THE STRUCTURE OF CROSSING SEPARATIONS IN MATROIDS

JEREMY AIKIN AND JAMES OXLEY

ABSTRACT. Oxley, Semple and Whittle described a tree decomposition for a 3-connected matroid M that displays, up to a natural equivalence, all non-trivial 3-separations of M. Crossing 3-separations gave rise to fundamental structures known as flowers. In this paper, we define a generalized flower structure called a k-flower, with no assumptions on the connectivity of M. We completely classify k-flowers in terms of the local connectivity between pairs of petals.

1. Introduction

For a matroid M, Cunningham and Edmonds [2] showed that if M is 2-connected, it has a corresponding tree that displays all 2-separations of M. In the same spirit, Oxley, Semple and Whittle [7] showed that, when M is 3-connected, there is an associated tree that displays, up to a natural equivalence, all non-trivial 3-separations of M. The interactions of crossing 3-separations in M were described by fundamental structures known as flowers. A *flower* in a 3-connected matroid M is a partition $(P_1, P_2, \ldots, P_n)$ of $E(M)$ in which each *petal* P_i is 3-separating having at least two elements, and the union of any two consecutive petals is 3-separating. All flowers are either *anemones* or *daisies* [7], that is, either every union of petals is 3-separating, or only consecutive such unions are. The classification of flowers was further refined by considering the local connectivity between pairs of petals.

In this paper, we make no assumptions about the connectivity of the matroid M, and analyze flowers that display exact k-separations. This theory relies only on the fact that the rank function r of a matroid on a set E is a polymatroid [3], that is, a non-negative, integer-valued, increasing, submodular function on 2^E whose value on $\emptyset$ is 0. Because of the potential for broader applicability of such a theory, we shall present it for an arbitrary polymatroid f on a finite set E. The reader whose sole interest is in matroids can, throughout this development, view f as the rank function of a matroid on E. The *connectivity function* λ_f of f is defined for all subsets

Date: May 17, 2007.
1991 *Mathematics Subject Classification.* 05B35.

X of E by $\lambda_f(X) = f(X) + f(E - X) - f(E)$; and the *local connectivity* $\sqcap_f(X, Y)$ between two subsets X and Y of E is given by $\sqcap_f(X, Y) = f(X) + f(Y) - f(X \cup Y)$. Clearly $\sqcap_f(X, E - X) = \lambda_f(X) = \lambda_f(E - X)$. We shall usually abbreviate λ_f and $\sqcap_f$ as λ and $\sqcap$. For a positive integer n, we write $[n]$ for $\{1, 2, \dots, n\}$.

Let f be a polymatroid on E. If $X \subseteq E$, then X is *k-separating* if $\lambda_f(X) \leq k - 1$. When $\lambda_f(X) = k - 1$, we say X is *exactly k-separating*. For an integer n exceeding one, we call $(P_1, P_2, \dots, P_n)$ a *k-flower* for f with *petals* $P_1, P_2, \dots, P_n$ if $(P_1, P_2, \dots, P_n)$ is a partition of E into non-empty sets such that each P_i is exactly k-separating and, when $n \geq 3$, each $P_i \cup P_{i+1}$ is exactly k-separating, where all subscripts are interpreted modulo n. It is also convenient to view (E) as a k-flower with a single petal. We call it a *trivial k-flower*. When f is the rank function of a 3-connected matroid, a 3-flower is what we defined to be a flower. Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ and I be a proper non-empty subset of $[n]$. Then Φ is a *k-anemone* if $\bigcup_{i \in I} P_i$ is exactly k-separating for all such I; and Φ is a *k-daisy* if $\bigcup_{i \in I} P_i$ is exactly k-separating for precisely those such subsets I whose members form a consecutive set in the cyclic order $(1, 2, \dots, n)$.

The theory of flowers that was developed for matroids in [7] assumed that the underlying matroid was 3-connected. Whittle (private communication) suggested that this assumption could be dropped and this is what we do here. In particular, for all $k \geq 1$, we develop a theory of k-flowers in arbitrary polymatroids and show that the classification of flowers in terms of local connectivity extends to k-flowers. For example, we prove the following result in Section 4.

Theorem 1.1. *Every k-flower is either a k-anemone or a k-daisy.*

Let $(P_1, P_2, \dots, P_n)$ be a flower Φ in a matroid with $n \geq 3$. When Φ is an anemone, Φ is a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct i, j in $[n]$; Φ is a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct i, j in $[n]$; and Φ is *spike-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all distinct i, j in $[n]$. When Φ is a daisy, it is *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and Φ is *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$. Matroid flowers with fewer than 4 petals can be viewed as anemones or daisies and we call Φ *unresolved* if $n = 3$, and $\sqcap(P_i, P_j) = 1$ for all distinct i, j in $\{1, 2, 3\}$. The same sort of ambiguity arises for general k-flowers with three petals. A k-flower with two petals is just an exactly k-separating partition. We shall spend the majority of this paper working with k-flowers that have at least three petals. It was proved in [7, Theorem 4.1] that every such matroid flower is of one of the types noted above.

Theorem 1.2. *If $(P_1, P_2, \ldots, P_n)$ is a flower Φ in a matroid, then Φ is either an anemone or a daisy. Moreover, if $n \geq 3$, then Φ is either a paddle or a copaddle, or is spike-like, swirl-like, Vámos-like, or unresolved.*

The main results of this paper are contained in the next two theorems, which generalize Theorem 1.2.

Theorem 1.3. *For some $n \geq 5$ and $k \geq 1$, let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a polymatroid. Then there are integers c and d with*

$$k - 1 \geq c \geq d \geq \max\{2c - (k - 1), 0\}$$

such that

- *(i) the local connectivity between distinct petals is c if the petals are consecutive and is d otherwise;*
- *(ii) Φ is a k-anemone if and only if $c = d$; and*
- *(iii) the local connectivity between any two sets of petals having disjoint index sets I and J can be expressed in terms of I, J, c, d, and k, and is invariant under the permutation $(1, 2, \ldots, n)$.*

Theorem 1.4. *If $k \geq 1$, then, for all pairs (c, d) of integers such that $k - 1 \geq c \geq d \geq \max\{2c - (k - 1), 0\}$ and, for all $n \geq 3$, there is a k-flower $(P_1, P_2, \ldots, P_n)$ in a matroid such that the local connectivity between pairs of distinct petals is c when the petals are consecutive and is d otherwise.*

We noted above that in a 3-flower (P_1, P_2, P_3, P_4), the values of $\sqcap(P_1, P_3)$ and $\sqcap(P_2, P_4)$ may differ. For general $k \geq 3$, there is an entire class of 4-petal k-flowers with the property that $\sqcap(P_1, P_3) \neq \sqcap(P_2, P_4)$. These flowers are studied in Section 6 where the following theorem, which corresponds to Theorem 1.4 for $n = 4$, is proved.

