THE STRUCTURE OF A 3–CONNECTED MATROID WITH A 3-SEPARATING SET OF ESSENTIAL ELEMENTS

JAMES OXLEY

Abstract. An element e of a 3–connected matroid M is *essential* if neither the deletion nor the contraction of e from M is 3–connected. Tutte's 1966 Wheels and Whirls Theorem proves that the only 3–connected matroids in which every element is essential are the wheels and whirls. It was proved by Oxley and Wu that if a 3–connected matroid M has a non-essential element, then it has at least two such elements. Moreover, the set of essential elements of M can be partitioned into classes where two elements are in the same class if M has a fan, a maximal partial wheel, containing both. In addition, if M has a fan with $2k$ or $2k+1$ elements for some $k \geq 2$, then M can be obtained by sticking together a $(k+1)$–spoked wheel and a certain 3–connected minor of M. In this paper, it is shown how a slight modification of these ideas can be used to describe the structure of a 3–connected matroid M having a 3–separation (A, B) such that every element of A is essential. The motivation for this study derives from a desire to determine when one can remove an element from M so as to both maintain 3–connectedness and preserve one side of the 3–separation.

1. Introduction

The terminology used here will follow Oxley [6] with two exceptions: the simplification and cosimplification of a matroid N will be denoted by $\mathrm{si}(N)$ and $\mathrm{co}(N)$, respectively. The property that a circuit and a cocircuit cannot have exactly one common element will be referred to as *orthogonality*. If X is a set in a matroid M and k is a positive integer, then X is *k-separating* [5] if $r(X)+r(E(M)-X)-r(M) \leq k-1$. Thus $(X, E(M)-X)$ is a k–separation of M if X is k–separating and $|X|, |E(M)-X| \geq k$. A basic structure in the study of 3–connected matroids is a chain of triangles and triads. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid N such that, for all i in $\{1, 2, \ldots, k-1\}$,

(i) $|T_i \cap T_{i+1}| = 2$;
(ii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty; and
(iii) in $\{T_i, T_{i+1}\}$, exactly one set is a triangle and exactly one set is a triad.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of N of *length* k with *links* $T_1, T_2, \ldots, T_k$. When this occurs, it is straightforward to show that N has $k+2$ distinct elements $x_1, x_2, \ldots, x_{k+2}$ such that $T_i = \{x_i, x_{i+1}, x_{i+2}\}$ for all i in $\{1, 2, \ldots, k\}$. When $k \geq 2$, the elements x_1 and x_{k+2} are the only elements of the chain that are in exactly one link. We call them the *ends* of the chain and call $x_2, x_3, \ldots, x_{k+1}$ the *internal elements* of the chain.

Date: February 14, 2002.
1991 *Mathematics Subject Classification.* 05B35.

In a 3–connected matroid other than a wheel or whirl, a maximal chain is called a *fan* [8]. If A is the set of internal elements of such a fan and $|A| \geq 3$, then $(A, E(M) - A)$ is a 3–separation of M in which every element of A is essential. We shall show here that if we begin with a 3–separation (A, B) in which every element of A is essential, then the structure of A relative to M can be described using a modified notion of a fan.

Let (A, B) be a partition of the ground set of a 3–connected matroid M such that $|A|, |B| \geq 2$. An *A-fan* is a chain $T_1, T_2, \ldots, T_k$ of triangles and triads of M of length at least two such that all internal elements of the chain are in A and both ends of the chain are in B. There are three types of A-fans: *type*-1 when both T_1 and T_k are triangles; *type*-2 when both T_1 and T_k are triads; and *type*-3 when one of T_1 and T_k is a triangle and the other is a triad. Evidently $\mathcal{F}$ is a type-1 A-fan of M if and only if it is a type-2 A-fan of M^*. We remark that, whereas the ends of a fan are non-essential elements, the ends of an A-fan are elements of B.

The following is the main result of the paper. It is proved in Section 3.

1.1. **Theorem.** *Let (A, B) be a 3–separation of a 3–connected matroid M in which every element of A is essential. Then there is a partition $\{A_1, A_2, \ldots, A_k\}$ of A such that a subset A' of A is the set of internal elements of an A-fan if and only if $A' \in \{A_1, A_2, \ldots, A_k\}$. Moreover, every A-fan has the same type t. If $t = 1$ or $t = 2$, then the ends of all A-fans are collinear in M or in M^*, respectively; if $t = 3$, then all A-fans have the same ends.*

This theorem and duality imply that we have two cases to consider: when all A-fans have type-1, and when all A-fans have type-3. Sections 4 and 5 give constructive descriptions, corresponding to these two cases, of all the 3–connected matroids having a 3–separation (A, B) in which every element is essential. These constructions will use a special case of the operation of generalized parallel connection introduced by Brylawski [3]. Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = \Delta$ where Δ is a triangle of both M_1 and M_2. Assume that M_1 is binary. The *generalized parallel connection* $P_\Delta(M_1, M_2)$ of M_1 and M_2 across Δ is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2.

This paper will rely heavily on [8]. Since the techniques used here are similar to those used in that paper and [9], a number of arguments from this paper will be abbreviated or omitted.

2. Preliminaries

This section presents three lemmas that will be used in the proof of the main result. The first two are known as Bixby's Lemma [2] and Tutte's Triangle Lemma [10], respectively.

