x \in f(I)$ if and only if $I \cup x$ is f-dependent.*

(4.3) LEMMA. *Let f be a vwI-operator on S and let B be a cofinite base of a subset Y of S. Then $f(B) = f(Y)$.*

From (4.2) and (4.3) the collection $\mathscr{I}$ of independent sets of a vwIwE-operator on S satisfies the following condition.

(4.1.2) *If $Y \subseteq S$, B is a maximal cofinite $\mathscr{I}$ subset of Y, and I is an $\mathscr{I}$ subset of Y, then, for x in $S \setminus Y$, $I \cup x \in \mathscr{I}$ if $B \cup x \in \mathscr{I}$.*

Moreover, we have the following result.

(4.4) THEOREM. *Let $\mathscr{I}$ be a collection of subsets of a set S. Then $\mathscr{I}$ is the collection of independent sets of some vwIwE-operator on S if and only if $\mathscr{I}$ satisfies* (4.1.1) *and* (4.1.2).

Proof. From above, we need only check the sufficiency of (4.1.1) and (4.1.2). Let $(S, \mathscr{I})$ be a preindependence space satisfying (4.1.2) and define $g\colon 2^S \to 2^S$ by

$$g(X) = \begin{cases} X \cup \{x\colon X \cup x \notin \mathscr{I}\}, & \text{if } X \in \mathscr{I}, \quad (4.4.1) \\ g(I_X), & \text{if } X \text{ has a maximal cofinite } \mathscr{I} \text{ subset } I_X, \quad (4.4.2) \\ S, & \text{otherwise.} \quad (4.4.3) \end{cases}$$

To check that g is well-defined, suppose that $X \subseteq S$ and I_X^1 and I_X^2 are maximal cofinite $\mathscr{I}$ subsets of X. Then, by (4.4.1), $X \subseteq g(I_X^i)$ for $i = 1, 2$. Now, if $x \in S \setminus X$, then, by (4.1.2), $I_X^1 \cup x \notin \mathscr{I}$ if and only if $I_X^2 \cup x \notin \mathscr{I}$. That is, by (4.4.1), $x \in g(I_X^1)$ if and only if $x \in g(I_X^2)$.

Next we show that g is an operator. Clearly g is enlarging. Now suppose that $X \subseteq Y \subseteq S$, then if $g(Y)$ is given by (4.4.3), $g(X) \subseteq g(Y) = S$. Alternatively, if $g(Y) = g(I_Y)$ for some maximal cofinite $\mathscr{I}$ subset I_Y of Y, then $g(X) = g(I_X)$ for some maximal cofinite $\mathscr{I}$ subset I_X of X. If $x \in g(X) \setminus Y$, then $I_X \cup x \notin \mathscr{I}$, so by (4.1.2), $I_Y \cup x \notin \mathscr{I}$ and hence $x \in g(I_Y)$. Thus $g(X) \subseteq g(Y)$.

To show that g satisfies (vwI), we show that g satisfies (wI). Suppose that $Y \in \mathscr{I}$ and $x \in g(Y)$. Then Y is a maximal cofinite $\mathscr{I}$ subset of $Y \cup x$ and so $g(Y \cup x) = g(Y)$. If $Y \notin \mathscr{I}$, $x \in g(Y)$, and $g(Y)$ is given by (4.4.2), then a similar argument gives that $g(Y \cup x) = g(Y)$.

For (wE), suppose that $x \in g(Y \cup p)$ and $x \notin g(Y)$. Then $g(Y) = g(I_Y)$ where I_Y is a maximal cofinite $\mathscr{I}$ subset of Y. Since $g(Y \cup p) \neq g(Y)$, $I_Y \cup p$ is a maximal $\mathscr{I}$ subset of $Y \cup p$. Thus $g(Y \cup p) = g(I_Y \cup p)$. Now, as $x \notin g(I_Y)$, $I_Y \cup x \in \mathscr{I}$. Moreover, $(I_Y \cup p) \cup x \notin \mathscr{I}$, hence, by (4.4.1), $p \in g(I_Y \cup x) = g(Y \cup x)$. That is, (wE) is satisfied.

