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On the Excluded Minors for Quaternary Matroids

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This paper strengthens the excluded-minor characterization of GF(4)-representable matroids. In particular, it is shown that there are only finitely many 3-connected matroids that are not GF(4)-representable and that have no $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minors. Explicitly, these matroids are all minors of $S(5, 6, 12)$ with rank and corank at least 4, and P''_8, the matroid that can be obtained from $S(5, 6, 12)$ by deleting two elements, contracting two elements, and then relaxing the only pair of disjoint circuit-hyperplanes. © 2000 Academic Press

1. INTRODUCTION

Kahn and Seymour had conjectured that the excluded minors for the class of GF(4)-representable matroids are $U_{2,6}$, $U_{4,6}$, P_6, the non-Fano matroid (F_7^-), and its dual; see [4, p. 205]. It turns out that the complete set of excluded minors for GF(4)-representability contains two more matroids, namely P_8 and P''_8; see [1]. However, Kahn and Seymour were almost right, as we show in the following theorem.

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THEOREM 1.1. *If M is a* 3-*connected non*-$GF(4)$-*representable matroid, then either*

(i) *M has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor,*

(ii) *M is isomorphic to P_8'', or*

(iii) *M is isomorphic to a minor of $S(5, 6, 12)$ with rank and corank at least* 4.

$S(5, 6, 12)$, which is discussed in detail in [4], is the matroid that is represented over GF(3) by the following matrix.

$$\left(\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & -1 & -1 & 1 \\ & 1 & 1 & 0 & 1 & -1 & -1 \\ I_6 & 1 & -1 & 1 & 0 & 1 & -1 \\ & 1 & -1 & -1 & 1 & 0 & 1 \\ & 1 & 1 & -1 & -1 & 1 & 0 \end{array}\right).$$

Evidently $S(5, 6, 12)$ is self-dual. Moreover, it has a 5-transitive automorphism group. P_8 is the matroid that is obtained by deleting two elements and contracting two elements from $S(5, 6, 12)$. Now P_8 has a unique pair of disjoint circuit-hyperplanes and P_8'' is obtained from P_8 by relaxing both of these circuit-hyperplanes. These observations and those made before the theorem imply that the matroids satisfying (i), (ii), or (iii) are not quaternary.

The following corollary is a reformulation of Theorem 1.1. For a collection $\mathcal{M}$ of matroids, we denote by $EX(\mathcal{M})$ the class of matroids that have no minors isomorphic to a member of $\mathcal{M}$.

COROLLARY 1.2. *$EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)$ can be constructed by taking direct sums and* 2-*sums of copies of P_8'', minors of $S(5, 6, 12)$, and quaternary matroids.*

We obtain Theorem 1.1 as a consequence of the excluded-minor characterization for quaternary matroids [1].

THEOREM 1.3. *A matroid M is $GF(4)$-representable if and only if M has no minor isomorphic to any of $U_{2,6}$, $U_{4,6}$, P_6, F_7^-, $(F_7^-)^*$, P_8, or P_8''.*

Theorem 1.1. is an immediate consequence of Theorem 1.3 and the following two theorems. Let $\mathcal{M}$ be a minor-closed family of matroids. A matroid M in $\mathcal{M}$ is called a *splitter* for $\mathcal{M}$ if no 3-connected matroid in $\mathcal{M}$ has a proper M-minor.

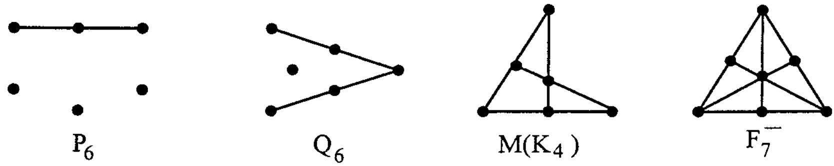

FIG. 1. Some rank-3 matroids.

THEOREM 1.4. *P_8'' is a splitter for $EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)$.*

THEOREM 1.5. *If M is a 3-connected matroid in $EX(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)$, and M has a P_8-minor, then M is isomorphic to a minor of $S(5, 6, 12)$.*

Using Seymour's Splitter Theorem [5, 4], Theorems 1.4 and 1.5 can be proved by a finite case check. While we use this approach, we have endeavoured to find elegant techniques to reduce the number of cases.