Theorem 1.5. *Let (P_1, P_2, P_3, P_4) be a k-flower Φ in a polymatroid. Then there are integers c, d_1, and d_2 with*

$$k - 1 \geq c \geq d_1 \geq d_2 \geq \max\{2c - (k - 1), 0\}, \tag{1.1}$$

such that

- *(i) the local connectivity between consecutive distinct petals is c; and*
- *(ii) $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{d_1, d_2\}$.*

Moreover, for all triples (c, d_1, d_2) with $c \geq d_1 > d_2 \geq 0$ and all k in $\{2c + 1 - d_2, 2c + 1 - d_2 + 1, \ldots, 2c + 1\}$, there is a 4-petal k-flower in a matroid such that (i) and (ii) hold.

In Section 3, we investigate the local connectivity between sets of petals of k-flowers. In particular, we prove part (i) of Theorem 1.3. In Section 4, we prove Theorem 1.1 and part (ii) of Theorem 1.3 enabling us to determine the type of a k-flower. This is followed by a section on constructing examples of k-anemones and k-daisies for all allowable values of k, c, and d, which will complete the proof of Theorem 1.4. We study 4-petal k-flowers in Section 6, where we prove Theorem 1.5. In Section 7, we compute the local connectivity between any two disjoint collections of petals in a k-flower and thereby prove part (iii) of Theorem 1.3. Finally, in Section 8, we briefly consider flowers for connectivity functions. To simplify our notation, if i and j are in $[n]$, we shall write $[i, j]$ for the set of integers $\{i, i+1, \ldots, j\}$. In addition, if $(A_1, A_2, \ldots, A_n)$ is a family of sets and I is a non-empty subset of $[n]$, we write A_I for $\bigcup_{i \in I} A_i$. Any unexplained notation throughout this paper will follow Oxley [5]. Finally, we remark that we could weaken the requirement that f is a polymatroid by dropping the assumption that $f(\emptyset) = 0$. This would not introduce any fundamentally different structures. Indeed, a k-flower in a polymatroid becomes a $(k+m)$-flower if $f(\emptyset)$ is m rather than 0.

2. Preliminaries

In this section, we present some more definitions along with some lemmas that will be needed in the proofs of the main results. Before computing local connectivity in k-flowers, we state two lemmas which give us useful properties of the connectivity and local connectivity functions. These results are stated for matroids in [7]. The extensions to polymatroids are straightforward and we prove the first of these as an illustration.

Lemma 2.1. *Let X_1, X_2, Y_1 and Y_2 be subsets of the ground set of a polymatroid f. If $X_1 \supseteq Y_1$ and $X_2 \supseteq Y_2$, then*

$$\sqcap(X_1, X_2) \geq \sqcap(Y_1, Y_2)$$

or, equivalently,

$$f(X_1) + f(X_2) - f(X_1 \cup X_2) \geq f(Y_1) + f(Y_2) - f(Y_1 \cup Y_2).$$

Proof. Let $m = |X_1 - Y_1| + |X_2 - Y_2|$. The result is immediate if $m = 0$. Assume it holds when $m < t$ and let $m = t > 0$. We may assume that $X_1 - Y_1$ contains an element e. Then, by the induction assumption,

$$f(X_1 - e) + f(X_2) - f((X_1 - e) \cup X_2) \geq f(Y_1) + f(Y_2) - f(Y_1 \cup Y_2).$$

Hence the lemma holds provided that

$$f(X_1) - f(X_1 \cup X_2) \geq f(X_1 - e) - f((X_1 - e) \cup X_2).$$

But, since f is a submodular, increasing function,

$$\begin{aligned} f(X_1) &+ f((X_1 - e) \cup X_2) \\ &\geq f(X_1 \cup [(X_1 - e) \cup X_2]) + f(X_1 \cap [(X_1 - e) \cup X_2]) \\ &\geq f(X_1 \cup X_2) + f(X_1 - e). \end{aligned}$$

The result follows. □

The next lemma is the most widely used result in this paper. In particular, it is frequently applied in Section 3 to get leverage on computing local connectivity and it is crucial in proving the main theorems.

Lemma 2.2. *Let A, B, C and D be subsets of the ground set E of a polymatroid f. Then the following hold:*

(i) $\sqcap(A \cup B, C \cup D) + \sqcap(A, B) + \sqcap(C, D)$

$$= \sqcap(A \cup C, B \cup D) + \sqcap(A, C) + \sqcap(B, D).$$

(ii) $\sqcap(A \cup B, C) + \sqcap(A, B) = \sqcap(A \cup C, B) + \sqcap(A, C)$.

(iii) $\sqcap(A \cup B, C) + \sqcap(A, B) \geq \sqcap(A, C) + \sqcap(B, C)$.

(iv) If $\{X, Y, Z\}$ is a partition of E, then

$$\lambda(X) + \sqcap(Y, Z) = \lambda(Z) + \sqcap(X, Y).$$

Hence, $\sqcap(X, Y) = \sqcap(Y, Z)$ if and only if $\lambda(X) = \lambda(Z)$.

The next lemma notes that λ itself is a submodular function. The proof is a straightforward consequence of the fact that f is submodular.

Lemma 2.3. *If X and Y are subsets of the ground set of a polymatroid, then*

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cup Y) + \lambda(X \cap Y).$$

3. Local Connectivity

In this section, we prove several local-connectivity results for k-flowers. In particular, we establish (i) of Theorem 1.3. Throughout this paper, whenever we deal with a k-flower $(P_1, P_2, \dots, P_n)$, all calculations on subscripts will be done modulo n. The arguments here generalize those in [7].

Lemma 3.1. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower. Then, for all t in $[n-1]$, every union of t consecutive petals is exactly k-separating.*

Proof. By the definition of a k-flower, the result is true if $t \in \{1, 2\}$. If $t \in [2, n-1]$, then

$$\lambda(P_{[1,t]}) + \lambda(P_{[t,t+1]}) \geq \lambda(P_t) + \lambda(P_{[1,t+1]}).$$

As $\lambda(P_{[t,t+1]}) = k - 1 = \lambda(P_t)$, we deduce that

$$\lambda(P_{[1,t]}) \geq \lambda(P_{[1,t+1]}).$$

By repeatedly applying the last inequality, we get

$$k - 1 = \lambda(P_{[1,2]}) \geq \lambda(P_{[1,3]}) \geq \cdots \geq \lambda(P_{[1,n-1]}) = \lambda(P_n) = k - 1.$$

Thus $\lambda(P_{[1,t]}) = k - 1$ and the lemma follows by symmetry. □

Lemma 3.2. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ. Then $\sqcap(P_i, P_{i+1}) = \sqcap(P_j, P_{j+1})$ for all i, j in $[n]$.*

Proof. This follows by making the obvious changes to the proof of [7, Lemma 4.5] and we omit the details. □

Lemma 3.3. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ with $n \geq 3$. Then*

$$\sqcap(P_1, P_2) = \sqcap(P_1, P_2 \cup P_I) = \sqcap(P_1, P_n \cup P_J)$$

for all proper subsets I and J of $[3, n-1]$.