It is straightforward to check that $\mathscr{I}$ is precisely the collection of g-independent sets.

Note that by Lemmas 4.2 and 4.3 any vwIwE-operator having $\mathscr{I}$ as its collection of independent sets must be defined as in (4.4.1) and (4.4.2) on sets containing maximal cofinite $\mathscr{I}$ subsets. It follows that g is maximal among such vwIwE-operators in the sense that if h is another such operator, then, for all $X \subseteq S$, $h(X) \subseteq g(X)$.

In the proof of Theorem 4.4, we showed that g satisfies (wI). It follows immediately that

(4.5) COROLLARY. *If $\mathscr{I}$ is a collection of subsets of a set S, then $\mathscr{I}$ is the collection of independent sets of a wIwE-operator on S if and only if $\mathscr{I}$ satisfies* (4.1.1) *and* (4.1.2).

In Theorem 4.4 (wE) cannot be replaced by (vwE), since one can easily find a vwIvwE-operator whose collection of independent sets is not a preindependence space.

Welsh (private communication, 1976) asks whether preindependence spaces can be described in operator terminology. Let P be Klee's set of conditions on operators. That is,

$$P = \{(\mathrm{vwI}), (\mathrm{wI}), (\mathrm{I}), (\mathrm{vwE}), (\mathrm{wE}), (\mathrm{E}), (\mathrm{C}), (\mathrm{C_F}), (\mathrm{H}), (\mathrm{H_F})\}.$$

If $Q \subseteq P$, an operator satisfying all of the conditions in Q will be called simply a *Q-operator*. Preindependence spaces cannot be described in terms of P. That is:

(4.6) THEOREM. *There is no subset K of P for which the following statement is true.*

(4.6.1) *The collection of independent sets of a K-operator on an arbitrary set S is a preindependence space on S and conversely, if $\mathscr{I}$ is a preindependence space on S, then there is some (not necessarily unique) K-operator on S having $\mathscr{I}$ as its collection of independent sets.*

To prove this result three examples will be given. The first is a preindependence space for which there is no vwIvwE-operator having the same collection of independent sets. From this and the relations between operators [8, p. 140], it follows that if (4.6.1) holds, then K does not contain both (vwI) and (vwE).

(4.7) EXAMPLE. Let $S = \{1, 2, 3, 4, \ldots\}$, and if $n_1, n_2, \ldots, n_m$ are distinct elements of S, then let $B_{n_1,n_2,\ldots,n_m}$ denote the set $S \setminus \{n_1, n_2, \ldots, n_m\}$. In addition, let

$$\mathscr{B} = \{B_{1,2,3}, B_{1,2,4}, B_{1,3,4}, B_{2,3,4}, B_{1,n}, B_{2,n}, B_{3,n}, B_{4,n} \text{ (for all } n \geqslant 5)\}$$

and

$$\mathscr{I} = \{X : X \subset\subset S \text{ or } X \subseteq B \text{ for some } B \text{ in } \mathscr{B}\}.$$

Now suppose that there is a vwIvwE-operator f on S having $\mathscr{I}$ as its collection of independent sets. Then f^* is a vwIvwE-operator on S and its set of spanning sets is

$$\{X : S \setminus X \subset\subset S\} \cup \{X : X \text{ contains } \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \text{ or } \{2,3,4\}\}$$
$$\cup \{X : X \text{ contains } \{1,n\}, \{2,n\}, \{3,n\}, \text{ or } \{4,n\} \text{ for some } n \geqslant 5\}.$$

Let $\{a,b\}$ be a two-element subset of $\{1,2,3,4\}$. Then $\{a,b\}$ is not f^*-spanning. Thus, either there is an element n of $S\setminus\{1,2,3,4\}$ such that $n \notin f^*(\{a,b\})$, or there is no such element. In the first case, since $\{b,n\}$ is spanning, $a \in f^*(\{b,n\})$. Moreover, $a \in f^*(b)$ as otherwise, by (vwE), $n \in f^*(\{a,b\})$, a contradiction. Similarly, since $\{a,n\}$ is spanning, $b \in f^*(a)$.

