BRIEFLY, WHAT IS A MATROID?

JAMES OXLEY

Abstract. Matroids were introduced in 1935 by Whitney and Nakasawa independently. These notes are intended to provide a brief introduction to the study of matroids beginning with two basic examples, matroids arising from graphs and matroids coming from matrices. Some aspects of the basic theory of matroids will be developed around these fundamental examples. No proofs will be included here. These may be found in the author's book.

1. Fundamental examples and definitions

In 1935, Hassler Whitney published a paper [18] entitled "On the abstract properties of linear dependence". The same year, Takeo Nakasawa published the first of a series of three papers dealing with similar ideas [6, 7, 8]. Both authors introduced what Whitney called "matroids". Whereas Whitney became a famous mathematician whose obituary appeared in *The New York Times*, Nakasawa died in obscurity at the age of 33 and has received only minimal recognition for his contributions [9].

Example 1.1. For the graph G in Figure 1, the edge set E is $\{1, 2, \ldots, 8\}$ and the set $\mathcal{C}$ of edge sets of cycles is $\{\{8\}, \{2, 3\}, \{2, 4, 5\}, \{3, 4, 5\}, \{4, 6, 7\}, \{2, 5, 6, 7\}, \{3, 5, 6, 7\}\}$. The pair $(E, \mathcal{C})$ is an example of a matroid. One way to begin to get intuition for matroids is to consider them as consisting of a finite set and a set of special subsets of that set that behave somewhat like the edge sets of cycles in a graph. Like most initial approximations, this notion will need considerable refinement.

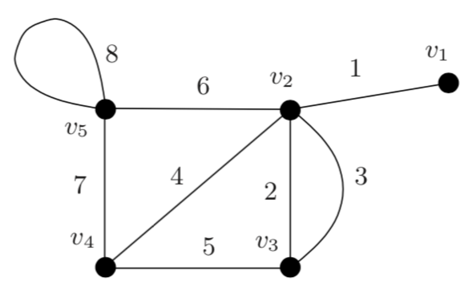

Figure 1. The graph G in Example 1.1.

Definition 1.2. A *matroid* M is a pair $(E, \mathcal{C})$ consisting of a finite set E, called the *ground set*, and a set $\mathcal{C}$ of subsets of E, called *circuits*, that obey the following three conditions.

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If C_1 and C_2 are in $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then $\mathcal{C}$ contains a member C_3 such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

In a matroid M, we often write $E(M)$ for the ground set and $\mathcal{C}(M)$ for the set of circuits, especially when several matroids are being considered.

Theorem 1.3. *Let G be a graph with edge set E and $\mathcal{C}$ be the set of edge sets of cycles of G. Then $(E, \mathcal{C})$ is a matroid.*

The proof of this result is straightforward. The matroid whose existence is asserted there is called the *cycle matroid* of the graph G and is denoted by $M(G)$.

In Example 1.1, both the loop 8 and the pair $\{2, 3\}$ of parallel edges correspond to circuits in $M(G)$. We call 8 a *loop*, and 2 and 3 *parallel elements* in the matroid $M(G)$.

In Example 1.1, the sets $\{1, 5, 7\}$ and $\{1, 4, 5, 6\}$ are both edge sets of forests of G. Indeed, a set X of edges in a graph H is the edge set of a forest if and only if no cycle of H has its edge set contained in X. This idea is generalized as follows.

Definition 1.4. Let M be the matroid $(E, \mathcal{C})$. A subset I of E is *independent* in M if no circuit of M is contained in I. A set that is not independent is called *dependent*. The set of independent sets of M is denoted by $\mathcal{I}(M)$.

Clearly a set C is a circuit in a matroid M if and only if C is a minimal dependent set of M. While the set $\mathcal{C}(M)$ of circuits of M certainly determines the set of independent sets of M, if we know the set of independent sets, then we know the set of dependent sets and the minimal ones of those are the circuits. Thus matroids are often described by listing the independent sets rather than the circuits.

