BOUNDING THE SIZE OF A CONNECTED MATROID

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Abstract. In 2001, we proved that, for a connected matroid M whose largest circuit and largest cocircuit have c and c^* elements, $|E(M)| \leq \frac{1}{2}\mathrm{c}\mathrm{c}^*$. In this paper, we revisit our proof of this result, shortening it by half. We also prove that $AG(3,2)$ is the unique 3-connected matroid attaining equality in the bound. This answers a 2007 question of Royle.

1. Introduction

Our terminology and notation follow [7]. Let M be a matroid. When M has a circuit, its *circumference*, $\mathrm{c}(M)$, is the cardinality of a largest circuit of M. When M has a cocircuit, $\mathrm{c}^*(M) = \mathrm{c}(M^*)$. We proved the following sharp bound on $|E(M)|$ when M is connected [3, Theorem 1.4].

Theorem 1.1. *Let M be a connected matroid having at least two elements. Then*

$$|E(M)| \leq \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor. \tag{1.1}$$

The proof of this theorem relied on the next two results. Let e be an element of a matroid M. When e is in a circuit, we denote by $\mathrm{c}_e(M)$ the cardinality of a largest circuit of M containing e. When e is in a cocircuit, $\mathrm{c}^*_e(M) = \mathrm{c}_e(M^*)$. Guoli Ding (private communication) significantly shortened our original proof of the next theorem. His proof can be found in [7, Theorem 4.3.13].

Theorem 1.2. *Let M be a connected matroid having at least two elements. For each element e of M,*

$$|E(M)| \leq (\mathrm{c}_e(M) - 1)(\mathrm{c}_e(M^*) - 1) + 1. \tag{1.2}$$

For a circuit C of a matroid M, let

$$\mathrm{c}^*(C, M) = \max\{|D^*| : D^* \in \mathcal{C}(M^*) \text{ and } |C \cap D^*| = 2\}.$$

Lemma 1.3. *Let M be a connected matroid having at least two elements. If C is a circuit of M with $|C| = c(M)$, then*

$$|E(M)| \leq \mathrm{c}(M) \left\lceil \frac{\mathrm{c}^*(C, M)}{2} \right\rceil.$$

We shall adapt and shorten the proof of this lemma [3, Lemma 3.2] to give a shorter proof of Theorem 1.1. The following is the main result of this paper.

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Theorem 1.4. *Let M be a connected matroid with at least two elements such that M is both simple and cosimple. Suppose that M is not isomorphic to $U_{2,4}$ or $AG(3,2)$. Then*

$$|E(M)| \leq \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor - 1.$$

The next corollary answers a question of Royle [8] (see also [7, Problem 15.4.5]).

Corollary 1.5. *Let M be a 3-connected matroid with at least five elements. Then*

$$|E(M)| = \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor$$

if and only if $M \cong AG(3,2)$.

We also sharpen Theorem 1.2. Let e be an element of a matroid M. When e is in some circuit, $\mathrm{g}_e(M)$ denotes the cardinality of a smallest circuit of M containing e. When e is in some cocircuit, $\mathrm{g}^*_e(M) = \mathrm{g}_e(M^*)$.

Theorem 1.6. *Let M be a connected matroid having at least two elements. For each element e of M,*

$$|E(M)| \leq (\mathrm{c}_e(M) - 1)(\mathrm{c}_e(M^*) - 1) - (\mathrm{c}_e(M) - \mathrm{g}_e(M)) - (\mathrm{c}^*_e(M) - \mathrm{g}^*_e(M)) + 1.$$

In Sections 2 and 3, we give the shortened proof of Theorem 1.1 and the proof of Theorem 1.6. Sections 4 and 5 review techniques that will be used throughout the paper. In particular, Section 4 introduces a theorem of Seymour [9] that gives conditions under which a k-separation of a restriction of a matroid can be extended to a k-separation of the whole matroid; Section 5 recalls techniques that Tutte [11] introduced for dealing with a matroid that is a union of circuits. In Section 6, we identify a specific structure that can arise in a connected matroid with two disjoint largest circuits. Understanding this structure is crucial in the proof of Theorem 1.4. Sections 7 and 8 develop further tools that are used in the proof of Theorem 1.4. That proof is completed in Section 9.

2. The shortened proof

In this section, we give a proof of Theorem 1.1 that is about half the length of the original proof. This proof is based on Lemma 2.2, a strengthening of Lemma 1.3. We begin with a preliminary result whose routine proof is omitted.

Lemma 2.1. *Let C be a circuit of a matroid M. If $Y' \subseteq Y \subseteq C$ and $|C - Y| \geq 2$, then*

$$\mathrm{c}^*(C - Y, M/Y) \leq \mathrm{c}^*(C - Y', M/Y') \leq \mathrm{c}^*(C, M) \leq \mathrm{c}^*(M).$$

Lemma 2.2. *Let C be a largest circuit of a connected matroid M such that $E(M) \neq C$. Let $\mathcal{F}$ be the family of sets Z that are properly contained in C such that M/Z is connected. If $Y' \in \mathcal{F}$ and Y is a maximal member of $\mathcal{F}$ containing Y', then $|C - Y| \geq 2$, and*

$$|E(M)| \leq \left\lfloor \frac{\mathrm{c}(M)[\mathrm{c}^*(C - Y, M/Y)]}{2} \right\rfloor \leq \left\lfloor \frac{\mathrm{c}(M)[\mathrm{c}^*(C - Y', M/Y')]}{2} \right\rfloor. \tag{2.1}$$

Proof. By Lemma 2.1, it suffices to prove that

$$|E(M)| \leq \left\lfloor \frac{\mathrm{c}(M)[\mathrm{c}^*(C - Y, M/Y)]}{2} \right\rfloor. \tag{2.2}$$

As $E(M) \neq C$ and $Y \subsetneqq C$, we see that $|C - Y| \geq 2$ otherwise M/Y has a loop and at least one other element, so M/Y is disconnected, a contradiction. Let $N = M/Y$ and $D = C - Y$. The maximality of Y implies that N/d is disconnected for each d in D. Thus there are matroids H_d and N_d such that H_d/d is connected, $D \subseteq E(N_d)$, and $N = P_d(H_d, N_d)$, the parallel connection of H_d and N_d with respect to the basepoint d. The next two observations follow from this definition.

2.2.1. *If d_1 and d_2 are distinct elements of D, then*

$$]E(H_{d_1}/d_1) - d_1] \cap [E(H_{d_2}/d_2) - d_2] = \emptyset$$

unless $D = \{d_1, d_2\}$.

2.2.2. *Suppose M/C is connected. Then $|D| = 2$ and $H_d/d = H_e/e$ where $D = \{d, e\}$. Moreover, $E(N_d) = \{d, e\}$.*

To see this, observe from 2.2.1 that $|D| = 2$. Letting $D = \{d, e\}$, we have that M/Y has $\{d, e\}$ as a circuit, and $M/Y/d$ has exactly one component apart from $\{e\}$. Hence $H_d/d = H_e/e$ and $D = \{d, e\}$. Clearly, $E(N_d) = D$. Thus 2.2.2 holds.

Extending this, we also have the following.

2.2.3. *If M/C is disconnected, then all of the matroids H_d/d for d in D can be chosen to be distinct.*

The rest of the proof of the lemma deals separately with the cases when M/C is disconnected and when M/C is connected, starting with the former. For each d in D, let Q_d be a circuit of H_d containing d. Let $X = \cup_{d \in D} Q_d$ and $K = N|X$. Then $\{d\}$ and $Q_d - d$ are series classes of K for each d in D. Hence K can be obtained from the circuit D by attaching each Q_d to it via parallel connection across the basepoint d. Thus $X - D$ is a circuit C' of K. As N is connected, we have the following.

2.2.4. *There is a partition $\{Z, W\}$ of C' such that $N \backslash Z/W$ is connected and $|W|$ is a minimum.*

This choice of W means that it is independent. For each d in D, let $W_d = W \cap (Q_d - d)$. Then W is the disjoint union of the sets in $\{W_d : d \in D\}$. Moreover, $W_d \neq Q_d - d$ for each d otherwise we obtain the contradiction that d is a loop of $N \backslash Z/W$, which is connected having at least two elements. Thus every element of W is a coloop of $K \backslash Z$, so $K \backslash Z/W = K \backslash Z \backslash W$ and this matroid has D as its only circuit. As $K = N|X$, we have $r_N(D \cup W) = r_N(D) + |W|$. But $N = M/Y$, so

$$r_M(D \cup W \cup Y) = r_M(D \cup Y) + |W|.$$

Hence $r_M(C \cup W) = r_M(C) + |W|$. This gives the first part of the following.

2.2.5. *When M/C is disconnected, C is a circuit of M/W and so of $M/W \backslash Z$. Moreover, $M \backslash Z/W$ is connected.*

To prove the second part of this assertion, assume that $M \backslash Z/W$ is disconnected. As $N \backslash Z/W$ is connected and $N = M/Y$, it follows that $M \backslash Z/W$ has a connected component whose ground set is contained in Y. This gives a contradiction because $Y \subsetneqq C$ and C is a circuit of $M \backslash Z/W$. Thus 2.2.5 holds.

2.2.6. *If M/C is disconnected and*

$$|E(M \backslash Z/W)| \leq \left\lfloor \frac{\mathrm{c}(M \backslash Z/W)[\mathrm{c}^*(C - Y, [M \backslash Z/W]/Y)]}{2} \right\rfloor, \tag{2.3}$$

then (2.2) *holds.*

As C' is a circuit of M, and $\{Z, W\}$ is a partition of C',

$$|E(M)| - \mathrm{c}(M) \le |E(M)| - |C'| = |E(M\backslash Z/W)|. \tag{2.4}$$

Substituting into (2.3), we get that

$$|E(M)| - c(M) \le \left\lfloor \frac{\mathrm{c}(M\backslash Z/W)\mathrm{c}^*(C - Y, [M\backslash Z/W]/Y)}{2} \right\rfloor. \tag{2.5}$$

As every cocircuit of M/Y that meets $C - Y$ in exactly two elements also meets C' in at least two elements, it follows that

$$\mathrm{c}^*(C - Y, [M\backslash Z/W]/Y) \le \mathrm{c}^*(C - Y, M/Y) - 2. \tag{2.6}$$

We also have that

$$\mathrm{c}(M\backslash Z/W) \le c(M). \tag{2.7}$$

Substituting from (2.6) and (2.7) into (2.5), we get that

$$\begin{aligned} |E(M)| - \mathrm{c}(M) &\le \left\lfloor \frac{\mathrm{c}(M)[\mathrm{c}^*(C - Y, M/Y) - 2]}{2} \right\rfloor \\ &= \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(C - Y, M/Y)}{2} \right\rfloor - c(M). \end{aligned}$$

Thus

$$|E(M)| \le \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(C - Y, M/Y)}{2} \right\rfloor.$$

Therefore 2.2.6 holds.

2.2.7. *If M/C is connected, then* (2.2) *holds.*

As M/C is connected, it follows that $|D| = 2$. Letting $D = \{d, e\}$, we see that $E(N_d) = \{d, e\}$. Observe that

$$\mathrm{c}_e^*(M/Y\backslash d) = \mathrm{c}^*(\{d, e\}, M/Y) - 1. \tag{2.8}$$

Next we show that

$$2[\mathrm{c}_e(M/Y\backslash d) - 1] \le \mathrm{c}(M). \tag{2.9}$$

Let D' be a circuit of $M/Y\backslash d$ such that $e \in D'$ and $|D'| = c_e(M/Y\backslash d)$. Then $D' - e$ is a circuit of M/C. Letting $L = C \cup (D' - e)$, we see that $(M|L)^*$ is a connected rank-2 matroid having, say, $\{X_1, X_2, \dots, X_n\}$ as its parallel classes. Then $n \ge 3$. We may assume that $X_1 = D' - e$ and that $|X_2| \le |X_3|$. As $L - X_1$ and $L - X_2$ are circuits of M and $C = L - X_1$, we see that $|L - X_1| \ge |L - X_2|$, so $|X_1| \le |X_2|$. Then

$$2[c_e(M/Y\backslash d) - 1] = 2|D' - e| = 2|X_1| \le |X_2| + |X_3| \le |C| \le c(M),$$

and (2.9) holds.