Proof. Using Lemma 2.1, we get

$$\begin{aligned}\sqcap(P_1, P_2) &\le \sqcap(P_1, P_2 \cup P_I)\\ &\le \sqcap(P_1, P_2 \cup P_3 \cup \cdots \cup P_{n-1}) = \sqcap(P_1, P_n) = \sqcap(P_1, P_2).\end{aligned}$$

The second-last equality holds by Lemmas 2.2(iv) and 3.1 because

$$\lambda(P_2 \cup P_3 \cup \cdots \cup P_{n-1}) = k - 1 = \lambda(P_n).$$

The second equality in the lemma follows by symmetry. □

Lemma 3.4. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ with $n \ge 5$. Then $\sqcap(P_1, P_3) = \sqcap(P_i, P_j)$ for all distinct non-consecutive i, j in $[n]$.*

Proof. We first show that $\sqcap(P_1, P_m) = \sqcap(P_1, P_{m+1})$ for all m in $[3, n-2]$. By Lemma 2.2(ii),

$$\sqcap(P_1 \cup P_m, P_{m+1}) + \sqcap(P_1, P_m) = \sqcap(P_1 \cup P_{m+1}, P_m) + \sqcap(P_1, P_{m+1}).$$

By Lemma 3.3 and symmetry,

$$\sqcap(P_1 \cup P_m, P_{m+1}) = \sqcap(P_m, P_{m+1}) = \sqcap(P_{m+1}, P_m) = \sqcap(P_1 \cup P_{m+1}, P_m).$$

Hence $\sqcap(P_1, P_m) = \sqcap(P_1, P_{m+1})$ as asserted. Using this and symmetry, we have $\sqcap(P_1, P_3) = \sqcap(P_1, P_4) = \sqcap(P_2, P_4)$. The lemma follows without difficulty. □

Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ with $n \ge 2$. Define $c(\Phi) = \sqcap(P_1, P_2)$. Then, by Lemma 3.2, $c(\Phi)$ is the local connectivity between any two consecutive petals of Φ. When $n \ge 5$, let $d(\Phi) = \sqcap(P_1, P_3)$. By Lemma 3.4, $d(\Phi)$ is the local connectivity between any two non-consecutive petals of Φ. As noted already, if (P_1, P_2, P_3, P_4) is a k-flower, then $\sqcap(P_1, P_3)$ and $\sqcap(P_2, P_4)$ may differ and this will require us to introduce a new local connectivity parameter in Section 6. But for k-flowers with at least five petals, the two parameters $c(\Phi)$ and $d(\Phi)$ will suffice. When the underlying flower is clear, we shall frequently abbreviate $c(\Phi)$ and $d(\Phi)$ to c and d. For notational convenience, we shall call a k-flower with local connectivity parameters c and d a (k, c, d)-*flower*.

Lemma 3.5. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ with $n \ge 3$. For some m in $[3, n]$, let $\Phi' = (P_1, P_2, \ldots, P_{m-1}, P_{[m,n]})$. Then*

(i) Φ' is a k-flower;
(ii) $c(\Phi) = c(\Phi')$; and
(iii) if $m \ge 5$, then $d(\Phi) = d(\Phi')$.

Proof. Part (i) follows easily from Lemma 3.1 since each union of a consecutive pair of petals of Φ' is the union of a consecutive set of petals of Φ. To prove part (ii), note that

$$c(\Phi') = \sqcap(P_1, P_2) = c(\Phi).$$

The hypothesis that $m \ge 5$ in part (iii) guarantees that both Φ and Φ' will have at least 5 petals. Hence we can use Lemma 3.4 to get $d(\Phi) = \sqcap(P_1, P_3) = d(\Phi')$. □

We shall say that a k-flower Φ' that is obtained from Φ by combining some set of consecutive petals of the latter into a single petal has been obtained from Φ by *concatenation*. Next, we determine the local connectivity of $P_{[1,t]}$ relative to any union of petals disjoint from $P_{[1,t]}$.

Lemma 3.6. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ with $n \geq 5$. If $t \in [n-3]$, then, for all subsets I of $[t+2, n-1]$, the following hold:*

(i) $\sqcap(P_{[1,t]}, P_I) = d$, provided I is non-empty;
(ii) $\sqcap(P_{[1,t]}, P_{t+1} \cup P_I) = c$;
(iii) $\sqcap(P_{[1,t]}, P_{t+1} \cup P_n \cup P_I) = 2c - d$, provided $I \neq [t+2, n-1]$; and
(iv) $\sqcap(P_{[1,t]}, P_{[t+1,n]}) = k - 1$.

Proof. Lemma 3.1 immediately gives (iv). For (ii), let $\Phi' = (P_{[1,t]}, P_{t+1}, P_{t+2}, \ldots, P_n)$. By Lemma 3.5(ii), $c(\Phi) = c(\Phi')$ and (ii) follows by Lemma 3.3.

To prove (i), note that, as $n \geq 5$, either $|[1,t]| \geq 2$ or $|[t+2, n-1]| \geq 2$. We shall complete the proof in the former case, noting that a symmetric argument gives the latter case. Let $\Phi' = (P_{[1,t-1]}, P_t, P_{t+1}, P_{[t+2,n-1]}, P_n)$. Then, by Lemma 3.5, $c(\Phi') = c(\Phi) = c$ and $d(\Phi') = d(\Phi) = d$. By Lemma 2.2(iii),

$$\begin{aligned}\sqcap(P_{[1,t-1]} \cup P_t, P_{[t+2,n-1]}) + \sqcap(P_{[1,t-1]}, P_t) \\ = \sqcap(P_{[1,t-1]} \cup P_{[t+2,n-1]}, P_t) + \sqcap(P_{[1,t-1]}, P_{[t+2,n-1]}).\end{aligned}$$

By Lemma 3.3, $\sqcap(P_{[1,t-1]}, P_t) = c = \sqcap(P_{[1,t-1]} \cup P_{[t+2,n-1]}, P_t)$. Hence $\sqcap(P_{[1,t-1]} \cup P_t, P_{[t+2,n-1]}) = \sqcap(P_{[1,t-1]}, P_{[t+2,n-1]}) = d$, that is,

$$\sqcap(P_{[1,t]}, P_{[t+2,n-1]}) = d.$$

Therefore, if $i \in I$, then

$$d = \sqcap(P_1, P_i) \leq \sqcap(P_{[1,t]}, P_I) \leq \sqcap(P_{[1,t]}, P_{[t+2,n-1]}) = d,$$

so (i) holds.