THEOREM 1.6 (Splitter Theorem). *Let M and N be 3-connected matroids such that M is neither a wheel nor a whirl, N has at least four elements, and M contains a proper N-minor. Then M has an element x such that either $M\backslash x$ or M/x is 3-connected with an N-minor.*

We assume that readers are familiar with elementary notions in matroid theory, including representability, minors, duality, connectivity, and 1- and 2-sums. We use the notation and terminology of [4]. Figure 1 depicts some well-known matroids that are referred to in the paper. We will describe P_8 and P_8'' in more detail in the next section.

2. DEALING WITH P_8

In this section, we prove Theorem 1.5. We begin by describing some useful properties of P_8. This matroid has a very natural geometric representation; see Fig. 2. This representation is obtained by rotating a face of the cube by 45 degrees. It is obvious from this description that P_8 has a transitive automorphism group. (However, there are automorphisms of P_8 that are not apparent from this description.) By this transitivity, all single-element contractions of P_8 are isomorphic to $P_8/8$, which is isomorphic to the matroid P_7 depicted in Fig. 2. It is also not difficult to see that P_8 is self-dual; its dual is obtained by rotating the twisted face a further 90 degrees. Therefore every single-element deletion of P_8 is isomorphic to the dual of P_7.

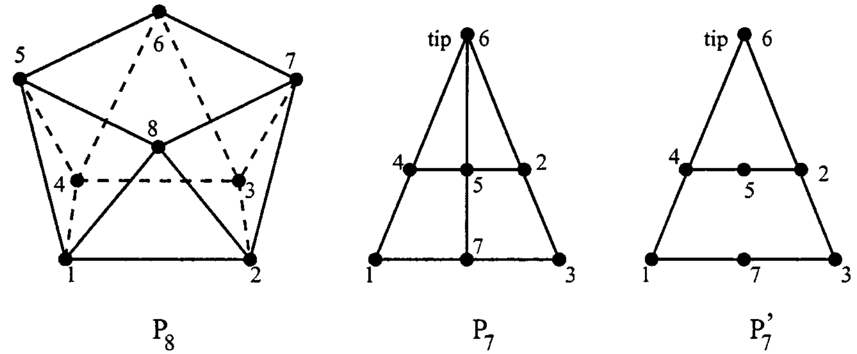

FIG. 2. Some interesting matroids.

P''_8 is the matroid obtained from P_8 by relaxing the circuit-hyperplanes $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. From this, it is readily seen that P''_8 is self-dual and has a transitive automorphism group, and that every single-element contraction is isomorphic to P'_7 (which is depicted in Fig. 2).

We use the following lemma [4, Proposition 11.2.16] and theorem [3] (see also [4, p. 367]). The matroid J is a rank-4 self-dual matroid that is not isomorphic to P_8.

LEMMA 2.1. *Let M be a 3-connected matroid having rank and corank at least three. Then M has a $U_{2,5}$-minor if and only if it has a $U_{3,5}$-minor.*

THEOREM 2.2. *If M is a 3-connected matroid in $Ex(U_{2,5}, U_{3,5}, M(K_4))$, then either M is a whirl, M is isomorphic to J, or M is isomorphic to a minor of $S(5, 6, 12)$.*

COROLLARY 2.3. *If M is a 3-connected matroid that is not isomorphic to a minor of $S(5, 6, 12)$, and M has a P_8-minor, then there is a minor N of either M or M^* and an element x of $E(N)$ such that $N\backslash x$ is isomorphic to P_8, and N contains either a $U_{2,5}$- or an $M(K_4)$-minor.*

Proof. Let M' be a minimal 3-connected minor of M that has a P_8-minor and a $U_{2,5}$-, $U_{3,5}$-, or $M(K_4)$-minor. By the Splitter Theorem, M' has an element x such that $M'\backslash x$ or M'/x is 3-connected and has a P_8-minor. By duality, we may assume that $M'\backslash x$ is 3-connected and has a P_8-minor. By minimality, $M'\backslash x$ has no $U_{3,5}$-, $U_{2,5}$-, or $M(K_4)$-minor. Therefore, by Theorem 2.2, $M'\backslash x$ is isomorphic to a minor of $S(5, 6, 12)$.