By Theorem 1.2, (2.8), and (2.9),

$$|E(M/Y\backslash d)| - 1 \le [\mathrm{c}_e(M/Y\backslash d) - 1][\mathrm{c}_e^*(M/Y\backslash d) - 1] \le \left\lfloor \frac{\mathrm{c}(M)[\mathrm{c}^*(\{d, e\}, M/Y) - 2]}{2} \right\rfloor.$$

As $C - Y = \{d, e\}$ and $|E(M/Y\backslash d)| - 1 = |E(M)| - |C| = |E(M)| - \mathrm{c}(M)$, we deduce that (2.2) holds. Hence 2.2.7 holds.

Now, assume that (2.2) fails and that (M, C, Y) is a counterexample for which $|E(M)| - |Y|$ is minimal. Then, by 2.2.6 and 2.2.7, we obtain a contradiction. $\square$

Proof of Theorem 1.1. Since $c^*(C, M) \leq c^*(M)$, the result follows from Lemma 2.2 by taking Y to be $\emptyset$. $\square$

3. A BOUND THAT IS INVARIANT UNDER DUALITY

The next two bounds are consequences of the proof of [3, Theorem 2.4].

$$\begin{aligned} |E(M)| - 1 &\leq (\mathrm{c}_e(M) - 1)(\mathrm{c}_e^*(M) - 1) - (\mathrm{c}_e(M) - \mathrm{g}_e(M)), \\ |E(M)| - 1 &\leq (\mathrm{c}_e(M) - 1)(\mathrm{c}_e^*(M) - 1) - (\mathrm{c}_e^*(M) - \mathrm{g}_e^*(M)). \end{aligned}$$

Theorem 1.6 merges these bounds into one that is invariant under duality. This bound is sharp for each value that we choose for $\mathrm{c}_e(M), \mathrm{c}_e^*(M), \mathrm{g}_e(M)$, and $\mathrm{g}_e^*(M)$ with $\min\{\mathrm{c}_e(M), \mathrm{c}_e^*(M)\} \geq 2$ and $\min\{\mathrm{c}_e(M) - \mathrm{g}_e(M), \mathrm{c}_e^*(M) - \mathrm{g}_e^*(M)\} \geq 0$.

Lemma 3.1. *Let e be an element of a connected matroid M. For a circuit C and a cocircuit C^* of M such that $e \in C \cap C^*$,*

$$|E(M)| - |C \cup C^*| \leq (\mathrm{c}_e(M) - 2)(\mathrm{c}_e^*(M) - 2).$$

Proof. If $E(M) = C \cup C^*$, then the result follows because $\mathrm{c}_e(M) \geq 2$ and $\mathrm{c}_e^*(M) \geq 2$. Assume that $E(M) \neq C \cup C^*$. Then $(C \cup C^*) - e$ has a partition $\{Z, W\}$ such that $M \backslash Z / W$ is a connected matroid N with $|E(N)| \geq 2$. Note that $\mathrm{c}_e(N) \leq \mathrm{c}_e(M) - 1$ and $\mathrm{c}_e^*(N) \leq \mathrm{c}_e^*(M) - 1$. Applying Theorem 1.2 to N, we conclude that

$$|E(M)| - |C \cup C^*| = |E(N)| - 1 \leq (\mathrm{c}_e(N) - 1)(\mathrm{c}_e^*(N) - 1) \leq (\mathrm{c}_e(M) - 2)(\mathrm{c}_e^*(M) - 2).$$

$\square$

Proof of Theorem 1.6. In Lemma 3.1, choose each of C and C^* to have minimum size. By orthogonality, $|C \cap C^*| \geq 2$, so $|C \cup C^*| \leq \mathrm{g}_e(M) + \mathrm{g}_e^*(M) - 2$ and the theorem follows. $\square$

4. SEYMOUR'S ARCS THEOREM

A result of Seymour [9] that gives conditions to extend a k-separation of a restriction of a matroid to the matroid itself will be fundamental in this paper. To state this result, we need some more definitions. Let M be a matroid. For $F \subseteq E(M)$, an *F-arc* [9, Section 3] is a minimal non-empty subset A of $E(M) - F$ such that there is a circuit C of M with $C - F = A$ and $C \cap F \neq \emptyset$. Such a circuit C is called an *F-fundamental for A*. Suppose A is an F-arc and $P \subseteq F$. Then $A \to P$ if there is an F-fundamental for A contained in $A \cup P$. Thus $A \not\to P$ denotes that there is no such F-fundamental. Note that A is an F-arc if and only if $A \in \mathcal{C}(M/F) - \mathcal{C}(M)$. We shall use the next three results from Seymour's paper [9, (3.5)–(3.7)].

Lemma 4.1. *If $P \subseteq Z$ and A_1 and A_2 are Z-arcs such that $A_1 \cap A_2 \neq \emptyset$ and $A_2 \not\to P$, then, for every $x \in A_1$ there is a Z-arc A with $x \in A \subseteq A_1 \cup A_2$ such that $A \not\to P$.*

Lemma 4.2. *If A_1, A_2 are Z-arcs with $A_1 \cap A_2 \neq \emptyset$, and $P_1, P_2 \subseteq Z$ are such that $A_1 \not\to P_1$ and $A_2 \not\to P_2$, then there is a Z-arc A that is contained in $A_1 \cup A_2$ such that $A \not\to P_1$ and $A \not\to P_2$.*

Proposition 4.3. *Let M be a connected matroid, and let Z be a non-empty subset of $E(M)$. Let (X_1, X_2) be a partition of $E(M)$ for which no Z-arc intersects both X_1 and X_2, and such that, for each $i \in \{1, 2\}$, if X_i contains a Z-arc A, then X_i contains a Z-fundamental for A. Then $\lambda_M(X_1, X_2) = \lambda_{M|Z}(X_1 \cap Z, X_2 \cap Z)$.*

If M is a matroid, $Z \subseteq E(M)$, and (Y_1, Y_2) a partition of Z, we set

$$\begin{array}{rcl} \mathcal{A} & = & \{A \subseteq E(M) - Z : A \text{ is a } Z\text{-arc of } M\}, \\ \mathcal{A}_0 & = & \{A \in \mathcal{A} : A \not\to Y_1 \text{ and } A \not\to Y_2\}, \\ \mathcal{A}_1 & = & \{A \in \mathcal{A} : A \to Y_1 \text{ and } A \not\to Y_2\}, \\ \mathcal{A}_2 & = & \{A \in \mathcal{A} : A \not\to Y_1 \text{ and } A \to Y_2\}, \\ \mathcal{A}_3 & = & \{A \in \mathcal{A} : A \to Y_1 \text{ and } A \to Y_2\}. \end{array}$$

Clearly $\{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a partition for $\mathcal{A}$. For each $k \in \{1, 2, 3\}$, let

$$U_k = \bigcup_{A \in \mathcal{A}_k} A.$$

Lemma 4.4. *If $\mathcal{A}_0 = \emptyset$, then*

(i) *$U_1 \cap U_2 = \emptyset$; and*
(ii) *for each $i \in \{1, 2\}$, if $A_1 \in \mathcal{A}_3$ and $A_1 \cap U_i \neq \emptyset$, then $A_1 \subseteq U_i$.*

Proof. To prove (i), suppose that $U_1 \cap U_2 \neq \emptyset$. There are Z-arcs $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ such that $A_1 \cap A_2 \neq \emptyset$. By definition, $A_1 \not\to Y_2$ and $A_2 \not\to Y_1$. By Lemma 4.2, $\mathcal{A}_0 \neq \emptyset$, a contradiction. Hence (i) holds.

To prove (ii), we may assume, by symmetry, that $i = 1$. Since $A_1 \cap U_1 \neq \emptyset$, there is an $A_2 \in \mathcal{A}_1$ such that $A_1 \cap A_2 \neq \emptyset$. In particular, $A_2 \not\to Y_2$. By Lemma 4.1, for each $x \in A_1$, there is a Z-arc A with $x \in A \subseteq A_1 \cup A_2$ such that $A \not\to Y_2$. As $\mathcal{A}_0 = \emptyset$, it follows that $A \in \mathcal{A}_1$, so $x \in U_1$. Hence $A_1 \subseteq U_1$ since x was arbitrarily chosen in A_1. Thus (ii) holds. □

Next we state Seymour's Arcs Theorem [9, (3.8)].

Theorem 4.5. *Let M be a matroid, let $Z \subseteq E(M)$, and let (Y_1, Y_2) be a partition of Z. Then exactly one of the following holds.*

(i) *There is a Z-arc A such that $A \not\to Y_1$ and $A \not\to Y_2$.*
(ii) *There is a partition (X_1, X_2) of $E(M)$ such that $X_i \cap Z = Y_i$ for each $i \in \{1, 2\}$, and $\lambda_M(X_1, X_2) = \lambda_{M|Z}(Y_1, Y_2)$.*

The proof of Theorem 4.5 can be adapted to establish the next result. In it, we explicitly describe the partition (X_1, X_2) appearing in Theorem 4.5(ii). This proof, which is essentially Seymour's, is included for completeness.

Theorem 4.6. *Let M be a matroid, let $Z \subseteq E(M)$, and let (Y_1, Y_2) be a partition of Z. If $A \to Y_1$ or $A \to Y_2$ for each Z-arc A, and*

$$U_1 = \bigcup \{A : A \subseteq E(M) - Z \text{ and } A \text{ is a } Z\text{-arc such that } A \not\to Y_2\},$$

then, when $X_1 = Y_1 \cup U_1$ and $X_2 = E(M) - X_1$,

$$\lambda_M(X_1, X_2) = \lambda_{M|Z}(Y_1, Y_2).$$

Proof. We argue by contradiction. Assume the result fails and choose a counterexample M such that $|E(M)|$ is a minimum. It is straightforward to check that

$$Z \neq \emptyset.$$

Next we establish that

$$M \text{ is connected.} \tag{4.1}$$

Suppose that there are non-empty matroids M_1 and M_2 such that $M = M_1 \oplus M_2$. For $\{i, j\} \subseteq \{1, 2\}$ and $k \in \{1, 2, 3\}$, set $(Z_i, Y_{ji}) = (Z \cap E(M_i), Y_j \cap E(M_i))$ and $\mathcal{A}'_i = \{A \subseteq E(M_i) - Z_i : A \text{ is a } Z_i\text{-arc of } M_i\}$. Also let $U_{ki} = U_k \cap E(M_i)$ and $\mathcal{A}_{ki} = \{A \in \mathcal{A}_k : A \subseteq E(M_i)\}$. Then

(a) $\{\mathcal{A}'_1, \mathcal{A}'_2\}$ is a partition of $\mathcal{A}$; and
(b) for each $k \in \{1, 2, 3\}$, the set $\mathcal{A}_k$ has $\{\mathcal{A}_{k1}, \mathcal{A}_{k2}\}$ as a partition, and $\{U_{k1}, U_{k2}\}$ is a partition of U_k.