2.1. **Lemma.** *Let e be an element of a 3–connected matroid M. Then either $\mathrm{co}(M \backslash e)$ or $\mathrm{si}(M/e)$ is 3–connected.*

2.2. **Lemma.** *Let $\{e, f, g\}$ be a triangle of a 3–connected matroid M. If neither $M \backslash e$ nor $M \backslash f$ is 3–connected, then M has a triad containing e and exactly one of f and g.*

2.3. **Lemma.** *Let (A, B) be a 3–separation of a 3–connected matroid M. Suppose that $|\mathrm{cl}(A) \cap B| \geq j$ for some j in $\{2, 3\}$. Then $M|B$ is j-connected.*

Proof. Let (X, Y) be a k–separation of $M|B$ for some k with $1 \leq k \leq j-1$. Then

$$r(X) + r(Y) - r(B) \leq k - 1 \leq j - 2. \tag{1}$$

Without loss of generality, we may assume that $|X \cap \mathrm{cl}(A)| \geq \lceil \frac{j}{2} \rceil$. Thus, as j is in $\{2, 3\}$, it follows that $r(X \cap \mathrm{cl}(A)) \geq \lceil \frac{j}{2} \rceil$. Let $t = r(X \cap \mathrm{cl}(A))$. Then

$$\begin{aligned} r(X \cup A) &= r(\mathrm{cl}(X) \cup \mathrm{cl}(A)) \\ &\leq r(\mathrm{cl}(X)) + r(\mathrm{cl}(A)) - r(\mathrm{cl}(X) \cap \mathrm{cl}(A)) \\ &= r(X) + r(A) - r(\mathrm{cl}(X) \cap \mathrm{cl}(A)). \end{aligned}$$

Hence

$$r(X \cup A) \leq r(X) + r(A) - t. \tag{2}$$

Thus

$$\begin{aligned} r(X \cup A) + r(Y) - r(M) &\leq r(X) + r(A) - t + r(Y) - r(M) && \text{by (2),} \\ &= r(A) + (r(X) + r(Y)) - r(M) - t \\ &\leq r(A) + r(B) + j - 2 - r(M) - t && \text{by (1),} \\ &= (r(A) + r(B) - r(M)) - 2 + j - t. \end{aligned}$$

Hence

$$r(X \cup A) + r(Y) - r(M) \leq j - t \leq j - \lceil \frac{j}{2} \rceil = \lfloor \frac{j}{2} \rfloor = 1. \tag{3}$$

As M is certainly connected, equality holds throughout (3), so $t = \lceil \frac{j}{2} \rceil$. Since M is 3–connected, we deduce that

$$|Y| = 1. \tag{4}$$

Now suppose that $j = 2$. Then, as $|\mathrm{cl}(A) \cap B| \geq 2$ and $r(X \cap \mathrm{cl}(A)) = 1$, we deduce that $r(Y \cap \mathrm{cl}(A)) \geq 1$. Thus we may interchange X and Y in the argument that produced (3), noting that t becomes equal to $r(Y \cap \mathrm{cl}(A))$. It follows, since M is 3–connected, that $|X| = 1$. Hence $|B| = |X| + |Y| = 2$; a contradiction. We conclude that the lemma holds for $j = 2$. In particular, $M|B$ is 2-connected. Now let $j = 3$. Then, since (X, Y) is a k–separation of $M|B$ for some k with $1 \leq k \leq j - 1$, it follows that $k = 2$. Thus $|Y| \geq 2$; a contradiction to (4). Hence the lemma also holds when $j = 3$. □

3. Proof of Theorem 1.1

Throughout this section, we shall assume that (A, B) is a 3–separation of a 3–connected matroid M in which every element of A is essential. When this occurs, a result of Tutte [10, 7.1] asserts that every element of A is in a triangle or a triad. We extend this to show that every element of A is in an A-fan. To prove this, we shall use the following result.

3.1. **Lemma.** *If $a \in A$, then a is in a triangle or a triad of M that contains at least two elements of A.*

Proof. By Bixby's Lemma (2.1) and duality, we may assume that $\mathrm{si}(M/a)$ is 3–connected. But M/a is not 3–connected. Thus a is in a triangle T of M. We may assume that $T - a \subseteq B$ otherwise the lemma holds. Thus $a \in \mathrm{cl}(B)$. Now let

$$r_{M/a}(A - a) + r_{M/a}(B) - r(M/a) = t.$$

Then

$$r(A) - 1 + r(B \cup a) - 1 - r(M) + 1 = t.$$

But $r(B \cup a) = r(B)$ so $r(A) + r(B) - r(M) = t + 1$. Hence $t = 1$. Since M/a is vertically 3–connected, we deduce that $r_{M/a}(A - a) \leq 1$ or $r_{M/a}(B) \leq 1$. Thus $r(A) = 2$ or $r(B) = 2$. In the first case, since $|A| \geq 3$, we deduce that a is contained in a triangle of M contained in A, so the lemma holds. In the second case, $r(A) = r(M)$ and B is contained in a line of M that contains a. Since this line contains $B \cup a$, it has at least four elements. It follows that $M \backslash a$ is 3–connected; a contradiction. □

3.2. **Corollary.** *Every element of A is in an A-fan.*

Proof. Let $a \in A$. By Lemma 3.1, a is in a triangle or a triad that contains at least two elements of A.

Assume first that a is in a 3-set T that is contained in A and is a triangle or a triad. Take a maximal chain $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\}$ of triangles and triads using T such that every element is in A. By using duality, we may assume that $\{a_1, a_2, a_3\}$ is a triangle. By Tutte's Triangle Lemma (2.2), a_1 is in a triad T^* that contains exactly one of a_2 and a_3. Suppose that $T^* = \{a_1, a_2, x\}$. Then $x \neq a_3$ as T^* is not both a triangle and a triad since $|E(M)| \geq 6$. Moreover, $x \neq a_4$ otherwise the triads $\{a_1, a_2, a_4\}$ and $\{a_2, a_3, a_4\}$ imply that the triangle $\{a_1, a_2, a_3\}$ is also a triad. Orthogonality implies that $x \notin \{a_5, a_6, \ldots, a_n\}$. Hence $x \notin \{a_1, a_2, \ldots, a_n\}$, so adjoining T^* to the beginning of the original chain gives a longer chain. Therefore if T^* contains $\{a_1, a_2\}$, then its third element is in B.

Now suppose that T^* contains $\{a_1, a_3\}$. Then either the original chain has just two links, or that chain has at least five elements. In the former case, we may interchange a_2 and a_3 and reduce to the situation where T^* contains $\{a_1, a_2\}$. In the latter case, by [8, Lemma 3.4], we have a contradiction. We conclude that if a is in a triangle or a triad contained in A, then we can obtain an A-fan containing a by adjoining links to the beginning and end of the original chain. Moreover, this A-fan has at least five elements.