As M' has rank and corank at least 3, by Lemma 2.1, M' has either a $U_{2,5}$- or $M(K_4)$-minor. Suppose that M' has rank at least 5. Then there exists $y \in E(M') - x$ such that M'/y has a $U_{2,5}$- or $M(K_4)$-minor. Now $M'\backslash x/y$ is isomorphic to a minor of $S(5, 6, 12)$ with rank and corank at least four. Hence, as $S(5,6,12)$ has a 5-transitive automorphism group, $M'\backslash x/y$ is 3-connected and has a P_8-minor. Now M'/y is an extension of the 3-connected matroid $M'/y\backslash x$, and, since $M'/y\backslash x$ has no $U_{2,5}$- or $M(K_4)$-minor, x is not in parallel with any element of M'/y. Hence M'/y is 3-connected. Moreover, M'/y has a P_8-minor and a $U_{2,5}$- or $M(K_4)$-minor, contradicting the minimality of M'. Therefore, M' has rank 4.

An argument similar to that in the last paragraph establishes that the corank of M' is 5. Therefore, taking N to be equal to M', we see that the theorem holds. ▮

Theorem 1.5. is implied by Corollary 2.3 and the following two results.

LEMMA 2.4. *If $M\backslash x=P_8$ and M has an $M(K_4)$-minor, then M has a $U_{2,5}$-, F_7^--, or $(F_7^-)^*$-minor.*

LEMMA 2.5. *If $M\backslash x=P_8$ and M has a $U_{2,5}$-minor, then M has a $U_{2,6}$-, $U_{4,6}$-, P_6, F_7^--, or $(F_7^-)^*$-minor.*

Proof of Lemma 2.4. Suppose that M is in $Ex(U_{2,5}, F_7^-, (F_7^-)^*)$. Since M has no $U_{2,5}$-minor and M has rank and corank at least 3, Lemma 2.1 implies that M has no $U_{3,5}$-minor.

We show next that $E(M)-x$ contains an element y such that M/y has an $M(K_4)$-minor. Suppose not. Then, as M has corank 4, M/x has an $M(K_4)$-minor. Therefore, $E(M)-x$ contains elements a and b such that $M/x\backslash a,b\cong M(K_4)$. Now $M\backslash a,b,x$ is isomorphic to a matroid obtained from P_7^* by deleting an element. Thus $M\backslash a,b,x$ has either two or three disjoint series pairs. Now $M\backslash a,b$ has no series pairs, otherwise we could contract an element other than x leaving an $M(K_4)$-minor. Therefore, x is in either two or three 3-element cocircuits of $M\backslash a,b$, and any two such cocircuits have only x in common. Thus, as $M\backslash a,b,/x\cong M(K_4)$, it follows that $M\backslash a,b$ is isomorphic to F_7^* or $(F_7^-)^*$. Now M certainly has no $(F_7^-)^*$-minor. Thus $M\backslash a,b\cong F_7^*$ and so, for every y in $E(M)\backslash\{x,a,b\}$, the matroid M/y has an $M(K_4)$-minor. This contradiction implies that there is, indeed, an element y of $E(M)-x$ such that M/y has an $M(K_4)$-minor. As P_8 has a transitive automorphism group, we may assume that $y=8$.

Now $M/8$ is an extension of P_7 that has no $U_{3,5}$-minor. Furthermore, as P_7 has no $M(K_4)$-minor and $M/8$ has an $M(K_4)$-minor, $M/8$ is 3-connected. It is not difficult to check that there are just three 3-connected extensions of P_7 that have no $U_{3,5}$- and hence no $U_{2,5}$-minor; these are depicted in Fig. 3. Note that $M_2\backslash a\cong F_7^-$, and M_3 has no $M(K_4)$-minor. Hence $M/8\cong M_1$. Thus, in M, the element x lies on the intersection of the planes spanned by the circuit-hyperplanes $\{2,4,5,8\}$ and $\{1,3,7,8\}$ of P_8. We may assume that x is not in the plane of M spanned by $\{1,2,3,4\}$,