Example 1.5. Let A be the following matrix over the field $\mathbb{R}$ of real numbers.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Evidently, A is the vertex-arc incidence matrix for the directed graph that is obtained from the graph G in Example 1.1 by directing each edge v_iv_j with $i \le j$ from v_i to v_j. Thus the column corresponding to the loop 8 is the zero vector. Now let $E = \{1, 2, \ldots, 8\}$ and let $\mathcal{C}$ be the set of subsets X of E such that the multiset of columns labelled by X is not a linearly independent set but, for every proper subset X' of X, the multiset of columns labelled by X' is a linearly independent set. Then, for example, $\{2, 3\} \in \mathcal{C}$ since the columns labelled by 2 and 3 are equal. Noting that $\{8\}$ is in $\mathcal{C}$ but that no member of $\mathcal{C}$ contains 1, it is straightforward to check that $\mathcal{C} = \{\{8\}, \{2, 3\}, \{2, 4, 5\}, \{3, 4, 5\}, \{4, 6, 7\}, \{2, 5, 6, 7\}, \{3, 5, 6, 7\}\}$. Thus the pair $(E, \mathcal{C})$ is precisely the matroid we considered in Example 1.1. Hence the particular matrix A chosen above gives rise to a matroid. The next theorem notes that every matrix over every field yields a matroid in precisely this way.

Expressing the theorem in terms of independent sets rather than circuits yields a somewhat cleaner statement.

Theorem 1.6. *Let E be the set of column labels of an $m \times n$ matrix over a field $\mathbb{F}$ and let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labelled by X is a set that is linearly independent over $\mathbb{F}$. Then $\mathcal{I}$ is the set of independent sets of a matroid on E.*

The matroid whose existence is asserted by the last theorem is called the *vector matroid* of the matrix A and is denoted by $M[A]$.

The next theorem characterizes precisely which sets of subsets of a set can be the set of independent sets of a matroid.

Theorem 1.7. *Let $\mathcal{I}$ be a set of subsets of a finite set E. Then $\mathcal{I}$ is the set of independent sets of a matroid on E if and only if $\mathcal{I}$ satisfies the following conditions.*

(I1) *$\mathcal{I}$ is non-empty.*

(I2) *Every subset of a member of $\mathcal{I}$ is also in $\mathcal{I}$.*

(I3) *If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\}$ is in $\mathcal{I}$.*

By **(I2)**, to specify a matroid M we need not list all of its independent sets; it suffices to list the maximal independent sets. These maximal independent sets are the *bases* of M. Observe that, by **(I3)**, all bases of M have the same cardinality. This cardinality is the *rank* $r(M)$ of M. Thus, for example, $r(M[A]) = 4$ for the real matrix A in Example 1.5. This coincides with the rank of the matrix A. This is one of many instances where the terminology of linear algebra is carried over into matroid theory.

We observed that the cycle matroid $M(G)$ of the graph G in Example 1.1 and the vector matroid of the matrix A in Example 1.5 are equal since they have the same ground sets and the same sets of circuits. More generally, two matroids are isomorphic if they have the same structure.

Definition 1.8. Let M_1 and M_2 be the matroids $(E_1, \mathcal{C}_1)$ and $(E_2, \mathcal{C}_2)$. We say that M_1 and M_2 are *isomorphic* and write $M_1 \cong M_2$ if there is a bijection $\varphi : E_1 \to E_2$ such that a subset C of E_1 is a circuit of M_1 if and only if $\varphi(C)$ is a circuit of M_2.

Definition 1.9. A matroid M is *graphic* if $M \cong M(G)$ for some graph G.

Definition 1.10. A matroid M is $\mathbb{F}$-*representable* if $M \cong M[A]$ for some matrix A over the field $\mathbb{F}$. When the latter occurs, the matrix A is called an $\mathbb{F}$-*representation* of M. We call M *binary* if it is $GF(2)$-representable; M is *ternary* if it is $GF(3)$-representable.

Example 1.11. Take the matrix A from Example 1.5 and view it over $GF(2)$ instead of over $\mathbb{R}$ (so $-1 = 1$) to get the matrix

$$A_2 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Then one can check that $M[A_2] = M(G) = M[A]$. Thus $M(G)$ is binary. In fact, if we view the original matrix A over any field $\mathbb{F}$, then $M[A] = M(G)$. Such a matroid is called regular.

Definition 1.12. A matroid M is *regular* if M is representable over every field.

The examples above illustrate two results that hold in general.

Theorem 1.13. *Every graphic matroid is binary.*

Theorem 1.14. *Every graphic matroid is regular.*

The reader may want to contemplate what property of the matrix A in our example means that $M[A]$ is regular. This question will be answered later.