To prove (iii), let $j \in [t+2, n-1] - I$. Then

$$\sqcap(P_{[1,t]}, P_{t+1} \cup P_n) \leq \sqcap(P_{[1,t]}, P_{t+1} \cup P_n \cup P_I) \leq \sqcap(P_{[1,t]}, P_{[t+1,n]} - P_j).$$

By parts (i) and (ii) above and Lemma 2.2(iii),

$$\begin{aligned}\sqcap(P_{[1,t]}, P_{[t+1,n]} - P_j) &= \sqcap(P_{[1,t]}, P_{[t+1,j-1]} \cup P_{[j+1,n]}) \\ &= \sqcap(P_{[1,t]} \cup P_{[t+1,j-1]}, P_{[j+1,n]}) + \sqcap(P_{[1,t]}, P_{[t+1,j-1]}) \\ &\qquad - \sqcap(P_{[t+1,j-1]}, P_{[j+1,n]}) \\ &= c + c - d = 2c - d.\end{aligned}$$

Similarly, $\sqcap(P_{[1,t]}, P_{t+1} \cup P_n) = 2c - d$, so $\sqcap(P_{[1,t]}, P_{t+1} \cup P_n \cup P_I) = 2c - d$, as required. □

4. k-anemones and k-daisies

In this section, we prove Theorem 1.1, which identifies the two main types of k-flowers. The proof extends the argument used to establish Lemma 4.4 of [7].

Proof of Theorem 1.1. The result is trivial if $n \leq 3$. By Lemma 3.1, all unions of consecutive non-empty proper sets of petals are exactly k-separating. Assume that Φ is not a k-daisy. The main part of the proof of this theorem is contained in the proofs of the next two lemmas.

Lemma 4.1. *The k-flower Φ has a pair of non-consecutive petals whose union is exactly k-separating.*

Proof. Since Φ is not a k-daisy, there is certainly a non-consecutive set of petals whose union B is exactly k-separating. Assume that such a set B is chosen to contain the minimum number p of petals. Then we may suppose that $p \geq 3$. Let $G = E - B$. We call the petals contained in B *black* and those contained in G *grey.* This coloring breaks $(P_1, P_2, \ldots, P_n)$ into a collection of monochromatic *arcs*, that is, maximal collections of consecutive petals all of which are the same color. Suppose that P_i and P_j are black but each of $P_{i+1}, P_{i+2}, \ldots, P_{j-1}$ is grey. Let $G_0 = P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{j-1}$. Traversing the petals of Φ from P_j to P_i cyclically in the direction avoiding G_0, we see alternating black and grey arcs beginning with a black one. Let the union of the petals in these arcs, in order, be $B_1, G_1, B_2, G_2, \ldots, B_{m+1}$ where $P_i \subseteq B_{m+1}$. Since B is not the union of a consecutive set of petals, $m \geq 1$. Moreover, $E = G_0 \cup B_1 \cup G_1 \cup \cdots \cup B_m \cup G_m \cup B_{m+1}$. Let $C = B_1 \cup G_1 \cup \cdots \cup B_m \cup G_m$. Clearly C is the union of a consecutive set of petals of Φ. An illustration of this situation is shown in Figure 1.

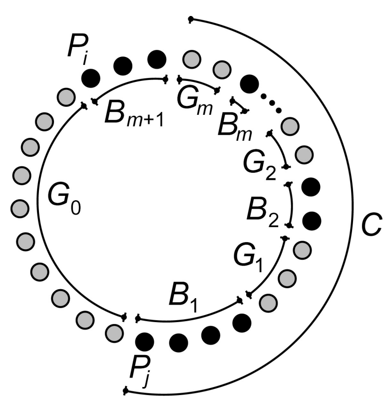

Figure 1. Monochromatic arcs in Φ.

By assumption, $\lambda(B) = k - 1$ and, by construction, $\lambda(C) = k - 1 = \lambda(B \cup C)$. Therefore, by submodularity, $\lambda(B \cap C) \leq k - 1$. We show next that $\lambda(B \cap C) = k - 1$. By submodularity, we see that

$$\lambda(B \cap C) + \lambda(B_1 \cup G_1 \cup \cdots \cup B_{m-1} \cup G_{m-1}) \geq \lambda(C - G_m) + \lambda(B_1 \cup \cdots \cup B_{m-1})$$

and

$$\lambda(B_1 \cup \cdots \cup B_{m-1}) + \lambda(B_1 \cup G_1 \cup \cdots \cup B_{m-2} \cup G_{m-2}) \geq \lambda(B_1 \cup G_1 \cup \cdots \cup G_{m-2} \cup B_{m-1}) + \lambda(B_1 \cup \cdots \cup B_{m-2}).$$

But

$$\begin{aligned}\lambda(B_1 \cup G_1 \cup \cdots \cup B_{m-1} \cup G_{m-1}) &= \lambda(C - G_m) \\ &= \lambda(B_1 \cup G_1 \cup \cdots \cup B_{m-2} \cup G_{m-2}) \\ &= \lambda(B_1 \cup G_1 \cup \cdots \cup G_{m-2} \cup B_{m-1}) \\ &= k - 1.\end{aligned}$$

By continuing this process, we get the following chain of inequalities:

$$\begin{aligned}\lambda(B \cap C) \geq \lambda(B_1 \cup \cdots \cup B_{m-1}) &\geq \lambda(B_1 \cup \cdots \cup B_{m-2}) \\ &\geq \lambda(B_1 \cup \cdots \cup B_{m-3}) \\ &\geq \cdots \geq \lambda(B_1 \cup B_2) \geq \lambda(B_1) = k - 1.\end{aligned}$$

We conclude that $\lambda(B \cap C) = k - 1$.

If $m > 1$, then $B \cap C$ contradicts the choice of B. Hence $m = 1$. Now let $C' = P_i \cup G_0 \cup P_j$. This situation is illustrated in Figure 2.

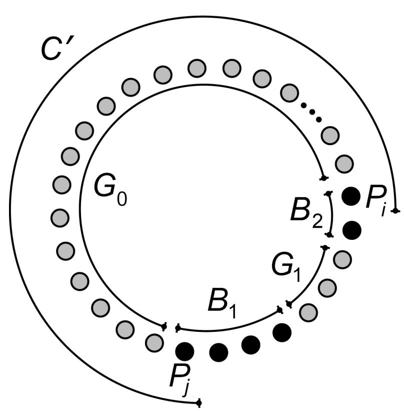

FIGURE 2. Φ has two black arcs and two grey ones.