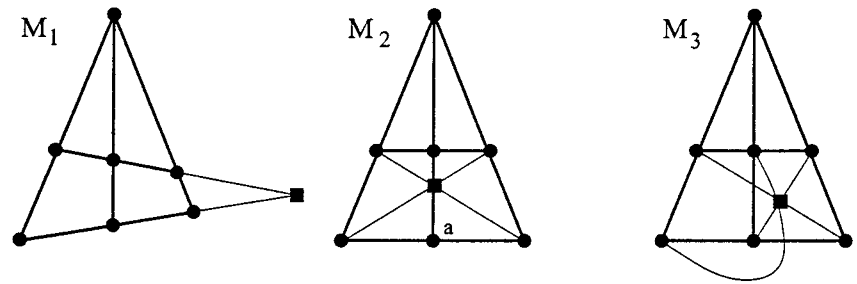

FIG. 3. Extensions of P_7 with no $U_{3,5}$-minor.

otherwise $M\backslash 1, 2 \cong (F_7^-)^*$. To see this, observe that if x is in the plane spanned by $\{1, 2, 3, 4\}$, then the planes $\{1, 2, 3, 4, x\}$ and $\{2, 4, 5, 8, x\}$ of M imply that $\{2, 4, x\}$ is a circuit of M. Similarly, the planes $\{1, 2, 3, 4, x\}$ and $\{1, 3, 7, 8, x\}$ of M imply that $\{1, 3, x\}$ is a circuit of M. We deduce that $\{2, 4, 6, 7, x\}$, $\{1, 3, 5, 6, x\}$, $\{3, 4, 5, 7\}$, and $\{5, 6, 7, 8\}$ are hyperplanes of M, and it is now not difficult to obtain the contradiction that $M\backslash 1, 2 \cong (F_7^-)^*$.

Next we show that either $M/1$ or $M/3$ is 3-connected. Assume the contrary and note that $M/1\backslash x$ is isomorphic to P_7, which is 3-connected. Now, as x lies on the plane spanned by $\{1, 3, 7, 8\}$, since $M/1$ is not 3-connected, x is parallel to 3, 7, or 8 in $M/1$. However, as x is not in the plane spanned by $\{1, 2, 3, 4\}$, the element x is not parallel to 3 in $M/1$. Also x is not parallel to 8 in $M/1$ since $M/8$ is 3-connected. Thus x is parallel to 7 in $M/1$, and hence $\{1, x, 7\}$ is a line in M. By symmetry, as $M/3$ is not 3-connected, $\{3, x, 7\}$ is a line in M. Thus $\{1, 3, 7\}$ is a line in P_8. This contradiction completes the proof that either $M/1$ or $M/3$ is 3-connected. But P_8 has an automorphism that swaps 1 and 2 with 3 and 4, respectively, while fixing all other elements. Therefore, we may assume that $M/1$ is 3-connected.

Now $M/1$ is a 3-connected extension of P_7 with no $U_{3,5}$-minor. Furthermore, the point x of the extension is on a 4-point line with the tip of P_7. Thus, $M/1$ is isomorphic to M_2 of Fig. 3. However, $M_2\backslash a \cong F_7^-$; a contradiction. ∎

To prove Lemma 2.5, we employ methods used in [1]. For a field **F**, two $r \times n$ matrices over **F** whose sequences of column labels coincide are *equivalent* **F**-representations of a matroid if one matrix can be obtained from the other by elementary row operations, column scalings, and applying automorphisms of **F**. A matroid is *uniquely representable* over **F** if any two **F**-representations of it are equivalent. If the $r \times n$ matrix $(I_r \mid D)$ with columns labelled $e_1, e_2, ..., e_n$ represents a matroid M over **F**, it is common to abbreviate this **F**-representation by specifying just the matrix D labelling its rows and columns by $e_1, e_2, ..., e_r$ and $e_{r+1}, e_{r+2}, ..., e_n$, respectively. Such a matrix D will be called a *standard* **F**-representation of M.