By the choice of M, for each $i \in \{1, 2\}$, there is a partition (X_{1i}, X_{2i}) of $E(M_i)$ with $X_{1i} = Y_{1i} \cup U_{1i}$ and $X_{2i} = E(M_i) - X_{1i}$ such that

$$\lambda_{M_i}(X_{1i}, X_{2i}) = \lambda_{M_i|Z_i}(Y_{1i}, Y_{2i}). \tag{4.2}$$

By (b), we get that

$$X_{11} \cup X_{12} = [Y_{11} \cup U_{11}] \cup [Y_{12} \cup U_{12}] = Y_1 \cup U_1 = X_1.$$

Hence $X_{21} \cup X_{22} = X_2$. By (4.2),

$$\begin{aligned}\lambda_M(X_1, X_2) &= \lambda_{M_1}(X_{11}, X_{21}) + \lambda_{M_2}(X_{12}, X_{22}) \\ &= \lambda_{M_1|Z_1}(Y_{11}, Y_{21}) + \lambda_{M_2|Z_2}(Y_{12}, Y_{22}) \\ &= \lambda_{M|Z}(Y_1, Y_2),\end{aligned}$$

a contradiction. Therefore (4.1) follows.

We now show that

$$E(M) - Z = U_1 \cup U_2 \cup U_3. \tag{4.3}$$

To see this, take e in $E(M) - Z$ and f in Z. As M is connected, it has a circuit C that contains $\{e, f\}$. Then $C - Z$, which is a subset of $C - f$, is a union of Z-arcs. Hence $e \in U_1 \cup U_2 \cup U_3$, so (4.3) holds.

By Lemma 4.4(i), for each Z-arc A with $A \subseteq U_1$, we have that$A \to Y_1$. Moreover, U_1 is disjoint from $U_2 \cup (U_3 - U_1)$. By Lemma 4.4 and (4.3), we see that $(Y_1 \cup U_1, Y_2 \cup U_2 \cup (U_3 - U_1))$ is a partition (X_1, X_2) of $E(M)$ for which the hypotheses of Proposition 4.3 hold. By that result,

$$\lambda_M(Y_1 \cup U_1, E(M) - (Y_1 \cup U_1)) = \lambda_{M|Z}(Y_1, Y_2),$$

a contradiction. □

The following is an immediate consequence of the last theorem.

Corollary 4.7. *Let M be a matroid, let $Z \subseteq E(M)$, and let (Y_1, Y_2) be a partition of Z. If $A \to Y_2$ for each Z-arc A, then*

$$\lambda_M(Y_1, E(M) - Y_1) = \lambda_{M|Z}(Y_1, Y_2).$$

Moreover, when Y_1 is contained in a series class of $M|Z$, then Y_1 is also contained in a series class of M.

Proof. By the hypothesis, $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_3$. Hence $U_1 = \emptyset$. The partition (X_1, X_2) described in Theorem 4.6 is equal to $(Y_1, E(M) - Y_1)$. The first part of the corollary follows. The second part follows from the first. □

5. TUTTE'S GEOMETRY

In this section, we present some results from Tutte's geometry, a powerful tool for dealing with circuits. A set F in a matroid M is a *Tutte-flat* of M if F is the union of some set of circuits of M, that is, if $E(M)-F$ is a flat of M^*. A Tutte-flat F of M is *connected* if $M|F$ is a connected matroid; otherwise F is *disconnected*. The *dimension* $\dim_M F$ of a Tutte-flat F of M is defined as $|F|-r(F)-1$. Thus a Tutte-flat of M has dimension 0 if and only if it is a circuit of M. Tutte-flats of dimension one and two are called, respectively, *Tutte-lines* and *Tutte-planes* of M. For a Tutte-flat F of M, the partition of $M|F$ into its series classes is called the *canonical partition* of F; it is denoted by $\Pi_M(F)$. For $F' \subseteq F$, observe that F' is a Tutte-flat of M such that $\dim_M F' = \dim_M F - 1$ if and only if $F - F' \in \Pi_M(F)$. In particular, we have the following result, which will be used implicitly throughout the paper.

Proposition 5.1. *Let $\{L_1, L_2, \dots, L_k\}$ be the canonical partition of a Tutte-line L of a matroid M. Then C is a circuit of M contained in L if and only if $C = L - L_i$ for some i in $\{1, 2, \dots, k\}$.*

Tutte [11, 4.23 and 4.22] and [10, (2.3)] proved the following useful results.

Proposition 5.2. *A disconnected Tutte-line of a matroid M contains exactly two circuits of M; a connected Tutte-line of M contains at least three circuits of M.*

Proposition 5.3. *If C_1 and C_2 are different circuits contained in a Tutte-line L of a matroid M, then $L = C_1 \cup C_2$.*

Proposition 5.4. *Let F be a Tutte-flat of dimension d in a matroid M and suppose that $f \in F$. Let F' be the union of all of the circuits of M that are contained in $F - f$. Then F' is a Tutte-flat of dimension $d-1$. Moreover, $F' = F - S$ where S is the series class of $M|F$ containing f.*

The next proposition is implicit in Tutte's paper [11]. For a proof, see Lemos [2, Proposition 2.5].

Proposition 5.5. *Suppose that F is a connected Tutte-flat of a connected matroid M. If $e \in E(M) - F$, then there is a connected Tutte-flat F' of M such that $F \cup e \subseteq F'$ and $\dim_M F' = \dim_M F + 1$.*

Tutte [11, 5.35]. characterized binary matroids in terms of Tutte-lines as follows.

Theorem 5.6. *A matroid M is binary if and only if each connected Tutte-line of M contains exactly three circuits.*

Although the next result is well known, it provides a helpful link between the concepts introduced above.

Lemma 5.7. *Let C be a circuit in a matroid M. A set A is a C-arc if and only if $M|(C \cup A)$ is a connected Tutte-line.*

Lemma 5.8. *Let L be a Tutte-line in a matroid M such that $|\mathcal{C}(M|L)| = m \geq 3$. Suppose that $|X_1| \geq |X_2| \geq \cdots \geq |X_m|$ where $\{X_1, X_2, \dots, X_m\}$ is the canonical partition of L. If C is a largest circuit of M such that $C \subseteq L$, then*

$$|X_m| \leq \frac{1}{m-1} c(M) \text{ and } |L| \leq \frac{m}{m-1} c(M).$$

Moreover, if $c(M) = 2k + \delta$ for integers k and δ such that $\delta \in \{0, 1\}$, then

(i) *$|X_m| \leq k$; and*
(ii) *if $|X_m| = k \geq 2$, then $m = 3$, $|X_1| = k + \delta$ and $|X_2| = k$.*

Proof. We may assume that $C = L - X_m$ because $|L - X_m| \geq |L - X_i|$ for every $i \in [m]$. Consequently $L = C \cup X_m$. Thus

$$c(M) = |C| = |X_1| + |X_2| + \cdots + |X_{m-1}| \geq (m-1)|X_m|. \tag{5.1}$$

Therefore

$$|X_m| \leq \frac{1}{m-1}c(M) \text{ and so } |L| = |C| + |X_m| \leq \frac{m}{m-1}c(M). \tag{5.2}$$

Now, as c$(M) = 2k + \delta$ and $m \geq 3$, it follows, by (5.2), that

$$|X_m| \leq \frac{2k+\delta}{m-1} \leq \frac{2k+\delta}{2} = k + \frac{\delta}{2}.$$

Hence (i) follows. Now, when $|X_m| = k \geq 2$, we need only to establish that $m = 3$ to get (ii). Assume that $m \geq 4$. By (5.1), $2k + 1 \geq 2k + \delta = |C| \geq (m-1)k$ and so $1 \geq (m-3)k$. Thus $k \leq 1$, a contradiction. ☐

Lemma 5.9. *Let C be a largest circuit of a connected matroid M. If $c(M) = 2k+\delta$ for integers k and δ such that $k \geq 1$ and $\delta \in \{0, 1\}$, then $|A| \leq k$ for every C-arc A of M. Moreover, if $k \geq 2$, then $|A| = k$ if and only if $C \cup A$ is a connected Tutte-line of M having exactly three sets in its canonical partition, two of cardinality k and one of cardinality $k + \delta$.*

Proof. If A is a C-arc of M, then $C \cup A$ is a connected Tutte-line of M containing m circuits of M for some $m \geq 3$. The result follows from Lemma 5.8 by taking $X_m = A$. ☐

Let C and D be circuits of a matroid M. We say that D is *C-anchored* if $C \cup D$ is a Tutte-line of M and $|D - C| \geq |D \cap C| > 0$.

Lemma 5.10. *Let C and D be circuits of a matroid M such that D is C-anchored. If C is a largest circuit of M, then $|D - C| = |D \cap C| \leq |C - D|$ and $M|(C \cup D)$ is binary. In particular, $\mathcal{C}(M|(C \cup D)) = \{C, D, C \triangle D\}$.*

Proof. By definition, $C \cup D$ is a connected Tutte-line L because $|D \cap C| > 0$. Let $\{X_1, X_2, \dots, X_m\}$ be the canonical partition of L where $X_m = D - C$. We may assume that $X_{m-1} = C - D$. Then $C \cap D = X_1 \cup X_2 \cup \cdots \cup X_{m-2}$. Because C is a largest circuit of M, for all $i < m$, we have $|L - X_i| \leq |L - X_m| = |C|$, so $|X_i| \geq |X_m|$. By hypotheses, D is C-anchored. Hence

$$|X_m| = |D - C| \geq |D \cap C| = |X_1 \cup \cdots \cup X_{m-2}| = \sum_{i=1}^{m-2} |X_i| \geq (m-2)|X_m|.$$

Therefore $m = 3$ so, by Theorem 5.6, $M|(C \cup D)$ is binary. Moreover, $|D \cap C| = |X_1| = |X_3| = |D - C|$. Now $D = X_1 \cup X_3$ and $C = X_1 \cup X_2$. As $|X_1| = |X_3|$ and $|D| \leq |C|$, it follows that $|X_2| \geq |X_1| = |X_3|$, so $|C - D| \geq |D \cap C| = |D - C|$. ☐

Lemma 5.11. *Let C and D be circuits of a matroid M such that $C \cup D$ is a connected Tutte-line of M. If C is a largest circuit of M, then $|D - C| \leq |D \cap C|$.*

Proof. If this result fails, then $|D - C| > |D \cap C|$ and so D is C-anchored. By Lemma 5.10, $|D - C| = |D \cap C|$, a contradiction. ☐

The proof of the next lemma will rely on the following result of Lemos [1, (3.1)].