Definition 1.15. Let E be an n-element set and r be an integer with $0 \leq r \leq n$. The set of subsets of E with at most r elements is the set of independent sets of a matroid $U_{r,n}$ with ground set E. This matroid has rank r and is called the *uniform matroid* of rank r on an n-element set.

Example 1.16. Observe that $U_{0,n}$, $U_{1,n}$, $U_{n-1,n}$, and $U_{n,n}$ are the cycle matroids of n-edge graphs consisting of, respectively, n loops, n parallel edges, an n-cycle, and a tree with n edges. What about $U_{2,4}$? It has ground set $\{1,2,3,4\}$ and its set of circuits is $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. It is not binary. Why? It is ternary since it is represented over $GF(3)$ by the following matrix.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Thus not every matroid is binary.

2. Some questions and how to answer them

We begin this section with four natural questions.

Question 2.1. Which matroids are binary?

Question 2.2. Which matroids are graphic?

Question 2.3. Which matroids are regular?

Question 2.4. Is every matroid representable over some field?

For the first three of these questions, we have not specified what form an answer should take. As a clue, consider Kuratowski's answer [5] to the question: Which graphs are planar?

Theorem 2.5 (Kuratowski 1930)**.** *A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

If we want to answer these questions analogously, then we need the concept of a minor for a matroid. To create a *minor of a graph*, we are allowed sequences of the following three operations:

(i) deletion of an edge;
(ii) contraction of an edge; and
(iii) deletion of an isolated vertex.

If we add three new degree-0 vertices v_6, v_7, and v_8 to the graph G in Figure 1, then we do not change the cycle matroid of the graph because the cycle matroid only notices the edges and the edge sets of cycles. Thus, for a minor of a matroid, we need only consider sequences of two operations:

(i) deletion of an element; and
(ii) contraction of an element.

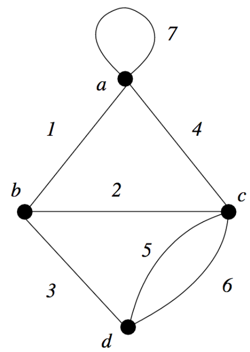

FIGURE 2. A graph G.

For the graph G in Figure 2, the deletion $G\backslash 4$ and the contraction $G/4$ are shown in Figure 3. Clearly the edge sets of cycles in $G\backslash 4$ are exactly the edge sets of cycles in G that do not use 4, while the edge sets of cycles in $G/4$ are the minimal sets of the form $C-\{4\}$ where C is the edge set of a cycle of G. This guides us as to how to define deletion and contraction for matroids. This definition is consistent with what happened for graphs. Thus the deletion of 4 from $M(G)$ is $M(G)\backslash 4 = M(G\backslash 4)$; the contraction of 4 from $M(G)$ is $M(G)/4 = M(G/4)$.

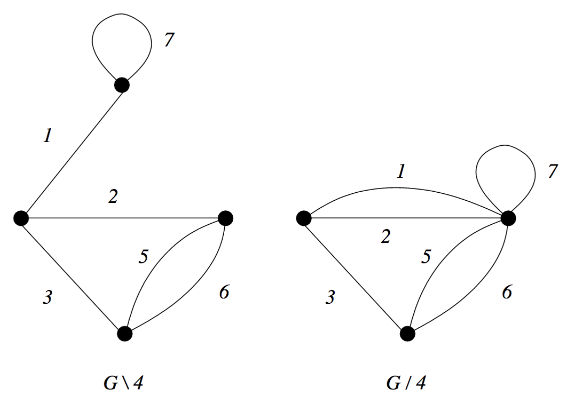

FIGURE 3. $G\backslash 4$ and $G/4$.

Definition 2.6. Let M be the matroid $(E, \mathcal{C})$ and suppose $x \in E$. The *deletion* $M\backslash x$ of x from M is the matroid $(E-\{x\}, \{C \in \mathcal{C}(M) : x \notin C\})$. The *contraction* M/x of x from M is the matroid $(E-\{x\}, \mathcal{C}')$ where $\mathcal{C}'$ consists of the minimal members of $\{C-\{x\} : C \in \mathcal{C}(M)\}$ unless $\{x\}$ is a circuit, in which case, $M/x = M\backslash x$.

It is not difficult to check that these operations do actually give matroids. The operations behave well, so, for example, they commute with each other and with themselves. We can extend the above definition to the deletion $M\backslash T$ and the contraction M/T of any subset T of E. We leave the reader to formalize the definitions of these operations.