Since $\lambda(B) = \lambda(C') = \lambda(B \cup C') = k - 1$, it follows by submodularity that $\lambda(B \cap C') = \lambda(P_i \cup P_j) \leq k - 1$. To see that $\lambda(P_i \cup P_j) = k - 1$, we observe that $\lambda(C' - P_i) = \lambda(P_j) = k - 1$ and $\lambda(P_i \cup P_j) + \lambda(C' - P_i) \geq \lambda(C') + \lambda(P_j)$. We conclude that Lemma 4.1 holds. □

Lemma 4.2. *Every union of a pair of petals of Φ is exactly k-separating.*

Proof. From Lemma 4.1, we know that $\lambda(P_i \cup P_j) = k-1$ for some non-consecutive i and j. We first show that $\lambda(P_i \cup P_{j-1}) = k-1$ by getting inequalities in both directions. By submodularity,

$$\lambda(P_i \cup P_j) + \lambda(P_{j-1} \cup P_j) \geq \lambda(P_j) + \lambda(P_i \cup P_j \cup P_{j-1})$$

and

$$\lambda(P_i \cup P_{j-1} \cup P_j) + \lambda(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_j) \geq \lambda(P_{j-1} \cup P_j) + \lambda(P_i \cup P_{i+1} \cup \cdots \cup P_j).$$

Thus $\lambda(P_i \cup P_{j-1} \cup P_j) = k-1$. Now,

$$\lambda(P_i \cup P_{j-1}) + \lambda(P_i \cup P_j) \geq \lambda(P_i \cup P_{j-1} \cup P_j) + \lambda(P_i).$$

Therefore, $\lambda(P_i \cup P_{j-1}) \geq k-1$. Moreover, the inequality

$$\lambda(P_i \cup P_{i+1} \cup \cdots \cup P_{j-1}) + \lambda(P_i \cup P_{j-1} \cup P_j) \geq \lambda(P_i \cup P_{i+1} \cup \cdots \cup P_j) + \lambda(P_i \cup P_{j-1})$$

implies that $\lambda(P_i \cup P_{j-1}) \leq k-1$. Hence $\lambda(P_i \cup P_{j-1}) = k-1$. By a symmetric argument, one can prove that $\lambda(P_i \cup P_{j+1}) = k-1$. Thus every union of two petals containing P_i is exactly k-separating. By symmetry, every union of two petals containing P_j is exactly k-separating. It is now straightforward to show, as in the proof of Lemma 4.4 of [7], that every union of two petals is exactly k-separating. □

To complete the proof, we note that every cyclic ordering of the petals of Φ is a k-flower. Therefore, by Lemma 3.1, every union of a proper non-empty set of petals is exactly k-separating. Hence Φ is a k-anemone. □

The following lemma proves statement (ii) of Theorem 1.3.

Lemma 4.3. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ with $n \geq 4$. Then Φ is a k-anemone if and only if $\sqcap(P_i, P_j) = c$ for all distinct i, j in $[n]$.*

Proof. Suppose $\sqcap(P_i, P_j) = c$ for all distinct i, j in $[n]$. By Lemma 2.2(ii) and Lemma 3.6(i), $\sqcap(P_1 \cup P_3, P_2) = c$ and $\sqcap(P_4 \cup \cdots \cup P_n, P_2) = c$. Moreover, by Lemma 2.2(iv),

$$\lambda(P_4 \cup \cdots \cup P_n) + \sqcap(P_2, P_1 \cup P_3) = \lambda(P_1 \cup P_3) + \sqcap(P_4 \cup \cdots \cup P_n, P_2),$$

so $\lambda(P_1 \cup P_3) = k-1$. Thus, by Theorem 1.1, Φ is a k-anemone.

Conversely, let Φ be a k-anemone. If i and j are distinct elements of $[n]$ and $j \in \{i-1, i+1\}$, then $\sqcap(P_i, P_j) = c$. If $j \notin \{i-1, i+1\}$, then, as Φ is a k-anemone, we may re-order the petals of Φ and retain a k-flower. By making P_{i-1}, P_i, and P_j consecutive, we see that $\sqcap(P_i, P_j) = \sqcap(P_{i-1}, P_i) = c$. □

If Φ is not a k-anemone, then Theorem 1.1 tells us that Φ is a k-daisy. Therefore, the k-daisies are precisely those k-flowers that have non-consecutive petals P_i and P_j with $\sqcap(P_i, P_j) = d \neq c$. The next lemma gives us a lower bound for d. In the next section, we will give a method for constructing k-daisies for all allowed values of d. We note that, from the inequality in the next lemma, one can determine precisely how many k-anemones and k-daisies there are for a fixed value of k.

Lemma 4.4. *Let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ with $n \geq 5$. Then*

$$k-1 \geq c \geq d \geq \max\{2c-(k-1), 0\}. \tag{4.1}$$

Proof. Since $c = \sqcap(P_1 \cup P_2, P_3) \geq \sqcap(P_1, P_3) = d$, we see that $c \geq d$. Since f is submodular and non-negative, $d \geq 0$. Also, by Lemma 2.2(iii),

$$\sqcap(P_1, P_3) \geq \sqcap(P_1, P_2) + \sqcap(P_3, P_2) - \sqcap(P_1 \cup P_3, P_2).$$

As $\sqcap(P_1 \cup P_3, P_2) \leq \sqcap(E - P_2, P_2) = k-1$, it follows that

$$d = \sqcap(P_1, P_3) \geq 2c - (k-1).$$ □

5. Constructions of k-anemones and k-daisies

In this section, we prove Theorem 1.4. In particular, we provide a method for constructing examples of k-anemones and k-daisies for all values of c and d satisfying the inequalities in the theorem. All of the examples we construct will be matroids. The method for constructing these examples is similar to the methods used to construct paddles, copaddles, spike-like, and swirl-like flowers in [7], but it also relies heavily on the matroid operation of *truncation*. The truncation $T(M)$ of a matroid M is the matroid that is obtained by freely extending M by an element p, and then contracting p. In particular, for $X \subseteq E(M)$, we have

$$r_{T(M)}(X) = \begin{cases} r(X), & \text{if } r(X) < r(M); \\ r(X)-1, & \text{if } r(X) = r(M). \end{cases} \tag{5.1}$$

We omit the routine proof of the next lemma.

Lemma 5.1. *For some $n \geq 3$, let $(P_1, P_2, \dots, P_n)$ be a (k, c, d)-flower Φ in a matroid M. If $r(E(M) - P_i) < r(M)$ for all i, then Φ is a $(k+1, c, d)$-flower in $T(M)$.*

The core of the proof of Theorem 1.4 is contained in the following result.

Lemma 5.2. *Let $c, d, n,$ and m be non-negative integers such that $c \geq d$ and $n \geq 4$. Then, for $k = 2c - d + 1$, there is a k-flower $(P_1, P_2, \dots, P_n)$ in a matroid M with $\sqcap(P_1, P_2) = c$ and $\sqcap(P_1, P_3) = d$ such that $r(M) - r(E(M) - P_i) = m$ for all i.*

Proof. Begin with a basis B for an $(n(c-d) + nm + d)$-dimensional vector space V over $\mathbb{R}$. Partition B into n subsets, $A_1, A_2, \dots, A_n$, each of size $c-d$; n subsets, $E_1, E_2, \dots, E_n$, each of size m; and one subset D of size d. For all i in $[n]$, let $F_i = A_i \cup A_{i+1} \cup E_i \cup D$. Then $|F_i| = 2(c-d) + m + d$. Let N be the vector matroid on $V|B$. Then each F_i is a flat of N. For each i, freely add a set G_i of $|F_i|$ elements to F_i. Then G_i spans F_i. Now delete $A_1 \cup A_2 \cup \dots \cup A_n \cup D$ to get a matroid M whose ground set E

is $P_1 \cup P_2 \cup \cdots \cup P_n$, where $P_i = G_i \cup E_i$ for all i. Then $r(M) = |B| = n(c-d)+nm+d$, while $|P_i| = 2(c-d)+2m+d$ and $r(P_i) = 2(c-d)+m+d$.

