ON THE MINOR-MINIMAL 2–CONNECTED GRAPHS HAVING A FIXED MINOR

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Abstract. Let H be a graph with κ_1 components and κ_2 blocks, and let G be a minor-minimal 2-connected graph having H as a minor. This paper proves that $|E(G)| - |E(H)| \le \alpha(\kappa_1 - 1) + \beta(\kappa_2 - 1)$ for all (α, β) such that $\alpha + \beta \ge 5, 2\alpha + 5\beta \ge 20$, and $\beta \ge 3$. Moreover, if one of the last three inequalities fails, then there are graphs G and H for which the first inequality fails.

1. Introduction

A telephone network in a town is disrupted when one of the optical-fibre cables is accidentally cut. The telephone company wishes to augment its network to ensure that it will still function in such a situation, or when a node fails after, say, a lightning strike. Modelling the existing network by a graph H, we seek a 2-connected graph G that has H as a subgraph. Moreover, in order to minimize cost, we want G to be a minimal such graph. What can be said about $|E(G)| - |E(H)|$? As another example, let H be the vertex-disjoint union of a collection of cliques, cycles, and stars, and let G be a 2-connected graph that is minor-minimal having H as a minor. Again, what can be said about $|E(G)|-|E(H)|$? Both of these problems are special cases of the problem of finding a sharp upper bound on $|E(G)|-|E(H)|$ when G is a minor-minimal n-connected graph having some fixed graph H as a minor. In this paper, we completely solve this problem in the case that $n = 2$. When $n = 1$, it is not difficult to see that $|E(G)| - |E(H)|$ can be bounded by a linear function in $\kappa_1(H)$, the number of connected components of H. In particular, $|E(G)| - |E(H)| = \kappa_1(H) - 1$. When $n = 2$, we again seek a linear bound, this time in $\kappa_1(H)$ and $\kappa_2(H)$, where the latter is the number of blocks of H. By considering several families of examples, we derive certain necessary conditions on the coefficients in such a bound. Our main result is that these necessary conditions are also sufficient.

1.1. **Theorem.** *Let α and β be real numbers. Then, for all graphs G and H such that G is a minor-minimal 2-connected graph having H as a minor,*

$$|E(G)| - |E(H)| \le \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$$

if and only if

$$\alpha + \beta \ge 5, \tag{C1}$$

$$2\alpha + 5\beta \ge 20, \text{ and} \tag{C2}$$

$$\beta \ge 3. \tag{C3}$$

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A *block* of a graph is a maximal connected subgraph H of G such that every two distinct edges of H lie in a cycle. In particular, each loop is a block of G as is each isolated vertex. It is well-known (see, for example, [6, Proposition 4.1.8]) that, for a graph G with at least three vertices, G is a block if and only if G is 2-connected and loopless.

The three inequalities (C1)–(C3) define an unbounded convex polyhedron A in the $\alpha\beta$-plane (see Figure 1). The following is a variant of the first theorem.

1.2. **Theorem.** *Let α and β be real numbers. Then, for all graphs G and H such that G is a minor-minimal block having H as a minor,*

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$$

if and only if $(\alpha, \beta) \in A$.

For all (α, β) not in the polyhedron A, we shall describe examples in which the bound on $|E(G)| - |E(H)|$ fails. We remark that both of the last two theorems remain valid if we insist that G and H are simple graphs. Both theorems will be derived from a more general, but slightly technical, result, which will be stated in the next section (Theorem 2.1). We now address a technicality that has been glossed over in the last two theorems. A *minor* of a graph G is a graph that can be obtained from G by a sequence of edge deletions, edge contractions, and vertex deletions. We shall say that such a minor H' *equals* some fixed graph H if H' and H are the same up to vertex labels or, more precisely, $E(H') = E(H)$ and there is a bijection $f : V(H') \to V(H)$ such that an edge e in H' joins vertices u and v if and only if e joins $f(u)$ and $f(v)$ in H.

The polyhedron A has exactly two vertices, namely $(\frac{5}{3}, \frac{10}{3})$ and $(\frac{5}{2}, 3)$. We get the next result by applying Theorem 1.1 to the two vertices of A. As we shall see, the fact that the bound on $|E(G)| - |E(H)|$ holds for these two points implies that it holds for all (α, β) in A. The difficulty of proving the main results of this paper is increased significantly because A has two vertices instead of just one. However, we believe that the curious, and apparently counterintuitive, shape of A increases the interest of the main theorems.

1.3. **Corollary.** *For all graphs G and H such that G is a minor-minimal 2-connected graph having H as a minor,*

$$|E(G)| - |E(H)| \leq \tfrac{5}{3}\kappa_1(H) + \tfrac{10}{3}\kappa_2(H) - 5 \text{ and}$$

$$|E(G)| - |E(H)| \leq \tfrac{5}{2}\kappa_1(H) + 3\kappa_2(H) - 5.$$

Part of the motivation for seeking a bound on $|E(G)| - |E(H)|$ that is linear in $\kappa_1(H)$ and $\kappa_2(H)$ derives from the solution to the corresponding matroid problem, which we state in the next result [5].

1.4. **Theorem.** *Let N be a matroid having k 2-connected components and M be a minor-minimal 2-connected matroid having N as a minor. Then*

$$|E(M)| - |E(N)| \leq 2k - 2$$

unless N or its dual is free, in which case,

$$|E(M)| - |E(N)| \leq k - 1.$$

Moreover, these bounds are attained for all choices of N.

FIGURE 1. The unbounded polyhedron A.

When M is a non-empty graphic matroid, $M \cong M(G)$ for some graph G having no isolated vertices. Moreover, M is 2-connected if and only if G is a block. Thus if H is a graph without isolated vertices, then the number of blocks of H equals the number k of 2-connected components of the matroid $M(H)$. Suppose that every connected component of the graph H is also a block. Then a minor-minimal 2-connected matroid having $M(H)$ as a minor has at most $2k-2$ more elements than H. This may suggest that a minor-minimal block having H as a minor should satisfy the bound

$$|E(G)| - |E(H)| \leq 2\kappa_2(H) - 2.$$

However, this is not so. For example, consider the graph G in Figure 2 that is constructed from the vertex-disjoint union of n 6-cycles where $n \geq 2$. Let X be the set of dashed edges, Y be the set of dotted edges, and $H = G\backslash X/Y$, the graph that is obtained from G by deleting X and contracting Y. Then H is the vertex-disjoint union of two 5-cycles and $n-2$ 4-cycles. It is straightforward to see that G is a minor-minimal block having H as a minor and

$$|E(G)| - |E(H)| = 4(n-1) = 4\kappa_2(H) - 4.$$

As we shall show in Theorem 3.5, the last bound holds for all graphs H having $\kappa_1(H) = \kappa_2(H)$ provided G is a minor-minimal 2-connected graph having H as a minor.

The disparity above between the graph and matroid bounds arises because the matroids of two graphs are equal provided the graphs have the same blocks. This does not mean that the graphs themselves must be equal. Indeed, the precise relationship between the graphs is described in Whitney's 2-Isomorphism Theorem [9] (see, for example, [6, Theorem 5.3.1]). In our example above, a minor-minimal 2-connected graphic matroid having $M(H)$ as a minor is the cycle matroid of the

FIGURE 2. The graph G.

graph that is obtained from G by contracting all dashed edges and then deleting one edge from each resulting 2-cycle.

The reader may feel that, instead of the bound in our main results, we should be seeking a more general linear bound of the form

$$|E(G)| - |E(H)| \leq \alpha\kappa_1 + \beta\kappa_2 + \gamma. \tag{1}$$

But if, for example, in Theorem 1.2, the graph H is a block, then $G = H$ and $\kappa_1(H) = \kappa_2(H) = 1$. Thus, the more general bound yields $-\alpha - \beta \geq \gamma$. The bound in Theorem 1.2 has $-\alpha - \beta = \gamma$ and so is at least as sharp as the bound in (1)

For graphs, considerable effort has been expended on the problems of determining the minimum number of edges that need to be added to a graph H to obtain a graph G with specified edge- or vertex-connectivity, and of algorithmically finding G (see, for example, [4, 3, 8]). In particular, Eswaran and Tarjan [3] solved the problem of bounding $|E(G)| - |E(H)|$ when G is required to be 2-connected. This differs from the problem we solve in two significant ways. Firstly, this variant of the problem requires that H is a spanning subgraph, rather than an arbitrary minor, of G. Secondly, and more significantly, this problem imagines a friendly constructor who wants to minimize the number of edges that need to be added to H to achieve 2-connectedness. The corresponding subgraph version of our problem imagines an adversarial constructor who wants to maximize the number of edges that can be added while still achieving a 2-connected graph that is minimal with the properties of being 2-connected and having H as a spanning subgraph.

2. PRELIMINARIES

The graph and matroid terminology used here will follow Bondy and Murty [1] and Oxley [6], respectively. For a graph G, we denote by $L(G)$ and $\iota(G)$ the set of loops of G and the number of isolated vertices of G. Moreover, if Z is a non-empty subset of $V(G)$ or of $E(G)$, then $G[Z]$ denotes the subgraph of G *induced* by Z.

In order to be able to prove Theorems 1.1 and 1.2 at the same time, we shall prove a more general result that has both theorems as special cases. Let H be a graph and L be a subset of $L(H)$. We denote by $\mathcal{G}_L(H)$ the class of all minor-minimal graphs G having the following properties:

(a) $G \backslash L(G)$ is a block;
(b) G has H as a minor; and
(c) $L(G) \subseteq L$.

When $L = \emptyset$ and H is not the graph consisting of a single loop, a graph $G \in \mathcal{G}_L(H)$ if and only if G is a minor-minimal block having H as a minor. When $L = L(H)$ and $|V(H)| \geq 3$, a graph $G \in \mathcal{G}_L(H)$ if and only if G is a minor-minimal 2-connected graph having H as a minor.

The next result is the main result of the paper.

2.1. **Theorem.** *Let α and β be real numbers. Then, for all graphs G and H such that $G \in \mathcal{G}_L(H)$ and L is a set of loops of H,*

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$$

if and only if $(\alpha, \beta) \in A$.

We observe that if H is a simple graph and $G \in \mathcal{G}_L(H)$, then G must also be simple. Thus the theorem remains valid if we add the requirement that both G and H are simple.

We now outline the structure of the paper. In the remainder of this section, we note some useful preliminary lemmas. Section 3 bounds $|E(G)| - |E(H)|$ when $G \in \mathcal{G}_L(H)$ and H is either a deletion or a contraction of G. In Section 5, we describe examples to prove that it is necessary that (α, β) lie in A for the specified bound on $|E(G)| - |E(H)|$ to hold for all G in $\mathcal{G}_L(H)$. These examples are based on constructions introduced in Section 4. The proof that (α, β) being in A is sufficient to yield the specified bound on $|E(G)| - |E(H)|$ will make frequent use of a decomposition described in Section 6, while Section 7 contains three technical lemmas that will be needed in the proof. In Section 8, we begin the proof that, when $(\alpha, \beta) \in A$,

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$$

for all G in $\mathcal{G}_L(H)$. The proof begins by establishing that it is sufficient to prove this result when (α, β) is one of the two vertices of A. It then chooses a counterexample G that is minimal with respect to some carefully chosen criteria, and shows that both G and H are loopless and that Y is non-empty where $H = G\backslash X/Y$. As one would expect from the shape of A, the rest of the proof is quite complex; an outline of it is given in Section 9.

The following elementary but useful graph-theoretic result is a special case of a well-known matroid result [7] (see, for example, [6, Theorem 4.3.1]).

2.2. **Lemma.** *If G is a block and $e \in E(G)$, then $G\backslash e$ or G/e is a block.*

The next three lemmas will be used repeatedly throughout the paper. The first shows that H can be obtained in just one way from a member of $\mathcal{G}_L(H)$.

2.3. **Lemma.** *Let H be a graph and L be a subset of $L(H)$. If $G \in \mathcal{G}_L(H)$, then there are unique subsets X and Y of $E(G)$ such that $H = G\backslash X/Y$. Hence $G[Y]$ is a forest and X does not contain a loop of G/Y.*

Proof. We know that H can be obtained from G by a sequence of edge deletions, edge contractions, and vertex deletions. By choosing such a sequence in which the number of vertex deletions is minimized, it is not difficult to show that $H = G\backslash X/Y$ for some subsets X and Y of $E(G)$.

Now suppose that there is an edge e of G such that H is a minor of both $G\backslash e$ and G/e. Then $e \notin L(G)$, so $L(G\backslash e) = L(G)$. Now either $(G\backslash L(G))\backslash e$ is or is not a block. In the first case, $(G\backslash e)\backslash L(G\backslash e)$ is a block and the choice of G is contradicted. In the second case, by Lemma 2.2, $(G\backslash L(G))/e$ is a block and, since $(G\backslash L(G))\backslash e$ is not, $L(G/e) = L(G)$. Hence $(G/e)\backslash L(G/e)$ is a block contradicting the choice of G. We conclude that G has no edge e such that H is a minor of both $G\backslash e$ and G/e. Hence the sets X and Y are unique. It follows immediately from this that $G[Y]$ is a forest and that X does not contain a loop of G/Y. □

2.4. **Lemma.** *Suppose that $G \in \mathcal{G}_L(H)$ and $H = G\backslash X/Y$. If G' is a connected component of $G\backslash X$, then G' has no pendant edge that belongs to Y.*

Proof. Suppose that G' has a pendant edge f that belongs to Y. Let v be a degree-1 vertex in G' incident with f. Then $E_v - f \subseteq X$ where E_v is the set of edges of G meeting v. Let $H' = G\backslash(X \cup f)/(Y - f)$. Then H' can be obtained from H by adjoining v as an isolated vertex. Now suppose we can choose e in $E_v - f$, and let $H'' = G\backslash[(X \cup f) - e]/[(Y \cup e) - f]$. Then the only difference between H'' and H' is that v is an isolated vertex of the latter. Thus $H'' = H$. This contradiction to the uniqueness of X and Y implies that $E_v - f = \emptyset$. In that case, G/f contradicts the minimality of G. □

2.5. **Lemma.** *Suppose that G and G' are blocks and that there are unique subsets X' and Y' of $E(G)$ such that $G' = G\backslash X'/Y'$. Then, for all x in X' and all y in Y', both $G\backslash x$ and G/y are blocks.*

Proof. Suppose that $G\backslash x$ is not a block for some x in X'. Then $G\backslash x$ has an endblock that contains no edges of G'. Since G' arises uniquely from G and G is a block, it follows that this endblock is a path P, one end of which is adjacent to x in G. Clearly $P \subseteq Y'$. Choose $y \in P$. Then G' also arises from G by deleting $(X' - x) \cup y$ and contracting $(Y' - y) \cup x$; a contradiction. We conclude that $G\backslash x$ is a block for all x in X'.