A matroid is *stable* if it cannot be expressed as the direct sum or 2-sum of two nonbinary matroids. For our purposes, the most important examples of stable matroids are those matroids which simplify to 3-connected matroids. Kahn [2] proved that a quaternary matroid has a unique GF(4)-representation if and only if it is stable. The following corollary of Kahn's theorem is established in [1].

PROPOSITION 2.6. *Let M be a matroid, and u, v be a coindependent pair of elements of M such that M/u, M/v, and $M/u, v$ are all stable, and $M/u, v$ is connected and nonbinary. If M/u and M/v are both quaternary, then there is a unique quaternary matroid N such that $N/u = M/u$ and $N/v = M/v$.*

Proof of Lemma 2.5. Since M has rank four, $E(M) - x$ contains an element a such that M/a has a $U_{2,5}$-minor. Now M/a is isomorphic to an extension of P_7. We assert that $E(M) - x - a$ contains an element b such that $M/a, b$ has a $U_{2,5}$-restriction. Suppose not. Then $M/a, x$ has a $U_{2,5}$-restriction. Hence there are at least three elements of M/a that are not on a 3- or 4-point line with x. Let b be one of these points, other than the tip of $M/a\backslash x$. Then $M/a, b$ has a $U_{2,5}$-restriction, as asserted.

We may assume that M has no $U_{2,6}$-minor, so $M/a, b$ simplifies to $U_{2,5}$. Hence $M/a, b$ is stable. Now M/a and M/b are both isomorphic to extensions of P_7. Hence M/a and M/b are both stable. We may assume that M/a and M/b are both quaternary. Then, by Proposition 2.6, there is a unique quaternary matroid N such that $N/a = M/a$ and $N/b = M/b$.

Every pair of points of P_8 is equivalent, under automorphism, to either (1, 2), (1, 3) or (1, 8). Now $P_8/1, 3$ is binary, so no extension of $P_8/1, 3$ has a $U_{2,5}$-restriction. Hence we may assume that (a, b) is either (1, 2) or (1, 8).

Case 1. *Suppose that $a = 1$ and $b = 2$.* Note that $M\backslash x/1$, $M\backslash x/2$, and $M\backslash x/1, 2$ are all stable, and that $M\backslash x/1, 2$ is connected and nonbinary. So, by Proposition 2.6, $N\backslash x$ is the unique quaternary matroid such that $N\backslash x/1 = M\backslash x/1$ and $N\backslash x/2 = M\backslash x/2$. Therefore, $N\backslash x$ has the following standard GF(4)-representation (where $w^2 = w + 1$).

$$\begin{array}{c} \\ 1 \\ 2 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & w \\ 1 & 0 & w & 1 \\ 1 & 1 & 1 & 1 \\ w+1 & w+1 & 1 & 1 \end{pmatrix} \end{array}.$$

Recall that $N/1, 2$ has a $U_{2,5}$-minor, so a GF(4)-representation for N can be obtained by appending the column $(\alpha, \beta, 1, w)^T$ to the above matrix, where α and β are yet to be determined.

First suppose that $\alpha = \beta = 0$, and consider the following standard GF(4)-representation of $N \backslash 6$.

$$\begin{array}{c} \\ 1 \\ 2 \\ x \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & w \\ 1 & 0 & w & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & w+1 & w+1 \end{pmatrix} \end{array}.$$

We see that $N \backslash 6/1$, $N \backslash 6/2$, and $N \backslash 6/1, 2$ are all stable, connected, and nonbinary. Furthermore, $N \backslash 6/1 = M \backslash 6/1$, $N \backslash 6/2 = M \backslash 6/2$, and $N \backslash 6/1, 2 = M \backslash 6/1, 2$. So, by Proposition 2.6, $N \backslash 6$ is the unique GF(4)-representable matroid such that $N \backslash 6/1 = M \backslash 6/1$ and $N \backslash 6/2 = M \backslash 6/2$. Now, $\{x, 6, 8\}$ is a triangle of N. It is also a triangle of M otherwise both $\{1, x, 6, 8\}$ and $\{2, x, 6, 8\}$ are circuits of M implying the contradiction that $\{1, 2, 6, 8\}$ is dependent in M. Moreover, $\{5, 6, 7, 8\}$ is a circuit in M but not in N. Hence, $\{5, 7, x, 8\}$ is dependent in $M \backslash 6$ although it is independent in $N \backslash 6$. In particular, $N \backslash 6 \neq M \backslash 6$, so, by uniqueness, $M \backslash 6$ is not GF(4)-representable. Now $M \backslash 4, 6, 7/1 = N \backslash 4, 6, 7/1 \cong U_{3,5}$, so $M \backslash 6$ is not isomorphic to P_8 since the last matroid is ternary. Also $M \backslash 6, x \cong P_7^*$, so $M \backslash 6$ is not isomorphic to P_8''. Therefore, by Theorem 1.3, $M \backslash 6$ has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor, as required.