We may now assume, using duality, that a is in neither a triangle nor a triad contained in A. Using duality again, we may assume that M has a triangle $\{b, a, a'\}$ where $b \in B$ and $a' \in A$. By Tutte's Triangle Lemma, a is contained in a triad T^* of M that contains exactly one of a' and b. If T^* contains a', then it must equal $\{a, a', b'\}$ for some b' in B. In this case, $\{b, a, a'\}, \{a, a', b'\}$ is a type-2 A-fan containing a and having four elements, and the lemma holds. Thus we may assume that T^* contains b, say $T^* = \{b, a, x\}$. A similar argument establishes that $\{b, a', y\}$ is a triad of M for some element y. If x or y is in A, then b is a coloop of $M|B$ that is in the closure of A, hence $(A \cup b, B - b)$ is a 2-separation of M; a contradiction. Thus $\{x, y\} \subseteq B$, so both a and a' are coloops of $M|A$. Therefore

$$r(A - \{a, a'\}) + r(B \cup \{a, a'\}) - r(M) \leq r(A) - 2 + r(B) + 1 - r(M) \leq 1.$$

Hence $|A - \{a, a'\}| = 1$, so $|A| = 3$ and A is a triangle or a triad containing a. This contradiction completes the proof of the corollary. □

The combination of the next two lemmas and duality proves Theorem 1.1.

3.3. **Lemma.** *Suppose that M has a type-3 A-fan with ends b and b' in B. Then*

(i) *every A-fan has type-3 and has ends b and b'; and*
(ii) *every element of A is in a unique A-fan.*

Proof. Let $\{b, a_0, a_1\}, \{a_0, a_1, a_2\}, \{a_1, a_2, a_3\}, \dots, \{a_{2m-2}, a_{2m-1}, b'\}$ be a type-3 A-fan $\mathcal{F}$ of M and assume, without loss of generality, that $\{b, a_0, a_1\}$ is a triad. Then $\{a_{2m-2}, a_{2m-1}, b'\}$ is a triangle, $b' \in B \cap \mathrm{cl}(A)$ and b is a coloop of $M|B$. Hence, by Lemma 2.3, b' is the unique element of $B \cap \mathrm{cl}(A)$. By duality, it follows that b is the unique element of $B \cap \mathrm{cl}^*(A)$.

An A-fan of type-1 has distinct ends both of which are in B. Thus the existence of such a fan implies that $|B \cap \mathrm{cl}(A)| \geq 2$; a contradiction. We conclude that M has no type-1 A-fans and, by duality, M has no type-2 A-fans. Thus every A-fan has type-3. Moreover, the ends of every such fan must be b and b' otherwise $|B \cap \mathrm{cl}(A)| \geq 2$ or $|B \cap \mathrm{cl}^*(A)| \geq 2$. Hence (i) holds.

To prove (ii), first note that, since one easily checks that $M \not\cong M(K_4)$, it follows, by [8, Lemma 3.4], that none of $a_1, a_2, \dots, a_{2m-2}$ is in any triangles or triads other than those in $\mathcal{F}$.

Suppose that a_0 is in a triad T^* that contains at most one element of B and is different from $\{b, a_0, a_1\}$. From the first paragraph, $\mathrm{cl}(A) \cap B = \{b'\}$ and $\mathrm{cl}^*(A) \cap B = \{b\}$. By orthogonality, T^* must contain an element of the triangle of $\mathcal{F}$ containing $\{a_0, a_1\}$. Thus if $m > 1$, then T^* contains a_1 or a_2; a contradiction. Hence $m = 1$ and either

(a) T^* contains a_1; or
(b) T^* contains b'.

Consider case (a), letting $T^* = \{a_0, a_1, z\}$. Since $\mathrm{cl}^*(A) \cap B = \{b\}$, we deduce that $z \in A$. By the dual of Tutte's Triangle Lemma, z is in a triangle with exactly one of a_0 and a_1. The triad $\{b, a_0, a_1\}$ implies that $\{z, a_0, b\}$ or $\{z, a_1, b\}$ is a triangle. In each case, we deduce that $b \in \mathrm{cl}(A) \cap B$; a contradiction. In case (b), $T^* = \{a_0, b', z\}$, where $z \in A$ since T^* contains at most one element of B. Thus $b' \in \mathrm{cl}^*(A) \cap B = \{b\}$; a contradiction.

Now assume that a_0 is in a triangle T that contains at most one element of B and is different from the triangle T_1 of $\mathcal{F}$ that contains $\{a_0, a_1\}$. By orthogonality, T contains a_1 or b. If T contains a_1, then $T \cup T_1$ is a 4-element set of which every 3-element subset is a triangle. Thus $(T \cup T_1) - a_0$ is a triangle meeting $\{b, a_0, a_1\}$ in a single element. This contradiction implies that $a_1 \notin T$. Thus $b \in T$. But this implies that $b \in \mathrm{cl}(A) \cap B = \{b'\}$; a contradiction. We conclude that T_1 is the unique triangle that contains a_0 and at most one element of B. Thus $\mathcal{F}$ is the unique A-fan containing a_0. By symmetry, $\mathcal{F}$ is the unique A-fan containing a_{2m-1}. But we noted earlier that none of $a_1, a_2, \dots, a_{2m-2}$ is in a triangle or triad other than those in $\mathcal{F}$. We conclude that (ii) holds. □

3.4. **Lemma.** *Suppose that M has a type-1 A-fan with ends b and b' in B. Then*

(i) *every A-fan has type-1, has at least five elements, and has its ends on the line of M spanned by b and b'; and*
(ii) *if two distinct A-fans contain a common element of A, then they contain exactly the same elements of A, and the restriction of M to the union of these two A-fans is isomorphic to $M(K_4)$.*

Proof. Evidently $\{b, b'\} \subseteq \mathrm{cl}(A)$. Thus, by Lemma 2.3, $M|B$ is connected. If M has an A-fan that is not of type-1, then $M|B$ has a coloop; a contradiction. Thus every A-fan of M has type-1. Moreover, $r(\mathrm{cl}(A) \cap \mathrm{cl}(B)) \leq 2$. Since $\{b, b'\} \subseteq \mathrm{cl}(A) \cap B$, it follows that $\{b, b'\}$ spans $\mathrm{cl}(A) \cap \mathrm{cl}(B)$. Thus the ends of every A-fan, which must be in $\mathrm{cl}(A) \cap \mathrm{cl}(B)$, lie on the line of M spanned by b and b'.