In the second case, there is an element n of $S\setminus\{1,2,3,4\}$ such that $n \in f^*(\{a,b\})$. It follows that $\{a,b\}$ is not independent, since otherwise, by (vwI), $f^*(\{a,b,n\}) = f^*(\{a,b\})$ which implies that $\{a,b\}$ is spanning, a contradiction. Therefore, $a \in f^*(b)$ or $b \in f^*(a)$.

From the above, for all two-element subsets $\{a,b\}$ of $\{1,2,3,4\}$, either $a \in f^*(b)$ or $b \in f^*(a)$. Hence there are at least six ordered pairs (x,y) such that x and y are distinct elements of $\{1,2,3,4\}$ and $x \in f^*(y)$.

Next let a, b, and c be distinct elements of $\{1,2,3,4\}$ and suppose that $a \in f^*(c)$ and $b \in f^*(c)$. If $\{c\}$ is independent, then, by (vwI), $f^*(\{a,b,c\}) = f^*(c)$ and so $\{c\}$ is spanning, a contradiction. On the other hand, if $\{c\}$ is dependent, then $c \in f^*(\emptyset)$ and so $f^*(c) = f^*(\emptyset)$. It follows that $\{a,b,c\} \subseteq f^*(\emptyset)$, and hence, by (vwI), that $f^*(\{a,b,c\}) = f^*(\emptyset)$. Thus $\emptyset$ is spanning, a contradiction.

Therefore, for a, b, c, and d, distinct elements of $\{1,2,3,4\}$, at most one of the statements $a \in f^*(d)$, $b \in f^*(d)$, and $c \in f^*(d)$ is true. Hence there are at most four ordered pairs (x,y) such that x and y are distinct elements of $\{1,2,3,4\}$ and $x \in f^*(y)$. However, earlier it was shown that there are at least six such pairs.

We conclude that there is no vwIvwE-operator on S having $\mathscr{I}$ as its collection of independent sets.

To complete the proof of Theorem 4.6, we give examples of an $\mathrm{ECHC_FH_F}$-operator and of an $\mathrm{ICHC_FH_F}$-operator whose collections of independent sets do not form preindependence spaces.

(4.8) Example. Let $S = \{1,2,3\}$ and define $f_1\colon 2^S \to 2^S$ by $f_1(\emptyset) = \emptyset$, $f_1(1) = \{1,3\}$, $f_1(2) = \{2,3\}$, $f_1(3) = \{1,2,3\}$, and $f_1(X) = \{1,2,3\}$ for $|X| \geqslant 2$. Then it is easily checked that f_1 is an $\mathrm{ECHC_FH_F}$-operator on S. Its collection $\mathscr{I}$ of independent sets is $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\}$. Clearly $(S, \mathscr{I})$ is not a preindependence space.

(4.9) Example. Again let $S = \{1,2,3\}$ and define $f_2\colon 2^S \to 2^S$ by

$$f_2(X) = \begin{cases} X, & \text{if } |X| < 2 \text{ and } 3 \notin X, \\ S, & \text{otherwise.} \end{cases}$$

Then it is easily checked that f_2 is an $\mathrm{ICHC_FH_F}$-operator on S having the same collection of independent sets as f_1.

Next let $\mathscr{A} = (A_i : i \in I)$ be an arbitrary family of subsets of an arbitrary set S and let $\tau(\mathscr{A})$ be the collection of partial transversals of $\mathscr{A}$. Then, as is well known, $(S, \tau(\mathscr{A}))$ is a preindependence space (see, for example, [11]). Moreover:

(4.10) LEMMA (Brualdi and Scrimger [4, Theorem 2]). *$(S, \tau(\mathscr{A}))$ is an exchange system satisfying* (2.1.4).

$(S, \tau(\mathscr{A}))$ may be put into the operator framework set up earlier.

(4.11) THEOREM. *If $\mathscr{A} = (A_i : i \in I)$ is a family of subsets of a set S, then there is some wIwE-operator on S having $\tau(\mathscr{A})$ as its collection of independent sets.*

Proof. It will be shown that $(S, \tau(\mathscr{A}))$ satisfies (4.1.1) and (4.1.2). The required result then follows by Corollary 4.6.