We may now assume that either $\alpha \neq 0$ or $\beta \neq 0$. Using the automorphism of P_8 that swaps 1, 4, and 5 with 2, 3 and 7, respectively, we may assume that $\alpha \neq 0$. Then, it is easy to check that $N \backslash 4/1$, $N \backslash 4/2$ and $N \backslash 4/1, 2$ are all stable, connected, and nonbinary. Furthermore, $N \backslash 4/1 = M \backslash 4/1$, $N \backslash 4/2 = M \backslash 4/2$, and $N \backslash 4/1, 2 = M \backslash 4/1, 2$. So, by Proposition 2.6, $N \backslash 4$ is the unique GF(4)-representable matroid such that $N \backslash 4/1 = M \backslash 4/1$ and $N \backslash 4/2 = M \backslash 4/2$. Now, $\{5, 6, 7, 8\}$ is a circuit in $M \backslash 4$ but not in $N \backslash 4$. In particular, $N \backslash 4 \neq M \backslash 4$, so, by uniqueness, $M \backslash 4$ is not GF(4)-representable. However, $M \backslash 4, x \cong P_7^*$, so $M \backslash 4$ is not isomorphic to P_8''. Also $M \backslash 4, 5/1, 2 = N \backslash 4, 5/1, 2 \cong U_{2,5}$, so $M \backslash 4$ is not isomorphic to P_8. Therefore, by Theorem 1.3, $M \backslash 4$ has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor, as required.

Case 2. *Suppose that $a = 1$ and $b = 8$.* Note that $M \backslash x/1$, $M \backslash x/8$, and $M \backslash x/1, 8$ are all stable, and that $M \backslash x/1, 8$ is connected and nonbinary. So, by Proposition 2.6, $N \backslash x$ is the unique quaternary matroid such that

$N\backslash x/1 = M\backslash x/1$ and $N\backslash x/8 = M\backslash x/8$. Therefore, $N\backslash x$ has the following standard GF(4)-representation.

$$\begin{array}{c} \\ 1 \\ 2 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & w \\ 1 & 0 & w+1 & 1 \\ 1 & 1 & 1 & 1 \\ w & w & 1 & 1 \end{pmatrix} \end{array}.$$

Recall that $N/1, 8$ has a $U_{2,5}$-minor, so a GF(4)-representation for N can be obtained by appending the column $(\alpha, w, 1, \beta)^T$ to the above matrix, where α and β are yet to be determined.

Now, $N\backslash 5, x/1$ and $N\backslash 5, x/8$ are both 3-connected. So, it is easy to check that $N\backslash 5/1$, $N\backslash 5/8$ and $N\backslash 5/1, 8$ are all stable, connected, and nonbinary. Furthermore, $N\backslash 5/1 = M\backslash 5/1$, $N\backslash 5/8 = M\backslash 5/8$, and $N\backslash 5/1, 8 = M\backslash 5/1, 8$. So, by Proposition 2.6, $N\backslash 5$ is the unique GF(4)-representable matroid such that $N\backslash 5/1 = M\backslash 5/1$ and $N\backslash 5/8 = M\backslash 5/8$. Now, $\{2, 4, 6, 7\}$ is a circuit in $M\backslash 5$ but not in $N\backslash 5$. In particular, $N\backslash 5 \neq M\backslash 5$, so, by uniqueness, $M\backslash 5$ is not GF(4)-representable. However, $M\backslash 5, x \cong P_7^*$, so $M\backslash 5$ is not isomorphic to P_8''. It is left to the reader to check that, for any $\alpha \in GF(4)$, the matroid $N/8\backslash 5$ is not isomorphic to P_7. Hence, $M/8\backslash 5$ is not isomorphic to P_7, so $M\backslash 5$ is not isomorphic to P_8. Therefore, by Theorem 1.3, $M\backslash 5$ has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor, as required. ∎

3. DEALING WITH P_8''

In this section, we prove Theorem 1.4. The techniques are very similar to those used in the previous section.