Now let $a \in A$. Then, by Corollary 3.2, a is contained in a type-1 A-fan $\mathcal{F}$ of M having at least five elements. Let $\mathcal{F}$ be $\{b, a_0, d_1\}, \{a_0, d_1, a_1\}, \ldots, \{d_n, a_n, b'\}$. If a is in a type-1 A-fan having exactly three elements, then a is in a triangle meeting A in $\{a\}$. But, from considering $\mathcal{F}$, we know that a is in a triad contained in A. This contradiction to orthogonality implies that every type-1 A-fan has at least five elements.

By [8, Lemma 3.4], either

(a) $M \cong M(K_4)$; or
(b) $M \ncong M(K_4)$ and $d_1, a_1, d_2, a_2, \ldots, a_{n-1}, d_n$ are in no triangles or triads other than those in the designated A-fan $\mathcal{F}$.

In case (a), A is a triad of M and B, which equals $E(M) - A$, is a triangle. Moreover, for each 2-element subset B' of B, there is a unique A-fan whose ends are in B'. We may now assume that (b) holds. Then $\mathcal{F}$ is the unique A-fan meeting $\{d_1, a_1, d_2, a_2, \ldots, a_{2n-1}, d_n\}$. By symmetry, to complete the proof it suffices to consider the triangles and triads containing a_0.

Assume that M has a triangle T that contains a_0 and is different from $\{b, a_0, d_1\}$. Then, by orthogonality, since T does not contain d_1, it follows that $n = 1$, and T contains a_1. Thus $T = \{a_0, a_1, z\}$ for some element z of M that is not in $\mathcal{F}$. Then $\{b, b', z\}$ is contained in $\mathrm{cl}(\{a_0, a_1, d_1\}) \cap [E(M) - \{a_0, a_1, d_1\}]$. Thus, by submodularity, $\{b, b', z\}$ is a triangle of M.

Suppose that $M\backslash z$ is not 3–connected. Then, by applying Tutte's Triangle Lemma to $\{a_0, a_1, z\}$, we deduce that M has a triad containing z and exactly one of a_0 and a_1. By orthogonality, this triad must be $\{z, a_0, b\}$ or $\{z, a_1, b'\}$. It follows, by [8, Lemma 2.4], that M is isomorphic to a wheel or whirl of rank three. Since M has four triangles, we deduce that $M \cong M(K_4)$; a contradiction.

We may now assume that $M\backslash z$ is 3–connected. Thus $z \in B$. By orthogonality, the only triad of M containing $\{a_0, a_1\}$ is $\{a_0, a_1, d_1\}$, and the only triangles containing $\{a_1, d_1\}$ and $\{a_0, d_1\}$ are $\{a_1, d_1, b'\}$ and $\{a_0, d_1, b\}$, respectively. Thus a_0 is in two A-fans other than $\mathcal{F}$. Each of these A-fans contains $\{a_0, a_1, d_1\}$, and one contains $\{b', z\}$ while the other contains $\{b, z\}$. It follows easily that the restriction of M to the union of two distinct A-fans containing a_0 is isomorphic to $M(K_4)$.

Finally, suppose that a_0 is in a triad other than $\{a_0, d_1, a_1\}$. This triad cannot contain d_1. Thus, by orthogonality, it contains b. By Lemma 2.3, since $M|B$ is connected, this triad has its third element in B. Thus $\{a_0, d_1, a_1\}$ is the only triad that contains a_0 and at most one element of B. The lemma follows. □

By duality and the last two lemmas, we have two cases to consider: when all A-fans have type-1, and when all A-fans have type-3. The first is somewhat easier to handle. In each case, we shall present results that describe how to construct all matroids with the specified property. These two cases will be considered in the next two sections. Note that Theorem 1.1 follows immediately by combining the last two lemmas.

4. M HAS A TYPE-1 A-FAN

In this section, we shall prove the following constructive description of all the matroids with a type-1 A-fan.

4.1. **Theorem.** *A matroid M is 3–connected, has a 3–separation (A, B) in which every element of A is essential, and has a type-1 A-fan if and only if M can be constructed as follows.*

(i) *Let N_0 be a 3–connected matroid with a distinguished line L having at least three points.*
(ii) *For some positive integer k, let $N_1, N_2, \ldots, N_k$ be a collection of wheels of rank at least three such that $E(N_0), E(N_1), E(N_2), \ldots, E(N_k)$ are disjoint.*
(iii) *Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be a collection of triangles contained in L and let $\Delta'_1, \Delta'_2, \ldots, \Delta'_k$ be triangles in $N_1, N_2, \ldots, N_k$, respectively.*
(iv) *For each i in $\{1, 2, \ldots, k\}$, take a bijection from Δ'_i to Δ_i and relabel each element of Δ'_i by the corresponding element of Δ_i.*
(v) *Let $H_0 = N_0$ and, for all i in $\{1, 2, \ldots, k\}$, let $H_i = P_{\Delta_i}(N_i, H_{i-1})$.*
(vi) *Let Z be a subset of L such that, for all i in $\{1, 2, \ldots, k\}$,*
   (a) $|Z \cap \Delta_i| \leq 1$;
   (b) *if N_i has rank at least four, then Z does not contain a spoke of N_i; and*
   (c) *if $N_0 \cong U_{2,3}$, then $Z = \emptyset$.*
(vii) *Let $M = H_k \backslash Z$, let $A = \cup_{i=1}^{k}(E(N_i) - \Delta_i)$, and let $B = E(M) - A$.*

Figure 1.

Proof. Suppose that M is a 3–connected matroid having a 3–separation (A, B) such that every element of A is essential and M has a type-1 A-fan $\mathcal{F}$ with ends b and b'. By Lemma 3.4, every A-fan of M has type-1 and has its ends on the line spanned by $\{b, b'\}$. Let $\mathcal{F}$ be such an A-fan, $\{b_0, a_0, d_1\}, \{a_0, d_1, a_1\}, \{d_1, a_1, d_2\}, \ldots, \{d_n, a_n, b'\}$. Then, by [8, Theorem 1.11], $M = P_\Delta(M(\mathcal{W}_{n+2}), M_1)\backslash z$ where $\Delta = \{b, b', z\}$; the wheel $\mathcal{W}_{n+2}$ is labelled as in Figure 1; and M_1 is obtained from $M/a_0, a_1, \ldots, a_{n-1}\backslash d_1, d_2, \ldots, d_n$ by relabelling a_n as z. Moreover, either M_1 is 3–connected; or z is in a unique 2–circuit $\{z, h\}$ of M_1, and $M_1\backslash z$ is 3–connected.