Lemma 5.12. *Suppose that a circuit C in a matroid M spans elements a and b. If each of $M|(C \cup a)$ and $M|(C \cup b)$ is binary, then so is $M|(C \cup \{a, b\})$.*

Lemma 5.13. *Let C, D_1, and D_2 be circuits of a matroid M such that D_i is C-anchored for each $i \in \{1, 2\}$. Suppose that C is a largest circuit of M. If $D_1 - C$ and $D_2 - C$ are different series classes of $M|(C \cup D_1 \cup D_2)$, then $D_1 \cap C \subseteq D_2 \cap C$, or $D_2 \cap C \subseteq D_1 \cap C$, or $D_1 \cap D_2 = \emptyset$.*

Proof. By Lemma 5.10, for each i, the set $C \cup D_i$ is a connected Tutte-line of M having $\{D_i - C, D_i \cap C, C - D_i\}$ as its canonical partition. Moreover,

$$|D_i - C| = |D_i \cap C| \leq |C - D_i|. \tag{5.3}$$

Let $N = M|(C \cup D_1 \cup D_2)$. By assumption, N has $D_1 - C$ and $D_2 - C$ as series classes. Take e_i in $D_i - C$ for each i and contract $[(D_1 - e_1) \cup (D_2 - e_2)] - C$ from N to get a matroid N' that has C as a circuit that has e_1 and e_2 in its closure. By Lemma 5.12, N' is binary, so N is binary. Moreover, $r^*(N') = 3 = r^*(N)$

Assume the lemma fails. Then there are elements d_1, d_2, and d_3 such that

$$d_1 \in (D_1 - D_2) \cap C \text{ and } d_2 \in (D_2 - D_1) \cap C \text{ and } d_3 \in (D_1 \cap D_2) \cap C. \tag{5.4}$$

Note that $(D_1 \cap D_2) - C = \emptyset$ because $D_1 - C$ and $D_2 - C$ are different series classes of $M|(C \cup D_1 \cup D_2)$ and so $D_1 \cap D_2 \subseteq C$. Observe that

$$\begin{aligned} |C - (D_1 \cup D_2)| &= |C| - |(D_1 \cup D_2) \cap C| \\ &= |C| - (|D_1 \cap C| + |D_2 \cap C| - |D_1 \cap D_2|) \\ &= |C| - |D_1 \cap C| - |D_2 \cap C| + |D_1 \cap D_2| \\ &\geq 1 \end{aligned}$$

where the inequality holds because, by (5.3), $|D_i \cap C| \leq \frac{|C|}{2}$ for each $i \in \{1, 2\}$ and, by (5.4), $D_1 \cap D_2 \neq \emptyset$. Since $C - (D_1 \cup D_2)$ is nonempty, it follows by (5.4) that N has at least four series classes contained in C. Thus N has at least six different series classes. As $r^*(N) = 3$, it follows that the cosimplification of N is isomorphic to $M(K_4)$ or F_7^*. It cannot be F_7^* because the cosimplification of N contains a spanning circuit and F_7^* does not.

Now

$$D_1 \triangle D_2 \triangle C = (D_1 - C) \cup (D_2 - C) \cup [C - (D_1 \cup D_2)] \cup (D_1 \cap D_2).$$

Since $D_1 \triangle D_2 \triangle C$ is a nonempty disjoint union of circuits of N and $M(K_4)$ has no two disjoint circuits, we deduce that $D_1 \triangle D_2 \triangle C$ is a circuit $\widetilde{D}$ of N. Clearly

$$|\widetilde{D}| = |D_1 - C| + |D_2 - C| + [|C| - |(D_1 \cup D_2) \cap C|] + |D_1 \cap D_2|. \tag{5.5}$$

We see that

$$|(D_1 \cup D_2) \cap C| = |D_1 \cap C| + |D_2 \cap C| - |D_1 \cap D_2| = |D_1 - C| + |D_2 - C| - |D_1 \cap D_2|$$

where the last equality follows by (5.3). Replacing $|(D_1 \cup D_2) \cap C|$ by $|D_1 - C| + |D_2 - C| - |D_1 \cap D_2|$ in (5.5), we get that $|\widetilde{D}| = |C| + 2|D_1 \cap D_2|$, a contradiction because $D_1 \cap D_2 \neq \emptyset$. □

6. TWO DISJOINT LARGEST CIRCUITS WITH A CLEAN INTERACTION

Let C and D be disjoint largest circuits of a connected matroid M. We say that C *forms a flower in* M *with nucleus contained in* D provided that

- (i) there is a partition $\{C_1, C_2, \dots, C_n\}$ of C with $n \geq 2$ such that $C_1, C_2, \dots, C_n$ are series classes of $M|(C \cup D)$; and
- (ii) there are elements $d_1, d_2, \dots, d_n$ of D and circuits $D_1, D_2, \dots, D_n$ of $M|(C \cup D)$ such that, for each $i \in [n]$,
  - (a) $D_i - D = C_i$; and
  - (b) $D_i \cap \{d_1, d_2, \dots, d_n\} = \{d_i\}$.

We say that $\{d_1, d_2, \dots, d_n\}$ is the *nucleus* and $C_1, C_2, \dots, C_n$ are the *petals* of this flower. We keep this notation fixed throughout this section.

Recall that, for $n \geq 2$, the graph C_n^2 is obtained from an n-edge cycle by, for each edge, adding a single edge in parallel to it. A *subdivision* of a matroid N is any matroid that can be obtained from N by a sequence of single-element series extensions.

Lemma 6.1. *$[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$ is connected. Moreover,*

- (i) *$C_i \cup d_i$ is a circuit of $[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$ for each i in $[n]$;*
- (ii) *$[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$ has $C_1, C_2, \dots, C_n$, $\{d_1\}, \{d_2\}, \dots, \{d_n\}$ as its series classes and has C as a circuit; and*
- (iii) *$[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$ is a subdivision of $M(C_n^2)$.*

Proof. Let $N = [M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$. Assume that N is disconnected. As $\{d_1, d_2, \dots, d_n\}$ is a circuit D' of N, it follows that N has a connected component N_1 such that $D' \subseteq E(N_1)$. If N_2 is another connected component of N, then $E(N_2) \subseteq C$. There is an i in $[n]$ such that $C_i \subseteq E(N_2)$ because $C_1, C_2, \dots, C_n$ are series classes of N. Observe that D' is a circuit of $N/E(N_2)$. By (ii), $\{d_i\}$ is a loop of N/C_i and so of $N/E(N_2)$. Thus $D' = \{d_i\}$, a contradiction because $n \geq 2$. We conclude that N is connected. Parts (i)–(iii) follow without difficulty. □

The matroid $[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_8\})$ is obtained from the cycle matroid of the graph in Figure 1 by replacing each edge labelled C_i by a path with edge set C_i. Our goal is to show that $M|(C \cup D)$ is obtained from $[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$ by adding elements in series with the elements of $\{d_1, d_2, \dots, d_n\}$, thereby establishing that $M|(C \cup D)$ is also a subdivision of $M(C_n^2)$.

Lemma 6.2. *$|D_i \cap D| \geq |C_i|$ for each i in $[n]$.*

Proof. As C_i is a series class of $M|(C \cup D)$, it follows that $C_i \cup D$ is a Tutte-line of M. Now $C_i \cup D = D_i \cup D$ and $D_i \cup D$ is connected because C_i is not a circuit of M. The result follows from Lemma 5.11. □

Lemma 6.3. *$C \cup D_i$ is a Tutte-line of M for each i in $[n]$.*

Proof. Let $N = [M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_n\})$. By Lemma 6.1(i) and (ii), C and each $C_i \cup d_i$ are circuits of N. As $(C_i \cup d_i) - C = \{d_i\}$, it follows that $r^*(N|(C \cup d_i)) = 2$. Hence $r^*(M|[(C \cup d_i) \cup (D - \{d_1, d_2, \dots, d_n\})]) = 2$ because $D - \{d_1, d_2, \dots, d_n\}$ is independent in M. Note that $r^*(M|(C \cup D_i)) \leq 2$ since $C \cup D_i \subseteq (C \cup d_i) \cup (D - \{d_1, d_2, \dots, d_n\})$. Thus $r^*(M|(C \cup D_i)) = 2$ as C and D_i are different circuits of M. We conclude that $C \cup D_i$ is a Tutte-line of M. □

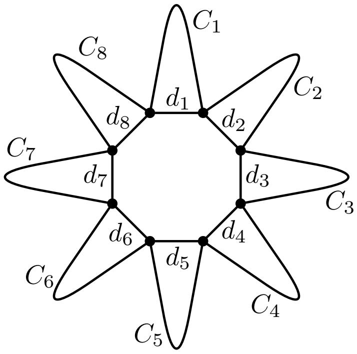

FIGURE 1. $[M|(C \cup D)]/(D - \{d_1, d_2, \dots, d_8\})$ is the cycle matroid of this graph where each edge labelled C_i corresponds to a path.

Lemma 6.4. *For each $i \in [n]$, the set $C \cup D_i$ contains exactly three circuits of M, namely, C, D_i, and $C \triangle D_i$. Moreover, $|D_i - C| = |D_i \cap D| = |D_i \cap C| = |C_i| = |D_i - D|$, and D_i is C-anchored.*

Proof. By Lemma 6.3, $C \cup D_i$ is a Tutte-line of M. By Lemma 6.2, $|D_i - C| = |D_i \cap D| \geq |C_i| = |D_i \cap C|$ and so D_i is C-anchored. The result follows from Lemma 5.10. □

Lemma 6.5. *$D_i \cap D_j = \emptyset$ for all distinct i and j.*

Proof. Assume that $i = 1$ and $j = 2$. By Lemma 6.4, $|D_k \cap D| = |D_k - D|$ for each $k \in \{1, 2\}$. Now, $D_1 - D = C_1$ and $D_2 - D = C_2$. It follows that D_1 and D_2 are D-anchored. As $D_1 - D$ and $D_2 - D$ are different series classes of $M|(D \cup D_1 \cup D_2)$, it follows, by Lemma 5.13, that $D_1 \cap D \subseteq D_2 \cap D$ or $D_2 \cap D \subseteq D_1 \cap D$ or $D_1 \cap D_2 = \emptyset$. The first two possibilities cannot occur since $d_1 \in (D_1 \cap D) - (D_2 \cap D)$ and $d_2 \in (D_2 \cap D) - (D_1 \cap D)$, so the lemma holds. □

Lemma 6.6. *There is a partition $\{Y_1, Y_2, \dots, Y_n\}$ of D such that, for each $i \in [n]$, the set Y_i is a series class of $M|(C \cup D)$ that has cardinality $|C_i|$ and contains d_i, and $D_i = C_i \cup Y_i$. Moreover, the cosimplification of $M|(C \cup D)$ is isomorphic to $M(C_n^2)$.*

Proof. For $i \in [n]$, set $Y_i = D_i - C_i$. By Lemma 6.4, $|C_i| = |Y_i|$. By Lemma 6.5, $Y_1, Y_2, \dots, Y_n$ are pairwise disjoint subsets of D. But

$$|D| = |C| = \sum_{i=1}^{n} |C_i| = \sum_{i=1}^{n} |Y_i| = |Y_1 \cup Y_2 \cup \cdots \cup Y_n| \leq |D|,$$

so $\{Y_1, Y_2, \dots, Y_n\}$ is a partition of D. To complete the proof of the lemma, we need only establish that Y_i is a series class of $M|(C \cup D)$ for each $i \in [n]$. Assume that Y_1, say, is not a series class of $M|(C \cup D)$. Then there is a circuit C' of $M|(C \cup D)$ such that $\emptyset \neq C' \cap Y_1 \neq Y_1$. Suppose that $a \in C' \cap Y_1$ and $b \in Y_1 - C'$. Choose C' so that $|C \cap C'|$ is a minimum.

Next we show that

$$C' \cap (C - C_1) = \emptyset. \tag{6.1}$$

If (6.1) fails, then $C_j \cap C' \neq \emptyset$, for some $j \in [n] - \{1\}$; say $c \in C_j \cap C'$. Now $a \in Y_1$, so $a \in D_1 \cap D$. Hence $a \notin D_j$. By the strong circuit elimination axiom, there is a circuit C'' of $M|(C \cup D)$ such that

$$a \in C'' \subseteq (C' \cup D_j) - c.$$

Observe that $C'' \cap C_j = \emptyset$ because C_j is a series class of $M|(C \cup D)$ and $c \notin C''$. Hence $C'' \cap (C - C_1) \subsetneqq C' \cap (C - C_1)$. Evidently $a \in C'' \cap Y_1$. Moreover, $b \in Y_1$, so $b \notin D_j$. Thus $b \in Y_1 - C''$. Hence C'' contradicts the choice of C'. Thus (6.1) holds.