Suppose that G/y is not a block for some y in Y'. Then, as G' is a block, G/y has a block G'' that contains no edges of G'. Since G' arises uniquely from G and G is a block, G'' must be a loop z at the vertex that arises from identifying the endpoints of y. But then G' can be obtained as a minor of both $G\backslash z$ and G/z; a contradiction. □

3. The deletion and contraction cases

In this section, we first bound $|E(G)| - |E(H)|$ when G is a minor-minimal 2-connected graph having H as a subgraph. This result will be deduced from a more general theorem about $\mathcal{G}_L(H)$. We omit the proof of the following elementary result.

3.1. **Lemma.** *Let e be an edge of a graph K. If $K\backslash e$ has more connected components than K, then $\kappa_2(K) = \kappa_2(K\backslash e) + 1 + [\iota(K) - \iota(K\backslash e)]$.*

3.2. **Theorem.** *Let H be a graph and L be a subset of $L(H)$. If $G \in \mathcal{G}_L(H)$ and $H = G\backslash X$, then $|X| \leq \kappa_1(H) + \kappa_2(H) - 2$.*

Proof. As every loop of H must be a loop of G, it follows that $L = L(H) = L(G)$. Clearly

$$\kappa_1(H) = \kappa_1(H\backslash L) \text{ and } \kappa_2(H) \geq \kappa_2(H\backslash L). \tag{2}$$

Observe that $G\backslash L \in \mathcal{G}_\emptyset(H\backslash L)$. Thus, by (2), we need only to prove that $|X| \leq \kappa_1(H\backslash L) + \kappa_2(H\backslash L) - 2$. Hence we may assume that neither H nor G has loops.

We prove the theorem by induction on $|X|$. Evidently it holds when $|X| = 0$ for, in that case, $G = H$ and $\kappa_1(H) = \kappa_2(H) = 1$. Assume the result holds for $|X| < n$ and let $|X| = n \geq 1$. Let e be an edge in X and let v and w be its endpoints. We distinguish the following three cases:

(i) v and w belong to the same component K of H;
(ii) v is an isolated vertex of H; and

(iii) v and w belong to different components of H both having at least two vertices.

In case (i), $\kappa_2(K+e) < \kappa_2(K)$ otherwise v and w belong to the same block of H so $G\backslash e$ is a block that contradicts the choice of G. Thus $\kappa_2(H+e) < \kappa_2(H)$. Moreover, $\kappa_1(H+e) = \kappa_1(H)$. Hence, by the induction assumption,

$$|X-e| \leq \kappa_1(H+e) + \kappa_2(H+e) - 2 < \kappa_1(H) + \kappa_2(H) - 2.$$

Thus, in case (i), $|X| \leq \kappa_1(H) + \kappa_2(H) - 2$, as required.

In case (ii), $\kappa_1(H+e) = \kappa_1(H) - 1$ and, by Lemma 3.1,

$$\kappa_2(H+e) = \kappa_2(H) + 1 + [\iota(H+e) - \iota(H)] \leq \kappa_2(H).$$

Thus, by the induction assumption,

$$|X-e| \leq \kappa_1(H+e) + \kappa_2(H+e) - 2 \leq [\kappa_1(H) - 1] + \kappa_2(H) - 2.$$

Hence, in case (ii), $|X| \leq \kappa_1(H) + \kappa_2(H) - 2$, as required.

In case (iii), let $G' = G/e$ and let $H' = G/e\backslash(X-e)$, so H' is a spanning subgraph of G'. Since $G\backslash e$ is not a block, Lemma 2.2 implies that G' is a loopless block. Now suppose that $G'\backslash f$ is a block for some f in $X-e$. Then $G/e\backslash f$ is a block but $G\backslash f$ is not. Thus e is a pendant edge of $G\backslash f$ and hence of $H+e$; a contradiction. We conclude that $G'\backslash f$ is not a block. Thus G' is a minor-minimal block having H' as a minor. Evidently $\kappa_1(H') = \kappa_1(H) - 1$ and $\kappa_2(H') = \kappa_2(H)$. Thus, by applying the induction assumption to the subgraph H' of G', we deduce that $|X-e| \leq \kappa_1(H') + \kappa_2(H') - 2 = [\kappa_1(H) - 1] + \kappa_2(H) - 2$ and again, just as in the first two cases, it follows that $|X| \leq \kappa_1(H) + \kappa_2(H) - 2$, as required. □

The next result follows immediately from the last theorem by using the remarks following the definition of $\mathcal{G}_L(H)$.

3.3. Corollary. *Let H be a graph. If G is a 2-connected graph that is minimal having H as a subgraph, then $|E(G)| - |E(H)| \leq \kappa_1(H) + \kappa_2(H) - 2$.*

Next we bound $|E(G)| - |E(H)|$ when $G \in \mathcal{G}_L(H)$ and H is a contraction of G.

3.4. Theorem. *Let H be a graph and L be a subset of $L(H)$. If $G \in \mathcal{G}_L(H)$ and $H = G/Y$, then*

$$|Y| \leq \kappa_2(H) - 1$$

unless H is the graph consisting of a single loop and $L = \emptyset$.

Proof. Since G is connected, so too is H. The proof can be completed by arguing by induction on $|Y|$. In particular, one shows, for any edge e of Y, that $\kappa_2(G/(Y-e)) < \kappa_2(G/Y)$. The details are omitted. □

The reader may suspect that the general result bounding $|E(G)| - |E(H)|$ when $G \in \mathcal{G}_L(H)$ may be obtained by combining the contraction case above with the deletion case considered earlier in the section. This approach, which is successfully applied in the special case considered in the next result, turns out to be problematic in general with much of the difficulty stemming from the possible presence of isolated vertices.

3.5. Theorem. *If $G \in \mathcal{G}_L(H)$ and $\kappa_1(H) = \kappa_2(H)$, then*

$$|E(G)| - |E(H)| \leq 4\kappa_2(H) - 4.$$

Proof. Recall that $H = G\backslash X/Y$. We get the result by summing separate bounds on $|X|$ and $|Y|$. Suppose that $G\backslash X$ has connected components $G_1, G_2, \dots, G_k$. Since $\kappa_1(H) = \kappa_2(H)$, it follows that $G_i/(Y \cap E(G_i))$ is a block for all i. By Lemmas 2.3 and 2.4, $G\backslash X$ has no cycles with edge-set contained in Y and has no pendent edges belonging to Y. Thus each G_i is a block. Hence $\kappa_2(G\backslash X) = \kappa_2(H)$. Now, by Theorem 3.2, $|X| \leq \kappa_1(G\backslash X) + \kappa_2(G\backslash X) - 2$. Thus

$$|X| \leq 2\kappa_2(H) - 2. \tag{3}$$

To get the bound on $|Y|$, we shall use the bound from the contraction case (Theorem 3.4). Thus we want a bound on $\kappa_2(G/Y)$. Clearly G/Y is connected. If B is a block of G/Y, then we have two possibilities for it:

(i) B contains an edge of H. Then $B\backslash[X \cap E(B)]$ contains some block of H.
(ii) B does not contain any edge of H. Then $E(B) \subseteq X$.

Let b be the number of blocks of G/Y of the second type. Then

$$\kappa_2(G/Y) \leq \kappa_2(H) + b.$$

Now observe that a block B whose edge-set is contained in X must have at least two edges, otherwise this block is an isthmus in G/Y and so in G. Hence, by (3), $b \leq \frac{|X|}{2} \leq \kappa_2(H) - 1$. Thus,

$$\kappa_2(G/Y) \leq 2\kappa_2(H) - 1.$$

Now, by Theorem 3.4, $|Y| \leq \kappa_2(G/Y) - 1$. Hence

$$|Y| \leq 2\kappa_2(H) - 2. \tag{4}$$

The lemma follows by summing the bounds on $|X|$ and $|Y|$ in (3) and (4). □

To see that the bound in the last theorem is sharp, consider the example given in Figure 2.

4. Replacements

Throughout this section, G will be a graph in $\mathcal{G}_\emptyset(H)$ where $H = G\backslash X/Y$ and $L(H) = \emptyset$. The graphs and the constructions that are described in this section will be used in the next section to prove that (α, β) must be in the polyhedron A if $|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$ for all graphs G and H with $G \in \mathcal{G}_L(H)$.

In the next paragraphs, we set more notation that we shall use in this section.

Suppose that $e \in Y$, say $e = uv$. Then G/e is not a block. We can write G as the union of two blocks G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{u, v\}, E(G_1) \cap E(G_2) = \{e\}$ and, for $i \in \{1, 2\}$, G_i/e has at least one block of G/e as a block. We say that (G_1, G_2) is an *admissible decomposition of G with respect to e*. We define $X_i = X \cap E(G_i), Y_i = Y \cap E(G_i)$ and $H_i = G_i\backslash X_i/Y_i$, for $i \in \{1, 2\}$. Observe that H is the union of H_1 and H_2, provided that the vertices in these three graphs that arise after the contraction of e are considered to be the same.

An *element* of a graph is a vertex or an edge of the graph. Now let F be a graph and let X_F and Y_F be disjoint subsets of $E(F)$ such that e is an edge of F joining u and v, and e, u, and v are the only common elements of G and F. Suppose that $e \in Y_F$ and let $H_F = F\backslash X_F/Y_F$. We say that G' *is obtained from G by the replacement of* (G_1, X_1, Y_1) *by* (F, X_F, Y_F) if G' is the union of F and G_2. In this case, we define $X' = X_F \cup X_2, Y' = Y_F \cup Y_2$, and $H' = G'\backslash X'/Y'$. Note that H'

is the union of H_F and H_2, provided that the vertices in these three graphs that arise after the contraction of e are identified. For each lemma in this section, we shall choose a graph F to replace G_1.

We say that (S, X_S, Y_S) is a *snake* on e if S is a 4-cycle labelled as follows: $V(S) = \{w, x, u, v\}$, $E(S) = \{wx, wv, xu, uv\}$, $X_S = \emptyset$, and $Y_S = \{e\}$.

4.1. **Lemma.** *If G' is obtained from G by replacing (G_1, X_1, Y_1) by (S, X_S, Y_S), then $G' \in \mathcal{G}_\emptyset(H')$.*

Proof. Suppose that $G' \backslash X''/Y'' = H'$. We shall show first that $X'' = X_2$ and $Y'' = Y_2$. The edges xu and wv are adjacent in H' so the vertices u and v must be identified in $G' \backslash X''/Y'' = H'$. Suppose that $e \notin Y''$. Then $e \in X''$ and there is a path from u to v in $G_2 \backslash e$ all of whose edges are in Y''. Now H_1 can be obtained from $G_1 \backslash X_1/(Y_1 - e)$ by identifying u and v and deleting e. It follows that $[G \backslash X_1/(Y_1 - e)] \backslash X''/Y''$ is the union of H_1 and H_2, when we use the same label for the vertex that we get after the contraction of e in these two graphs. Thus $G \backslash (X_1 \cup X'')/((Y_1 - e) \cup Y'') = G \backslash X/Y$; a contradiction to the fact that H occurs uniquely as a minor of G. Thus $e \in Y''$. But again $G \backslash (X_1 \cup X'')/(Y_1 \cup Y'') = G \backslash X/Y$ and so $X'' = X_2$ and $Y'' = Y_2$.

It is not difficult to see that, for all x in X_2 and all y in Y_2, both $G' \backslash x$ and G'/y have cut-vertices that prevent either graph from having a block containing $E(H')$. Hence $G' \in \mathcal{G}_\emptyset(H')$. □

FIGURE 3. (a) A dog. (b) A pig.

Let P be the graph in Figure 3(b), so $V(P) = \{u_1, v_1, w_1, u_2, v_2, u', v', u, v\}$ and $E(P)$ is partitioned into subsets X_P, Y_P, and Z_P, where $X_P = \{u'v, v'u, v'w_1, vv_1\}$, $Y_P = \{uv, u'v', vw_1\}$, and $Z_P = \{u'u_2, u_2v_2, v_2v', u_1v_1, v_1w_1, w_1u_1\}$. Observe that the edges of X_P, Y_P, and Z_P are, respectively, dashed, dotted, and solid. We shall call (P, X_P, Y_P) a *pig on* $e = uv$ and say that $T_P = \{u_1v_1, v_1w_1, w_1u_1\}$ is the *head* of the pig which *is at* v. Note that v is not a vertex of T_P; it is a vertex of e.

We say that (D, X_D, Y_D) is a *dog on* $e = uv$ if D is a single-edge deletion of K_4 labelled as follows: $V(D) = \{u, v, w, x\}$, $E(D) = \{wx, wv, xu, xv, uv\}$, $X_D = \{xu\}$, and $Y_D = \{uv\}$ (see Figure 3(a)). The triangle $T_D = \{wx, wv, xv\}$ is said to be *the head* of the dog which is *at* v.

G' is obtained from G by replacing a dog by a pig on e if $(G_1, X_1, Y_1) = (D, X_D, Y_D)$ and this is replaced by (P, X_P, Y_P). Note that both the dog and the pig must have their heads at the same vertex of e. Observe also that $(P[\{u_1, v_1, w_1, v\}], \{v_1v\}, \{w_1v\})$ is a dog on w_1v. Thus, we can repeat the process of replacing a dog by a pig as many times as we wish. The next lemma asserts that the replacement of a dog by a pig creates a graph that still belongs to the family that we are interested in studying. The proof will use the notation of the last two paragraphs.

FIGURE 4. (a) A bull. (b) A (symmetric) rhino.

4.2. **Lemma.** *If G' is obtained by the replacement of a dog by a pig on e, then $G' \in \mathcal{G}_\emptyset(H')$.*

Proof. Let $G'\backslash X''/Y'' = H'$. We shall show first that $X'' = X'$ and $Y'' = Y'$. Observe that $v'w_1 \in X''$ because v' and w_1 are incident to edges of Z_P which are not adjacent in H'. Now consider the connected component Q_e containing e of the subgraph of $G\backslash X$ induced by Y. Since $G[Y]$ is a forest, Q_e is a tree. As $G\backslash X$ has no pendent edges belonging to Y, every degree-one vertex of Q_e is incident with an edge of H. It follows that when the edges of Q_e are contracted in the formation of H, the connected component of H that contains T_D must have at least two blocks. Now H' can be obtained from H by identifying the edges of T_D with the edges of T_P and adding a new connected component, which is a triangle. Thus the connected component of H' that contains T_P must have at least two blocks. Hence, as $\{w_1v, v_1v\} \subseteq X'' \cup Y''$, at least one of w_1v and v_1v is in Y''. If both w_1v and v_1v are in Y'', then the triangle T_P is destroyed. Thus one of w_1v and v_1v is in Y'' and the other is in X''. Since both ends of u_1v_1 have degree two in H', it follows that $v_1v \in X''$ and $w_1v \in Y''$. By considering $G\backslash\{v'w_1, vv_1\}/\{vw_1\}$, we deduce, since the edges v_1w_1 and $u'u_2$ do not become adjacent in H', that $u'v \in X''$. Then, since $u'u_2$ and $v'v_2$ are adjacent in H', it follows that $u'v' \in Y''$.