To prove that M can be constructed as described, we argue by induction on the number of *internally disjoint A-fans*, that is, the number of pairwise disjoint subsets A' of A for which A' is the set of internal elements of an A-fan. This number would be equal to the number of A-fans except for the possibility that some three-element subsets of A can be the internal elements of three different A-fans. Suppose that $A = \{a_0, d_1, a_1, d_2, \ldots, d_n, a_n\}$. If z is in a 2–circuit of M_1, then take N_0 to be $M_1 \backslash z$ and $Z = \emptyset$. If z is not in a 2-circuit of M_1, then take N_0 to be M_1 and $Z = \{z\}$. In each case, take N_1 to be $M(\mathcal{W}_{n+2})$. Then M can certainly be constructed as in (i)–(vii).

Now assume that M can be constructed as in (i)–(vii) when M has fewer than k internally disjoint A-fans and suppose that M has exactly k internally disjoint A-fans one of which is $\mathcal{F}$. Let $A_1 = A - \{a_0, a_1, \ldots, a_n, d_1, d_2, \ldots, d_n\}$. Then it is not difficult to show that $M \backslash (A - A_1)$, which equals $M_1 \backslash z$, is 3–connected and has (A_1, B) as a 3–separation. Moreover, a straightforward argument, whose details we omit, shows that the A_1-fans of $M \backslash (A - A_1)$ are exactly the A-fans of M other than $\mathcal{F}$. By the induction assumption, $M \backslash (A - A_1)$ can be constructed as in (i)–(vii). If z is in a 2–circuit $\{z, h\}$ of M_1, then, by [8, Theorem 1.8], $M = P_{\Delta'}(M(\mathcal{W}_{n+2}), M \backslash (A - A_1))$ where $\Delta' = \{b, b', h\}$, and $\mathcal{W}_{n+2}$ is labelled as in Figure 1 with z relabelled as h. It follows that, in this case, M itself can be constructed as in (i)–(vii).

We may now suppose that z is not in a 2–circuit of M_1. Then M_1, which is 3–connected, has $\{b, b', z\}$ as a circuit and so has $(A_1, B \cup z)$ as a 3–separation. Moreover, the A_1-fans of M_1 are exactly the A-fans of M other than $\mathcal{F}$. Thus, by the induction assumption, M_1 can be constructed as in (i)–(vii). Hence so too can $P_\Delta(M(\mathcal{W}_{n+2}), M_1)$ where $N_k = M(\mathcal{W}_{n+2})$ and $\Delta_k = \Delta$. To obtain M from $P_\Delta(M(\mathcal{W}_{n+2}), M_1)$, we delete z. If $Z(M_1)$ is the set of elements that are deleted in (vi) of the construction of M_1, then to prove that M can also be constructed in the specified manner, it suffices to show that (vi) holds for the set $Z(M)$ obtained by adjoining z to $Z(M_1)$. Clearly $|Z(M) \cap \Delta_k| = 1$, and, if $n \geq 2$, then $Z(M)$ does not contain a spoke of N_k. If z is a spoke of some N_i for $i < k$ such that $r(N_i) \geq 4$, then the rim element of N_i that is in A and is adjacent to this spoke can be contracted from M to produce a 3–connected matroid; a contradiction. If $r(N_i) = 3$ and $|Z(M) \cap \Delta_i| = 2$ for some $i < k$, then one of the elements of $A \cap E(N_i)$ can be contracted from M to produce a 3–connected matroid; a contradiction. We conclude that $Z(M)$ satisfies (vi), and it follows that M can be constructed as in (i)–(vii).

It is straightforward to check that every matroid M that can be constructed as in (i)–(vii) has (A, B) as a 3–separation, has every element in A essential, and has a type-1 A-fan. □

A *contraction-minimally* 3*–connected matroid* is a 3–connected matroid of which no single-element contraction is 3–connected. The structure of such matroids in which the set of non-essential elements has rank two was described in [9]. The next result, whose straightforward proof is omitted, relates such matroids to those 3–connected matroids having a type-1 A-fan.

4.2. **Corollary.** *Let M be a 3–connected matroid having a 3–separation (A, B) in which every element of A is essential. Suppose that M has a type-1 A-fan and let M' be obtained from M by freely adding an element x on the line of M spanned by $\mathrm{cl}(A) \cap B$. Then $M'|(B \cup x)$ is 3–connected and $M'|(\mathrm{cl}(A) \cup x)$ is either a whirl or a contraction-minimally 3–connected matroid in which the set of non-essential elements has rank two.*

5. M HAS A TYPE-3 A-FAN

The constructive description of 3–connected matroids having type-3 A-fans will be presented in two cases: when there is a unique such fan; and when there is more than one such fan. In the first case, the description is somewhat more explicit and is closer to the situation when we have a type-1 A-fan. The proof of this result is not difficult to derive from [8, Corollaries 2.7 and 2.8, Theorem 4.7] and is omitted.

5.1. **Theorem.** *Let N be a 3–connected matroid having a triangle $\Delta = \{x, y, z\}$. Construct $P_\Delta(M(\mathcal{W}_n), N)$ for some $n \geq 4$, letting $\{y, z, w\}$ be a triad of $M(\mathcal{W}_n)$.*

(i) *Let $M_1 = P_\Delta(M(\mathcal{W}_n), N)$, let $A = E(M(\mathcal{W}_n)) - \{x, y, z, w\}$, and let $B = E(N) \cup w$.*
(ii) *Assuming $|E(N)| \geq 4$, let $u \in \{y, z\}$, let $M_2 = P_\Delta(M(\mathcal{W}_n), N)\backslash u$, let $A = E(M(\mathcal{W}_n)) - \{x, y, z, w\}$, and let $B = (E(N) - u) \cup w$.*
(iii) *Assuming $|E(N)| \geq 4$ and N has no triad containing $\{y, z\}$, let $M_3 = P_\Delta(M(\mathcal{W}_n), N)\backslash\{y, z\}$, let $A = E(M(\mathcal{W}_n)) - \{x, y, z, w\}$, and let $B = (E(N) - \{y, z\}) \cup w$.*

Then, for each i in $\{1, 2, 3\}$, the matroid M_i is 3–connected having (A, B) as a 3–separation and having a unique A–fan. Moreover, this A-fan has type-3 and contains all the elements of A.