Definition 2.7. A *minor* of M is any matroid that can be obtained from M by a sequence of deletions and contractions.

Example 2.8. For a field $\mathbb{F}$, consider the matrix

$$A_{\mathbb{F}} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

The deletion $M[A_{\mathbb{F}}]\backslash 1$ is the matroid of the matrix that is obtained from $A_{\mathbb{F}}$ by deleting the column of A labelled by 1, namely,

$$\begin{array}{c} \begin{matrix} 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

One can check that the contraction $M[A_{\mathbb{F}}]/1$ is the matroid of the following matrix, which is obtained from $A_{\mathbb{F}}$ by deleting the row containing the unique non-zero entry in column 1 and also deleting column 1:

$$\begin{array}{c} \begin{matrix} 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

To contract an element x for which the corresponding column has more than one non-zero entry, we first do elementary row operations on the matrix so that the column corresponding to x has just one non-zero entry. The reader should check that these elementary row operations do not change the matroid.

By generalizing the examples above, it is not difficult to show the following.

Theorem 2.9. *Every minor of a graphic matroid is graphic.*

Theorem 2.10. *For every field $\mathbb{F}$, every minor of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.*

This means that we can try to characterize the classes of binary, graphic and regular matroids by listing the minor-minimal matroids not in the class. In particular, we can answer Question 2.1. We noted earlier that $U_{2,4}$ is not binary, so any matroid with a $U_{2,4}$-minor is not binary. Tutte [15] proved that the converse of this is also true.

Theorem 2.11 (Tutte, 1958)**.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

3. Duality and more answers

Duality is a useful operation for plane graphs and in coding theory. Matroid duality encompasses both of these familiar notions.

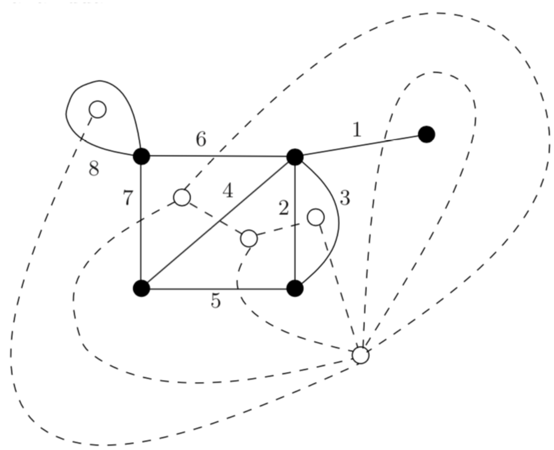

FIGURE 4. Constructing the dual of a plane graph.

Example 3.1. The construction of the dual G^* of a plane graph G is well known. An example of a plane graph with its dual superimposed is shown in Figure 4.

In Figure 5, we show G and G^* separately, where a different plane embedding of G^* is used to more clearly show its cycles. What do the cycles in G^* correspond to in G? The reader can easily check that these cycles are exactly the bonds in G where a *bond* in a graph is a minimal edge cut.

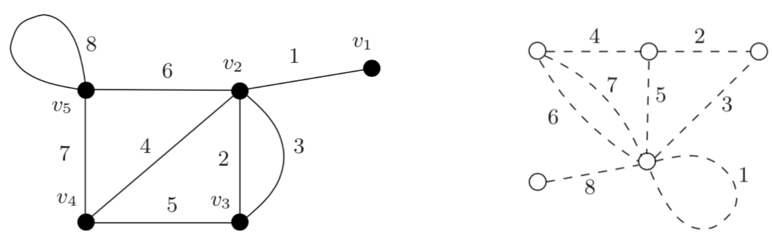

FIGURE 5. A plane graph G and its dual G^*.

The construction in the last example can be generalized to all graphs, planar or otherwise.

Theorem 3.2. *Let G be a graph and $\mathcal{C}^*$ be the collection of bonds of G. Then $(E(G), \mathcal{C}^*)$ is a matroid.*

The matroid identified in the last theorem is denoted by $M^*(G)$ and is called the *bond matroid* of G. It is the dual matroid of the cycle matroid $M(G)$. Thus, although non-planar graphs do not have duals as graphs, they do have duals as matroids.

Looking at $M(G)$ in our example, we see that $\{1, 2, 4, 6\}$ is the edge set of a spanning tree of G, so it is a basis in $M(G)$. Observe that its complement, $\{3, 5, 7, 8\}$, is the edge set of a spanning tree in G^*, so it is a basis in $M(G^*)$, that is, in $M^*(G)$.