As noted above, $(S, \tau(\mathscr{A}))$ satisfies (4.1.1). Now suppose that $Y \subseteq S$, B is a cofinite base of Y, and I is an independent subset of Y. Then, since $I \setminus B$ is finite, we have, on applying (2.1.4) to $\tau(A) | Y$, that there is a subset I' of $B \setminus I$ such that $|I'| = |I \setminus B|$ and $(B \setminus I') \cup (I \setminus B)$ is a base of Y. Let $B' = (B \setminus I') \cup (I \setminus B)$. Clearly $B' \supseteq I$.

If $x \in S \setminus Y$, $B \cup x \in \tau(\mathscr{A})$, and $I \cup x \notin \tau(\mathscr{A})$, then, since $B' \cup x \notin \tau(\mathscr{A})$, $B \cup x$ and B' are bases of $Y \cup x$. Applying (2.1.4) again gives that there is an element y of $B' \setminus (B \cup x)$ such that $[(B \cup x) \setminus x] \cup y$ is a base of $Y \cup x$. But $B \subset B \cup y \subseteq Y$ and this is a contradiction.

5. B-matroids

In this section we characterize those families of sets which can occur as the collection of independent sets of a B-matroid and note that a B-matroid is uniquely determined by this collection. In addition we show that B-matroids satisfy (2.4.1)–(2.4.3). A collection $\mathscr{A}$ of subsets of a set S is called a *clutter* if no element of $\mathscr{A}$ properly contains another.

From [5, Proposition (10)], if (S, f) is a B-matroid, its collection $\mathscr{B}$ of bases satisfies the conditions:

(5.1.1) *$\mathscr{B}$ is a clutter; and*

(5.1.2) *if $B_1, B_2 \in \mathscr{B}$ and $A \subseteq C \subseteq S$ where $A \subseteq B_1$ and $B_2 \subseteq C$, then there is an element B of $\mathscr{B}$ such that $A \subseteq B \subseteq C$.*

Higgs [6] asks the following question.

(5.2) *If $\mathscr{B} \neq \emptyset$ and $\mathscr{B}$ satisfies* (5.1.1) *and* (5.1.2), *then does there exist a B-matroid having $\mathscr{B}$ as its collection of bases?*

The next example shows that such a B-matroid need *not* exist.

(5.3) EXAMPLE. Let $S = \mathbf{Z}\setminus\{0\}$, the set of non-zero integers. Then S is the disjoint union of $\mathbf{Z}^+$, the set of positive integers and $\mathbf{Z}^-$, the set of negative integers.

Suppose that $\mathscr{B}$ is the collection of subsets of S consisting of $\mathbf{Z}^+$ together with all sets of the form $[\mathbf{Z}^+\setminus\{i_1, i_2, \ldots, i_n\}] \cup \{-i_1, -i_2, \ldots, -i_n\}$ where $i_1, i_2, \ldots, i_n$ are n distinct elements of $\mathbf{Z}^+$ and n takes all the values $1, 2, 3, \ldots$. Clearly $\mathscr{B}$ is a clutter.

Suppose that A is contained in an element of $\mathscr{B}$, C contains an element of $\mathscr{B}$, and $A \subseteq C$. Every element of $\mathscr{B}$ contains only finitely many elements of $\mathbf{Z}^-$ and all but finitely many elements of $\mathbf{Z}^+$. Moreover, if $x \in \mathbf{Z}^+ \cap (S\setminus C)$, then $-x \in C$. If $y \in \mathbf{Z}^- \cap A$, then $-y \notin A$. Thus suppose that $\mathbf{Z}^- \cap A = \{-j_1, -j_2, \ldots, -j_m\}$ and that

$$\mathbf{Z}^+ \cap (S\setminus C) = \{j_{k_1}, j_{k_2}, \ldots, j_{k_p}, u_1, u_2, \ldots, u_q\}$$

where $\{k_1, k_2, \ldots, k_p\} \subseteq \{1, 2, \ldots, m\}$. Then

$$C \supseteq [\mathbf{Z}^+\setminus\{j_1, j_2, \ldots, j_m, u_1, u_2, \ldots, u_q\}] \cup \{-j_1, -j_2, \ldots, -j_m, -u_1, -u_2, \ldots, -u_q\} \supseteq A$$

and hence $\mathscr{B}$ satisfies (5.1.2).