We prove next that $v'u \in X''$. Assume the contrary. Then $v'u \in Y''$. Consider the graph $J = G'\backslash\{v'w_1, vv_1, u'v\}/\{vw_1, u'v', v'u\}$. This graph can be obtained from $G\backslash xu$ by identifying the edges of T_D with the edges of T_P and adding a new block, which is a triangle T'' and which has u as its only common element with

$G\backslash xu$. Now J has H' as a minor. As T_D and T'' do not have a common vertex in H', it follows that e is not contracted in producing H' from J. Since H' is the disjoint union of H with the triangle T'', it follows that H can be obtained as a minor of $G\backslash xu$ without contracting e. This contradicts the fact that H is uniquely obtainable from G and implies that $v'u \in X''$. A similar argument using $G'\backslash\{v'w_1, vv_1, u'v, v'u\}/\{vw_1, u'v'\}$ in place of J establishes that $uv \in Y''$. We conclude that if $G'\backslash X''/Y'' = H'$, then $X'' = X'$ and $Y'' = Y'$.

To complete the proof that $G' \in \mathcal{G}_\emptyset(H')$, it suffices, by Lemma 2.5, to show that if $x \in X'$ and $y \in Y'$, then neither $G'\backslash x$ nor G'/y is a block. This is not difficult to check if x or y is in the pig, and it follows if x or y is in $E(G_2) - e$ because neither $G\backslash x$ nor G/y is a block. □

We call (R, X_R, Y_R) a *rhino on* $e' = zw_2$ if R is a graph with $V(R) = \{z, w_2, u'_1, v'_1, w'_1, u'_2, v'_2, w'_2, u'_3, v'_3, w'_3, u'_4, v'_4, w'_4, z'\}$ and the set of edges of R is partitioned into three sets X_R, Y_R, and Z_R, where $Y_R = \{e', w'_1z', w'_2z', w'_3w_2, w'_4z\}$, $Z_R = \cup_{i=1}^4\{u'_iv'_i, u'_iw'_i, v'_iw'_i\}$, and $X_R = \{u'_1z', u'_2z', u'_3w_2, u'_4z, w'_1z, w'_3z', w'_4z'\} \cup \{w'_2a\}$, where a is either w_2 or z. The rhino R is *symmetric* if $a = w_2$ (see Figure 4(b)) and *assymmetric* otherwise.

We say that (B, X_B, Y_B) is a *bull on* e if B is the graph in Figure 4(a) with $V(B) = \{u, v, u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3, z\}$ and the set of edges of B is partitioned into three subsets X_B, Y_B, and Z_B, where $Z_B = \cup_{i=1}^3\{u_iv_i, u_iw_i, v_iw_i\}$, $X_B = \{u_1z, u_2z, u_3v, w_1v, w_2v, uz, w_3z\}$, and $Y_B = \{e, w_1z, w_2z, w_3v\}$. *The head of* B is $\{u_3v_3, u_3w_3, v_3w_3\}$ which is *at* v.

Both bulls and rhinos will feature prominently in the proof of the main theorem. Next we combine a bull B with a symmetric rhino R to produce a graph that will be important in the next section. Suppose that $B - \{u_2, v_2\}$ and R have z, w_2, and $e' = zw_2$ as their only common elements. The union M of R and $B - \{u_2, v_2\}$ is called a *monster* on e (see Figure 5). We say that $\{u_3v_3, u_3w_3, v_3w_3\}$ is *the head* of M which is *at* v. We set $X_M = X_R \cup [E(B - \{u_2v_2\}) \cap X_B]$ and $Y_M = Y_R \cup [E(B - \{u_2v_2\}) \cap Y_B]$.

G' is obtained from G by replacing a dog by a monster on e if $(G_1, X_1, Y_1) = (D, X_D, Y_D)$ and this is replaced by (M, X_M, Y_M). Note that both the dog and the monster must have their heads at the same vertex of e. Observe also that $(M[\{u_1, v_1, w_1, z\}], \{u_1z\}, \{w_1z\})$ is a dog on w_1z. Thus, we can repeat the process of replacing a dog by a monster as many times as we wish. The next lemma asserts that the replacement of a dog by a monster creates a graph that still belongs to the family that we are interested in studying. The proof will use the notation of the last three paragraphs.

4.3. **Lemma.** *If G' is obtained by the replacement of a dog by a monster on e, then $G' \in \mathcal{G}_\emptyset(H')$.*

Proof. Suppose that $G'\backslash X''/Y'' = H'$. To show that H' is uniquely determined as a minor of G', one first shows, by arguing as in the last proof, that $w'_1z \in X''$. Next one shows that $X_R \subseteq X''$ and $Y_R - e' \subseteq Y''$ and then that $X_B - \{u_2z, zu\} \subseteq X''$ and $Y_B - e \subseteq Y''$. The straightforward details of these arguments are omitted.

To complete the proof that H' is uniquely determined as a minor of G', let $G_0 = G'\backslash(X_M - zu)/(Y_M - e)$. We shall show that, to produce H' from G_0, we must contract e and delete zu. Observe that the connected component G'_0 of G_0 that contains the edge e is obtained from G_2 by adding five new blocks: one

FIGURE 5. A monster.

triangle incident with v, the edge zu, and three triangles incident with z. Observe that $G\backslash X_1$ is obtained from G_2 by adding a block, which is a triangle incident with v, because the dog G_1 has head at v. There is just one way of getting H from $G\backslash X_1$: by deleting X_2 and contracting Y_2. Since we can view $G\backslash X_1$ as a subgraph of G'_0, it follows that we must contract e from G_0 to get H'. Finally, we must delete zu to produce H' otherwise the three blocks incident with z in G'_0 have a common vertex with the head of the monster. Hence H' is indeed uniquely determined as a minor of G'.

To get the result, we need only to prove that $G'\backslash x$ and G'/y are not blocks, for every $x \in X'$ and $y \in Y'$. But this is clearly true when $x \in X_M$ and $y \in Y_M$. Thus we may suppose that this is not the case. But, for x in X_2 and y in Y_2, we must have that neither $G_2\backslash x$ nor G_2/y is 2-connected since neither $G\backslash x$ nor G/y is 2-connected. Hence neither $G'\backslash x$ nor G'/y is 2-connected and the lemma holds. □

We say that F is *a snake, a dog, a bull, or a rhino on* $e \in Y$ *with respect to* (G, X, Y), when (F, X_F, Y_F) is a snake, a dog, a bull, or a rhino on e, respectively, and there is an admissible decomposition (G_1, G_2) of G with respect to e such that $(G_1, X_1, Y_1) = (F, X_F, Y_F)$.

5. NECESSARY BOUNDS

We shall break the proof of the main theorem into two parts. In this section, we establish that conditions (C1)–(C3) are necessary for the specified bound on $|E(G)| - |E(H)|$ to hold for all G in $\mathcal{G}_L(H)$.

For real numbers α and β, a graph H and a set L of loops of H, define

$$\mathcal{G}_L^{(\alpha,\beta)}(H) = \{G \in \mathcal{G}_L(H) : |E(G) - E(H)| > \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)\}.$$

We are looking for necessary conditions on α and β such that $\mathcal{G}_L^{(\alpha,\beta)}(H) = \emptyset$, for every H and L.

5.1. **Theorem.** *If α and β are real numbers such that*

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$$

for all graphs G and H such that $G \in \mathcal{G}_L(H)$ and L is a set of loops of H, then

$$\alpha + \beta \geq 5, \tag{C1}$$

$$2\alpha + 5\beta \geq 20, \text{ and} \tag{C2}$$

$$\beta \geq 3. \tag{C3}$$

Proof. To obtain (C1), we start with a graph with six vertices and then we replace a dog by a pig, repeating this operation n times to get our graph. Let G be the graph having a vertex-set $\{u_1, v_1, w_1, u_2, v_2, w_2\}$ and edge-set $\{u_1v_1, u_1w_1, v_1w_1, u_2v_2, u_2w_2, v_2w_2, u_1u_2, v_1u_2, u_1v_2\}$. Let $X = \{v_1u_2, u_1v_2\}$ and $Y = \{u_1u_2\}$. Observe that $G \in \mathcal{G}_\emptyset(H)$, where $H = G\backslash X/Y$. Moreover, the edge u_1u_2 has two dogs with respect to (G, X, Y). By Lemma 4.2, we can replace a dog by a pig getting a graph G' such that $G' \in \mathcal{G}_\emptyset(H')$, where $H' = G'\backslash X'/Y'$, for some disjoint subsets X' and Y' of $E(G')$. Observe that this pig has an edge in Y' with a dog with respect to (G', X', Y'). Thus, we can continue replacing dogs by pigs. After n such replacements, we get a graph $G_2^\#$ such that $G_2^\# \in \mathcal{G}_\emptyset(H_2^\#)$ for some minor $H_2^\#$ of $G_2^\#$. Observe that

$$|E(G_2^\#)| - |E(H_2^\#)| = 5n + 3, \kappa_1(H_2^\#) = n + 1, \text{ and } \kappa_2(H_2^\#) = n + 2,$$

since, at each replacement, we increase the number of connected components of the minor by one, the number of blocks by one, and the difference between the numbers of edges of the graph and the minor by five. As $G_2^\# \notin \mathcal{G}_\emptyset^{(\alpha,\beta)}(H_2^\#)$, it follows that

$$|E(G_2^\#)| - |E(H_2^\#)| \leq \alpha(\kappa_1(H_2^\#) - 1) + \beta(\kappa_2(H_2^\#) - 1)$$

Hence, we get $5n + 3 \leq \alpha n + \beta(n + 1)$ Dividing this inequality by n and taking the limit as n goes to infinity, we obtain (C1).

To get (C2), we start with the same 6-vertex graph that we used to get (C1). Instead of replacing dogs by pigs, we shall replace dogs by monsters. At each replacement, we increase the number of connected components by two, the number of blocks by five, and the number of edges that belong to the graph and do not belong to the minor by twenty. As in the previous paragraph, we repeat this operation n times. At the end, we get a graph $G_3^\#$ such that $G_3^\# \in \mathcal{G}_\emptyset(H_3^\#)$, for some minor $H_3^\#$ of $G_3^\#$. Observe that

$$|E(G_3^\#)| - |E(H_3^\#)| = 20n + 3, \kappa_1(H_3^\#) = 2n + 1, \text{ and } \kappa_2(H_3^\#) = 5n + 2.$$

As $G_3^\# \notin \mathcal{G}_\emptyset^{(\alpha,\beta)}(H_3^\#)$, it follows that $20n + 3 \leq \alpha(2n) + \beta(5n + 1)$. Dividing this inequality by n and taking the limit as n goes to infinity, we get (C2).

To obtain (C3), consider the graph $G_4^\#$ constructed as follows. Begin with $n+1$ vertex-disjoint copies of K_3 with vertex-sets $\{u_0, v_0, w_0\}, \{u_1, v_1, w_1\}, \ldots, \{u_n, v_n, w_n\}$. The set of edges of $G_4^\#$ that join vertices belonging to different K_3's is partitioned into two sets X and Y, where $Y = \{u_iu_0 : 1 \leq i \leq n\}$ and $X = \cup_{i=1}^n \{v_iu_0, u_iv_0\}$. Let $H_4^\# = G_4^\#\backslash X/Y$ and $L = \emptyset$. Observe that $H_4^\#$ has just one connected component and has $n + 1$ blocks all of which are triangles. Moreover, $G_4^\# \in \mathcal{G}_L(H_4^\#)$. As $G_4^\# \notin \mathcal{G}_L^{(\alpha,\beta)}(H_4^\#)$, it follows that $3n \leq \beta n$ and (C3) follows. □

6. Decompositions

In this section, we begin with a graph G in $\mathcal{G}_\emptyset(H)$ where $H = G\backslash X/Y$ and we produce related graphs J_2 and H_2' such that $J_2 \in \mathcal{G}_\emptyset(H_2')$. These constructions will be used repeatedly in the proof of the main theorem.

Suppose that $e \in Y$, say $e = uv$. Let (G_1, G_2) be an admissible decomposition of G with respect to e. We say that (G_1, G_2) has *type-k with respect to G_i* if there are exactly k vertices in $\{u, v\}$ that meet edges in $E(G_i) \cap (E(H) \cup Y)$. By convention, when we say that (G_1, G_2) has type-k we shall mean that (G_1, G_2) has type-k with respect to G_1.

For the next three lemmas, let (G_1, G_2) be an admissible decomposition of G with respect to $e = uv$. For i in $\{1, 2\}$, recall that $X_i = X \cap E(G_i)$ and $Y_i = Y \cap E(G_i)$. Define $H_1 = G_1\backslash X_1/Y_1$. We shall define two graphs J_2 and H_2' which depend on the type of (G_1, G_2). When (G_1, G_2) has type-0, let $H_2' = H_2 = G_2\backslash X_2/Y_2$. In this case, we shall define J_2 after the next lemma.

6.1. **Lemma.** *If (G_1, G_2) has type-0, then G_2 or G_2/e belongs to $\mathcal{G}_\emptyset(H_2')$.*

Proof. First, we shall prove that if $H_2' = G_2\backslash X'/Y'$, then $X' = X_2$ and $Y' = Y_2$. Observe that both when $e \in Y'$ and when $e \in X'$, the graph $G\backslash(X_1 \cup X')/(Y_1 \cup Y')$ is the union of the vertex-disjoint graphs H_1 and H_2. But this union is equal to H. Hence, as H can be obtained in a unique way as a minor of G, we conclude that $X' = X_2$ and $Y' = Y_2$. Thus H_2' is obtainable in a unique way as a minor of G_2.

Suppose that $G_2\backslash X''/Y''$ belongs to $\mathcal{G}_\emptyset(H_2')$. Then $G_2\backslash X''/Y''$ has H_2' as a minor, so $X'' \subseteq X_2$ and $Y'' \subseteq Y_2$. Now $G_2\backslash X''/Y''$ is a block. Therefore, whether or not $e \in Y''$, either (i) $G_2\backslash X''/(Y'' - e)$ is a block, or (ii) $G_2\backslash X''/(Y'' - e)$ has e as a loop or isthmus. Suppose that (ii) occurs. If e is a loop of $G_2\backslash X''/(Y'' - e)$, then $G_2\backslash e$ has H_2' as a minor; a contradiction. Thus we may assume that e is an isthmus of $G_2\backslash X''/(Y'' - e)$. Let w be the unique endpoint of e that has degree one in $G_2\backslash X''/(Y'' - e)$. Then some x'' in X'' is incident in G_2 with w or with some vertex in the tree in $G_2[Y'' - e]$ that is contracted to produce the vertex w. Thus w is incident only with x'' and e in $G_2\backslash(X'' - x'')/(Y'' - e)$. Observe that x'' cannot be a loop in this graph otherwise it could be contracted instead of being deleted when H_2' is obtained. Therefore $G_2\backslash[(X'' - x'') \cup e]/[(Y'' - e) \cup x''] = G_2\backslash X''/Y''$; a contradiction. We conclude that (ii) does not occur.