Theorem 3.3. *Let M be a matroid and $\mathcal{B}^*(M)$ be $\{E(M)-B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^*(M)$ is the set of bases of a matroid on $E(M)$.*

The matroid in the last theorem whose ground set is $E(M)$ and whose set of bases is $\mathcal{B}^*(M)$ is called the *dual matroid* of M and is denoted by M^*. Clearly

$$(M^*)^* = M.$$

Example 3.4. Since the bases of $U_{r,n}$ are the r-element subsets of an n-element set E, the bases of the dual matroid are the $(n-r)$-element subsets of E. Hence $U^*_{r,n} = U_{n-r,n}$.

In coding theory, if the $r \times n$ matrix $[I_r|D]$ is a generator matrix of a linear code over $GF(q)$, then $[-D^T|I_{n-r}]$ is a generator matrix for the dual (or orthogonal) code. Using elementary row operations and column permutations, it is straightforward to prove the following.

Lemma 3.5. *Let A be a non-zero matrix with n columns. Then $M[A] = M[I_r|D]$ where r is the rank of the matrix A, and D is some $r \times (n-r)$ matrix.*

The next result implies that the class of $\mathbb{F}$-representable matroids is closed under taking duals.

Theorem 3.6. *Let M be the vector matroid of the matrix $[I_r|D]$ where the columns of this matrix are labelled, in order, $e_1, e_2, \ldots, e_n$ and $1 \le r < n$. Then M^* is the vector matroid of the matrix $[-D^T|I_{n-r}]$ where its columns are also labelled $e_1, e_2, \ldots, e_n$ in that order.*

The dual pair of matroids $U_{0,n}$ and $U_{n,n}$ are not covered by the last theorem. Each is representable over all fields, the first by a zero matrix with n columns, the second by the matrix I_n. Combining this observation with the last theorem gives the following.

Corollary 3.7. *If a matroid is representable over a field $\mathbb{F}$, then M^* is also representable over $\mathbb{F}$.*

The vector matroids of $[-D^T|I_{n-r}]$ and $[D^T|I_{n-r}]$ are equal. The significance of using $[-D^T|I_{n-r}]$ in Theorem 3.6 is that it highlights the link between matroid duality and vector-space orthogonality.

Definition 3.8. Let W be a subspace of $V(n,\mathbb{F})$, the n-dimensional vector space over the field $\mathbb{F}$. Let $W^\perp$ be the set of vectors of $V(n,\mathbb{F})$ whose inner product with every vector in W is zero. Then $W^\perp$ is a subspace of $V(n,\mathbb{F})$ called the *orthogonal subspace* of W. If A is an $m \times n$ matrix over $\mathbb{F}$, the *row space* $\mathcal{R}(A)$ of A is the subspace of $V(n,\mathbb{F})$ generated by the rows of A.

Theorem 3.9. *Let $[I_r|D]$ be an $r \times n$ matrix over a field $\mathbb{F}$ where $1 \le r \le n-1$. Then the orthogonal subspace of $\mathcal{R}[I_r|D]$ is $\mathcal{R}[-D^T|I_{n-r}]$.*

We have now seen that matroid duality simultaneously generalizes duality in graph theory and orthogonality in coding theory. In Figure 6, we illustrate the fact that deletion and contraction are dual operations in graphs. In particular, we see that

$$G/3 = (G^*\backslash 3)^*.$$

This link between deletion, contraction, and duality generalizes to matroids.

Theorem 3.10. *For a subset T of the ground set of a matroid M,*

$$(M\backslash T)^* = M^*/T \text{ and } (M/T)^* = M^*\backslash T.$$

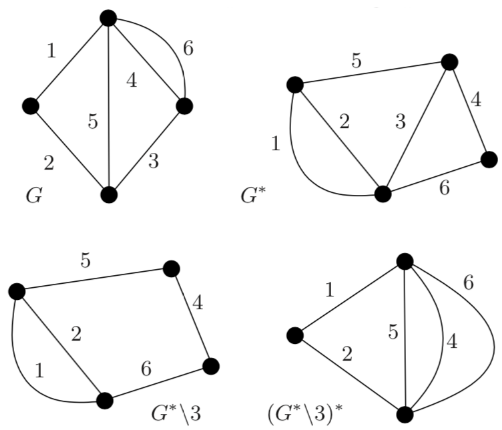

FIGURE 6. Deletion and contraction are dual operations in graphs.