Furthermore, if M is a 3–connected matroid with a 3–separation (A, B) in which every element of A is essential, and M has exactly one A-fan, which is of type-3, then there is a 3–connected matroid N and a wheel $M(\mathcal{W}_n)$ such that M is one of the matroids M_1, M_2, and M_3 constructed above.

The next theorem describes constructively how to obtain every 3–connected matroid M that has a 3–separation (A, B) in which every element of A is essential and has at least two type-3 A-fans. The construction we describe will be less explicit than in the last theorem in that we shall add elements on lines in the matroid where the positions of these elements need not be fixed.

5.2. **Theorem.** *A matroid M is 3–connected, has a 3–separation (A, B) in which every element of A is essential, and has at least two type-3 A-fans if and only if M can be constructed as follows.*

(i) *Let N be a 3–connected matroid having a triangle $\{b_0, x, y\}$.*
(ii) *Add b_1 in parallel with x.*
(iii) *For some $k \geq 2$, add $a_1, a_2, \ldots, a_k$ in series with b_1.*
(iv) *For each i in $\{1, 2, \ldots, k\}$, add a'_i so that $\{b_0, a_i, a'_i\}$ is a triangle. Let the resulting matroid be N_0.*
(v) *For each i in $\{1, 2, \ldots, k\}$, let N_i be a wheel of rank at least two having both b_0 and a_i as spokes and having $\{b_0, a_i, a'_i\}$ as a triangle Δ_i.*
(vi) *Let $H_0 = N_0$ and, for all i in $\{1, 2, \ldots, k\}$, let $H_i = P_{\Delta_i}(N_i, H_{i-1})\backslash a'_i$ and let $A_i = E(N_i) - \{b_0, a'_i\}$.*
(vii) *Let Z be a subset of $\{x, y\}$ such that*
  (a) *if $|E(N)| = 3$, then $Z = \emptyset$; and*
  (b) *if $\{x, y\}$ is in a triad of N, then $Z \neq \{x, y\}$.*
(viii) *Let $M = H_k\backslash Z$, let $A = \cup_{i=1}^k A_i$, and let $B = E(M) - A$.*

In order to simplify the proofs when M has a type-3 A-fan, we first shrink one such A-fan so that it contains exactly two elements of A. More precisely, we have the following result, which follows without difficulty from [8, Theorem 1.8].

5.3. **Lemma.** *Let* $\{b, a_0, a_1\}, \{a_0, a_1, a_2\}, \{a_1, a_2, a_3\}, \ldots, \{a_{2n-2}, a_{2n-1}, b'\}$ *be a type-*3 *A-fan* $\mathcal{F}$ *of* M *for some* $n \geq 2$ *where* $\{b, b'\} \subseteq B$ *and* $\{b, a_0, a_1\}$ *is a triad. Then*

(i) $M = P_\Delta(M(\mathcal{W}_{n+1}), M_1)\backslash a'_1$ *where* $\Delta = \{a_0, a'_1, b'\}$ *and* M_1 *is obtained from* $M/a_3, a_5, \ldots, a_{2n-1}\backslash a_2, a_4, \ldots, a_{2n-2}$ *by relabelling* a_1 *as* a'_1.
(ii) M_1 *is* 3*–connected.*
(iii) *If* $A_1 = [A - \{a_1, a_2, \ldots, a_{2n-1}\}] \cup \{a'_1\}$*, then, either* $|A_1| = 2$*, or* (A_1, B) *is a* 3*–separation of* M_1 *in which each element of* A_1 *is essential.*
(iv) *If* $\mathcal{F}'$ *is an A-fan different from* $\mathcal{F}$*, then* $\mathcal{F}'$ *is an* A_1*-fan of* M_1*; and* $\{b, a_0, a'_1\}$,
$\{a_0, a'_1, b'\}$ *is an* A_1*-fan of* M_1.

The next result indicates how to remove an A-fan containing exactly two elements of A while maintaining all remaining A-fans.

5.4. **Theorem.** *Let* N *be a* 3*–connected matroid having a* 3*–separation* (A, B) *such that every element of* A *is essential. Let* $\{b, a, a'\}, \{a, a', b'\}$ *be an A-fan* $\mathcal{F}$ *of* N *where* $\{b, b'\} \subseteq B$ *and* $\{b, a, a'\}$ *is a triad. Then* $N\backslash a/a'$ *is* 3*–connected. Moreover,* $A - \{a, a'\}$ *is a* 3*–separating set of* $N\backslash a/a'$ *in which every element is essential, and every A-fan of* N *other than* $\mathcal{F}$ *is an* $(A - \{a, a'\})$*-fan of* $N\backslash a/a'$.

The proof of this theorem will use the next lemma. Since $|A| \geq 3$, it follows that there are at least two A-fans. Hence, by Lemma 3.3, $|A| \geq 4$. Using this, it is straightforward to complete the proof of the lemma and the details are omitted.

5.5. **Lemma.** *N has no triangle containing b.*

Proof of Theorem 5.4. We shall prove the theorem in the case that all A-fans contain exactly two elements of A. The general case will follow from this by using Lemma 5.3. We begin by proving that $N\backslash a/a'$ is 3–connected. Assume the contrary. Now $N\backslash a/a' \cong N\backslash a/b$. Since b is not in a triangle of N, it follows by [1] that N/b is 3–connected. Let (J, K) be a 2–separation of $N\backslash a/b$. Suppose that $|J| = 2$. Then $J \cup a$ is a triad of N/b and hence of N. By the dual of the last lemma, b' is not in a triad of N. Thus, by orthogonality, a is in a unique triad of N, namely $\{a, a', b\}$. Hence $J = \{a, a', b\}$; a contradiction. We conclude that

$$|J| > 2 \tag{5}$$

and, similarly,

$$|K| > 2. \tag{6}$$

Now, as N/b is 3–connected,

$$r_{N/b}(J \cup a) = r_{N/b}(J) + 1 \tag{7}$$

and

$$r_{N/b}(K \cup a) = r_{N/b}(K) + 1. \tag{8}$$

Hence $\{a', b'\}$ is not contained in either J or K. Thus, we may assume, without loss of generality, that $a' \in J$ and $b' \in K$.