We now show that $(P_1, P_2, \ldots, P_n)$ is the required k-flower. If I is the union of t consecutive elements in the cyclic order $(1, 2, \ldots, n)$, then

$$r(P_I) = (t+1)(c-d) + tm + d \text{ if } t \leq n-1.$$

Thus, for all such I and all i in $[n]$, we have $\lambda(P_I) = 2c - d = \lambda(P_i)$. Also $\sqcap(P_i, P_{i+1}) = c$ and $\sqcap(P_i, P_{i+t}) = d$ for all t such that $2 \leq t \leq n-2$. We conclude that $(P_1, P_2, \ldots, P_n)$ is a (k, c, d)-flower with $k = 2c-d+1$. Finally, we note that $r(M) - r(E(M) - P_i) = m$ for all i. □

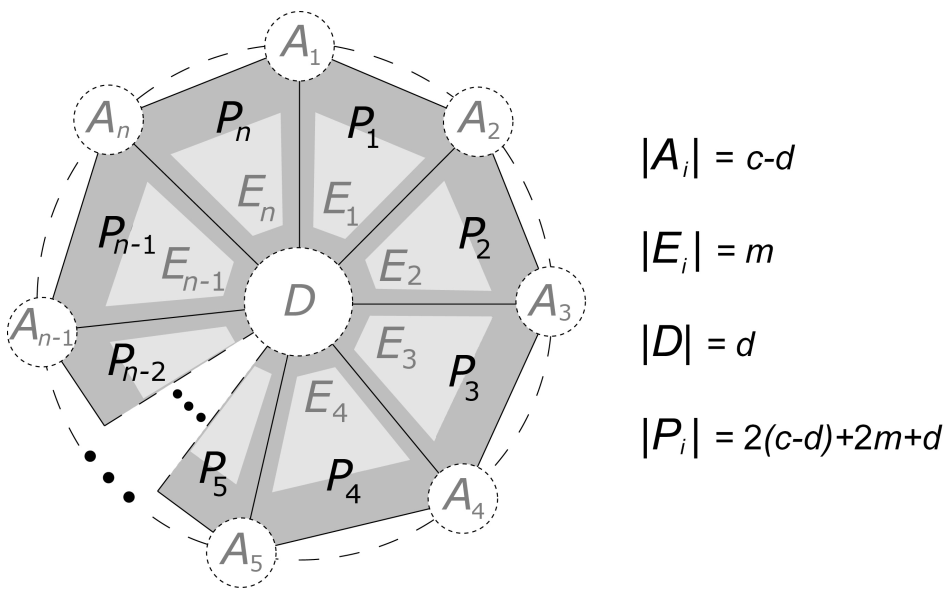

FIGURE 3. Constructing a (k, c, d)-flower.

An illustration, which might aid in visualizing the construction in Lemma 5.2, is given in Figure 3. Note that, when $c > d$, each A_i is non-empty and the construction produces a k-daisy; when $c = d$, each A_i is empty and we get a k-anemone.

Proof of Theorem 1.4. In the last lemma, we constructed $(P_1, P_2, \ldots, P_n)$, a $(2c-d+1, c, d)$-flower Φ in a matroid M such that $r(M)-r(E(M)-P_i) = m$ for all i. We note that $2c - d + 1$ is the smallest value of k allowed by inequality (4.1). To obtain a (k, c, d)-flower for a larger value of k, take $m = k - (2c - d + 1)$. Then, by Lemma 5.1, if we truncate M m times, we obtain a matroid in which Φ is a $(2c - d + 1 + m, c, d)$-flower, that is, a (k, c, d)-flower. □

6. k-FLOWERS WITH FOUR PETALS

Let Φ be a k-flower $(P_1, P_2, \dots, P_n)$. If $n \geq 5$, then Lemma 3.4 establishes that the local connectivity between any two non-consecutive petals is a well-defined invariant of Φ. In this section, we consider what happens when $n = 4$. In this case, we define $\sqcap(P_1, P_3) = d_1(\Phi)$ and $\sqcap(P_2, P_4) = d_2(\Phi)$ where we may assume that $d_1(\Phi) \geq d_2(\Phi)$. As before, $c(\Phi) = \sqcap(P_1, P_2)$ and, when the underlying flower is clear, we abbreviate these parameters to d_1, d_2 and c. If Φ is a k-daisy, then $d_1 = d_2 = d$. But, for example, if Φ is a Vámos-like flower in a matroid [7], then $(c, d_1, d_2) = (1, 1, 0)$. Such a flower is an example of a $(3, 1, 1, 0)$-flower, where we call a k-flower with parameters c, d_1, and d_2 a (k, c, d_1, d_2)-*flower*. In this section, we prove Theorem 1.5, showing, in particular, that, for all $k \geq 3$, there is a matroid having a 4-petal (k, c, d_1, d_2)-flower with $d_1 > d_2$. The last inequality will be assumed throughout this section. The following theorem [6] will be used to verify that our constructions do, in fact, yield matroids.

Theorem 6.1. *Let $\mathcal{C}$ be a collection of subsets of a set E and m be a non-negative integer. Then $\mathcal{C}$ is the set of non-spanning circuits of a rank-m matroid on E if and only if $\mathcal{C}$ has the following properties:*

- *(i) No member of $\mathcal{C}$ properly contains another.*
- *(ii) If $e \in C_1 \cap C_2$ where C_1 and C_2 are distinct members of $\mathcal{C}$ and $|(C_1 \cup C_2) - e| \leq m$, then $(C_1 \cup C_2) - e$ contains a member of $\mathcal{C}$.*
- *(iii) All members of $\mathcal{C}$ have at most m elements.*
- *(iv) E has an m-element subset that contains no member of $\mathcal{C}$.*

The proof of Theorem 1.5 will also use the Higgs lift, the dual operation of truncation. Formally, for a matroid M, its *Higgs lift* $L(M)$ is $(T(M^*))^*$. To construct $L(M)$ directly, we first freely coextend M by a non-loop element p, and then delete p. The rank function of $L(M)$ is

$$r_{L(M)}(X) = \begin{cases} r(X), & \text{if } r(X) = |X|; \\ r(X) + 1, & \text{if } r(X) < |X|. \end{cases} \tag{6.1}$$

The next lemma shows how the Higgs lift can be used to transform a (k, c, d_1, d_2)-flower into a $(k + 1, c + 1, d_1 + 1, d_2 + 1)$-flower. We omit the routine proof.

Lemma 6.2. *For some $n \geq 3$, let $(P_1, P_2, \dots, P_n)$ be a k-flower Φ in a matroid M with $n \geq 4$. If every petal of Φ is dependent, then Φ is a $(k+1)$-flower in $L(M)$. Moreover, $\sqcap_{L(M)}(P_i, P_j) = \sqcap_M(P_i, P_j) + 1$ for all distinct i and j in $[n]$.*

Proof of Theorem 1.5. Inequality (1.1) follows by the same argument used to prove Lemma 4.4.