4. More answers

Recall the following matrix over the field $\mathbb{F}$ from Example 2.8.

$$A_{\mathbb{F}} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

When $\mathbb{F} = GF(2)$, we see that the columns of the matrix consist of all non-zero vectors of length three over $GF(2)$. The matroid $M[A_{GF(2)}]$ is the *Fano matroid* F_7. It can be represented geometrically as in Figure 7. In this diagram, three collinear points form a dependent set as do all sets of four coplanar points. Since this diagram exists in the plane, all sets of four points are dependent. In general, suppose we have a set S of points in the plane and a collection of subsets of S called *lines* such that, whenever two lines meet, they do so in at most one point. We get a matroid with ground set S in which the circuits consist of all sets of three points that lie in some line along with all sets of four points that contain no three in a common line.

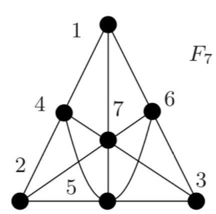

FIGURE 7. The Fano matroid.

Observe that, in $A_{GF(2)}$, the set $\{4, 5, 6\}$ is linearly dependent. This set corresponds to the curved line in Figure 7, which indicates that $\{4, 5, 6\}$ is a circuit in F_7. By contrast, in $A_{GF(3)}$, the set $\{4, 5, 6\}$ is linearly independent, so $\{4, 5, 6\}$ is not a circuit in $M[A_{GF(3)}]$. We denote the latter matroid by F_7^- and call it the *non-Fano matroid*. It is represented geometrically by Figure 8. Note the absence of the curved line $\{4,5,6\}$ from this diagram.

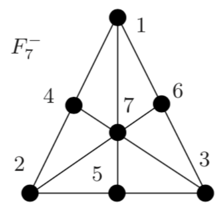

FIGURE 8. The non-Fano matroid.

The following result will enable us to answer Question 2.4.

Theorem 4.1. *For a field $\mathbb{F}$, the Fano matroid is representable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is two; the non-Fano matroid is representable over $\mathbb{F}$ if and only if characteristic of $\mathbb{F}$ is not two.*

In view of this, if we can create a matroid having both the Fano and non-Fano matroids as minors, we will have built a non-representable matroid.

Theorem 4.2. *Let M_1 and M_2 be matroids with disjoint ground sets E_1 and E_2. Then there is a matroid with ground set $E_1 \cup E_2$ whose set of circuits consists of the union of the sets of circuits of M_1 and M_2.*

The matroid whose existence is established by the last theorem is denoted by $M_1 \oplus M_2$ and is called the *direct sum* of M_1 and M_2. By Theorems 2.10 and 4.1, $F_7 \oplus F_7^-$ is not representable over any field. Of course, this matroid has 14 elements. The reader may want to consider constructing a smallest non-representable matroid. It is known that all matroids with at most seven elements are representable.

We defined a matroid to be regular if it is representable over all fields. We noted earlier that the matrix A in Example 1.5, which is the vertex-arc incidence matrix for an orientation of the graph G in Example 1.1, represents $M(G)$ over all fields. We also raised the question as to what property of the matrix A ensured that it represents the same matroid over all fields. We will now answer that question.

Definition 4.3. A matrix over $\mathbb{R}$ is *totally unimodular* if every square submatrix of it has its determinant in $\{0,1,-1\}$.

Tutte [15] proved the following.

Lemma 4.4. *A matroid M is regular if and only if there is a totally unimodular matrix A such that $M = M[A]$.*

Tutte [15] also identified all of the excluded minors for regular matroids thereby answering our Question 2.3.

Theorem 4.5 (Tutte 1958)**.** *A matroid is regular if and only if it has no minor isomorphic to $U_{2,4}, F_7$, or F_7^*.*

The following year, Tutte [16] generalized Kuratowski's Theorem by identifying all of the excluded minors for graphic matroids thereby answering our Question 2.2.

Theorem 4.6 (Tutte 1959)**.** *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}, F_7$, F_7^*, $M^*(K_5)$, or $M^*(K_{3,3})$.*

We saw in Theorem 2.11 that $U_{2,4}$ is the unique excluded minor for the class of binary matroids. In 1979, Bixby [2] and Seymour [13] independently proved the following excluded-minor characterization of the class of ternary, that is, $GF(3)$-representable matroids.