We may now suppose that $G_2\backslash X''/(Y'' - e)$ is a block. This block has e as an edge so its union G' with G_1 is also a block. Clearly G' has H as a minor. Thus $G' = G$, so $X'' = Y'' - e = \emptyset$. We conclude that G_2 or G_2/e belongs to $\mathcal{G}_\emptyset(H_2')$. □

When (G_1, G_2) is of type-0, let J_2 be the graph in $\{G_2, G_2/e\}$ that belongs to $\mathcal{G}_\emptyset(H_2')$.

Next we define J_2 when (G_1, G_2) has type-1. Without loss of generality, we may suppose that, in $G_1\backslash e$, every edge incident with u is in X, while some edge incident with v is not. Let J_2 be obtained from G_2 by adding two new vertices w and x and the edges wx, wv, xv, and xu. We define $X_2' = X_2 \cup xu$ and $H_2' = J_2\backslash X_2'/Y_2$. Observe that $(J_2[\{u, v, w, x\}], \{xu\}, \{uv\})$ is a dog on e having its head at v.

6.2. **Lemma.** *If (G_1, G_2) has type-1, then J_2 belongs to $\mathcal{G}_\emptyset(H_2')$.*

Proof. Suppose that $J_2\backslash X'/Y' = H_2'$. We shall show first that $X' = X_2'$ and $Y' = Y_2$. Since H_2' is obtained from H_2 by adjoining a triangle at v, we can obtain H from H_2' by replacing this triangle by H_1. Suppose that $e \in Y'$. Then $xu \in X'$.

It follows that $G\backslash[(X_1 \cup X') - xu]/(Y_1 \cup Y') = H$. As H is uniquely determined as a minor of G, we conclude that $X' = X_2$ and $Y' = Y_2$. Thus we may assume that $e \in X'$. Now consider the graph H_2'' that equals $G_2\backslash(X' - \{e, xu\})/(Y' - \{xu\})$. Observe that $E(H_2'') = E(H_2) \cup \{e\}$. Now e is not a loop of H_2'' otherwise it is not difficult to see that H can be obtained as a minor of G both from the deletion and the contraction of e; a contradiction. We shall show next that $xu \in X'$. Suppose that u is incident only with e in H_2''. Then either (i) $H_2''\backslash e = H_2$, or (ii) $H_2''\backslash e$ is obtained from H_2 by adding an isolated vertex, namely u. In the first case, $G\backslash[(X_1 \cup X') - \{xu\}]/[(Y_1 \cup Y') - \{e, xu\}] = H$ and we have a contradiction to the fact that H is uniquely obtained as a minor of G. Thus (ii) holds. In that case, $G\backslash[(X_1 \cup X') - \{xu\}]/[(Y_1 \cup Y') - \{e, xu\}]$ equals the graph that is obtained by adjoining u to H as an isolated vertex. We could eliminate this isolated vertex by contracting, rather than deleting, some edge of X_1 incident with u in G. Let f be such an edge. Then $G\backslash[(X_1 \cup X') - \{f, xu\}]/[(Y_1 \cup Y' \cup \{f\}) - \{e, xu\}] = H$, so H can be obtained in more than one way as a minor of G; a contradiction. We conclude that u must be incident with some edge g of $E(H_2)$ in H_2''. It follows that $xu \in X'$, as asserted, otherwise xw is adjacent to g in H_2'; a contradiction. Since $\{xu, e\} \subseteq X'$ and $H_2' = J_2\backslash X'/Y'$, it follows that

$$H = G\backslash[(X_1 \cup X') - \{xu\}]/[(Y_1 \cup Y') - \{e\}].$$

This is a contradiction since we have now obtained H as a minor of $G\backslash e$. We conclude that we do indeed have $X' = X_2'$ and $Y' = Y_2$.

We now show that $J_2 \in \mathcal{G}_\emptyset(H_2')$. If this is not so, then we can obtain a block having H_2' as a minor by contracting some subset Y_3 of Y_2 and deleting some subset X_3 of $X_2 \cup xu$. Clearly we cannot delete xu or contract e to produce this block. Thus $G_2\backslash X_3/Y_3$ is a block containing e and having H_2 as a minor, so $G\backslash X_3/Y_3$ is a block having H as a minor, so $X_3 = \emptyset = Y_3$. Hence $J_2 \in \mathcal{G}_\emptyset(H_2')$. □

When (G_1, G_2) has type-2, J_2 is the graph obtained from G_2 by adding two new vertices w and x and the edges wx, wv, and xu; and H_2' is $J_2\backslash X_2/Y$. Observe that $(J_2[\{u, v, w, x\}], \emptyset, \{uv\})$ is a snake on e with respect to (J_2, X_2, Y).

6.3. Lemma. *If (G_1, G_2) has type-2, then J_2 belongs to $\mathcal{G}_\emptyset(H_2')$.*

Proof. The result follows from Lemma 4.1, since we get J_2 from G by replacing G_1 by a snake. □

7. Some technical lemmas

In this section, we shall prove three technical lemmas that will be used in the proof of the main result. Throughout, G is a graph in $\mathcal{G}_\emptyset(H)$ where $H = G\backslash X/Y$.

In the next lemma, the labelling on the bull is the same as that in Section 4.

7.1. Lemma. *Let e be an edge in Y and suppose that (B, X_B, Y_B) is a bull on e with respect to (G, X, Y). Then e is not pendent in $[G - (V(B) - V(e))]\backslash(X - X_B)$.*

Proof. Let $e = uv$ and assume that e is a pendent edge in $[G - (V(B) - V(e))]\backslash(X - X_B)$. Suppose that the head T of the bull is at v and that vw_3 is an edge of Y_B, for $w_3 \in V(T)$. We show next that $d_{[G-(V(B)-\{u,v\})]\backslash(X-X_B)}(v) = 1$. If not, then $d_{[G-(V(B)-\{u,v\})]\backslash(X-X_B)}(u) = 1$. But e is the only edge in the bull that is incident with u and does not belong to X. Thus $d_{G\backslash X}(u) = 1$ and so the edge e of Y is pendent in $G\backslash X$. This contradiction implies that $d_{[G-(V(B)-\{u,v\})]\backslash(X-X_B)}(v) = 1$.

Thus, in G, every edge incident with v, with the exception of vw_3 and e, belongs to X. Observe that H can be obtained from G by contracting all the edges of X_B that join different connected components of $B\backslash X_B$, deleting all the other edges in $X_B \cup Y_B$, and then deleting all the edges in $X - [X_B \cup Y_B]$ and contracting all the edges in $Y - [X_B \cup Y_B]$. This is a contradiction since we have shown that H can be obtained in two different ways as a minor of G. □

7.2. **Lemma.** *Suppose that $e' \in Y$ and there is a dog or a rhino P on e' with respect to (G, X, Y). Then there is no connected component of $[G - (V(P) - V(e'))]\backslash(X - X_P)$ whose edge set is $\{e'\}$.*

Proof. Suppose that there is a connected component of $[G - (V(P) - V(e'))]\backslash(X - X_P)$ whose edge-set is $\{e'\}$. Observe that P is a connected component of $G\backslash(X - X_P)$, since this graph is the union of P with $[G - (V(P) - V(e'))]\backslash(X - X_P)$. When P is a dog, we arrive at a contradiction because e' is a pendent edge in $G\backslash X$, since e' is a pendent edge in $P\backslash(X \cap E(P))$. Suppose now that P is a rhino R. Recall that H_R, which equals $R\backslash X_R/Y_R$, is a graph having two connected components, each with two blocks both of which are triangles. Note also that H_R can be obtained as a minor of R in a different way: contract e' and all the edges of X_R that join different connected components of $R\backslash X_R$; and delete all the other edges belonging to $X_R \cup Y_R$. Thus H_R can be obtained in two different ways as a minor of the connected component R of $G\backslash(X - X_R)$. Hence H can be obtained in two different ways as a minor of G; a contradiction. □

Let $(D, \{t\}, \{e\})$ be a dog on $e = uv$ having head at v. We call t the *tail* of D and say that it is *at* u. If there is exactly one edge vy in $E(G) - E(D)$ meeting v, and vy is in X, then vy is called the *lead* of the dog and we say it is *at* y. The next lemma asserts that we can remove a dog and its lead and stay in the desired class whenever we have two dogs with tails at the same vertex and leads at the same vertex provided some minor technical condition holds.

7.3. **Lemma.** *Let uv_1 and uv_2 be edges e_1 and e_2 in Y where $v_1 \neq v_2$ and suppose that G has a vertex y such that, for each $i \in \{1, 2\}$,*

(i) *$(D_i, \{t_i\}, \{e_i\})$ is a dog on e_i having head T_i at v_i, and $y \notin V(D_i)$; and*
(ii) *$d_G(v_i) = 4$ and $v_i y \in X$.*

If the connected component of H that has T_1 and T_2 as blocks has at least one more block, then $G - (V(D_1) - u) \in \mathcal{G}_\emptyset(H - (V(T_1) - v_1))$.

Proof. Let $G' = G - (V(D_1) - u)$ and $H' = H - (V(T_1) - v_1)$ and suppose that $G'\backslash X'/Y' = H'$. Since the connected component of H having T_1 and T_2 as blocks has another block, it is not difficult to see that Y' must contain v_2u or v_2y. Moreover, since v_1 and v_2 both have degree four in G, it follows that G' is a block.

In this paragraph, we shall prove that $X' = X \cap E(G')$ and $Y' = Y \cap E(G')$. We have two cases to consider: (a) $v_2u \in Y'$; and (b) $v_2y \in Y'$. Assume (a) holds. Let $G'' = G\backslash(X' \cup \{v_1y, v_1u, t_1\})/Y'$. Then G'' is the vertex-disjoint union of the graphs H' and T_1. As v_1u joins v_1 to the vertex that v_2 has been contracted to in H', it follows that $(G'' + v_1u)/v_1u$, which equals $G\backslash(X' \cup \{v_1y, t_1\})/(Y' \cup v_1u)$ is equal to H. But H is uniquely obtained as a minor of G. Hence $X = X' \cup \{v_1y, t_1\}$ and $Y = Y' \cup v_1u$. Thus $X' = X \cap E(G')$ and $Y' = Y \cap E(G')$ in case (a). Now assume that (b) holds. Then, since v_2 is contracted to y in H', it follows that

$G\backslash(X' \cup \{v_1u, t_1\})/(Y' \cup v_1y)$ equals H and again $X' = X \cap E(G')$ and $Y' = Y \cap E(G')$.

Now suppose that, for some x' in X' and some y' in Y', one of $G'\backslash x'$ and G'/y' is a block. Then we obtain the contradiction that $G\backslash x'$ or G/y' is a block unless u and y have been identified in $G'\backslash x'$ or G'/y'. The exceptional case can only occur if y' joins y and u. Then $G'\backslash[(X' \cup v_2u) - v_2y]/[(Y' \cup v_2y) - v_2u] = H'$; a contradiction. We conclude that if $x' \in X'$ and $y' \in Y'$, then neither $G'\backslash x'$ and G'/y' is a block. Hence, by Lemma 2.5, $G' \in \mathcal{G}_\emptyset(H')$. □

8. The beginning of the main proof

In this section, we begin the proof of the second part of the main theorem of the paper. This proof is quite complex and we will need to take some detours, which will appear in separate sections, before we can complete it. An outline of the strategy of the proof will be given in the next section. Theorem 5.1 established that if every G in $\mathcal{G}_L(H)$ obeys the inequality

$$|E(G) - E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1),$$

then (α, β) must lie in A. Our main theorem establishes that, provided $(\alpha, \beta) \in A$ the desired inequality on $|E(G)| - |E(H)|$ holds.

8.1. **Theorem.** *Suppose that α and β are real numbers such that $(\alpha, \beta) \in A$. If G and H are graphs, L is a set of loops of H, and $G \in \mathcal{G}_L(H)$, then*

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1).$$

Proof. We show first that, to verify the theorem, it suffices to prove it for $(\frac{5}{3}, \frac{10}{3})$ and $(\frac{5}{2}, 3)$, the two vertices of the polyhedron A. To establish this, we show that if the theorem holds for $(\alpha, \beta) \in \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$, then it also holds for:

(i) (α_3, β_3) where $\alpha_3 \geq \alpha_1$ and $\beta_3 \geq \beta_1$;
(ii) $(\alpha_1 - c, \beta_1 + c)$ where $c \geq 0$; and
(iii) $a(\alpha_1, \beta_1) + b(\alpha_2, \beta_2)$ where $a + b = 1$ and $a, b \geq 0$.

The fact that the theorem holds for (i) follows because both $\kappa_1(H)$ and $\kappa_2(H)$ are positive. To see that the theorem holds for (ii), it suffices to observe that $\kappa_2(H) \geq \kappa_1(H)$. Finally, it is straightforward to verify that the theorem holds for (iii). We conclude that, as asserted, we need only verify the theorem when $(\alpha, \beta) \in \{(\frac{5}{3}, \frac{10}{3}), (\frac{5}{2}, 3)\}$.

We shall assume that the theorem fails, that is, we suppose that $\mathcal{G}_L^{(\alpha,\beta)}(H) \neq \emptyset$ for some triple (G, H, L), where $H = G\backslash X/Y$. We choose a triple (G, H, L) such that $G \in \mathcal{G}_L^{(\alpha,\beta)}(H)$ and $(\kappa_2(H), -\theta(H))$ is minimal in the lexicographic order, where $\theta(H)$ denotes the number of blocks of H that are triangles.

The next two lemmas establish that neither G nor H has any loops.

8.2. **Lemma.** *G has no loops.*

Proof. Suppose that l is a loop of G. Then l is also a loop of H so we cannot simply delete l. Assume first that l is adjacent to some edge h in $E(H)$. It is not difficult to show that $(G\backslash l, H\backslash l, L - l)$ violates our choice of (G, H, L).

Next assume that l is adjacent in G to an edge e of Y. Let G' be obtained by taking the union of $G\backslash l$ and a snake on e. Take $H' = G'\backslash X/Y$. Then it is straightforward to check that $(G', H', L - l)$ contradicts the choice of (G, H, L).

We may now assume that l is incident to a vertex v of G that is incident only with loops and edges of X. Then it follows from the first paragraph that l is the unique loop incident with v otherwise l is adjacent to a loop h, which must be in H. As $G \in \mathcal{G}_L(H)$, it is not difficult to show that $|X| \geq 2$. In that case, we construct a new graph G'' as follows. First delete l. Then take an edge x in X joining v to, say, v' and replace it by a path v, u, v' labelling vu as e and uv' by x. Let the resulting graph be G'. Finally, let G'' be union of G' with a snake on $e = uv$ that has e, u, and v as its only common elements with G' (see Figure 6). Let $H'' = G''\backslash X/(Y \cup e)$.