Now N has an A-fan other than $\mathcal{F}$. Let the A-elements of it be a_1 and a'_1. If $\{a_1, a'_1\} \subseteq J$, then, as $\{a, a', a_1, a'_1\}$ is a union of circuits of N/b, we have a contradiction to (7). We conclude that $\{a_1, a'_1\} \not\subseteq J$. Thus, we may assume, without loss of generality, that $a_1 \in K$. Now $(J \cup a, K)$ is a 3–separation of N/b. Suppose that $a'_1 \in J$. Then, as $\{a, a', a_1, a'_1\}$ is a cocircuit and a union of circuits of N/b, we deduce, since $|K| \geq 3$, that $(J \cup a \cup a_1, K - a_1)$ is a 2–separation of N/b; a contradiction. We conclude that $a'_1 \in K$. Now $(J, K \cup a)$ is a 3–separation of N/b. Again the fact that $\{a, a', a_1, a'_1\}$ is a cocircuit and a union of circuits in N/b implies, since $|J| \geq 3$, that $(J - a', K \cup a \cup a')$ is a 2–separation of N/b. This contradiction completes the proof that $N\backslash a/a'$ is 3–connected.

To check that all the A-fans of N other than $\mathcal{F}$ are $(A - \{a, a'\})$-fans of $N\backslash a/a'$, we note that every triangle and every triad of an A-fan other than $\mathcal{F}$ contains a circuit or cocircuit of $N\backslash a/a'$. Since the last matroid is 3–connected, all the triangles and triads of the A-fans other than $\mathcal{F}$ remain triangles and triads of $N\backslash a/a'$. Thus every A-fan of N other than $\mathcal{F}$ is an $(A - \{a, a'\})$-fan of $N\backslash a/a'$.

Finally, it is straightforward to check that $A - \{a, a'\}$ is 3–separating in $N\backslash a/a'$ and we omit the details. □

The proof of Theorem 5.2 will use a fundamental class of 3–connected matroids called spikes [4]. For $n \geq 3$, an *n-spike with tip p* is a rank-n matroid N whose ground set is the union of n three-point lines, $L_1, L_2, \dots, L_n$, all containing p, such that, for all k in $\{1, 2, \dots, n-1\}$, the union of any k of these lines has rank $k+1$. Let $L_1 = \{p, x, q\}$. It follows from the properties of spikes [4] that $N\backslash x$ is isomorphic to its dual. We call it a *spike with tip p and cotip q*.

To prove Theorem 5.2, we shall prove the case when all A-fans contain exactly two elements of A. The general case will follow by combining that theorem with Lemma 5.3, and again we omit the details. In the former case, we have a 3–connected matroid M having a 3–separation (A, B) in which every element of A is essential, and M has at least two type-3 A-fans each of which has exactly four elements. Loosely speaking, such a matroid is obtained from a spike with tip and cotip and another 3–connected matroid by sticking these two matroids together along a 3-point line L of the spike and then possibly deleting some non-tip elements from L. The tip and the cotip serve as the ends of all the A-fans where A consists of all the elements of the spike except the tip, the cotip, and the elements of L.

5.6. **Theorem.** *Suppose that M is a 3–connected matroid whose ground set has a partition (A, B) in which $|A| \geq 2$, $|B| \geq 3$, every element of A is essential, and M has exactly k type-3 A-fans each of which has exactly four elements. Then there is a matroid N_1 and a subset Z of $E(N_1)$ such that $M = N_1\backslash Z$, where N_1 can be constructed from a 3–connected matroid N_0 having a triangle $\{b_0, x, y\}$ by*

(i) *adding an element b_1 in parallel with x;*
(ii) *for some $k \geq 1$, adding elements $a_1, a_2, \dots, a_k$ in series with b_1; and*
(iii) *for each i in $\{1, 2, \dots, k\}$, adding an element a'_i so that $\{b_0, a_i, a'_i\}$ is a triangle;*

and Z is a subset of $\{x, y\}$ such that

(a) *$Z = \emptyset$ if $|E(N_0)| = 3$; and*
(b) *$Z \neq \{x, y\}$ if $\{x, y\}$ is in a triad of N_0.*

Furthermore, for every pair (N_1, Z) satisfying these conditions, the restriction N_2 of N_1 to $\{b_0, b_1, x, y, a_1, a'_1, a_2, a'_2, \dots, a_k, a'_k\}$, is a spike with tip b_0 and cotip b_1, and $N_1\backslash Z$ is a 3–connected matroid. Moreover, if $A = \{a_1, a'_1, a_2, a'_2, \dots, a_k, a'_k\}$, then A is a 3–separating set of $N_1\backslash Z$ in which every element is essential; and $N_1\backslash Z$ has exactly k A-fans, each of type-3, and each having exactly four elements.

Proof. Let N_1 be a matroid and Z be a subset of $E(N_1)$ that satisfy the specified conditions. Let N'_2 be obtained from N_2 by freely adding z on the line through

b_0 and b_1. The resulting matroid N_2' is a $(k+2)$-spike with tip b_0. Thus, by the properties of spikes [4], N_2' and N_2 are 3–connected. Hence, by [7], N_1 is 3–connected since both $N_1|E(N_0)$ and $N_1|E(N_2)$ are 3–connected and $|E(N_0) \cap E(N_2)| \geq 3$.