Now $M|(C_1 \cup D) = M|(D_1 \cup D)$ and this matroid is a Tutte-line. By Lemma 6.4, $|D_1 - D| = |D_1 \cap D|$, so D_1 is D-anchored. By Lemma 5.10, $M|(D_1 \cup D)$ contains exactly three circuits, D_1, D, and $D \triangle D_1$. Because C' contains some but not all of the elements of $D_1 \cap D$, we conclude that C' is not a circuit of $M|(D_1 \cup D)$. This contradicts (6.1) since the latter implies that $C' \subseteq D_1 \cup D$. □

7. An important lemma

The goal of this section is to prove Lemma 7.3, which will be an important step in the proof of Theorem 1.4. This lemma features the class $\mathcal{B}_3(e)$ of matroids that attain equality in the bound in Theorem 1.2. That class was determined in [3, Theorem 4.5 and (4.5.1)]. The statement of that result, which is our next theorem, will require some preliminaries.

A matroid M is *uniform with respect to an element* f if every circuit containing f has the same cardinality, and every cocircuit containing f has the same cardinality. A matroid M is a *uniform series connection across* e of k matroids $M_1, M_2, \dots, M_k$ for some $k \geq 2$ if all of the following hold.

(i) $M_1, M_2, \dots, M_k$ are connected matroids each having at least two elements;
(ii) $E(M_i) \cap E(M_j) = \{e\}$ for every 2-subset $\{i, j\}$ of $[k]$;
(iii) $\mathrm{c}_e^*(M_1) = \mathrm{c}_e^*(M_2) = \cdots = \mathrm{c}_e^*(M_k)$;
(iv) all of $M_1, M_2, \dots, M_k$ are uniform with respect to e; and
(v) M is $S_e(M_1, M_2, \dots, M_k)$, the series connection with basepoint e of M_1, M_2,$\dots$, M_k.

Theorem 7.1. *$\mathcal{B}_3(e)$ is the minimal class $\mathcal{M}$ of matroids with the following properties.*

(i) *Every circuit that contains e and has at least two elements is in $\mathcal{M}$.*
(ii) *The dual of every member of $\mathcal{M}$ belongs to $\mathcal{M}$.*
(iii) *The uniform series connection across e of a collection of members of $\mathcal{M}$ belongs to $\mathcal{M}$.*

Moreover, if $M \in \mathcal{B}_3(e)$, then M is uniform with respect to e, and M or M^* is a circuit or is a uniform series connection of a collection of matroids in $\mathcal{B}_3(e)$.

We shall also use the following result of Wu [12, Theorem 1.1].

Theorem 7.2. *Let M be a connected matroid with at least two elements. For each element e of M,*

$$\mathrm{c}(M) \leq 2[\mathrm{c}_e(M) - 1].$$

Lemma 7.3. *Let M be a connected matroid for which $|E(M)| = \lfloor \mathrm{c}(M)\mathrm{c}^*(M)/2 \rfloor$. Suppose that $M = M_1 \oplus_2 M_2$ with $E(M_1) \cap E(M_2) = \{e\}$. If $\mathrm{c}_e(M_1) \leq \mathrm{c}_e(M_2)$ or $\mathrm{c}_e^*(M_1) \leq \mathrm{c}_e^*(M_2)$, then $M_1 \in \mathcal{B}_3(e)$.*

Proof. Taking the dual when necessary, we may assume that

$$\mathrm{c}_e^*(M_1) \le \mathrm{c}_e^*(M_2). \tag{7.1}$$

Let C be a circuit of M_1 such that $e \in C$ and $|C| = \mathrm{c}_e(M_1)$. Assume that $\mathrm{c}_e^*(M_1) = k+1$ for some $k \ge 1$. Let M_1' be a matroid obtained from $M_1|C$ by adding $k-1$ elements in parallel to each element of $C - e$. This can be done so that $E(M_1') \cap E(M_2) = \{e\}$. Set $M' = M_1' \oplus_2 M_2$. Observe that

$$\mathrm{c}_e(M_1) = \mathrm{c}_e(M_1') \text{ and } \mathrm{c}_e^*(M_1) = \mathrm{c}_e^*(M_1'). \tag{7.2}$$

The structure of M' implies that, for each circuit of M' of cardinality exceeding two, there is a circuit of M of the same cardinality. Thus

$$\mathrm{c}(M) \ge \mathrm{c}(M'). \tag{7.3}$$

Next we show that

$$\mathrm{c}^*(M) = \max\{\mathrm{c}^*(M_2), \mathrm{c}_e^*(M_1) + \mathrm{c}_e^*(M_2) - 2\}. \tag{7.4}$$

Let C^* be a cocircuit of M such that $|C^*| = \mathrm{c}^*(M)$. We may assume that $C^* \subseteq E(M_1) - e$ otherwise the result holds. By the dual of Theorem 7.2, $|C^*| \le 2[\mathrm{c}_e^*(M_1) - 1]$. By (7.1), $2[\mathrm{c}_e^*(M_1) - 1] \le \mathrm{c}_e^*(M_1) + \mathrm{c}_e^*(M_2) - 2$. Since (7.4) also holds in this case, it holds in general.

By (7.1) and (7.2), we can apply (7.4) to M'. Therefore

$$\mathrm{c}^*(M') = \max\{\mathrm{c}^*(M_2), \mathrm{c}_e^*(M_1') + \mathrm{c}_e^*(M_2) - 2\}. \tag{7.5}$$

Combining (7.2), (7.4), and (7.5), we get that

$$\mathrm{c}^*(M) = \mathrm{c}^*(M'). \tag{7.6}$$

By (7.3) and (7.6), we obtain the first inequality in the next display; by (1.1) applied to M', we obtain the second. Thus

$$|E(M)| = \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor \ge \left\lfloor \frac{\mathrm{c}(M')\mathrm{c}^*(M')}{2} \right\rfloor \ge |E(M')|. \tag{7.7}$$

Observe that

$$|E(M')| = |E(M)| - [|E(M_1)| - 1] + [\mathrm{c}_e(M_1) - 1][\mathrm{c}_e^*(M_1) - 1] \ge |E(M)|, \tag{7.8}$$

where the inequality is obtained by applying Theorem 1.2 to M_1. Clearly we must have equality in both (7.7) and (7.8). Hence

$$|E(M_1)| - 1 = [\mathrm{c}_e(M_1) - 1][\mathrm{c}_e^*(M_1) - 1],$$

so $M_1 \in \mathcal{B}_3(e)$. □

8. When equality holds in Theorem 1.1

We begin this section by showing that if equality holds in (1.1), then equality holds in (2.1). Then, in a subsection, we analyze when equality holds in (2.1).

Let C be a largest circuit of a connected matroid M. Let Y' be a subset of C for which $|C - Y'| \ge 2$ and M/Y' is connected. By (2.1),

$$|E(M)| \le \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(C - Y', M/Y')}{2} \right\rfloor. \tag{8.1}$$

We call (M, C, Y') an *optimal triple* if the above conditions hold and equality holds in (8.1).

Lemma 8.1. *Let C be a largest circuit of a connected matroid M and suppose that $|C| \geq 4$. When $|C - Y'| \geq 2$ and M/Y' is connected, if M attains the bound in* (1.1)*, then (M, C, Y') is an optimal triple, and*

$$\mathrm{c}^*(M) = \mathrm{c}^*(C, M). \tag{8.2}$$

Proof. By Lemmas 2.1 and 2.2,

$$|E(M)| \leq \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(C - Y', M/Y')}{2} \right\rfloor \leq \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor.$$

As the first and last terms are equal, equality holds in (8.1), so (M, C, Y') is an optimal triple. As $\mathrm{c}(M) \geq 4$, it follows by Lemma 2.1 that (8.2) holds. □

8.1. **Properties of a matroid attaining equality in** (2.1)**.** Throughout this subsection, we assume that the hypotheses of Lemma 2.2 hold, that is, C is a largest circuit of a connected matroid M such that $E(M) \neq C$, the family $\mathcal{F}$ consists of those sets Z that are properly contained in C such that M/Z is connected, Y' is in $\mathcal{F}$, and Y is a maximal member of $\mathcal{F}$ containing Y'. We also assume that $|C| \geq 4$ and that the notation used in the proof of Lemma 2.2 is in effect. Suppose that (M, C, Y') is an optimal triple. By Lemma 2.2, (M, C, Y) is also an optimal triple. We also assume that

$$M/C \text{ is disconnected.} \tag{8.3}$$

Let $N = M/Y$ and $D = C - Y$. The maximality of Y means that N/d is disconnected for each $d \in D$. Thus there are matroids H_d and N_d such that H_d/d is connected, $D \subseteq E(N_d)$, and $N = P_d(H_d, N_d)$. By 2.2.3, $H_d/d \neq H_e/e$ for each 2-subset $\{d, e\}$ of D. Now, for each d in D, we arbitrarily chose a circuit Q_d of H_d that contains d and we let $X = \cup_{d \in D} Q_d$. Then $\{d\}$ and $Q_d - d$ are series classes of $N|X$. Moreover, $X - D$ is a circuit C' of $N|X$.

By 2.2.6, since we have equality in (2.1), we must have equality in (2.4), so $|C'| = \mathrm{c}(M)$. We must also have equality in (2.5) and in (2.6). Hence we have equality in (2.3). Thus, using Lemma 2.1, we get

$$|E(M \backslash Z/W)| = \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(C - Y, [M \backslash Z/W]/Y)}{2} \right\rfloor \text{ and}$$

$$\mathrm{c}^*(C - Y, [M \backslash Z/W]/Y) = \mathrm{c}^*(C - Y, M/Y) - 2 = \mathrm{c}^*(C - Y', M/Y') - 2.$$

The next lemma, which summarizes the structure of $M|(C \cup C')$, will be used frequently.

Lemma 8.2. *The circuits C and C' are disjoint largest circuits of M, the cosimplification of $M|(C \cup C')$ is isomorphic to $M(C^2_{|D|})$, and C' forms a flower in M for which the nucleus D is contained in C and the set of petals is $\{Q_d - d : d \in D\}$. In particular,*

- (i) *$\{Q_d - d : d \in D\}$ is a partition of C' into series classes of $M|(C \cup C')$;*
- (ii) *there is a partition $\{Y_d : d \in D\}$ of C into series classes of $M|(C \cup C')$;*
- (iii) *for all d in D, both $(Q_d - d) \cup Y_d$ and $(C - Y_d) \cup (Q_d - d)$ are circuits of $M|(C \cup C')$; and*
- (iv) *$|Q_d - d| = |Y_d|$ for all d in D.*

Proof. As $\{Q_d - d : d \in D\}$ is a partition of C' into series classes of $[M|(C \cup C')]/Y$, and C' is a circuit of M, this partition is also a partition of C' into series classes of $M|(C \cup C')$. As C and C' are disjoint largest circuits of M, and Q_d is a circuit of

M/Y for each d in D, we see from Section 6 that C' forms a flower in M for which the nucleus D is contained in C and the set of petals is $\{Q_d - d : d \in D\}$. The rest of the lemma follows by Lemma 6.6. □

The next result follows immediately from the fact that the circuit Q_d was an arbitrarily chosen circuit of H_d containing d.