Now suppose that there is a B-matroid (S,f) having $\mathscr{B}$ as its set of bases. Then the collection, $\mathscr{I}$, of independent sets of (S,f) is given by $\mathscr{I} = \{X : X \subseteq B \text{ for some } B \text{ in } \mathscr{B}\}$. Consider $(\mathbf{Z}^-, f_{\mathbf{Z}^-})$. By [5, Proposition (13)], this is a B-matroid. Its collection of independent sets is $\mathscr{I}|\mathbf{Z}^- = \{X : X \in \mathscr{I}, X \subseteq \mathbf{Z}^-\} = \{X : X \subset\subset \mathbf{Z}^-\}$. Clearly $\mathscr{I}|\mathbf{Z}^-$ does not satisfy the maximal condition and this contradicts the definition of a B-matroid.

The proof of the next lemma is straightforward.

(5.4) LEMMA. *Let f be an I-operator on S and let B be a base of a subset Y of S. Then $f(B) = f(Y)$.*

It follows from this lemma that, in contrast to wIwE-operators, B-matroids *are* uniquely determined by their collections of independent sets. The next proposition should be compared with Theorem 4.4.

(5.5) PROPOSITION. *The collection $\mathscr{I}$ of independent sets of a B-matroid on a set S satisfies the following conditions.*

(5.5.1) *$(S, \mathscr{I})$ is a preindependence space.*

(5.5.2) *Every restriction of $\mathscr{I}$ satisfies the maximal condition.*

(5.5.3) *If $Y \subseteq S$, B is a maximal $\mathscr{I}$ subset of Y and I is an $\mathscr{I}$ subset of Y, then, for x in $S\setminus Y$, $I \cup x \in \mathscr{I}$ if $B \cup x \in \mathscr{I}$.*

Conversely, if $\mathscr{I}$ is a collection of subsets of S satisfying (5.5.1)–(5.5.3), *then there is a unique B-matroid on S having $\mathscr{I}$ as its collection of independent sets.*

Proof. The first part is straightforward. For the second part the required IE-operator is defined by

$$f(X) = \begin{cases} X \cup \{x : X \cup x \notin \mathscr{I}\}, & \text{if } X \in \mathscr{I}, \\ f(I_X), & \text{if } I_X \text{ is a maximal } \mathscr{I} \text{ subset of } X. \end{cases}$$

The next result characterizes B-matroids in terms of their collections of bases. Notice the similarity to the corresponding result for finite matroids (see Welsh [**12**]). If $\mathscr{A}$ is a collection of subsets of a set S and $X \subseteq S$, then define $\mathscr{A}(X) = \{Y : Y \text{ is maximal of the form } A \cap X \text{ where } A \in \mathscr{A}\}$.

(5.6) Theorem. *A collection $\mathscr{B}$ of subsets of a set S is the set of bases of a B-matroid on S if and only if $\mathscr{B}$ satisfies the following conditions.*

(5.6.1) *$\mathscr{B}$ is non-empty.*

(5.6.2) *If $Y \subseteq X \subseteq S$ and $Y \subseteq B$ for some B in $\mathscr{B}$, then $Y \subseteq B_X$ for some B_X in $\mathscr{B}(X)$.*

(5.6.3) *If $X \subseteq S$, $B_1, B_2 \in \mathscr{B}(X)$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathscr{B}(X)$.*

Proof. The necessity of (5.6.1) and (5.6.2) is clear. For (5.6.3), use Proposition 5.5.