Theorem 4.7 (Bixby, 1979; Seymour, 1979)**.** *A matroid is ternary if and only if it has no minor isomorphic to* $U_{2,5}$*,* $U_{3,5}$*,* F_7*, or* F_7^**.*

By combining this theorem with Theorems 2.11 and 4.5, one can show the following.

Corollary 4.8. *A matroid is regular if and only if it is both binary and ternary.*

In 1970, Rota [12] made the following conjecture and, soon thereafter, this conjecture became the major unsolved problem in matroid theory.

Conjecture 4.9 (Rota 1970)**.** For all finite fields $GF(q)$, the set of minor-minimal matroids that are not $GF(q)$-representable is finite.

The conjecture was verified in 2000 by Geelen, Gerards, and Kapoor [3] in the case when $q = 4$. For this work, they won the Fulkerson Prize. In 2013, Geelen, Gerards, and Whittle [4] announced a proof of Rota's Conjecture for all q.

5. Decomposing regular matroids

We have seen earlier that all graphic matroids are regular. Hence so are the duals of all such matroids, which are called *cographic matroids*. There is a special 10-element regular matroid R_{10}, which was initially identified by Bixby [1]. It is neither graphic nor cographic but, for every element e,

$$R_{10}\backslash e \cong M(K_{3,3}) \text{ and } R_{10}/e \cong M^*(K_{3,3}).$$

The matroid R_{10} is isomorphic to its dual. It is represented over $GF(2)$ by the ten vectors of length five in which every vector has exactly three ones. The following totally unimodular matrix represents R_{10} over $\mathbb{R}$ and, of course, over all fields.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Seymour [14] showed that all regular matroids could be built from graphic matroids, cographic matroids and copies of R_{10} using three basic operations. We have seen the first of these operations, direct sum. In Figure 9, two graphs G_1 and G_2 are shown sharing a single common edge p.

In Figure 10, we show two 2-sums G_3 and G_4 of the graphs G_1 and G_2 obtained by sticking together G_1 and G_2 along p in one of the two possible ways and then deleting p from the result. It is straightforward to specify the (edge sets of the) cycles of G_3 and G_4 in terms of the cycles of G_1 and G_2. An element of a matroid M is a *coloop* if it is a loop in M^*.

Theorem 5.1. *Let* M_1 *and* M_2 *matroids whose ground sets meet in the element* p *where* p *is neither a loop nor a coloop of* M_1 *or* M_2*. Then there is a matroid*

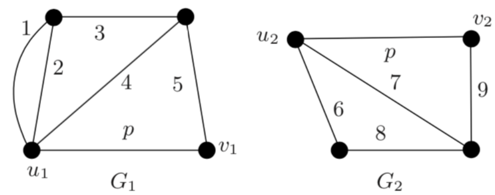

FIGURE 9. Two graphs.

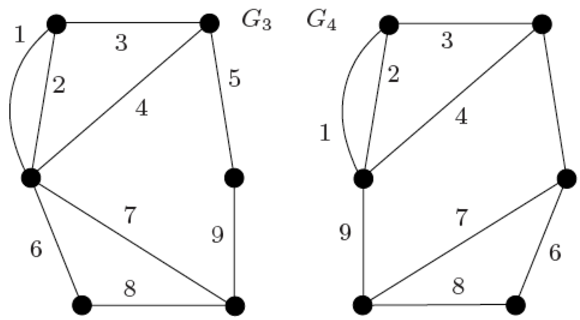

FIGURE 10. Two 2-sums of G_1 and G_2.

$M_1 \oplus_2 M_2$ with ground set $(E(M_1) \cup (E(M_2)) - \{p\})$ whose set of circuits is the union of $\mathcal{C}(M_1 \backslash p) \cup \mathcal{C}(M_2 \backslash p)$ with $\{(C_1 - p) \cup (C_2 - p) : p \in C_i \in \mathcal{C}(M_i)$ for each $i \in \{1, 2\}\}$.

The matroid whose existence is asserted by the last theorem is called the 2-*sum* of M_1 and M_2 with respect to the *basepoint* p.

Whereas the operations of direct sum and 2-sum can be applied to arbitrary matroids, Seymour's third operation, 3-sum, is only defined for binary matroids. It is analogous to the graph operation of the same name in which two graphs are stuck together across a common triangle and then the elements of that triangle are deleted. Recall that, for sets X_1 and X_2, their symmetric difference $X_1 \triangle X_2$ is $(X_1 \cup X_2) - (X_1 \cap X_2)$.