We assert that $(G'', H'', L - l)$ contradicts the choice of (G, H, L). The main step in the proof of this is to show that e must be contracted in order to obtain H'' from G''. From this, it follows that H'' arises uniquely as a minor of G'': we must delete X and contract $Y \cup e$. Finally, it is straightforward to show that $G'' \in \mathcal{G}^{(\alpha,\beta)}_{L-l}(H'')$ and thence to deduce that $(G'', H'', L - l)$ contradicts the choice of (G, H, L). Therefore G has no loops. □

Figure 6. The replacement in the proof of Lemma 8.2.

8.3. **Lemma.** *H has no loops.*

Proof. If l is a loop of H, then $(G', H', L - l)$ contradicts the choice of (G, H, L), where G' and H' are obtained from G and H, respectively, by replacing l by a path of length three. □

8.4. **Lemma.** *Y is non-empty.*

Proof. If $Y = \emptyset$, then, by Theorem 3.2, $|E(G)| - |E(H)| \leq \kappa_1(H) + \kappa_2(H) - 2$. Thus $|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)$ for $(\alpha, \beta) \in \{(\frac{5}{3}, \frac{10}{3}), (\frac{5}{2}, 3)\}$; a contradiction. □

9. An outline of the main proof

The beginning of the proof of Theorem 8.1 given in the last section is relatively direct. The rest of the proof is far less so and we shall outline it here.

Let y be an edge of Y. Next we define the *depth* of y inductively. If G has an admissible decomposition (G^1_y, G^2_y) with respect to y such that G^1_y/y is a block and $(Y - y) \cap E(G^1_y)$ is empty, then y has depth 0. For $k \geq 1$, the edge y has depth k if y does not have depth less than k and G has an admissible decomposition (G^1_y, G^2_y)

such that G^1_y/y is a block and all edges of $Y - y$ in $E(G^1_y)$ have depth less than k. There are six main steps in the proof, the first of which has already been done in Lemma 8.4.

(S1) G has at least one depth-0 edge.
(S2) On every depth-0 edge of G, there is a dog or a snake with respect to (G, X, Y).
(S3) G has at least one depth-1 edge.
(S4) On every depth-1 edge of G, there is a rhino or a bull with respect to (G, X, Y).
(S5) G has at least one depth-2 edge.
(S6) G cannot have a depth-2 edge.

The proofs of steps (S2)–(S6) appear in Lemmas 12.1–12.5. The proofs of steps (S2), (S4), and (S6) are very similar, as are the proofs of steps (S3) and (S5). To avoid repetitive arguments, we shall prove two general but technical lemmas, 11.4 and 11.5 respectively, which combine the common features of these two sets of situations.

If $G \in \mathcal{G}^{(\alpha,\beta)}_{\emptyset}(H)$ where $H = G\backslash X/Y$ and (C1)–(C3) hold, then in Section 11 we study a certain subgraph G_1 of G in order to deal simultaneously with the following two cases.

Case I. (G_1, G_2) is an admissible decomposition of G with respect to an edge e in Y such that G_1/e is a block. In this case, $X_1 = X \cap E(G_1)$, $Y_1 = Y \cap E(G_1)$, and we set $Y'' = \{e\}$.

Case II. $G_1 = G$. In this case, $X_1 = X$, $Y_1 = Y$, and we set $Y'' = \emptyset$.

Note that, in both cases,

$$G_1/Y'' \text{ is a block.}$$

We also assume throughout that section that the following hold in both Cases I and II.

(H1) $E(G_1) \cap (Y - Y'')$ contains only depth-0 or depth-1 edges of G.
(H2) Every depth-0 edge in $E(G_1) \cap (Y - Y'')$ has a dog or a snake with respect to (G, X, Y).
(H3) Every depth-1 edge in $E(G_1) \cap (Y - Y'')$ has a bull or a rhino with respect to (G, X, Y).

Note that if e is a depth-0 edge of Y and we are in Case I, then (H1)–(H3) hold. Moreover, once (S2) is proved, (H1)–(H3) hold if (S3) fails and we are in Case II. In this manner, hypotheses (H1)–(H3) enable us to prove (S2)–(S6) one after the other.

Much of the argument in Section 11 focuses on the graph that we get by breaking off the bulls, rhinos, snakes, and dogs whose existence is guaranteed by (H1)–(H3).

10. An auxiliary lemma

In this section, we detour from the proof of Theorem 8.1 to prove a technical lemma that will be fundamental to the proof of that theorem. This lemma has numerous hypotheses. The motivation for these will be made clear in the next section. We begin by defining a slight modification of the function κ_2. Let $\kappa_2^{>}(\widetilde{G}, \widetilde{X}, \widetilde{Y})$ be the number of blocks of $\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}$ with at least one edge plus the number of isolated vertices of $\widetilde{G}\backslash\widetilde{X}$. Thus $\kappa_2^{>}(\widetilde{G}, \widetilde{X}, \widetilde{Y})$ is $\kappa_2(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y})$ minus the number of isolated vertices of $\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}$ that arise from the contraction of a connected component of $\widetilde{G}\backslash\widetilde{X}$ whose edge-set is non-empty and is contained in $\widetilde{Y}$.

10.1. **Lemma.** *Suppose that*

(i) *$\widetilde{G}$ is a block and $\widetilde{X}$ and $\widetilde{Y}$ are disjoint subsets of $E(\widetilde{G})$ such that $|E(\widetilde{G})| \neq 1$ or $E(\widetilde{G}) \neq \widetilde{Y}$;*
(ii) *$\widetilde{G}\backslash x$ is not a block, for every x in $\widetilde{X}$;*
(iii) *$\widetilde{Y}$ does not contain a cycle of $\widetilde{G}$;*
(iv) *$\widetilde{Y}$ does not span any edge of $\widetilde{X}$;*
(v) *$\widetilde{Y}$ has a subset Y_0, which may be empty, such that $\widetilde{G}/e$ is not a block for every e in Y_0.*

Then

$$|Y_0| \leq \kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) + \kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) - 2.$$

Proof. We shall argue by induction on $|Y_0|$. First, suppose that $|Y_0| = 0$. If $\kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) \geq 2$ or $\kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) \geq 1$, then the result follows. Thus we may suppose that $\kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) = 1$ and $\kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) = 0$. Therefore $\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}$ is a vertex and $E(\widetilde{G}) = \widetilde{X} \cup \widetilde{Y}$. Since $\kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) = 1$, it follows that $\widetilde{G}\backslash\widetilde{X}$ has just one connected component. By (iii), $\widetilde{G}\backslash\widetilde{X}$ is a tree. Thus, $\widetilde{Y}$ spans $\widetilde{X}$. By (iv), it follows that $\widetilde{X} = \emptyset$. Hence $E(\widetilde{G}) = \widetilde{Y}$ and $\widetilde{G}$ is a tree. As $\widetilde{G}$ is a block and $\kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) = 0$, it follows that $|E(\widetilde{G})| = 1$. Thus we have a contradiction to (i) since $|E(\widetilde{G})| = 1$ and $E(\widetilde{G}) = \widetilde{Y}$. Hence the lemma holds for $|Y_0| = 0$.

Suppose that $|Y_0| > 0$. Choose $e \in Y_0$. By (v), $\widetilde{G}/e$ is not a block. Thus, for some $n \geq 2$, there are n blocks $\widetilde{G_1}, \widetilde{G_2}, \dots, \widetilde{G_n}$ whose union is $\widetilde{G}$ such that each has at least two edges and, for $i \neq j$, the only common elements between $\widetilde{G_i}$ and $\widetilde{G_j}$ are the edge e and its vertices. For i in $\{1, 2, \dots, n\}$, set $X^i = \widetilde{X} \cap E(\widetilde{G_i}), Y^i = \widetilde{Y} \cap E(\widetilde{G_i})$, and $Y_0^i = (Y_0 \cap E(\widetilde{G_i})) - e$. Observe that $(\widetilde{G_i}, X^i, Y^i, Y_0^i)$ has the same properties as $(\widetilde{G},\widetilde{X},\widetilde{Y}, Y_0)$. By induction, we have that

$$|Y_0^i| \leq \kappa_1(\widetilde{G_i}\backslash X^i/Y^i) + \kappa_2^{>}(\widetilde{G_i}, X^i, Y^i) - 2,$$

for every i in $\{1, 2, \dots, n\}$. Hence

$$|Y_0 - e| = \sum_{i=1}^{n} |Y_0^i| \leq \sum_{i=1}^{n} \kappa_1(\widetilde{G_i}\backslash X^i/Y^i) + \sum_{i=1}^{n} \kappa_2^{>}(\widetilde{G_i}, X^i, Y^i) - 2n.$$

Observe that

$$\sum_{i=1}^{n} \kappa_2^{>}(\widetilde{G_i}, X^i, Y^i) = \kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) \text{ and } \sum_{i=1}^{n} \kappa_1(\widetilde{G_i}\backslash X^i/Y^i) = \kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) + n - 1,$$

where the last equality occurs because each of $\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}, \widetilde{G_1}, \widetilde{G_2}, \dots, \widetilde{G_n}$ has a component containing the vertex that results from the contraction of e. Thus

$$\begin{aligned} |Y_0| - 1 &\leq (\kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) + n - 1) + \kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) - 2n \\ &= \kappa_1(\widetilde{G}\backslash\widetilde{X}/\widetilde{Y}) + \kappa_2^{>}(\widetilde{G},\widetilde{X},\widetilde{Y}) - n - 1, \end{aligned}$$

and the result follows by induction since $n \geq 2$. □

11. Some basic inequalities

In this section, we assume that $G \in \mathcal{G}_{\emptyset}^{(\alpha,\beta)}(H)$ where $H = G\backslash X/Y$. We also assume that (C1)–(C3) from Theorem 1.1 hold, that one of Cases I and II defined in Section 9 occurs, and that hypotheses (H1)–(H3) defined at the end of Section 10 hold.

We now distinguish three disjoint subsets of $(Y \cap E(G_1)) - Y''$ each of which may be empty. Let Y_b be the set of depth-1 edges g in $(Y \cap E(G_1)) - Y''$ that have a bull with respect to (G, X, Y) and let D_g be one of these bulls. Let Y_r be the other depth-1 edges in $(Y \cap E(G_1)) - Y''$. By assumption, every such edge g has a rhino with respect to (G, X, Y). Let D_g be such a rhino. Let Y_{sd} be the set of edges of $(Y \cap E(G_1)) - Y''$ that do not belong to any of the graphs D_g for $g \in Y_b \cup Y_r$. By assumption, every such edge g is a depth-0 edge. In this case, we choose D_g to be a dog or a snake on g with respect to (G, X, Y).

Figure 7. Breaking off bulls, rhinos, dogs, and snakes.

Next we shall break off all the bulls, rhinos, dogs, and snakes that we have associated with edges of $(Y \cap E(G_1)) - Y''$. Let

$$G' = G_1 - \left(\bigcup_{g \in Y_b \cup Y_r \cup Y_{sd}} [V(D_g) - V(g)] \right).$$

An example of this construction is shown in Figure 7. In G_1, the type of each edge of Y is marked and, for each edge of Y that remains in G', we have indicated to which of the sets Y_{sd}, Y_b, Y_r, or Y'' it belongs. We observe that G_1 has a rhino at the top, a bull at the bottom, a snake on the left, and a dog on the right. It is not difficult to see that, in this example and in general, G' is a block. Define $X' = E(G') \cap X$ and $Y' = E(G') \cap Y$. These sets have the following properties:

(P1) Y' does not contain a cycle of G'.
(P2) Y' does not span any edge of X'.
(P3) $G' \backslash x$ is not a block for every x in X'.
(P4) No edge of Y_b is pendent in $G' \backslash X'$.
(P5) $E(G') \neq Y'$.

The first three parts of this follow from Lemma 2.3 and the fact that $G \in \mathcal{G}_{\emptyset}(H)$. For (P4) , we observe that, by Lemma 7.1, if $g \in Y_b$, then g is not pendent in $[G - (V(D_g) - V(g))] \backslash (X - X_{D_g})$ and, using this, it is not difficult to show that g is not pendent in $G' \backslash X'$.

To show (P5), suppose that $E(G') = Y'$. As G' is a block and $G'[Y']$ contains no cycle, it follows that $|E(G')| = 1$. If $Y' = Y''$, then $G' = G_1$, which contradicts the fact that (G_1, G_2) is an admissible decomposition of G with respect to e. Thus $Y'' = \emptyset$. But, in that case, G is a snake, a dog, a bull, or a rhino, and G/Y' is a block; a contradiction. We conclude that (P5) holds.

In this section, we shall apply Lemma 10.1 to get an upper bound on $|X_1| + |Y_1|$. We shall also get bounds on $\kappa_1(G_1 \backslash X_1/Y_1)$ and $\kappa_2(G_1 \backslash X_1/Y_1)$. These bounds will be used to derive two lemmas (11.4 and 11.5) that are fundamental in the proof of the main result.

Observe that

$$\kappa_1(G_1 \backslash X_1/Y_1) = \kappa_1(G' \backslash X'/Y') + |Y_r| + |Y_b|. \tag{5}$$

We also have that

$$\kappa_2(G_1 \backslash X_1/Y_1) = 4|Y_r| + 3|Y_b| + |Y_{sd}| + \kappa_2^{>}(G', X', Y') + \delta_1, \tag{6}$$

where $\delta_1 = 1$ when $|Y''| = 1$ and Y'' is the edge-set of a connected component of $G' \backslash X'$, and $\delta_1 = 0$ otherwise.

Now, we shall get an upper bound for $|X_1| + |Y_1|$. Let s be the number of edges g in Y_{sd} such that D_g is a snake. Observe that

$$\begin{aligned} |X_1| + |Y_1| &= (|X'| + 8|Y_r| + 7|Y_b| + |Y_{sd}| - s) + (|Y'| + 4|Y_r| + 3|Y_b|) \\ &= |X'| + 13|Y_r| + 11|Y_b| + 2|Y_{sd}| + |Y''| - s. \end{aligned} \tag{7}$$

Next we seek an upper bound for $|X'|$. We shall obtain this by applying Lemma 10.1 to a certain graph K. There are two cases to consider:

(a) Y'' is not a pendent edge of $G' \backslash X'$.
(b) Y'' is a pendent edge of $G' \backslash X'$.

Observe that (a) includes the possibility that Y'' is empty.