Lemma 8.3. *For $d \in D$, if Q'_d is a circuit of H_d containing d and $Q'_d \neq Q_d$, then $(C' - Q_d) \cup (Q'_d - d)$ and $(C - Y_d) \cup (Q'_d - d)$ are largest circuits of M and $(Q'_d - d) \cup Y_d$ is a circuit of M. In particular, all circuits of H_d that contain d have the same cardinality.*

Lemma 8.4. *For $d \in D$, if D' is a circuit of $H_d \backslash d$, then D' is a circuit of M.*

Proof. By Proposition 5.5, there is a connected Tutte-line L'' of H_d such that $D' \cup d \subseteq L''$. Let C''_1 and C''_2 be distinct circuits of $H_d|L''$ that contain d. By Lemma 8.3, $(C''_1 - d) \cup Y_d$ and $(C''_2 - d) \cup Y_d$ are circuits, D_1 and D_2, of M. Consider $P' = L'' \cup D$. Observe that $D - d$ is a series class of $(M/Y)|P'$, and $[(M/Y)|P'] \backslash (D - d) = (M/Y)|L''$. Thus P' is a Tutte-plane of M/Y. Let $P = P' \cup Y$. Then $r^*(M|P) = r^*((M/Y)|P')$ since Y is independent in M. As $P = D_1 \cup D_2 \cup C$, it follows that P is a Tutte-plane of M. Let $L = D_1 \cup D_2$. Then $P - L = C - Y_d$. By Proposition 5.3, L is a Tutte-line of M and so $C - Y_d$ is a series class of $M|P$. For $i \in \{1, 2\}$, let $D'_i = (C - Y_d) \cup (C''_i - d)$. By Lemma 8.3, D'_1 and D'_2 are circuits of $M|P$. Let $L' = D'_1 \cup D'_2$. Then L' is a Tutte-flat having dimension b, say. Since $L' \subseteq P$, it follows that $1 \leq b \leq 2$. But $b = 2$ if and only if $L' = P$. As $P - L' = Y_d$, it follows that $b = 1$, so L' is a Tutte-line of M, and Y_d is a series class of $M|P$. Choose d' in $C - Y_d$ and let

$$N' = (M|P)/(Y_d - d)/[(C - Y_d) - d']) = (M|P)/(C - \{d, d'\}).$$

As Y_d and $C - Y_d$ are series classes of $M|P$, it follows that N' is obtained from $(M|[P - (C - Y_d)])/(Y_d - d)$ by adding d' in parallel with d. Since D' is a circuit of $H_d \backslash d$, we deduce that D' is a circuit of $M|P$ and hence of M. □

Lemma 8.5. *For $d \in D$, if A is a C-arc of M such that $A \subseteq E(H_d) - d$, then*

(i) *$A \to Y_d$ and $A \to C - Y_d$.*
(ii) *Y_d and $C - Y_d$ are series classes of $M|[E(H_d) \cup C]$.*

Proof. As A is a circuit of M/C, it is a circuit of H_d/d. Thus A or $A \cup d$ is a circuit of H_d. If A is a circuit of H_d, then, by Lemma 8.4, A is a circuit of M, a contradiction. Hence $A \cup d$ is a circuit of H_d. By Lemma 8.3, $A \cup Y_d$ is a circuit of M, so $A \to Y_d$. Also $A \cup (C - Y_d)$ is a circuit of M, so $A \to C - Y_d$, and (i) holds. Part (ii) follows from (i) by two applications of Corollary 4.7. □

Lemma 8.6. *If $d \in D$, then $(E(H_d), E(M) - E(H_d))$ or $(E(H_d) - d, (E(M) - E(H_d)) \cup d)$ is a 2-separation of M.*

Proof. Suppose first that $|E(H_d)| = 2$. Then $Q_d = E(H_d)$. By Lemma 8.2(iv), $1 = |Q_d - d| = |Y_d|$, so $Y_d = \{d\}$. By Lemma 8.2(iii), $E(H_d)$ is a 2-circuit of M. As $|E(M)| \geq 5$, it follows that $(E(H_d), E(M) - E(H_d))$ is a 2-separation of M.

Now suppose that $|E(H_d)| \geq 3$ and let $U = E(H_d) - d$. As H_d/d is a connected component of N/d, it follows that U is a 2-separating set in N, that is,

$$r_N(U) + r_N(E(N) - U) - r(N) = 1. \tag{8.4}$$

Recall that $Y_d - Y = \{d\}$ and $(C - Y_d) - Y = D - d \neq \emptyset$. Since $U \cup C = E(H_d) \cup C$, Lemma 8.5 implies that Y_d and $C - Y_d$ are series classes of $M|(U \cup C)$. Hence Y is a set of coloops of $M|(U \cup Y) = [M|(U \cup C)]\backslash D$ because both Y_d and $C - Y_d$ meet D. Thus $N|U = [M|(U \cup Y)]/Y = [M|(U \cup Y)]\backslash Y = M|U$, so $r_N(U) = r_M(U)$. Substituting into (8.4), we get

$$r_M(U) + [r_M(E(M) - U) - r_M(Y)] - [r(M) - r_M(Y)] = 1$$

and so U, that is, $E(H_d) - d$, is a 2-separating set in M. Since $|U| \geq 2$ and $|E(M) - U| \geq |C| \geq 4$, it follows that $(E(H_d) - d, (E(M) - E(H_d)) \cup d)$ is a 2-separation of M.

□

9. The circumference decreases by at least two

McMurray, Reid, Wei, and Wu [6, Corollary 1.4] proved that if C is a largest circuit of a connected matroid M and $E(M) \neq C$, then $\mathrm{c}(M/C) \leq \mathrm{c}(M) - 1$. We sharpen this bound when $r(M) \geq 2$.

Theorem 9.1. *Let M be a connected matroid such that $r(M) \geq 2$. If C is a largest circuit of M and $E(M) \neq C$, then*

$$\mathrm{c}(M/C) \leq \begin{cases} \mathrm{c}(M) - 2, & \textit{when } \mathrm{c}(M) \textit{ is even or } \mathrm{c}(M) = 3; \\ \mathrm{c}(M) - 3, & \textit{when } \mathrm{c}(M) \textit{ is odd and } \mathrm{c}(M) \geq 5. \end{cases}$$

Moreover, if M is 3-connected and $\mathrm{c}(M) \geq 5$, then $\mathrm{c}(M/C) \leq \mathrm{c}(M) - 3$.

We need the next results from Tutte [11]. The first is 4.171 and the second is a special instance of 4.26.

Proposition 9.2. *Let P be a Tutte-plane of a matroid M. If L_1 and L_2 are different Tutte-lines of M contained in P, then there is a unique circuit of M contained in $L_1 \cap L_2$.*

Proposition 9.3. *Let P be a connected Tutte-plane of a matroid M. If C is a circuit of M contained in P, then there are connected Tutte-lines L_1 and L_2 of M contained in P such that $C \subseteq L_1 \cap L_2$.*

Let P be a Tutte-plane of a matroid M and C be a circuit of M contained in P. Let $\mathcal{L}$ and $\mathcal{L}'$ be the families of Tutte-lines of M contained in P that, respectively, contain and do not contain C. Then, by using Proposition 5.4 and the fact that a circuit contained in P is the union of all of the series classes of $M|P$ that are contained in that circuit, we see that C can be written in the following two ways:

$$C = \bigcap_{L \in \mathcal{L}} L = \bigcup_{L' \in \mathcal{L}'} (P - L'). \tag{9.1}$$

These two perspectives will be important in our argument below.

The next result is part of Theorem 1.3 of McMurray, Reid, Wei, and Wu [6].

Theorem 9.4. *If C_1 and C_2 are distinct circuits of a connected matroid M with $|C_1| + |C_2| \geq 2\mathrm{c}(M) - 1$, then $r(C_1 \cup C_2) \leq r(C_1) + r(C_2) - 1$.*

When C and D are disjoint circuits of a connected matroid M such that $M|(C \cup D)$ is not connected, Theorem 9.4 implies that

$$|C| + |D| \leq 2\mathrm{c}(M) - 2. \tag{9.2}$$

In the next result, we analyze what happens when equality is attained in (9.2).

Lemma 9.5. *Let C and D be disjoint circuits of a connected matroid such that $\mathrm{c}(M) = 2k + \delta$ for some $k \geq 1$ and some $\delta \in \{0, 1\}$. If $M|(C \cup D)$ is disconnected and $|C| + |D| = 2\mathrm{c}(M) - 2$, then the following hold.*

- (i) *There is a $(C \cup D)$-arc A of M such that $A \not\rightarrow C$ and $A \not\rightarrow D$.*
- (ii) *$|A| = 1$, for every $(C \cup D)$-arc A of M such that $A \not\rightarrow C$ and $A \not\rightarrow D$.*
- (iii) *$\{|C|, |D|\} = \{2k, 2k - 2\}$ when $\delta = 0$; and $\{|C|, |D|\} = \{2k\}$ when $\delta = 1$.*
- (iv) *If e is an element of M such that $M|(C \cup D \cup e)$ is connected, then there are partitions $\{X_e, X'_e\}$ and $\{Y_e, Y'_e\}$ of C and D, respectively, such that $|X_e| = |X'_e|$ and $|Y_e| = |Y'_e|$ while the series classes of $M|(C \cup D \cup e)$ are X_e, X'_e, Y_e, Y'_e, and $\{e\}$.*
- (v) *If $\{e, f\}$ be a 2-subset of $E(M) - (C \cup D)$ such that $M|(C \cup D \cup e)$ and $M|(C \cup D \cup f)$ are connected, then $\{X_e, X'_e\} = \{X_f, X'_f\}$ or $\{Y_e, Y'_e\} = \{Y_f, Y'_f\}$.*
- (vi) *M is not 3-connected.*

Proof. By Theorem 4.5, since M is connected but $M|(C \cup D)$ is not, there is a $(C \cup D)$-arc A of M such that $A \not\rightarrow C$ and $A \not\rightarrow D$, that is, (i) holds. Next, we establish (ii), letting A be an arbitrary $(C \cup D)$-arc of M such that $A \not\rightarrow C$ and $A \not\rightarrow D$. Set $P = C \cup D \cup A$. Observe that P is a connected Tutte-plane of M and $C \cup D$ is a disconnected Tutte-line of M. As $C \cup D = P - A$, we see that A is a series class of $M|P$. Each of the other series classes of $M|P$ is contained in C or in D. Let $\{X_1, X_2, \dots, X_m\}$ and $\{Y_1, Y_2, \dots, Y_n\}$ be the partitions of C and D, respectively, such that X_i and Y_j are series classes of $M|P$ for each $i \in [m]$ and $j \in [n]$. Note that $m \geq 2$ and $n \geq 2$ because $M|P$ is connected. Let $M_C = (M|P)/C$ and $M_D = (M|P)/D$. Then A is a series class of both M_C and M_D, and $(M|P)\backslash A$ has $M|C$ and $M|D$ as its connected components. Take a in A. If $|A| = 1$, then $M|P$ is the series connection of M_C and M_D with respect to the basepoint a. If $|A| > 1$, then we can obtain $M|P$ by contracting $A - a$ from each of M_C and M_D, then taking the series connection of the resulting two matroids with respect to a, and finally adding the elements of $A - a$ in series with a in this series connection. We can consider this operation on M_C and M_D as a generalization of series connection. Now $E(M_C)$ is a Tutte-line of M_C having $\{A, Y_1, Y_2, \dots, Y_n\}$ as its canonical partition. Hence $A \cup (D - Y_j)$ is a circuit of M_C for each $j \in [n]$. Likewise, $A \cup (C - X_i)$ is a circuit of M_D for each $i \in [m]$. Let $C_{ij} = (C - X_i) \cup (D - Y_j) \cup A$. Then C_{ij} is a circuit of M. As $C_{11} \cup C_{22} = C \cup D \cup A$ and $A \subseteq C_{11} \cap C_{22}$, it follows that

$$2\mathrm{c}(M) \geq |C_{11}| + |C_{22}| = |C_{11} \cup C_{22}| + |C_{11} \cap C_{22}| \geq |C| + |D| + 2|A|, \tag{9.3}$$

so $2\mathrm{c}(M) - 2|A| \geq |C| + |D| = 2\mathrm{c}(M) - 2$. Therefore $|A| = 1$, that is, (ii) holds.