Theorem 5.2. *Let M_1 and M_2 be binary matroids whose ground sets meet in a set T that is a triangle in both matroids. Then there is a matroid whose ground set is $(E(M_1) \cup E(M_2)) - T$ and whose set of circuits consists of the union of $\mathcal{C}(M_1 \backslash T)$, $\mathcal{C}(M_2 \backslash T)$, and the collection of minimal sets of the form $C_1 \triangle C_2$ where C_i is a circuit of M_i such that $C_1 \cap T = C_2 \cap T$ and the last set has exactly one element.*

The matroid whose existence is asserted by the last theorem is denoted $M_1 \triangle M_2$. We now impose some minor technical conditions on this operation to define matroid 3-sum. When both $E(M_1)$ and $E(M_2)$ exceed six and, for each M_i and each element t of T, there is a circuit of M_i that meets T in $\{t\}$, we call $M_1 \triangle M_2$ the 3-*sum* $M_1 \oplus_3 M_2$ of M_1 and M_2.

Theorem 5.3 (Seymour 1980). *A matroid is regular if and only if it can be constructed from graphic matroids, cographic matroids, and copies of R_{10} by using sequences of the operations of direct sum, 2-sum, and 3-sum.*

Apart from the intrinsic beauty of this theorem, it has broader significance because it has as a corollary the following result, which has very important consequences in integer programming.

Corollary 5.4. *There is a polynomial-time algorithm to test whether a given matrix over the integers is totally unimodular.*

6. Conclusion

These notes omit vast areas of matroid theory. They do, however, point to the highly influential role played by graph theory and linear algebra in guiding the developement of matroid theory. On the author's home page, there is a more detailed survey paper,"What is a matroid?" That paper [10] includes exercises. The reader interested in even more about matroids should consult the books of Welsh [17] and the author [11].

References

[1] Robert E. Bixby. Kuratowski's and Wagner's Theorems for matroids, *J. Combin. Theory Ser. B* **22** (1977), 21–53.
[2] Robert E. Bixby. On Reid's characterization of the ternary matroids, *J. Combin. Theory Ser. B* **26** (1979), 174–204.
[3] J. F. Geelen, A. M. H. Gerards, and A. Kapoor. The excluded minors for $GF(4)$-representable matroids, *J. Combin. Theory Ser. B* **79** (2000), 247–299.
[4] Jim Geelen, Bert Gerards, and Geoff Whittle. Solving Rota's Conjecture, *Notices Amer. Math. Soc.* **61** (2015), 736–743.
[5] K. Kuratowski. Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.
[6] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. I , *Sci. Rep. Tokyo Bunrika Daigaku Sect. A* **2** (1935), 129–149.
[7] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. II, *Sci. Rep. Tokyo Bunrika Daigaku Sect. A* **3** (1936), 77–90.
[8] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. III. Schluss, *Sci. Rep. Tokyo Bunrika Daigaku Sect. A* **3** (1936), 116–129.
[9] Hirokazu Nishimura and Susumu Kuroda (Eds.). *A Lost Mathematician, Takeo Nakasawa: The Forgotten Father of Matroid Theory*, Birkhäuser, Basel, 2009.
[10] James Oxley. What is a matroid? *Cubo* **5** (2003), 179–281. `https://www.math.lsu.edu/~oxley/survey4.pdf`
[11] James Oxley. *Matroid theory*, Second edition, Oxford University Press, Oxford, 2011.
[12] Gian-Carlo Rota. Combinatorial theory, old and new, in *Proc. Internat. Cong. Math.* (Nice, 1970), pp. 229–233, Gauthier-Villars, Paris, 1971.
[13] P. D. Seymour. Matroid representation over GF(3), *J. Combin. Theory Ser. B* **26** (1979), 159–173.
[14] P. D. Seymour. Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.
[15] W. T. Tutte. A homotopy theorem for matroids, I, II. *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
[16] W. T. Tutte. Matroids and graphs, *Trans. Amer. Math. Soc.* **890** (1959), 527–552.
[17] D. J. A. Welsh. *Matroid theory*, Academic Press, London, 1976. Reprinted Dover, New York, 2010.
[18] Hassler Whitney. On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935), 509–533.

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: `oxley@math.lsu.edu`