Consider (a). Since G_1/Y'' is a block, it follows that G'/Y'' is a block. Let Y_0 be a minimal subset of Y_b such that $(G'/Y'')/(Y_b - Y_0)$ is a block. In case (b), let Y_0 be a minimal subset of Y_b such that $G'/(Y_b - Y_0)$ is a block. Let

$$K = \begin{cases} (G'/Y'')/(Y_b - Y_0) & \text{in case (a);} \\ G'/(Y_b - Y_0) & \text{in case (b).} \end{cases}$$

Evidently K/g is not a block for every g in Y_0. We want to apply Lemma 10.1 to $(\widetilde{G}, \widetilde{X}, \widetilde{Y}, Y_0)$ where $\widetilde{G} = K$, $\widetilde{X} = X'$, and

$$\widetilde{Y} = \begin{cases} Y_r \cup Y_0 \cup Y_{sd} & \text{in case (a);} \\ Y_r \cup Y_0 \cup Y_{sd} \cup Y'' & \text{in case (b).} \end{cases}$$

11.1. **Lemma.** *With* $(\widetilde{G}, \widetilde{X}, \widetilde{Y}, Y_0) = (K, X', \widetilde{Y}, Y_0)$*, the hypotheses of Lemma 10.1 hold.*

Proof. All of the hypotheses except (i) and (ii) follow easily. We verify (ii) in case (a) noting that a similar argument applies in case (b). If $K\backslash x$ is a block for some x in $\widetilde{X}$, then, by (P1)–(P3) , it follows that $Y'' \cup (Y_b - Y_0)$ contains a pendent edge in $G'\backslash x$; a contradiction. Thus (ii) holds. To show (i), suppose that $|E(K)| = 1$ and $E(K) = \widetilde{Y}$. Then $E(G') = Y'$. By (P1) and the fact that G' is a block, we deduce that $|E(G')| = 1$; a contradiction to (P5) . Hence (i) holds. □

Applying Lemma 10.1, we get

$$|Y_0| = \kappa_1(K\backslash X'/\widetilde{Y}) + \kappa_2^{>}(K, X', \widetilde{Y}) - 2 - \delta_2, \tag{8}$$

for some $\delta_2 \geq 0$. Evidently $K\backslash X'/\widetilde{Y} = G'\backslash X'/Y'$, so

$$\kappa_1(K\backslash X'/\widetilde{Y}) = \kappa_1(G'\backslash X'/Y'). \tag{9}$$

We shall show next that

$$\kappa_2^{>}(K, X', \widetilde{Y}) = \kappa_2^{>}(G', X', Y'). \tag{10}$$

Certainly $\kappa_2(K\backslash X'/\widetilde{Y}) = \kappa_2(G'\backslash X'/Y')$. Moreover, if v is an isolated vertex of $G'\backslash X'$, then v is an isolated vertex of $K\backslash X'$. Now suppose that v is an isolated vertex of $K\backslash X'$ that is not an isolated vertex of $G'\backslash X'$. Then $G'\backslash X'$ has a component Z whose edge-set is non-empty and is contained in, respectively, $Y'' \cup (Y_b - Y_0)$ in case (a) or $Y_b - Y_0$ in case (b). Because Y' contains no cycle of $G'\backslash X'$, it follows that Z must contain a pendent edge. But this is a contradiction by (P4) and the fact that Y'' is not pendent in $G'\backslash X'$ when (a) holds. We conclude that (10) holds.

As $K\backslash x$ is not a block, for every x in X', it follows that $K \in \mathcal{G}_\emptyset(K\backslash X')$ so, by Theorem 3.2,

$$|X'| = \kappa_1(K\backslash X') + \kappa_2(K\backslash X') - 2 - \delta_3, \tag{11}$$

for some $\delta_3 \geq 0$. Evidently

$$\kappa_1(K\backslash X') = \kappa_1(K\backslash X'/\widetilde{Y}). \tag{12}$$

Next we show that

$$\kappa_2(K\backslash X') - |\widetilde{Y}| \leq \kappa_2^{>}(K, X', \widetilde{Y}). \tag{13}$$

Consider the blocks of $K\backslash X'$. They are of three types: isolated vertices, those with at least one edge that is not in $\widetilde{Y}$, and those with non-empty edge-set contained in $\widetilde{Y}$. Each block of the first type is counted in $\kappa_2^{>}(K, X', \widetilde{Y})$. The edge-set of each block of the second type contains the edge-set of at least one block of $K\backslash X'/\widetilde{Y}$ with non-empty edge-set. Such blocks of $K\backslash X'/\widetilde{Y}$ are counted in $\kappa_2^{>}(K, X', \widetilde{Y})$. No block of $K\backslash X'$ of the third type is counted in $\kappa_2^{>}(K, X', \widetilde{Y})$ and there are at most $|\widetilde{Y}|$ blocks of this type. Hence there are at most $\kappa_2(K\backslash X') - |\widetilde{Y}|$ blocks of $K\backslash X'$ of the first two types and (13) follows. Thus, by the definition of $\widetilde{Y}$, we have

$$\kappa_2(K\backslash X') = \kappa_2^{>}(K, X', \widetilde{Y}) + |Y_r| + |Y_0| + |Y_{sd}| + |Y''| - \delta_4, \tag{14}$$

where $\delta_4 \geq 0$. Indeed, $\delta_4 \geq 1$ unless Y'' is a pendent edge of $G'\backslash X'$ or $Y'' = \emptyset$. Substituting from (12) and (14) into (11), we get that

$$|X'| = \kappa_1(K\backslash X'/\widetilde{Y}) + \kappa_2^{>}(K, X', \widetilde{Y}) + |Y_r| + |Y_0| + |Y_{sd}| + |Y''| - \delta_4 - 2 - \delta_3.$$

Using (8) to replace $\kappa_1(K\backslash X'/\widetilde{Y}) + \kappa_2^{>}(K, X', \widetilde{Y})$ by $|Y_0| + 2 + \delta_2$, we get

$$|X'| = 2|Y_0| + |Y_r| + |Y_{sd}| + |Y''| + \delta_2 - \delta_3 - \delta_4.$$

Substituting from this equation for $|X'|$ into (7), we obtain

$$|X_1| + |Y_1| = 14|Y_r| + 11|Y_b| + 3|Y_{sd}| + 2|Y_0| + 2|Y''| + \delta_2 - s - \delta_3 - \delta_4. \tag{15}$$

By substituting for $\kappa_2^{>}(K, X', \widetilde{Y})$ from (8) into (5) and using (10), we can also get a new equation for $\kappa_1(G_1\backslash X_1/Y_1)$, namely,

$$\kappa_1(G_1\backslash X_1/Y_1) = |Y_r| + |Y_b| + |Y_0| - \kappa_2^{>}(G', X', Y') + 2 + \delta_2. \tag{16}$$

The proof of Theorem 8.1 will involve reducing to the case when $\kappa_2^{>}(G', X', Y')$ is 0. The next two lemmas gather together useful information about this case.

11.2. **Lemma.** *If $\kappa_2^{>}(G', X', Y') = 0$, then $G'[Y'] = G'\backslash X'$, $\kappa_1(G'\backslash X'/Y') \geq 2$ and*

$$|Y_r| + |Y_{sd}| + s + \delta_5 \geq 2\kappa_1(G'\backslash X'/Y') \geq 4,$$

where $\delta_5 = 0$ unless $|Y''| = 1$, in which case, δ_5 is 2 minus the type of (G_1, G_2).

Proof. Since $\kappa_2^{>}(G', X', Y') = 0$, every edge of G' is in X' or Y', and $G'\backslash X'$ has no isolated vertices. Thus $G'[Y'] = G'\backslash X'$. Now, since $E(G') \neq Y'$, it follows that $X' \neq \emptyset$. Thus, as Y' does not span any edge of X' and Y' contains no cycle of G', we deduce that $G[Y']$ is a forest having at least two connected components. Thus $\kappa_1(G'\backslash X'/Y') = \kappa_1(G'\backslash X') \geq 2$.

To determine

$$|Y_r| + |Y_{sd}| + s + \delta_5, \tag{17}$$

we shall consider the contribution of each connected component of $G'[Y']$ to this sum where, if $|Y''| = 1$, we view δ_5 as contributing to the component of $G'[Y']$ containing Y''. If every component of $G'[Y']$ contributes at least two to $|Y_r| + |Y_{sd}| + s + \delta_5$, then the required result holds. Thus we may assume that $G'[Y']$ has a component Z that contributes less than 2 to (17). Then no edge of Z has a snake on it. Thus every edge of $Z - Y''$ has a dog, a bull, or a rhino on it. By Lemma 7.1, no edge of Y_b is pendent in Z. Therefore every pendent edge of Z is in $Y_r \cup Y_{sd} \cup Y''$. We now suppose that Z has at least two edges. Then Z has at least two pendent edges. As Z has at most one pendent edge in $Y_r \cup Y_{sd}$, it follows that Z is a path one end of which is the edge in Y''. Thus we are in Case I and, since $E(G') = X' \cup Y'$, it follows that (G_1, G_2) has type-1, so $\delta_5 = 1$. In this case, Z contains Y'' and the contribution of Z to (17) is at least two; a contradiction.

It remains to consider the case when Z has exactly one edge. By Lemma 7.2 and the fact that no edge of Z has a snake on it, we deduce that the edge-set of Z is Y''. In that case, (G_1, G_2) has type-0 and so $\delta_5 = 2$, and Z contributes 2 to (17). This contradiction completes the proof of the lemma. □

A *star* is a tree in which there is a vertex incident with every edge. This vertex, the *center* of the star, is unique unless the star consists of a single edge. In the exceptional case, we are free to choose one of the two vertices to be the center of the star. The next lemma involves four of the seven parameters δ_1–δ_5, s, and t. In Table 1, which appears below, these seven parameters are summarized.

11.3. **Lemma.** *Suppose that*

$$\kappa_2^{>}(G', X', Y') = |Y_0| = |Y_b| = \delta_1 = \delta_2 = \delta_3 = s = 0.$$

Then $K\backslash X'$ has two connected components each being a star, and every edge of X' joins a pendent vertex of one connected component to the center of the other.

Proof. By definition, since $Y_b = Y_0 = \emptyset$, it follows that $K = G'$ unless Y'' is not a pendent edge of $G'\backslash X'$, in which case, $K = G'/Y''$. By (8) and (10), we have that $\kappa_1(K\backslash X'/\widetilde{Y}) = 2$. By (11), we have that

$$|X'| = \kappa_1(K\backslash X') + \kappa_2(K\backslash X') - 2. \tag{18}$$

Since $\delta_1 = 0$, either $|Y''| = 0$, or $|Y''| = 1$ and Y'' is not the edge-set of a connected component of $G'\backslash X'$. As $\kappa_2^{>}(G', X', Y') = 0$, the graph $G'\backslash X'$ has no isolated vertices. We deduce that, both when $|Y''| = 0$ and when $|Y''| = 1$, the set Y'' is not the edge-set of a connected component of $G'\backslash X'$. By the last lemma, $G'\backslash X' = G'[Y']$. Thus $K\backslash X'$ has no isolated vertices. Since K is G' or G'/Y'' with the latter occurring when Y'' is not a pendent edge of $G'\backslash X'$, it follows that

$$2 = \kappa_1(G'\backslash X'/Y') = \kappa_1(G'\backslash X') = \kappa_1(K\backslash X'). \tag{19}$$

Moreover, each component of $K\backslash X'$ is a tree. Let T_1 and T_2 be these two components. Then each edge of X' joins a vertex of T_1 to a vertex of T_2. Thus, by (18) and (19),

$$|X'| = \kappa_2(K\backslash X') = (|V(T_1)| - 1) + (|V(T_2)| - 1) = |V(K)| - 2. \tag{20}$$

Suppose that $|V(T_1)| = |V(T_2)| = 2$. Then, by (20), $|X'| = 2$ and it follows that K is a 4-cycle, and the lemma holds. Thus we may suppose that $|V(T_1)| \geq 3$. For i in $\{1, 2\}$, let P_i be the set of degree-one vertices of T_i. Then $|P_i| \geq 2$. Since K is a block, for each u in $P_1 \cup P_2$, there is an edge x_u in X' such that x_u meets u. Let $X'_u = \{x_u : u \in P_1 \cup P_2\}$. Now take v and w in P_1. Then T_1 has a path joining v and w, and so $K\backslash(X' - X'_u)$ has a cycle containing this path, x_v, x_w, and a subset of $E(T_2)$. It follows without difficulty that $K\backslash(X' - X'_u)$ is a block. But, as noted earlier, $K\backslash x$ is not a block for all x in X'. Hence $X' = X'_u$. Thus

$$|X'| = |X'_u| \leq |P_1| + |P_2| \leq (|V(T_1)| - 1) + (|V(T_2)| - 1) = |V(G)| - 2.$$

By (20), equality must hold throughout the last line. Thus $|P_i| = |V(T_i)| - 1$ for each i, so each T_i is a star. Since $|X'_u| = |P_1| + |P_2|$, it follows that $x_v \neq x_w$ if $v \neq w$. Therefore, provided $|V(T_2)| \geq 3$, every edge of X' is incident with the center of one of the stars T_i and the lemma follows. It remains to consider the case when $|V(T_2)| = 2$. In that case, T_2 has a vertex that is incident with all but one edge of X', otherwise $K\backslash x$ is a block for some x in X'. The result follows by taking that vertex to be the center of T_2. □

In the next two lemmas, we shall specialize the argument to consider Cases I and II separately. Thus assume that (G_1, G_2) is an admissible decomposition of G with respect to e such that G_1/e is a block. Now we follow Section 4 in defining J_2 and H'_2 depending on the type of (G_1, G_2). We also define the integer t. Recall that $X_2 = X \cap E(G_2)$ and $Y_2 = Y \cap E(G_2)$. When (G_1, G_2) has type-0, we let $H'_2 = H_2 = G_2\backslash X_2/Y_2$, let J_2 be the member of $\{G_2, G_2/e\}$ that is in $\mathcal{G}_\emptyset(H'_2)$, and let $t = 0$. When (G_1, G_2) has type-1, we let J_2 be obtained from G by the replacement of $(G_1, X_1, \{e\})$ by a dog (F, X_F, Y_F) with head at the end of e that meets $E(G_1) \cap [E(H) \cup (Y - e)]$; we let $H'_2 = J_2\backslash(X_2 \cup X_F)/(Y_2 \cup Y_F)$; and we let $t = 2$. When (G_1, G_2) has type-2, we let J_2 be obtained from G by the replacement of $(G_1, X_1, \{e\})$ by a snake $(F, \emptyset, Y_F)$; we let $H'_2 = J_2\backslash X_2/(Y_2 \cup Y_F)$; and we let $t = 1$. By Lemmas 6.1, 6.2, and 6.3, in every case, $J_2 \in \mathcal{G}_\emptyset(H'_2)$.