Since P_8'' is self-dual, it suffices to prove that every 3-connected single-element extension of P_8'' contains a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor. Suppose not. Then there is a 3-connected matroid M in $Ex(U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*)$ such that $M\backslash x = P_8''$.

3.1. *$M/1, 3$ has no $U_{2,5}$-restriction.*

Suppose to the contrary that $M/1, 3$ has a $U_{2,5}$-restriction. It is readily seen that $M/1$, $M/3$, and $M/1, 3$ are all stable, connected, and nonbinary. Then, by Proposition 2.6, there is a unique quaternary matroid N such that $N/1 = M/1$ and $N/3 = M/3$. It is easily checked that $N\backslash x$ is uniquely

Recall that $N/1, 8$ has a $U_{2,5}$-minor, so a GF(4)-representation for N can be obtained by appending the column $(\alpha, 1, w, \beta)^T$ to the above matrix, where α and β are yet to be determined.

Now, $N\backslash 2, x/1$ and $N\backslash 2, x/8$ are both 3-connected. So, it is easy to check that $N\backslash 2/1$, $N\backslash 2/8$ and $N\backslash 2/1, 8$ are all stable, connected, and non-binary. So, by Proposition 2.6, $N\backslash 2$ is the unique GF(4)-representable matroid such that $N\backslash 2/1 = M\backslash 2/1$ and $N\backslash 2/8 = M\backslash 2/8$. Now, $\{3, 4, 5, 7\}$ is a circuit in $M\backslash 2$ but not in $N\backslash 2$. In particular, $N\backslash 2 \neq M\backslash 2$, so, by uniqueness, $M\backslash 2$ is not GF(4)-representable. However, $M\backslash 2, x \cong (P'_7)^*$, so $M\backslash 2$ is not isomorphic to P_8. Therefore, by Theorem 1.3, either $M\backslash 2$ is isomorphic to P''_8, or $M\backslash 2$ has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor. Thus, we may assume that $M\backslash 2$ is isomorphic to P''_8. In particular, $M\backslash 2/1$ is isomorphic to P'_7. It is left to the reader to check that this implies that $\beta = w + 1$. Then $M/1, 3$ has a $U_{2,5}$-restriction, contradicting (3.1). This proves (3.2).

Let R be the matroid depicted in Fig. 4.

3.3. *If $M/1$ is 3-connected, then it is isomorphic to R, where the 4-point line is either $\{3, 7, 8, x\}$ or $\{3, 5, 6, x\}$.*

Suppose $M/1$ is 3-connected. Then it is isomorphic to a 3-connected extension of P'_7. Now 2 and 4 are each on only one 3-point line of $M/1\backslash x$. So, since neither $M/1, 2\backslash x$ nor $M/1, 4\backslash x$ has a $U_{2,6}$-restriction, each of 2 and 4 is on some 3- or 4-point line of $M/1$ with x. By (3.1), 3 is also on some 3- or 4-point line of $M/1$ with x. By (3.2) and symmetry, each of 5, 6, 7, 8 is on some 3- or 4-point line of $M/1$ with x. Hence x is on some 3- or 4-point line with every other element of $M/1$. It follows that x is on one 4-point line and two 3-point lines in $M/1$. There is, up to isomorphism, just one such extension of P'_7, namely R. This proves (3.3).

If x lies on three 3-point lines in M, say $\{x, a_1, a_2\}$, $\{x, b_1, b_2\}$ and $\{x, c_1, c_2\}$, then $\{a_1, a_2, b_1, b_2\}$, $\{a_1, a_2, c_1, c_2\}$, and $\{b_1, b_2, c_1, c_2\}$ are all hyperplanes of P''_8. However, P''_8 has no three such hyperplanes, so x is

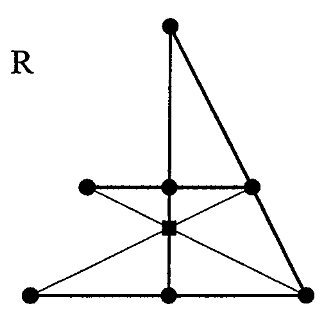

FIG. 4. An extension of P'_7.