For the converse, let $\mathscr{B}$ be a collection of subsets of S satisfying (5.6.1)–(5.6.3) and let $\mathscr{I} = \{X : X \subseteq B \text{ for some } B \text{ in } \mathscr{B}\}$. Then, from (5.6.1) and (5.6.3), $\mathscr{I}$ is a preindependence space. Moreover, by (5.6.2), every restriction of $\mathscr{I}$ satisfies the maximal condition. Now suppose that B is a maximal $\mathscr{I}$ subset of a subset Y of S. Let I be an $\mathscr{I}$ subset of Y and x be an element of $S \setminus Y$ such that $B \cup x \in \mathscr{I}$. If $I \cup x \notin \mathscr{I}$, then $I \subseteq B'$, a maximal $\mathscr{I}$ subset of $Y \cup x$ not containing x. Clearly $B \cup x$ is a maximal $\mathscr{I}$ subset of $Y \cup x$; hence by (5.6.3), since $x \in (B \cup x) \setminus B'$, there is an element y of $B' \setminus (B \cup x)$ such that $B \cup y$ is a maximal $\mathscr{I}$ subset of $Y \cup x$. But $B \cup y \subseteq Y$ and hence the choice of B is contradicted. We conclude that $\mathscr{I}$ satisfies (5.5.1)–(5.5.3) and the result follows by Proposition 5.5.

A consequence of (5.5) and (5.6) is the following:

(5.7) Corollary. *A collection $\mathscr{I}$ of subsets of a set S is the set of independent sets of a B-matroid on S if and only if $\mathscr{I}$ satisfies* (2.7.1)–(2.7.3).

The main result of this section comes from combining this corollary with [**5**, Propositions (9), (11), (12), and (13)].

(5.8) THEOREM. *If S is an infinite set, then the ~~unique~~ maximal class of preindependence spaces defined on S and all its non-empty subsets such that* (2.4.1)–(2.4.3) *are satisfied is the class of B-matroids.*

It follows from this theorem that (2.4.1) and the fact that Δ satisfies (2.2.2) are consequences of the other conditions on $\mathscr{D}_U$.

6. Some further links

It is straightforward to show that, in Example 5.3, $(S, \mathscr{I})$ is an inductive exchange system (see § 2). However, $(\mathbf{Z}^-, \mathscr{I}|\mathbf{Z}^-)$ does not satisfy the maximal condition. Thus the restriction of an inductive exchange system need not be inductive. The next theorem shows that B-matroids are exactly those exchange systems for which every restriction is inductive. The following lemma extends a result of Brualdi [2] for independence spaces.

(6.1) LEMMA. *Let B be a base of a B-matroid (S,f) and let a be an element of $S \setminus B$. Then there is a unique circuit C containing a, such that $C \subseteq B \cup a$. Moreover, if $b \in B$, then $(B \setminus b) \cup a$ is a base if and only if $b \in C$.*

Proof. As B is spanning, $a \in f(B)$. Thus, since f satisfies (C) [**5**, Proposition (16)], there is a minimal subset U of B such that $a \in f(U)$ and U is independent. By [8, Proposition 2], $a \cup U$ is a circuit and clearly this circuit satisfies the requirements of the lemma. The fact that it is unique follows by [8, Proposition 5] since f is an IECH-operator. Thus let $a \cup U = C$.

If $b \in B$ and $(B \setminus b) \cup a$ is a base, then $(B \setminus b) \cup a$ does not contain C, and hence $b \in C$. Conversely, if $b \in C \cap B$, then $b \in C \subseteq [(B \setminus b) \cup a] \cup b$ and hence, by [**8**, Proposition 5], $b \in f((B \setminus b) \cup a)$. Thus $(B \setminus b) \cup a$ is spanning. Moreover, $(B \setminus b) \cup a$ is independent, as otherwise, by Lemma 4.2, $a \in f(B \setminus b)$ and hence $B \setminus b$ is spanning, a contradiction. Therefore, for $b \in C \cap B$, $(B \setminus b) \cup a$ is a base.