Name	Definition	Range	Remarks
δ_1	1 when $\|Y''\| = 1$ and $G'[Y'']$ is a component of $G'\backslash X'$; 0 otherwise	$\{0, 1\}$	See equation (6)
δ_2	See equation (8)	$\{0, 1, 2, \ldots\}$	
δ_3	See equation (11)	$\{0, 1, 2, \ldots\}$	
δ_4	See equation (14)	$\{0, 1, 2, \ldots\}$	Positive unless Y'' is a pendent edge of $G'\backslash X'$
δ_5	2 minus the type of (G_1, G_2) when $\|Y''\| = 1$; 0 otherwise	$\{0, 1, 2\}$	See Lemma 11.2
s	The number of edges g of Y_{sd} for which D_g is a snake	$\{0, 1, 2, \ldots\}$	See equation (7)
t	0 if (G_1, G_2) has type-0; 2 if (G_1, G_2) has type-1; 1 if (G_1, G_2) has type-2	$\{0, 1, 2\}$	Defined in Case I

Table 1. A summary of certain non-negative integer parameters

In Table 1, for easy reference, we have summarized information about the seven parameters δ_1–δ_5, s, and t each of which must be a non-negative integer. Four of these parameters, t, δ_1, δ_4, and δ_5, change their values according to the case we are in. The other three parameters act as slack variables to turn inequalities into equalities. We need to know when certain inequalities become equations. We could not recover this information just from knowing that the parameters are non-negative integers since, at certain points, they are multiplied by non-integers. The kind of difficulty that would arise by avoiding the use of these parameters is exemplified in equation (15) where δ_2 and δ_3 have opposite signs. The information conveyed by equations (8) and (11), which define δ_2 and δ_3 is valuable at certain points in the proof.

11.4. **Lemma.** *If $J_2 \notin \mathcal{G}_{\emptyset}^{(\alpha,\beta)}(H_2')$, then*

$$0 > |Y_r| + \frac{2|Y_b|}{3} - \frac{|Y_0|+1}{3} + \frac{|Y_{sd}|}{3} + \frac{5\kappa_2^{\geq}(G',X',Y')}{3} + \frac{2\delta_2}{3} + \frac{10(\delta_1-1)}{3} + s + t + \delta_3 + \delta_4,$$

when $(\alpha,\beta) = (\frac{5}{3}, \frac{10}{3})$; and

$$0 > \frac{|Y_r|}{2} + \frac{|Y_b|}{2} + \frac{|Y_0|+1}{2} + \frac{\kappa_2^{\geq}(G',X',Y')}{2} + \frac{3\delta_2}{2} + 3(\delta_1 - 1) + s + t + \delta_3 + \delta_4,$$

when $(\alpha,\beta) = (\frac{5}{2}, 3)$. Moreover,

$$3(\delta_1 - 1) + t + \delta_4 \geq -1 \tag{21}$$

and this inequality is strict unless (G_1, G_2) has type-1 or type-2.

Proof. As $J_2 \notin \mathcal{G}_{\emptyset}^{(\alpha,\beta)}(H_2')$,

$$|E(J_2)| - |E(H_2')| \leq \alpha(\kappa_1(H_2') - 1) + \beta(\kappa_2(H_2') - 1). \tag{22}$$

Now

$$\begin{aligned} |E(G)| - |E(H)| &= |X_1| + |Y_1| + |X_2| + |Y_2| - 1 \\ &= |X_1| + |Y_1| + |E(G_2)| - |E(H_2)| - 1. \end{aligned} \tag{23}$$

We have

$$|E(G_2)| \leq |E(J_2)| + 1 \text{ if } (G_1, G_2) \text{ has type-0,}$$

and

$$|E(G_2)| = \begin{cases} |E(J_2)| - 4 & \text{if } (G_1, G_2) \text{ has type-1;} \\ |E(J_2)| - 3 & \text{if } (G_1, G_2) \text{ has type-2.} \end{cases}$$

Moreover,

$$|E(H_2)| = \begin{cases} |E(H'_2)| & \text{if } (G_1, G_2) \text{ has type-0;} \\ |E(H'_2)| - 3 & \text{if } (G_1, G_2) \text{ has type-1 or type-2.} \end{cases}$$

Thus

$$|E(G_2)| - |E(H_2)| - 1 \leq \begin{cases} |E(J_2)| - |E(H'_2)| & \text{if } (G_1, G_2) \text{ has type-0;} \\ |E(J_2)| - |E(H'_2)| - 2 & \text{if } (G_1, G_2) \text{ has type-1;} \\ |E(J_2)| - |E(H'_2)| - 1 & \text{if } (G_1, G_2) \text{ has type-2.} \end{cases}$$

Hence

$$|E(G_2)| - |E(H_2)| - 1 \leq |E(J_2)| - |E(H'_2)| - t. \tag{24}$$

Therefore, from (23) and (24),

$$|E(G)| - |E(H)| \leq |X_1| + |Y_1| + |E(J_2)| - |E(H'_2)| - t. \tag{25}$$

Now, with $H_1 = G_1 \backslash X_1 / Y_1$, it is clear that

$$\kappa_1(H) = \kappa_1(H_1) + \kappa_1(H'_2) - 1. \tag{26}$$

Moreover,

$$\kappa_2(H) = \kappa_2(H_1) + \kappa_2(H'_2) - 1, \tag{27}$$

where we note that if (G_1, G_2) has type-0, then H_1 has an isolated vertex that results from contracting e. Substituting from (22), (26), and (27) into (25), we get

$$\begin{aligned} |E(G)| - |E(H)| - (|X_1| + |Y_1| - t) \leq \alpha(\kappa_1(H) - \kappa_1(H_1)) \\ + \beta(\kappa_2(H) - \kappa_2(H_1)). \end{aligned} \tag{28}$$

Thus, letting

$$\Delta = [|E(G)| - |E(H)|] - [\alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1)],$$

it follows, since $G \in \mathcal{G}_{\emptyset}^{(\alpha,\beta)}(H)$ that $\Delta > 0$. Moreover, by (28),

$$\Delta \leq |X_1| + |Y_1| - t + \alpha(1 - \kappa_1(H_1)) + \beta(1 - \kappa_2(H_1)).$$

Substituting from (6), (15), and (16) into the last inequality and using the fact that $|Y''| = 1$ since we are in Case I, we get, after rearranging terms, that

$$\begin{aligned} 0 > -\Delta \geq |Y_r|(\alpha + 4\beta - 14) + |Y_b|(\alpha + 3\beta - 11) + (|Y_0| + 1)(\alpha - 2) + |Y_{sd}|(\beta - 3) \\ + \kappa_2^{>}(G', X', Y')(\beta - \alpha) + \delta_2(\alpha - 1) + \beta(\delta_1 - 1) + s + t + \delta_3 + \delta_4]. \end{aligned}$$

By substituting the two values for (α, β) into the last inequality, we obtain the two inequalities stated in the lemma.

It remains to check (21). Since $\delta_4 \geq 0$ and $t \in \{0, 1, 2\}$, the inequality certainly holds and, indeed, is strict if $\delta_1 \geq 1$. Thus we may assume that $\delta_1 = 0$. Then Y'' is not the edge-set of a connected component of $G' \backslash X'$. Therefore there are edges of $Y' - Y''$ incident with at least one of the endpoints of e. These edges are in $E(H) \cap E(G_1)$. Thus (G_1, G_2) is of type-1 or type-2. Since $t = 2$ in the former case, (21) certainly holds then. In the latter case, $t = 1$, and Y'' is not pendent in $G' \backslash X'$ so $\delta_4 \geq 1$ and again (21) holds. □

The next lemma deals with Case II.

11.5. **Lemma.** *If $G_1 = G$, then*

$$|Y_r| + \frac{2|Y_b|}{3} - \frac{|Y_0|}{3} + \frac{|Y_{sd}|}{3} + \frac{5}{3}(\kappa_2^{>}(G', X', Y') - 1) + \frac{2\delta_2}{3} + s + \delta_3 + \delta_4 < 0,$$

when $(\alpha, \beta) = (\frac{5}{3}, \frac{10}{3})$; and, when $(\alpha, \beta) = (\frac{5}{2}, 3)$,

$$\frac{|Y_r|}{2} + \frac{|Y_b|}{2} + \frac{|Y_0|}{2} + \frac{\kappa_2^{>}(G', X', Y') - 1}{2} + \frac{3\delta_2}{2} + s + \delta_3 + \delta_4 < 0.$$

Proof. By definition, $|Y''| = 0$ and $|E(G)| - |E(H)| = |X_1| + |Y_1|$. Thus, by (15),

$$|E(G)| - |E(H)| = 14|Y_r| + 11|Y_b| + 3|Y_{sd}| + 2|Y_0| + \delta_2 - s - \delta_3 - \delta_4. \tag{29}$$

Moreover, as $G \in \mathcal{G}_{\emptyset}^{(\alpha,\beta)}(H)$, we have $|E(G)| - |E(H)| > \alpha(\kappa_1(H_1) - 1) + \beta(\kappa_2(H_1) - 1)$. Substituting from (29), (6), and (16) and using the fact that $\delta_1 = 0$ because $|Y''| = 0$, we get, after some rearrangement of terms, that

$$\begin{aligned} 0 > |Y_r|(\alpha + 4\beta - 14) + |Y_b|(\alpha + 3\beta - 11) + |Y_0|(\alpha - 2) + |Y_{sd}|(\beta - 3) \\ + (\kappa_2^{>}(G', X', Y') - 1)(\beta - \alpha) + \delta_2(\alpha - 1) + s + \delta_3 + \delta_4. \end{aligned}$$

The lemma follows by substituting the appropriate values for α and β. □

12. The end of the main proof

In this section, we complete the proof of Theorem 8.1 and thereby finish the proof of Theorem 2.1. This is a continuation of the proof that we began in Section 8 so the assumptions we made there apply. In particular, $G \in \mathcal{G}_L^{(\alpha,\beta)}(H)$ where the triple (G, H, L) is chosen so that $(\kappa_2(H), -\theta(H))$ is lexicographically minimal where $\theta(H)$ is the number of blocks of H that are triangles. By Lemmas 8.2, 8.3, and 8.4, $L = L(H) = \emptyset$ and $Y \neq \emptyset$.

The proof of Theorem 8.1 will be completed by establishing the next five lemmas, the last two of which contradict each other.

12.1. **Lemma.** *On every depth-0 edge, there is a dog or a snake with respect to (G, X, Y).*

Proof. Let (G_1, G_2) be an admissible decomposition of G with respect to an edge e of Y such that $E(G_1) \cap (Y - e)$ is empty and G_1/e is a block. Then, as in Section 6, we construct graphs J_2 and H_2', depending on the type of (G_1, G_2), such that $J_2 \in \mathcal{G}_{\emptyset}(H_2')$.

Suppose that $\kappa_2(H_2') \geq \kappa_2(H)$. We shall show that, after suitable relabelling, $J_2 = G$. If (G_1, G_2) has type-0, then $\kappa_2(H_2') = \kappa_2(H_2) < \kappa_2(H)$; a contradiction. Thus (G_1, G_2) has type-1 or type-2 and

$$\kappa_2(H_2') = \kappa_2(H_2) + 1 \leq \kappa_2(H). \tag{30}$$

Therefore equality must hold here. Thus H has a single block that is not a block of H_2 and this block must meet the vertex that results from contracting e. It follows from this that $\kappa_1(H) = \kappa_1(H_2)$, so $\kappa_1(H) = \kappa_1(H_2')$. Thus

$$\alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1) = \alpha(\kappa_1(H_2') - 1) + \beta(\kappa_2(H_2') - 1). \tag{31}$$

Moreover, because $G \in \mathcal{G}_\emptyset(H)$, if (G_1, G_2) has type-2, then G_1 contains no edge of X, while if (G_1, G_2) has type-1, then G_1 contains a unique edge of X. Thus

$$|E(G)| - |E(H)| = |E(J_2)| - |E(H_2')|. \tag{32}$$

By (31) and (32), since $G \in \mathcal{G}_\emptyset^{(\alpha,\beta)}(H)$, it follows that $J_2 \in \mathcal{G}_\emptyset^{(\alpha,\beta)}(H_2')$. Since $\kappa_2(H_2') = \kappa_2(H)$, the fact that $(\kappa_2(H), -\theta(H))$ is lexicographically smaller than $(\kappa_2(H_2'), -\theta(H_2'))$ implies that $\theta(H) \geq \theta(H_2')$. But $\theta(H_2') = \theta(H_2) + 1$. Since equality holds in (30), it follows that the one block of H that is not a block of H_2 is a triangle. Therefore, since G_1 contains 0 or 1 edge of X depending on whether (G_1, G_2) has type-1 or type-2, it follows that, by labelling appropriately, we may assume that $J_2 = G$. Thus Lemma 12.1 holds if $\kappa_2(H_2') \geq \kappa_2(H)$.

We may now suppose that $\kappa_2(H_2') < \kappa_2(H)$. Then $J_2 \notin \mathcal{G}_\emptyset^{(\alpha,\beta)}(H_2')$ and we are in Case I from Section 9 so we may apply Lemma 11.4. Moreover, since $G' = G_1$, $X' = X_1$, and $Y' = Y_1$, we have that $\kappa_2^{>}(G', X', Y') = \kappa_2^{>}(G_1, X_1, Y_1) \neq 0$. Then, by (21),

$$3(\delta_1 - 1) + t + \delta_4 \geq -1.$$

Furthermore, $Y_r = Y_b = Y_{sd} = Y_0 = \emptyset$. Thus, when $(\alpha, \beta) = (\frac{5}{2}, 3)$, the second inequality in Lemma 11.4 gives

$$0 > \frac{1}{2} + \frac{\kappa_2^{>}(G_1, X_1, Y_1)}{2} + \frac{3\delta_2}{2} + s + \delta_4 - 1,$$

so $\kappa_2^{>}(G_1, X_1, Y_1) = 0$; a contradiction. Similarly, when $(\alpha, \beta) = (\frac{5}{3}, \frac{10}{3})$, the first inequality in Lemma 11.4 gives

$$0 > -\frac{1}{3} + \frac{5\kappa_2^{>}(G', X', Y')}{3} + \frac{2\delta_2}{3} + \frac{\delta_1}{3} - \frac{1}{3} + [3(\delta_1 - 1) + t + \delta_4] + s + \delta_3.$$

Using (21), we again obtain the contradiction that $\kappa_2^{>}(G_1, X_1, Y_1) = 0$. We conclude that Lemma 12.1 holds. □

Next, we shall prove the following.