Now assume that c, d_1, and d_2 are integers such that $c \geq d_1 > d_2 \geq 0$. First we construct a 4-petal (k, c, d_1, d_2)-flower with $k = 2c + 1$. Observe that $c \geq 1$, so $k \geq 3$. Begin with a $4(k - 1)$-element set E partitioned into four $(k-1)$-element sets P_1, P_2, P_3, and P_4. Let $\mathcal{F} = \{P_1 \cup P_2, P_2 \cup P_3, P_3 \cup P_4, P_4 \cup P_1\}$. Now let $\mathcal{C}$ be the collection of all $(3c+1)$-element subsets of members of $\mathcal{F}$ along with all $(4c - d_1 + 1)$-element subsets of $P_1 \cup P_3$ and, when $d_2 > 0$, all $(4c-d_2+1)$-element subsets of $P_2 \cup P_4$. We use Theorem 6.1 with $m = 4c$ to show that $\mathcal{C}$ is the set of non-spanning circuits of a rank-$(4c)$ matroid M on $P_1 \cup P_2 \cup P_3 \cup P_4$.

Clearly no member of $\mathcal{C}$ properly contains another. Next, we let C_1 and C_2 be distinct members of $\mathcal{C}$, with $e \in C_1 \cap C_2$. If C_1 and C_2 are contained in the same member of $\mathcal{F} \cup \{P_1 \cup P_3, P_2 \cup P_4\}$, then it is easily checked that (ii) holds. Therefore, by symmetry, we may assume that $C_1 \subseteq P_1 \cup P_2$ and C_2 is contained in one of $P_1 \cup P_4$, $P_1 \cup P_3$, and $P_2 \cup P_4$. Thus $|C_1 \cap C_2| \le |P_1| = |P_2| = 2c$ and $|C_2| \in \{3c+1, 4c - d_1 + 1, 4c - d_2 + 1\}$. Hence $|C_2| \ge 3c+1$ since $c \ge d_1 > d_2$. Therefore,

$$\begin{aligned} |(C_1 \cup C_2) - e| &= |C_1| + |C_2| - |C_1 \cap C_2| - 1 \\ &\ge (3c+1) + (3c+1) - 2c - 1 \\ &= 4c + 1 > 4c. \end{aligned}$$

We deduce that, in this case, the hypothesis in (ii) of Theorem 6.1 never holds, so (ii) holds vacuously. Statement (iii) of Theorem 6.1 clearly holds since all members of $\mathcal{C}$ have at most $4c$ members. Finally, M has a $(4c)$-element set that contains no member of $\mathcal{C}$; for example, we can obtain such a set by taking, for arbitrary i, the union of a c-element subset of P_{i+1}, a $(2c)$-element subset of P_{i+2}, and a c-element subset of P_{i+3}.

We conclude, by Theorem 6.1, that M is indeed a rank-$(4c)$ matroid on $P_1 \cup P_2 \cup P_3 \cup P_4$ having $\mathcal{C}$ as its set of non-spanning circuits. Thus, for all i, we have $r(P_i) = |P_i| = k-1 = 2c$ and $r(P_i \cup P_{i+1}) = 3c$. Hence $\lambda(P_i \cup P_{i+1}) = 2c$. Moreover, one easily checks that $(\sqcap(P_1, P_2), \sqcap(P_1, P_3), \sqcap(P_2, P_4)) = (c, d_1, d_2)$. Finally, we note, from the last sentence of the previous paragraph, that $r(P_{i+1} \cup P_{i+2} \cup P_{i+3}) = 4c = r(M)$ for all i. Hence $\lambda(P_i) = k-1$ and (P_1, P_2, P_3, P_4) is indeed a $(2c+1, c, d_1, d_2)$-flower.

We now construct a 4-petal (k, c, d_1, d_2)-flower with $k = 2c+1-j$ and j in $[d_2]$. First, as above, construct a $(k', c-j, d_1-j, d_2-j)$-flower (P_1, P_2, P_3, P_4) in a matroid M with $k' = 2(c-j)+1$. Then each petal P_i is independent. Form M' from M by, for all i in $\{1, 2, 3, 4\}$, freely adding a j-element set X_i of elements to P_i. Let $P'_i = P_i \cup X_i$. Then (P'_1, P'_2, P'_3, P'_4) is a $(2c+1-2j, c-j, d_1-j, d_2-j)$-flower Φ' in M'. Moreover, by Lemma 6.2, Φ' is a $(2c+1-j, c, d_1, d_2)$-flower in the matroid $L^j(M')$ that is obtained by performing a sequence of j Higgs lifts starting with M'. Since $k = 2c+1-j$, we conclude that Φ' is a (k, c, d_1, d_2)-flower in $L^j(M')$. □

An attractive property of a 4-petal k-flower in which the local connectivity between non-consecutive pairs of petals differs is that the matroid in which it is found must be non-representable. This theorem generalizes [7, Corollary 6.2], which shows that a matroid with a Vámos-like 3-flower is non-representable.

Theorem 6.3. *Let M be a matroid having a k-flower (P_1, P_2, P_3, P_4) such that $\sqcap(P_1, P_3) \neq \sqcap(P_2, P_4)$. Then M is non-representable.*

Proof. Ingleton [4] (see also [5, Exercise 6.1.8(v)]) proved a rank inequality that must hold for four subsets X_1, X_2, X_3, and X_4 of a representable matroid. This inequality can be rewritten in terms of local connectivity using Lemma 2.2(ii) as

$$\begin{aligned}\sqcap(X_1, X_2) + \sqcap(X_2, X_3) + \sqcap(X_1, X_4)&\\ \leq \sqcap(X_1 \cup X_3, X_2) + \sqcap(X_2 \cup X_4, X_1) + \sqcap(X_3, X_4).&\end{aligned}$$

We may assume that $\sqcap(P_1, P_3) > \sqcap(P_2, P_4)$. By taking $(P_1, P_2, P_3, P_4) = (X_1, X_3, X_2, X_4)$, we get a contradiction to Ingleton's inequality. □

7. A general formula for local connectivity

Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a polymatroid f on a set E. One can view the complete structural information associated with Φ as consisting of a listing of the values of $\sqcap(B, G)$ for all non-empty disjoint sets B and G each of which is a union of petals of Φ. In this section, we prove that this set of values is uniquely determined by the set of values $\sqcap(P_i, P_j)$, where i and j are distinct elements of $[n]$. In particular, when $n \geq 5$, the set of values $\sqcap(B, G)$ is uniquely determined by c and d, while, when $n = 4$, it is determined by c, d_1, and d_2. Part (iii) of Theorem 1.3 will follow from these results.

As in the proof of Theorem 1.1, we color the petals in B and G black and grey. Those petals in $E - (B \cup G)$ are colored *white*, respectively. As before, we shall be interested in the monochromatic arcs into which this coloring breaks the k-flower $(P_1, P_2, \ldots, P_n)$.