Recall that $N/1, 8$ has a $U_{2,5}$-minor, so a GF(4)-representation for N can be obtained by appending the column $(\alpha, 1, w, \beta)^T$ to the above matrix, where α and β are yet to be determined.

Now, $N\backslash 2, x/1$ and $N\backslash 2, x/8$ are both 3-connected. So, it is easy to check that $N\backslash 2/1$, $N\backslash 2/8$ and $N\backslash 2/1, 8$ are all stable, connected, and non-binary. So, by Proposition 2.6, $N\backslash 2$ is the unique GF(4)-representable matroid such that $N\backslash 2/1 = M\backslash 2/1$ and $N\backslash 2/8 = M\backslash 2/8$. Now, $\{3, 4, 5, 7\}$ is a circuit in $M\backslash 2$ but not in $N\backslash 2$. In particular, $N\backslash 2 \neq M\backslash 2$, so, by uniqueness, $M\backslash 2$ is not GF(4)-representable. However, $M\backslash 2, x \cong (P'_7)^*$, so $M\backslash 2$ is not isomorphic to P_8. Therefore, by Theorem 1.3, either $M\backslash 2$ is isomorphic to P''_8, or $M\backslash 2$ has a $U_{2,6}$-, $U_{4,6}$-, P_6-, F_7^--, or $(F_7^-)^*$-minor. Thus, we may assume that $M\backslash 2$ is isomorphic to P''_8. In particular, $M\backslash 2/1$ is isomorphic to P'_7. It is left to the reader to check that this implies that $\beta = w + 1$. Then $M/1, 3$ has a $U_{2,5}$-restriction, contradicting (3.1). This proves (3.2).

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If x lies on three 3-point lines in M, say $\{x, a_1, a_2\}$, $\{x, b_1, b_2\}$ and $\{x, c_1, c_2\}$, then $\{a_1, a_2, b_1, b_2\}$, $\{a_1, a_2, c_1, c_2\}$, and $\{b_1, b_2, c_1, c_2\}$ are all hyperplanes of P''_8. However, P''_8 has no three such hyperplanes, so x is

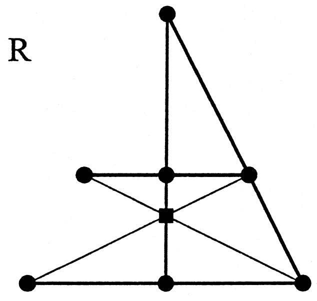

FIG. 4. An extension of P'_7.

on at most two 3-point lines of M. Furthermore, if $\{x, a_1, a_2\}$ and $\{x, b_1, b_2\}$ are 3-point lines, then $\{a_1, a_2, b_1, b_2\}$ is a hyperplane of P''_8. Therefore, since any two points of P''_8 lie on some circuit-hyperplane of P''_8, if M has any 3-point lines, then M is obtained by adding x to some circuit-hyperplane of P''_8, and all 3-point lines are contained in that hyperplane. The automorphisms of P''_8 act transitively on its circuit-hyperplanes, so we may assume that the 3-point lines of M use only points from the set $\{x, 2, 4, 6, 7\}$. Therefore, $M/1$, $M/3$, $M/5$ and $M/8$ are all 3-connected. Therefore, by (3.3) and symmetry, each of these matroids is isomorphic to R. Since $M/1$ is isomorphic to R, either $\{x, 1, 3, 5, 6\}$ or $\{x, 1, 3, 7, 8\}$ is a hyperplane of M. Using the automorphism of P''_8 that swaps 4, 5, and 6 with 2, 8, and 7, respectively, we may assume that $\{x, 1, 3, 5, 6\}$ is a hyperplane of M. Then $M/5$ is not isomorphic to R. This contradiction completes the proof of Theorem 1.4. ∎

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