Next, we show that (iv) holds. Observe that equality must hold throuhout (9.3). In particular, $C_{11} \cap C_{22} = A$ and $|C_{11}| = |C_{22}| = \mathrm{c}(M)$. From the first identity, we conclude that $m = n = 2$. We may assume that $|X_1| \leq |X_2|$ and $|Y_1| \leq |Y_2|$. As $|C_{11}| = |C_{22}| = \mathrm{c}(M)$, it follows that $|X_1| = |X_2|$ and $|Y_1| = |Y_2|$. We get (iv) by taking $A = \{e\}$ and noting that A is a $(C \cup D)$-arc such that $A \not\rightarrow C$ and $A \not\rightarrow D$. From that, we deduce that $|C|$ and $|D|$ are both even.

Now, we establish (iii). If $\delta = 1$, then $|C| \leq 2k = \mathrm{c}(M) - 1$ and $|D| \leq 2k = \mathrm{c}(M) - 1$, so $2\mathrm{c}(M) - 2 = |C| + |D| \leq 2\mathrm{c}(M) - 2$. We must have equality and (iii) follows in this case. We may now assume that $\delta = 0$. We may also assume that $|C| \geq |D|$. As $|C| + |D| = 2\mathrm{c}(M) - 2 = 4k - 2$ and $|C|$ and $|D|$ are both even, it follows that $|C| = 2k$ and $|D| = 2k - 2$. We conclude that (iii) holds.

Now, we prove (v). Suppose that $\{X_e, X'_e\} \neq \{X_f, X'_f\}$ and $\{Y_e, Y'_e\} \neq \{Y_f, Y'_f\}$. By (iv), $|X_e| = |X'_e| = |X_f| = |X'_f|$ and $|Y_e| = |Y'_e| = |Y_f| = |Y'_f|$. There is $a \in X'_e \cap X'_f$ and $b \in Y'_e \cap Y'_f$. As $r^*([M|(C \cup D \cup e \cup f)]\backslash\{a, b\}) = 2$, it follows that $(X_e \cup Y_e \cup e) \cup (X_f \cup Y_f \cup f)$ is a Tutte-line L_1 of M because $L_1 \subseteq (C \cup D \cup e \cup f) - \{a, b\}$ and $X_e \cup Y_e \cup e$ and $X_f \cup Y_f \cup f$ are different circuits of M. Note that L_1 is connected because $(X_e \cup Y_e \cup e) \cap (X_f \cup Y_f \cup f) \neq \emptyset$. From considering the rank-2 matroid $(M|L_1)^*$, we see that M has a circuit C_1 contained in L_1 such that

$$C_1 \supseteq (X_e \cup Y_e \cup e) \triangle (X_f \cup Y_f \cup f) = (X'_e \cap X_f) \cup (X_e \cap X'_f) \cup (Y'_e \cap Y_f) \cup (Y_e \cap Y'_f) \cup \{e, f\}.$$

Similarly, $(X_e \cup Y_e \cup e) \cup (X'_f \cup Y'_f \cup f)$ is a connected Tutte-line of M containing a circuit C_2 such that

$$C_2 \supseteq (X_e \cup Y_e \cup e) \triangle (X'_f \cup Y'_f \cup f) = (X'_e \cap X'_f) \cup (X_e \cap X_f) \cup (Y'_e \cap Y'_f) \cup (Y_e \cap Y_f) \cup \{e, f\}.$$

As $C \cup D \cup \{e, f\} = C_1 \cup C_2$ and $\{e, f\} \subseteq C_1 \cap C_2$, it follows that

$$2\mathrm{c}(M) \geq |C_1| + |C_2| = |C_1 \cup C_2| + |C_1 \cap C_2| \geq (|C| + |D| + 2) + 2 \geq 2\mathrm{c}(M) + 2.$$

This contradiction implies that (v) holds.

It remains to prove (vi). Let $\mathcal{A}$ be the set of all $(C \cup D)$-arcs of M. Take

$$\begin{aligned} \mathcal{A}_c &= \{A \in \mathcal{A} : A \to C\}, \\ \mathcal{A}_d &= \{A \in \mathcal{A} : A \to D\}, \text{and} \\ \mathcal{A}_m &= \{A \in \mathcal{A} : A \not\to C \text{ and } A \not\to D\}. \end{aligned}$$

If $A \in \mathcal{A}$, then $C \cup D \cup A$ is a Tutte-plane P of M. If P is not connected, then $M|P$ has two connected components because A is not a circuit of M. We have two possibilities for these components.

(1) Their ground sets are C and $D \cup A$. Then $A \in \mathcal{A}_d$ and $A \notin \mathcal{A}_c$.
(2) Their ground sets are $C \cup A$ and D. Then $A \in \mathcal{A}_c$ and $A \notin \mathcal{A}_d$.

If P is connected, then $A \in \mathcal{A}_m$. We deduce that $\{\mathcal{A}_c, \mathcal{A}_d, \mathcal{A}_m\}$ is a partition of $\mathcal{A}$.

By (ii), there is a subset Z of $E(M) - (C \cup D)$ such that $\mathcal{A}_m = \{\{z\} : z \in Z\}$. For a fixed $e \in Z$, we establish that

(3) $\{X_e, X'_e\} = \{X_f, X'_f\}$, for every $f \in Z$; or
(4) $\{Y_e, Y'_e\} = \{Y_f, Y'_f\}$, for every $f \in Z$.

Suppose that neither (3) nor (4) holds. Then $\{X_e, X'_e\} \neq \{X_f, X'_f\}$, for some $f \in Z - e$, and $\{Y_e, Y'_e\} \neq \{Y_g, Y'_g\}$, for some $g \in Z - e$. By (v), $\{Y_e, Y'_e\} = \{Y_f, Y'_f\}$ and $\{X_e, X'_e\} = \{X_g, X'_g\}$. Hence $\{X_f, X'_f\} \neq \{X_g, X'_g\}$ and $\{Y_f, Y'_f\} \neq \{Y_g, Y'_g\}$; a contradiction to (v). Therefore (3) or (4) holds.

By symmetry, we may assume that (3) holds. Next, we establish the following.

(9.4) If $e \in Z$ and $A \in \mathcal{A}_c$, then $A \to X_e$ or $A \to X'_e$.

Assume that $A \not\to X_e$ and $A \not\to X'_e$. For some $n \geq 2$, there is a partition $\{W_1, W_2, \dots, W_n\}$ of C such that $\{W_1, W_2, \dots, W_n, A\}$ is the canonical partition of the connected Tutte-line $A \cup C$. We now show that each W_i is contained in X_e or X'_e. Assume that $a \in W_i \cap X_e$ and $a' \in W_i \cap X'_e$. Then $(A \cup C) - W_i$ is a circuit C' of M avoiding $\{a, a'\}$. Moreover, $X_e \cup Y_e \cup e$ is a circuit C'' of M. Now let $K = M|(C \cup D \cup A \cup e)$. Then K is a Tutte-flat of M of dimension 3. Because $A \in \mathcal{A}_c$, we see by (2) and the fact that $\{\mathcal{A}_c, \mathcal{A}_d, \mathcal{A}_m\}$ is a partition of $\mathcal{A}$ that $M|(A \cup C)$ and $M|D$ are the components of $K\backslash e$. Thus $M|[(C' \cup C'') - e] = [M|(C' \cup X_e)] \oplus [M|Y_e]$ as $C' \cup X_e \subseteq (A \cup C) - a'$. The elements of Y_e are coloops of $M|[(C' \cup C'') - e]$ as are

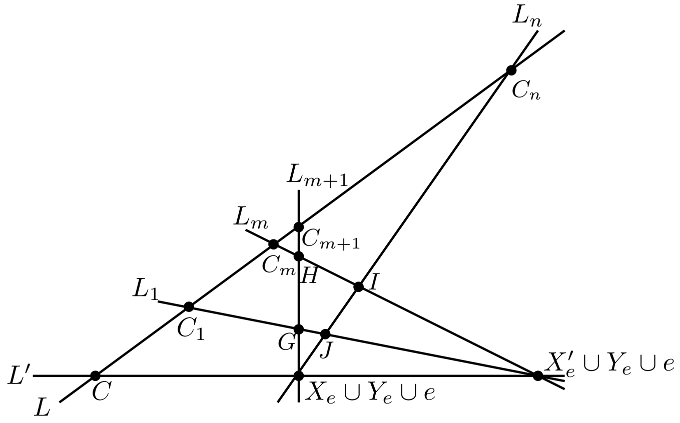

FIGURE 2. The geometry of the Tutte-plane P of M where $D_{1,m+1}, D_{m,m+1}, D_{m,n}$, and $D_{1,n}$ are labelled by G, H, I, and J.

the elements of $X_e - C'$. Thus C' is the unique circuit contained in $(C' \cup C'') - e$. It follows by Proposition 5.4 that $C' \cup C''$ is a Tutte-line. Thus $C' \bigtriangleup C''$, which equals $(W_i \cap X_e) \cup (X'_e - W_i) \cup A \cup Y_e \cup e$, is contained in a circuit C_1 of M. Similarly, there is a circuit C_2 of M containing $(W_i \cap X'_e) \cup (X_e - W_i) \cup A \cup Y_e \cup e$. Note that

$$|C_1|+|C_2| = |C|+2|A|+2|Y_e \cup e| = |C|+|D|+2|A|+2|\{e\}| \geq |C|+|D|+4 = 2\mathrm{c}(M)+2,$$

a contradiction. Thus W_i is contained in X_e or X'_e, for every $i \in [n]$, so X_e and X'_e have partitions $\{W_1, W_2, \ldots, W_m\}$ and $\{W_{m+1}, W_{m+1}, \ldots, W_n\}$, respectively. Moreover, as $A \not\rightarrow X_e$ and $A \not\rightarrow X'_e$, we see that $2 \leq m \leq n-2$. Since $C \cup A \cup Y_e \cup e$ is a Tutte-flat P, while $C \cup A$ is a Tutte-line and the elements of Y_e are coloops of $M|(C \cup A \cup Y_e)$, it follows by Proposition 5.4 that P is a Tutte-plane. Let $L = C \cup A$ and $L' = C \cup Y_e \cup e$. Because $\{A, Y_e \cup e\}$ is a partition of $P - C$, we see that L and L' are the only Tutte-lines contained in P that contain C. As $M|L$ and $M|L'$ are connected, so is $M|P$. For each i in $[n]$, let $C_i = L - W_i$. By Proposition 9.3, for each $i \in [n]$, there is a connected Tutte-line L_i of $M|P$ such that $L_i \neq L$ and $C_i \subseteq L \cap L_i$. By Proposition 9.2, M has a circuit D_i such that $D_i \subseteq L_i \cap L'$.

We show next that, when $i \in [m]$, we have $D_i = X'_e \cup Y_e \cup e$. By Proposition 5.3, $L_i = C_i \cup D_i$. Observe that L', which equals $C \cup Y_e \cup e$, contains exactly three circuits, namely, C, $X_e \cup Y_e \cup e$, and $X'_e \cup Y_e \cup e$. But $D_i \neq C$ since $C_i \cup C = L$. Moreover, $D_i \neq X_e \cup Y_e \cup e$ because $C_i \cup (X_e \cup Y_e \cup e) = P$. We conclude that $D_i = X'_e \cup Y_e \cup e$ as asserted. A symmetric argument gives that if $i \in [n] - [m]$, then $D_i = X_e \cup Y_e \cup e$.

For i in $[m]$ and j in $[n] - [m]$, Proposition 9.2 guarantees the existence of a circuit D_{ij} of M that is contained in both L_i and L_j. Because $L_i = C_i \cup D_{ij}$ and $L = C_i \cup C$, we see that D_{ij} is not contained in L. Moreover, D_{ij} is not contained in L' since $L' = C \cup (X'_e \cup Y_e \cup e) = C \cup (X_e \cup Y_e \cup e)$, while $L_i = C_i \cup (X'_e \cup Y_e \cup e)$ and $L_j = C_j \cup (X_e \cup Y_e \cup e)$. Figure 2 shows the geometry of the Tutte plane P.