Now, we shall show that $N_1\backslash Z$ is 3–connected. Let $u \in \{x, y\}$. It is straightforward to check that

$$(E(N_0) - u, \{b_1, a_1, a_1', a_2, a_2', \ldots, a_k, a_k'\})$$

is a vertical 2–separation of N_1/u. Thus, by Bixby's Lemma, $\text{co}(N_1\backslash u)$ is 3–connected. Thus $N_1\backslash u$ is 3–connected unless u is in a triad T^* of N_1. Assume that the exceptional case occurs. Then, by orthogonality, T^* contains $\{x, y\}$ or T^* contains b_0. In the former case, since $x \in T^*$, orthogonality implies that T^* meets $\{b_1, a_1, a_2, \ldots, a_k\}$. In the latter case, orthogonality implies that T^* meets each $\{a_i, a_i'\}$. In both cases, $N_1|E(N_0)$ has a cocircuit properly contained in T^*, so N_0 is not 3–connected because, by assumption, $N_0 \not\cong U_{2,3}$. This contradiction implies that $N_1\backslash u$ is 3–connected.

Next let $Z = \{x, y\}$. Then, by assumption, $\{x, y\}$ is not in a triad of N_0. One easily checks that $(E(N_0) - \{x, y\}, \{b_1, a_1, a_1', a_2, a_2', \ldots, a_k, a_k'\})$ is a vertical 2–separation of $N_1\backslash x/y$. Thus, since $N_1\backslash x$ is 3–connected, it follows that $\text{co}(N_1\backslash\{x, y\})$ is 3–connected. We deduce that $N_1\backslash Z$ is 3–connected unless $N_1\backslash x$ has a triad T^* containing y. Assume that the exceptional case occurs. Then T^* must contain an element of $E(N_0) - \{x, y\}$ otherwise $N_1|E(N_0)$, which equals N_0, has a cocircuit of size at most two. If $b_0 \in T^*$, then T^* must meet each $\{a_i, a_i'\}$. Thus $k = 1$ and $N_1|(E(N_0) - x)$ has $\{b_0, y\}$ as a cocircuit. Hence $\{b_0, x, y\}$ is a cocircuit of N_0, contradicting (b). We deduce that $b_0 \notin T^*$. Since $|T^* - E(N_0)| \leq 1$, it follows by orthogonality that T^* avoids $\{a_1, a_1', a_2, a_2', \ldots, a_k, a_k'\}$. If $b_1 \in T^*$, then $N_1|(E(N_0) - x)$ has a 2–cocircuit containing y, and again we obtain the contradiction that N_0 has a triad containing $\{x, y\}$. We conclude that T^* is a triad of $N_1\backslash x$ containing y but avoiding $\{b_0, b_1, a_1, a_1', a_2, a_2', \ldots, a_k, a_k'\}$. However, in N_1, the set $\{b_1, a_1, a_1', a_2, a_2', \ldots, a_k, a_k'\}$ spans b_0 and x, and so also spans y; a contradiction. It follows that $N_1\backslash\{x, y\}$ is 3–connected.

It is straightforward to check that $N_1\backslash Z$ has the properties specified in the last paragraph of the theorem and we omit the details of the argument.

Now suppose that M is a 3–connected matroid whose ground set has a partition (A, B) in which $|A| \geq 2$, $|B| \geq 3$, every element of A is essential, and M has exactly k type-3 A-fans each of which has exactly four elements. Then, by Lemma 3.3, we may assume that the A-fans are $\{b_0, a_i', a_i\}, \{a_i', a_i, b_1\}$ for $i = 1, 2, \ldots, k$ where $\{b_0, a_i', a_i\}$ is a triangle. Then, by repeatedly applying Theorem 5.4, we deduce that if $M' = M\backslash a_2, a_3, \ldots, a_k/a_2', a_3', \ldots, a_k'$, then M' is 3–connected having $\{b_0, a_1', a_1\}, \{a_1', a_1, b_1\}$ as an $\{a_1', a_1\}$-fan. Now the parallel classes of M'/b_1 containing a_1 and a_1' are the only posssible non-trivial parallel classes of M'/b_1. Let N_0 be obtained from $\text{si}(M'/b_1)$ by relabelling the elements corresponding to these parallel classes as x and y, respectively. Moreover, let Z consist of the subset of $\{x, y\}$ for which the corresponding parallel classes of M'/b_1 are trivial.

Certainly $\{b_0, x, y\}$ is a triangle of N_0 and, since M' is 3–connected, if $|E(N_0)| = 3$, then $Z = \emptyset$. Moreover, if $\{x, y\}$ is in a triad of N_0, then $Z \neq \{x, y\}$ otherwise M' has a triad that contains $\{a_1, a_1'\}$ and is different from $\{a_1, a_1', b_1\}$, and it is not difficult to get a contradiction to orthogonality. We conclude that the matroid M can be constructed as described in the theorem. □

Acknowledgements

The author thanks Geoff Whittle for drawing his attention to the problem considered here and for helpful discussions during the preparation of the paper. The author's research was partially supported by grants from the National Security Agency.

References

[1] Akkari, S. and Oxley, J., Some local extremal connectivity results for matroids, *Combin. Probab. Comput.* **2** (1993), 367–384.
[2] Bixby, R. E., A simple theorem on 3–connectivity, *Linear Algebra Appl.* **45** (1982), 123–126.
[3] Brylawski, T. H., Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.
[4] Ding, G., Oporowski, B., Oxley, J., and Vertigan, D., Unavoidable minors of large 3–connected matroids, *J. Combin. Theory Ser. B* **71** (1997), 244–293.
[5] Geelen, J. and Whittle, G., Matroid 4–connectivity: a deletion-contraction theorem, *J. Combin. Theory. Ser. B* **83** (2001), 15–37.
[6] Oxley, J., *Matroid Theory*, Oxford University Press, New York, 1992.
[7] Oxley, J. and Wu, H., On matroid connectivity, *Discrete Math.* **146** (1995), 321–324.
[8] Oxley, J. and Wu, H., On the structure of 3–connected matroids and graphs, *European J. Combin.* **21** (2000), 667–688.
[9] Oxley, J. and Wu, H., Matroids and graphs with few non-essential elements, *Graphs Combin.* **16** (2000), 199–229.
[10] Tutte, W. T., Connectivity in matroids. *Canad. J. Math.* **18** (1966), 1301–1324.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803–4918, USA

E-mail address: oxley@math.lsu.edu