Theorem 7.1. *For some $n \geq 5$, let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a polymatroid f on a set E. For all non-empty disjoint unions of petals B and G of Φ,*

$$\sqcap(B, G) = \begin{cases} (k-1) + (c-d)(b+g-2) & \text{if } w = 0; \\ d + (c-d)(b+g-w) & \text{if } w > 0; \end{cases}$$

where b, g, and w are the numbers of black, grey, and white arcs.

Proof. Suppose first that $w = 0$. Then $b = g$. In this case, we shall prove the result by induction on b. If $b = 1$, then (B, G) is an exact k-separation of f, so $\sqcap(B, G) = k - 1$ and the theorem holds. Now assume the result holds for $b < m$ and let $b = m \geq 2$. Let B' be the union of the petals in some black arc of Φ. Then, by Lemma 2.2(ii), we have

$$\sqcap(B, G) + \sqcap(B', B - B') = \sqcap(B' \cup G, B - B') + \sqcap(B', G).$$

Now, by Lemma 3.6, $\sqcap(B - B', B') = d$ and $\sqcap(B', G) = 2c - d$. To calculate $\sqcap(B' \cup G, B - B')$, we can apply the induction assumption to the recoloring of Φ in which the petals of B' are grey. This recoloring has $b-1$ black arcs and $g-1$ grey arcs. Hence

$$\sqcap(B' \cup G, B - B') = (k-1) + (c-d)((b-1) + (g-1) - 2),$$

so

$$\begin{aligned} \sqcap(B, G) &= (k-1) + (c-d)(b+g-4) + (2c-d) - d \\ &= (k-1) + (c-d)(b+g-2), \end{aligned}$$

as required. We deduce that the theorem holds for $w = 0$.

We complete the proof by arguing by induction on w. Assume the theorem holds for $w < m$ and let $w = m \geq 1$. Take a white arc, the union of whose petals is W' and consider the colors of the two arcs adjacent to it. Clearly there are two cases:

(a) these arcs differ in color;
(b) these arcs are the same color.

In case (a), we recolor the petals in W' black. In the new coloring, we have b black arcs, g grey arcs, and $w-1$ white arcs. By Lemma 2.2(ii), $\sqcap(B, G) = \sqcap(B \cup W', G) + \sqcap(B, W') - \sqcap(B \cup G, W')$. Now, by Lemma 3.6 and the induction assumption, $\sqcap(B, W') = c$ and

$$\begin{aligned} &\sqcap (B \cup W', G) - \sqcap(B \cup G, W') \\ &\qquad = \begin{cases} (k-1) + (c-d)(b+g-2) - (k-1) & \text{if } w = 1; \\ d + (c-d)(b+g-(w-1)) - (2c-d) & \text{if } w > 1. \end{cases} \end{aligned}$$

Thus $\sqcap(B \cup W', G) - \sqcap(B \cup G, W') = d + (c-d)(b+g-w) - c$ so, in case (a), $\sqcap(B, G) = d + (c-d)(b+g-w)$, as required.

Now consider case (b). Without loss of generality, we may assume that the two arcs adjacent to our distinguished white arc are both black. In this case, we recolor the petals in W' black. In the new coloring, we have $b-1$ black arcs, g grey arcs, and $w-1$ white arcs. By Lemma 2.2(ii) again, $\sqcap(B, G) = \sqcap(B \cup W', G) + \sqcap(B, W') - \sqcap(B \cup G, W')$. By Lemma 3.6 and the induction assumption, $\sqcap(B, W') = 2c - d$ and

$$\begin{aligned} &\sqcap (B \cup W', G) - \sqcap(B \cup G, W') \\ &\qquad = \begin{cases} (k-1) + (c-d)((b-1)+g-2) - (k-1) & \text{if } w = 1; \\ d + (c-d)((b-1)+g-(w-1)) - (2c-d) & \text{if } w > 1. \end{cases} \end{aligned}$$

Thus $\sqcap(B \cup W', G) - \sqcap(B \cup G, W') = d + (c-d)(b+g-w) - (2c-d)$ so, in case (b), $\sqcap(B, G) = d + (c-d)(b+g-w)$, and the theorem follows. □

The next result establishes part (iii) of Theorem 1.3. For $n \geq 5$, this corollary is an immediate consequence of the last theorem. For $n \in \{2, 3, 4\}$, the corollary is easily verified using Lemma 2.2 and we omit the details.

Corollary 7.2. *Let $(P_1, P_2, \ldots, P_n)$ be a k-flower Φ in a polymatroid f on a set E. Let I and J be disjoint non-empty subsets of $[n]$. Then $\sqcap(P_I, P_J)$*

can be expressed in terms of $I, J, k, \sqcap(P_1, P_2)$*, and* $\sqcap(P_1, P_3)$ *unless* $n = 4$. *In the exceptional case,* $\sqcap(P_2, P_4)$ *may also be required to specify* $\sqcap(P_I, P_J)$.

8. Flowers for connectivity functions

Let f be a polymatroid on a set E. The connectivity function λ of f is an integer-valued, submodular function such that $\lambda(\emptyset) = 0$ and $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$. Now let λ be an arbitrary function satisfying these conditions. A k-flower for λ is a partition $(P_1, P_2, \ldots, P_n)$ of E into petals $P_1, P_2, \ldots, P_n$ such that, for all i in $[n]$, both $\lambda(P_i)$ and $\lambda(P_i \cup P_{i+1})$ equal $k - 1$. As the reader can easily check, the proofs of Lemma 3.1 and Theorem 1.1 immediately give that, in a k-flower for λ, we have $\lambda(P_I) = k-1$ for all proper non-empty consecutive subsets I of $(1, 2, \ldots, n)$. Moreover, either this equation holds only for such consecutive subsets I, or it holds for all proper non-empty subsets I of $(1, 2, \ldots, n)$.

Acknowledgments

The authors thank Geoff Whittle for his suggestion to investigate this topic and for his many helpful comments during the research. The second author was partially supported by a grant from the National Security Agency.

References

[1] T.H. Brylawski, Constructions, in: Theory of Matroids (ed. N. White), Cambridge University Press, Cambridge, 1986, pp.127–223.

[2] W.H. Cunningham, J. Edmonds, A combinatorial decomposition theory, Canad. J. Math. 32 (1980) 734–765.

[3] J. Edmonds, Submodular functions, matroids, and certain polyhedra, in: Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970, pp.69–87.

[4] A.W. Ingleton, Representation of matroids, in: Combinatorial Mathematics and its Applications (ed. D. J. A. Welsh), Academic Press, London, 1971, pp.149–167.

[5] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[6] J. Oxley, G. Whittle, A note on the non-spanning circuits of a matroid, Europ. J. Combinatorics 12 (1991) 259–261.

[7] J. Oxley, C. Semple, G. Whittle, The structure of 3-separations of 3-connected matroids, J. Combin. Theory Ser. B 92 (2004) 257–293.

[8] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18(1966) 1301–1324.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana, 70803-4918

E-mail address: jaikin@math.lsu.edu

E-mail address: oxley@math.lsu.edu