We now show that L_i and L_j are the only Tutte-lines of $M|P$ containing D_{ij}. Let L'' be a Tutte-line of $M|P$ containing D_{ij}. By Proposition 9.2, $L'' \cap L' \supseteq D'$ for some circuit D'. By Proposition 5.3, $L'' = D_{ij} \cup D'$. Hence D' is one of the three circuits contained in L', namely, $C, X_e \cup Y_e \cup e$, and $X'_e \cup Y_e \cup e$. But

$D' \neq C$ as L and L' are the only Tutte-lines of $M|P$ containing C. If $D' = X'_e \cup Y_e \cup e$, then $L'' = L_i$, while if $D' = X_e \cup Y_e \cup e$, then $L'' = L_j$. Thus L_i and L_j are indeed the only Tutte-lines of $M|P$ containing D_{ij}. Observe that $L_i = C_i \cup (X'_e \cup Y_e \cup e) = [(C \cup A) - W_i] \cup (X'_e \cup Y_e \cup e) = P - W_i$. Similarly, $L_j = P - W_j$. By (9.1), $D_{ij} = P - (W_i \cup W_j)$. We can choose i and j such that $2|W_i| \leq |X_e|$ and $2|W_j| \leq |X'_e|$. Hence $|W_i| + |W_j| \leq |X_e| = |X'_e|$, so

$$|D_{ij}| = |P| - (|W_i| + |W_j|) \geq (|C| + |A| + |Y_e \cup e|) - |X_e| = |X_e| + |Y_e| + |A| + 1.$$

Since $|C| + |D| = 2\mathrm{c}(M) - 2$, it follows by that $|X_e| + |Y_e| = \mathrm{c}(M) - 1$ Therefore $|D_{ij}| \geq \mathrm{c}(M) + 1$, a contradiction. We conclude that (9.4) holds.

Recall that we are assuming that (3) holds. Observe that $\{X'_e, X_e \cup D \cup e\}$ is a 2-separation for $M|(C \cup D \cup e)$ for a fixed $e \in Z$. Assume that M is 3-connected. Then $|E(M)| \geq |C| + |D| + 2$. Moreover, by Theorem 4.5, there is a $(C \cup D \cup e)$-arc A such that $A \not\to X'_e$ and $A \not\to X_e \cup D \cup e$. Note that, as A is a $(C \cup D \cup e)$-arc, it is also a $(C \cup D)$-arc, so $A \in \mathcal{A}$. Hence $A \in \mathcal{A}_d \cup \mathcal{A}_c \cup \mathcal{A}_m$. If $A \in \mathcal{A}_d$, then $A \to D$, so $A \to X_e \cup D \cup e$. If $A \in \mathcal{A}_c$, then, by (9.4), $A \to X'_e$ or $A \to X_e$, so $A \to X'_e$ or $A \to X_e \subseteq X_e \cup D \cup e$. If $A \in \mathcal{A}_m$, then $A = \{f\}$ for some $f \in Z - e$. By (3), $X_f = X_e$, so, by (iv), $X_e \cup A \cup Y_f$ is a circuit of M. Thus $A \to X_e \cup D \cup e$. This contradiction completes the proof of (vi). □

The next three results are, respectively, Proposition 1, Proposition 3, and Theorem 8 of Maia [5].

Theorem 9.6. *Let M be a simple connected matroid such that $r(M) \geq 2$. If $e \in E(M)$, then the following are equivalent.*

(i) *M/e is connected and $\mathrm{c}_e(M) = 3$.*
(ii) $\mathrm{c}(M) = 3$.
(iii) *$r(M) = 2$.*
(iv) *$M \cong U_{2,|E(M)|}$.*

Proposition 9.7. *Let M be a simple connected matroid. If $\mathrm{c}(M) = 4$ and M is not 3-connected, then, for some $n \geq 2$, there are connected rank-2 matroids $M_1, M_2, \dots, M_n$ with $E(M_i) \cap E(M_j) = \{p\}$ for all $i \neq j$ such that*

$$M = P(M_1, M_2, \dots, M_n) \text{ or } M = P(M_1, M_2, \dots, M_n)\backslash p.$$

Theorem 9.8. *Let M be a 3-connected matroid such that $r(M) \geq 4$ and $\mathrm{c}(M) \leq 5$. If M has an element e such that $\mathrm{c}_e(M) = 4$, then M is isomorphic to F_7^* or $AG(3, 2)$.*

Proof of Theorem 9.1. Since M is connected and $r(M) \geq 2$, we have that $\mathrm{c}(M) \geq 3$. Suppose first that $\mathrm{c}(M) = 3$. Then, by Theorem 9.6, $r(M/C) = 0$, so $\mathrm{c}(M/C) = 1 = \mathrm{c}(M) - 2$ and the result holds in this case. Next suppose that $\mathrm{c}(M) = 4$ and M is not 3-connected. Then, by Proposition 9.7, each connected component of M/C has rank equal to 0 or 1, so $\mathrm{c}(M/C) \leq 2 = \mathrm{c}(M) - 2$ and again the result holds. Now, suppose that $\mathrm{c}(M) = 4$ and M is 3-connected. If $r(M) = 3$, then $\mathrm{c}(M/C) = 1 < \mathrm{c}(M) - 2$, so we may assume that $r(M) \geq 4$. Then, by Theorem 9.6, for e in $E(M)$, we have $\mathrm{c}_e(M) = \mathrm{c}(M) = 4$. By Theorem 9.8, M is isomorphic to F_7^* or $AG(3, 2)$. Therefore $\mathrm{c}(M/C) \leq 2 = \mathrm{c}(M) - 2$. We conclude that the result holds for $\mathrm{c}(M) < 5$.

Suppose that $\mathrm{c}(M) = 2k + \delta$ for some $k \geq 2$ and $\delta \in \{0, 1\}$ satisfying $(k, \delta) \neq (2, 0)$. Let D be a circuit of M/C such that $|D| = \mathrm{c}(M/C)$. We may assume that $|D| \geq \mathrm{c}(M) - 2 = 2k + \delta - 2$ otherwise the result holds. Now, we show that D

is not a C-arc of M. If D is a C-arc of M, then, by Lemma 5.9, $|D| \leq k$. As $|D| \geq 2k+\delta-2$, it follows that $2k+\delta-2 \leq k$ and so $k+\delta \leq 2$, a contradiction. Hence D is not a C-arc of M. Therefore D is a circuit of M such that $C \cup D$ is a disconnected Tutte-line of M. Then, by Theorem 9.4, $|D| = \mathrm{c}(M)-2$. By Lemma 9.5(vi), M is not 3-connected. By Lemma 9.5(iii), $|C| = 2k$, so $\mathrm{c}(M)$ is even and the result follows. □

Now, we can prove the following.

Proposition 9.9. *Let M be a 3-connected matroid such that M/C is connected for some largest circuit C of M. If $\mathrm{c}(M^*) \geq \mathrm{c}(M) \geq 5$, then*

$$|E(M)| \leq \left\lfloor \frac{[\mathrm{c}(M)-1]\mathrm{c}^*(M)}{2} \right\rfloor \leq \left\lfloor \frac{\mathrm{c}(M)\mathrm{c}^*(M)}{2} \right\rfloor - 2.$$

Proof. By Theorem 9.1, $\mathrm{c}(M/C) \leq \mathrm{c}(M)-3$. By Theorem 1.1,

$$|E(M/C)| \leq \frac{\mathrm{c}(M/C)\mathrm{c}^*(M/C)}{2} \leq \frac{[\mathrm{c}(M)-3]\mathrm{c}^*(M)}{2}.$$

Thus

$$2|E(M)| - 2|C| \leq [\mathrm{c}(M)-3]\mathrm{c}^*(M).$$

Hence

$$2|E(M)| \leq \mathrm{c}(M)\mathrm{c}^*(M) - 2[\mathrm{c}^*(M)-\mathrm{c}(M)] - \mathrm{c}^*(M),$$

so

$$2|E(M)| \leq \mathrm{c}(M)\mathrm{c}^*(M) - \mathrm{c}^*(M) = [\mathrm{c}(M)-1]\mathrm{c}^*(M).$$

The result follows since $\mathrm{c}^*(M) \geq 5$. □

Corollary 1.5 follows immediately from the next result.

Theorem 9.10. *Let M be a 3-connected matroid with at least two elements. If $|E(M)| = \lfloor \mathrm{c}(M)\mathrm{c}^*(M)/2 \rfloor$, then M is isomorphic to $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, $U_{2,4}$, or $AG(3,2)$.*

Proof. Taking the dual when necessary, we may suppose that $\mathrm{c}(M^*) \geq \mathrm{c}(M)$. By Proposition 9.9, we have two possibilities:

1. $\mathrm{c}(M) \leq 4$; or
2. M/C is not connected for every largest circuit C of M.

Suppose that (1) occurs. If $\mathrm{c}(M) = 2$, then $r(M) = 1$ and so M is isomorphic to $U_{1,2}$ or $U_{1,3}$. If $\mathrm{c}(M) = 3$, then, by Theorem 9.6, M is isomorphic to $U_{2,n}$ for some $n \geq 3$. In this case, $n \in \{3,4\}$ because $\mathrm{c}(U_{2,n}) = 3$ and $\mathrm{c}(U^*_{2,n}) = n-1$. Assume that $\mathrm{c}(M) = 4$. If $r(M) \geq 4$, then, by Theorems 9.6 and 9.8, M must be isomorphic to $AG(3,2)$. It remains to deal with the case when $r(M) = 3$. Let C^* be a largest circuit of M. As $\mathrm{c}(M) = 4$ and $|E(M)| = \lfloor \mathrm{c}(M)\mathrm{c}^*(M)/2 \rfloor$, we conclude that $|E(M)-C^*|+|C^*| = |E(M)| = 2|C^*|$, so $|E(M)-C^*| = |C^*|$. Note that $E(M)-C^*$ is a shortest line L of M, say $L = \{e_1, e_2, \dots, e_n\}$. Then $n \geq 3$ otherwise $M \cong U_{3,4}$, which is not 3-connected. Fix $e \in C^*$ and let L_i be the line of M containing e and e_i. Then $n = |L| \leq |L_i|$. As $C^* - e = \cup_{i \in [n]}(L_i - \{e, e_i\})$, we conclude that

$$n = |L| = |C^*| = 1 + \sum_{i=1}^{n}(|L_i|-2) \geq 1 + n(n-2) \geq n+1,$$

where the last inequality follows because $n \geq 3$. This contradiction implies that (1) does not hold, so M satisfies (2). Let C be a largest circuit of M. By Lemma 8.6, M has a 2-separation, a contradiction. $\square$

Proof of Theorem 1.4. Let M be a connected matroid with at least two elements. Suppose M has only trivial series and parallel classes and $|E(M)| = \lfloor \mathrm{c}(M)\mathrm{c}^*(M)/2 \rfloor$. By Theorem 9.10, either M is isomorphic to $U_{2,4}$ or $AG(3,2)$, or M is not 3-connected. Suppose that M is not 3-connected. Then $M = M_1 \oplus_2 M_2$ for some connected matroids M_1 and M_2 with $E(M_1) \cap E(M_2) = \{e\}$, say. We may assume that $\mathrm{c}_e(M_1) \leq \mathrm{c}_e(M_2)$. By Lemma 7.3, $M_1 \in \mathcal{B}_3(e)$. Since M has only trivial series and parallel classes, M_1 is neither a circuit nor a cocircuit. It follows by Theorem 7.1 that M_1 has a non-trivial series or parallel class avoiding e. This is a contradiction as this class is also a non-trivial series or parallel class of M. $\square$

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