(6.2) THEOREM. *Let $(S, \mathscr{I})$ be a preindependence space. Then $\mathscr{I}$ is the collection of independent sets of a B-matroid on S if and only if every restriction of $(S, \mathscr{I})$ is an inductive exchange system.*

Proof. Suppose that B_1 and B_2 are bases of a B-matroid (S,f) and that $b_1 \in B_1 \setminus B_2$. Let $S \setminus (B_1 \cap B_2) = T$. Then, by [**5**, Proposition (13)], (T, f^T) is a B-matroid which, by [**5**, Proposition (11)], has $B_1 \setminus B_2$ and $B_2 \setminus B_1$ as bases.

Now $b_1 \in f(B_2) \cap T$ and therefore, by Lemma 6.1, there is a unique f^T-circuit C such that $b_1 \in C \subseteq (B_2 \setminus B_1) \cup b_1$. Suppose that for all b in

$C \cap (B_2 \setminus B_1)$, $[(B_1 \setminus B_2) \setminus b_1] \cup b$ is dependent. Then

$$C \setminus b_1 \subseteq f^T((B_1 \setminus B_2) \setminus b_1)$$

and so, by [8, Proposition 5], $b_1 \in f^T(C \setminus b_1) \subseteq f^T((B_1 \setminus B_2) \setminus b_1)$. That is, $B_1 \setminus B_2$ if f^T-dependent, a contradiction. Therefore for some b_2 in $C \cap (B_2 \setminus B_1)$, $[(B_1 \setminus B_2) \setminus b_1] \cup b_2$ is independent. As $B_1 \setminus B_2$ is a base, $(B_1 \setminus B_2) \cup b_2$ is dependent, and so $b_1 \in f^T([(B_1 \setminus B_2) \setminus b_1] \cup b_2)$. Thus $[(B_1 \setminus B_2) \setminus b_1] \cup b_2$ is spanning and hence is an f^T-base. By Lemma 6.1, $[(B_2 \setminus B_1) \setminus b_2] \cup b_1$ is also an f^T-base and the required result follows.

The converse is an immediate consequence of Corollary 5.7.

Acknowledgements

The author thanks Dr D. J. A. Welsh for suggesting the study of infinite matroids and for much valuable advice during the preparation of this paper. The generous financial support of CSIRO (Australia) is also gratefully acknowledged.

REFERENCES

1. D. S. Asche, 'Minimal dependent sets', *J. Austral. Math. Soc.* 6 (1966) 259–62.
2. R. A. Brualdi, 'Comments on bases in dependence structures', *Bull. Austral. Math. Soc.* 1 (1969) 161–67.
3. —— 'Admissible mappings between dependence spaces', *Proc. London Math. Soc.* (3) 21 (1970) 296–312.
4. —— and E. B. Scrimger, 'Exchange systems, matchings and transversals', *J. Combinatorial Theory* 5 (1968) 244–57.
5. D. A. Higgs, 'Matroids and duality', *Colloq. Math.* 20 (1969) 215–20.
6. —— 'Equicardinality of bases in *B*-matroids', *Canad. Math. Bull.* 12 (1969) 861–62.
7. —— 'Infinite graphs and matroids', *Recent progress in combinatorics*, Proceedings Third Waterloo Conference on Combinatorics (Academic Press, New York, 1969), pp. 245–53.
8. V. Klee, 'The greedy algorithm for finitary and cofinitary matroids', *Combinatorics*, Proceedings Symposia in Pure Mathematics XIX (Amer. Math. Soc., Providence, R.I., 1971), pp. 137–52.
9. M. Las Vergnas, 'Sur la dualité en théorie des matroïdes', *Théorie des matroïdes*, Lecture Notes in Mathematics 211 (Springer-Verlag, Berlin, 1971), pp. 67–85.
10. J. H. Mason, 'A characterization of transversal independence spaces', *Combinatorial structures and their applications*, Proceedings Calgary International Conference (Gordon and Breach, New York, 1970), pp. 257–59.
11. L. Mirsky, *Transversal theory. An account of some aspects of combinatorial mathematics* (Academic Press, London, 1971).
12. D. J. A. Welsh, *Matroid theory*, London Mathematical Society Monographs 8 (Academic Press, London, 1976).

Mathematical Institute
24–29 *St Giles*
*Oxford OX*1 3*LB*