12.2. **Lemma.** *G has at least one depth-1 edge.*

Proof. Assume the lemma fails. Then $Y = Y' = Y_{sd}$ and $Y_r = Y_b = Y_0 = \emptyset$. Thus we are in Case II so $|Y''| = 0 = \delta_1$ and $K = G_1$. Moreover, by Lemma 11.5,

$$\frac{|Y_{sd}|}{3} + \frac{5}{3}(\kappa_2^{>}(G', X', Y') - 1) + \frac{2\delta_2}{3} + s + \delta_3 + \delta_4 < 0 \text{ or}$$

$$\frac{\kappa_2^{>}(G', X', Y') - 1}{2} + \frac{3\delta_2}{2} + s + \delta_3 + \delta_4 < 0$$

depending on the value of (α, β). In both cases, we must have that $\kappa_2^{>}(G', X', Y') = 0$. If $E(G') = Y'$, then $E(G') = Y$ so $|E(G')| = 1$ and (P5) is contradicted. Thus $E(G') \neq Y'$. Hence, by Lemma 11.2, we have that $G' \backslash X' = G'[Y']$ and

$$|Y_{sd}| + s \geq 2\kappa_2(G' \backslash X' / Y') \geq 4.$$

It is not difficult to check that, for both values of (α, β), we must have that $\delta_2 = \delta_3 = \delta_4 = s = 0$. By Lemma 11.3, we have that $K \backslash X'$, which equals $G' \backslash X'$ and

$G'[Y']$, has two connected components T_1 and T_2. Moreover, each T_i is a star with center v_i, say, and every edge of X' joins the center of one star to a pendent vertex of the other. Since $Y_{sd} = Y'$ but $s = 0$, it follows that, for every edge g in Y', the graph D_g is a dog. If D_g has its head at v_i for some i, then, in $G\backslash X$, the edge g of Y is pendent, contradicting Lemma 2.4 . Thus no dog D_g has its head at v_i. Now H can be obtained not only as $G\backslash X/Y$ but also as $G\backslash[Y \cup (X - X')]/X'$. This contradiction to the fact that H arises uniquely as a minor of G completes the proof of Lemma 12.2. □

The proof of the next lemma is quite long since it involves actually constructing a bull or a rhino.

12.3. **Lemma.** *On every depth-*1 *edge, there is a bull or a rhino with respect to* (G, X, Y).

Proof. Let e be a depth-1 edge with respect to (G, X, Y). Let (G_1, G_2) be an admissible decomposition of G with respect to e such that G_1/e is a block and $E(G_1) \cap (Y - e)$ is non-empty and contains only depth-0 edges. Then, as in Section 4, we construct graphs J_2 and H_2', depending on the type of (G_1, G_2), such that $J_2 \in \mathcal{G}_\emptyset(H_2')$.

In this paragraph, we shall prove that J_2 is lexicographically smaller than G or, more formally, that $(\kappa_2(H_2'), -\theta(H_2'))$ is lexicographically smaller than $(\kappa_2(H), -\theta(H))$. First we note that

$$\kappa_2(H) = \begin{cases} \kappa_2(H_2) + \kappa_2(G_1\backslash X_1/Y_1) - 1 & \text{if } (G_1, G_2) \text{ has type-0;} \\ \kappa_2(H_2) + \kappa_2(G_1\backslash X_1/Y_1) & \text{otherwise.} \end{cases}$$

To see this, we note that $G_1\backslash X_1/Y_1$ has an isolated vertex that results from contracting the edge e. But H_2 also has a block containing the vertex that results from contracting e. Since $H_2' = H_2$ if (G_1, G_2) has type-0, and H_2' has one more block than H_2 otherwise, we deduce that, in all cases,

$$\kappa_2(H) = \kappa_2(H_2') + \kappa_2(G_1\backslash X_1/Y_1) - 1.$$

Thus we may assume that $\kappa_2(G_1\backslash X_1/Y_1) = 1$ otherwise J_2 is lexicographically smaller than G. Now, on each edge in $(Y_1 - e) \cap E(G_1)$, there is a dog or a snake from which we get a block of $G_1\backslash X_1/Y_1$. Thus $|(Y_1 - e) \cap E(G_1)| \leq 1$. But, since e is a depth-1 edge, $|(Y_1 - e) \cap E(G_1)| \geq 1$. Hence equality holds here. Let f be the unique edge in $(Y_1 - e) \cap E(G_1)$. Then the only block of $G_1\backslash X_1/Y_1$ is a triangle and $Y_1 = \{e, f\}$. Hence (G_1, G_2) has type-1 or type-2. Thus, at least one endpoint of e is incident with f or an edge of H that is in the dog or snake on f. But the only vertices of a dog or snake on f that can be adjacent to edges not in the dog or snake are the endpoints of f. Hence e and f are adjacent in G_1. Now $G_1\backslash X_1/Y_1$ has no isolated vertices. Thus, since G_1 is a block, the ends of e and f that are different must be joined by an edge of X_1. This edge of X_1 is spanned by edges of Y_1; a contradiction. We conclude that J_2 is, indeed, lexicographically smaller than G.

We now know that $J_2 \notin \mathcal{G}_\emptyset^{(\alpha,\beta)}(H_2')$ and that we are in Case I, so we may apply Lemma 11.4. Evidently $Y_r = Y_b = Y_0 = \emptyset$ and $Y'' = \{e\}$. Thus $Y' = Y_1$. For both values of (α, β), we shall prove that

$$\kappa_2^>(G', X', Y') = s = \delta_1 = \delta_2 = \delta_3 = 0. \tag{33}$$

First, suppose that $(\alpha, \beta) = (\frac{5}{3}, \frac{10}{3})$. By Lemma 11.4, we have that

$$0 > -\frac{1}{3} + \frac{|Y_{sd}|}{3} + \frac{5\kappa_2^{>}(G', X', Y')}{3} + \frac{2\delta_2}{3} + \frac{10(\delta_1 - 1)}{3} + s + t + \delta_3 + \delta_4. \quad (34)$$

As $|Y_{sd}| \geq 1$, it follows that $\delta_1 = 0$. Thus, by (21), $t + \delta_4 - 2 \geq 0$. Using this and the fact that $\delta_1 = 0$, we get from (34) that

$$0 > \frac{|Y_{sd}|}{3} + \frac{5(\kappa_2^{>}(G', X', Y') - 1)}{3} + \frac{2\delta_2}{3} + s + \delta_3.$$

Thus $\kappa_2^{>}(G', X', Y') = 0$ and so

$$0 > \frac{(|Y_{sd}| + s) - 5}{3} + \frac{2\delta_2}{3} + \frac{2s}{3} + \delta_3. \quad (35)$$

Now $E(G') \neq Y'$ since G' is a block having at least two edges and Y' contains no cycle of G'. Hence, by Lemma 11.2, $|Y_r| + |Y_{sd}| + s \geq 4 - \delta_5$. Observe that (G_1, G_2) does not have type-0 because $\delta_1 = 0$. Hence $\delta_5 \in \{0, 1\}$. As $Y_r = \emptyset$, it follows that $|Y_{sd}| + s \geq 3$. Using this inequality in (35), we obtain

$$0 > -\frac{2}{3} + \frac{2\delta_2}{3} + \frac{2s}{3} + \delta_3.$$

Hence all the integers $\delta_2, s,$ and δ_3 are non-positive. As these integers are non-negative, they must be equal to zero. Thus (33) holds when $(\alpha, \beta) = (\frac{5}{3}, \frac{10}{3})$. Now, suppose that $(\alpha, \beta) = (\frac{5}{2}, 3)$. By Lemma 11.4,

$$0 > \frac{1}{2} + \frac{\kappa_2^{>}(G', X', Y')}{2} + \frac{3\delta_2}{2} + 3(\delta_1 - 1) + s + t + \delta_3 + \delta_4.$$

Observe that $\delta_1 = 0$ because $\kappa_2^{>}(G', X', Y'), \delta_2, \delta_1, s, t, \delta_3, \delta_4$ are all non-negative integers. By rewriting the last inequality and using (21), we have

$$0 > \frac{1}{2} + \frac{\kappa_2^{>}(G', X', Y')}{2} + \frac{3\delta_2}{2} + \delta_3 + s + [-1].$$

Hence

$$0 > \frac{\kappa_2^{>}(G', X', Y') - 1}{2} + \frac{3\delta_2}{2} + \delta_3 + s.$$

As all of $\kappa_2^{>}(G', X', Y'), \delta_2, \delta_3,$ and s are non-negative integers, it follows that $\kappa_2^{>}(G', X', Y') = 0$. Hence $\delta_2 = \delta_3 = s = 0$. Thus, again, we get (33).

By (33) and the fact that $Y_b = Y_0 = \emptyset$, the hypotheses of Lemma 11.3 hold so $K \backslash X'$ has two connected components T_1 and T_2. Moreover, each T_i is a star with center v_i, and every edge of X' joins the center of one star to a pendent vertex of the other. As $\kappa_2^{>}(G', X', Y') = 0$ and $E(G') \neq Y'$, Lemma 11.2 implies that $G[Y'] = G' \backslash X'$. In addition, since $\delta_1 = 0$, (G_1, G_2) has type-1 or type-2 .

Next we relate the connected components of $G' \backslash X'$ to those of $K \backslash X'$. Suppose first that (G_1, G_2) has type-1. Then e is a pendent edge of $G' \backslash X'$ so $K = G'$. Let $T'_1 = T_1$ and $T'_2 = T_2$ where $e \in E(T_1)$. Then T'_1 and T'_2 are the connected components of $G' \backslash X'$. Evidently e is a pendent edge of T'_1. Moreover, as (G_1, G_2) has type-1, it follows that $|E(T'_1) - e| \geq 1$.

Suppose next that (G_1, G_2) has type-2. Then $K = G'/e$. Thus G' has two connected components T'_1 and T'_2, each a tree, where $e \in E(T'_1)$ and $T'_1/e = T_1$. As e is not pendent in T'_1, it follows that the vertex of T_1 that results from contracting e must be v_1, the center of the star T_1. Thus, as $e = uv$, there is a partition

$\{E_u, E_v\}$ of $E(T_1)$ such that, for each w in $\{u, v\}$, the set E_w is the set of edges of T_1 that meet w.

Now, $E(G_1) \cap (Y - e) = Y' - e$. Since $s = 0$ and every edge of $Y' - e$ is a depth-0 edge, it follows by Lemma 12.1 that D_f is a dog on f for every f in $Y' - e$. Moreover, since f is not pendent in $G \backslash X$, if the head of the dog is at h_f, then $h_f \notin \{v_1, v_2, u, v\}$. Also X' contains a unique edge x_f incident with h_f in $G'[Y']$. This edge is the lead of the dog D_f.

We shall show next that, both when (G_1, G_2) has type-1 and when it has type-2, $|E(T_2)| = 2$. We begin by proving that $|E(T_2)| \leq 2$. Suppose that $|E(T_2)| \geq 3$. Then there are different edges f and g of T_2 such that x_f and x_g are adjacent to the same vertex z of T_1' where z is v_1 if (G_1, G_2) has type-1, and z is in $\{u, v\}$ when (G_1, G_2) has type-2 . Now D_f and D_g both have their tails incident with v_2. Moreover, D_f and D_g have their leads at the same vertex. As $|E(T_2)| \geq 3$, the connected component of H that contains the heads of D_f and D_g contains at least one more block. Therefore, by Lemma 7.3, $G - [V(D_f) - v_2] \in \mathcal{G}_\emptyset(H - [V(D_f) - V(f)])$. As we shall see, this will imply a contradiction to the minimality of G. Clearly $\kappa_1(H) = \kappa_1(H - [V(D_f) - V(f)])$ and $\kappa_2(H) = \kappa_2(H - [V(D_f) - V(f)]) + 1$. The last equation implies that $G - [V(D_f) - v_2] \notin \mathcal{G}_\emptyset^{(\alpha,\beta)}(H - [V(D_f) - V(f)])$. Thus, if t_f is the tail of the dog D_f, then

$$|E(G)| - |E(H)| - |\{f, x_f, t_f\}| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 2).$$

Therefore, $|E(G)| - |E(H)| \leq \alpha(\kappa_1(H) - 1) + \beta(\kappa_2(H) - 1) + [3 - \beta]$. Since $\beta \geq 3$, this implies the contradiction that $G \in \mathcal{G}_\emptyset^{(\alpha,\beta)}(H)$. We conclude that $|E(T_2)| \leq 2$. If $|E(T_2)| \leq 1$, then $|E(T_2)| = 1$ because $\kappa_2^{>}(G', X', Y') = 0$. Therefore we have a contradiction to Lemma 7.2. We deduce that we do indeed have $|E(T_2)| = 2$.

Next we prove that T_1' is a path which has length two when (G_1, G_2) has type-1 and has length three when (G_1, G_2) has type-2. To establish this, it suffices to show that $|E(T_1) - e| = 1$ when (G_1, G_2) has type-1, and $|E_u| = |E_v| = 1$ when (G_1, G_2) has type-2. Thus assume that $|E(T_1) - e| > 1$ if (G_1, G_2) has type-1, and $|E_u| > 1$ when (G_1, G_2) has type-2. In each case, there are at least two dogs D_f having leads at v_2 and having tails at z where z is v_1 or u depending on whether (G_1, G_2) has type-1 or type-2, respectively. Now the component of H that contains the head of these two dogs contains the head of a third dog if (G_1, G_2) has type-2 and contains a block with edge-set in $E(G_2)$ if (G_1, G_2) has type-1. Thus we may apply Lemma 7.3 as in the previous paragraph to obtain a contradiction. We conclude that T_2' is indeed a path of length two or three.

Assembling the information obtained above enables us to conclude that G_1 is, respectively, a bull or a rhino on e when (G_1, G_2) has type-1 or type-2. This contradiction completes the proof of Lemma 12.3. □

12.4**. Lemma.** *G has a depth-*2 *edge.*

Proof. Assume that this is not the case. Then every edge of Y is a depth-0 or depth-1 edge. By Lemma 12.1, every depth-0 edge of G has a dog or a snake with respect to (G, X, Y) and, by Lemma 12.3, every depth-1 edge of G has a bull or a rhino with respect to (G, X, Y). Thus we are in Case II so $Y'' = \emptyset$ and we can apply Lemma 11.5. When (α, β) is equal to $(\frac{5}{3}, \frac{10}{3})$, we have that

$$|Y_r| + \frac{2|Y_b|}{3} - \frac{|Y_0|}{3} + \frac{|Y_{sd}|}{3} + \frac{5}{3}(\kappa_2^{>}(G', X', Y') - 1) + \frac{2\delta_2}{3} + s + \delta_3 + \delta_4 < 0. \quad (36)$$

As $Y_0 \subseteq Y_b$ and $\kappa_2^{>}(G', X', Y'), \delta_2, s, \delta_3, \delta_4$ are non-negative integers, it follows that $\kappa_2^{>}(G', X', Y') = 0$. By Lemma 11.2, since $|Y''| = 0$, we have that $\delta_5 = 0$, so

$$|Y_r| + |Y_{sd}| + s - 4 \geq 0. \tag{37}$$

Substituting the value of $\kappa_2^{>}(G', X', Y')$ into (36) and reordering, we obtain

$$\frac{|Y_r|}{3} + \frac{|Y_r| + |Y_b| - 1}{3} + \frac{|Y_r| + |Y_{sd}| + s - 4}{3} + \frac{|Y_b| - |Y_0|}{3} + \frac{2\delta_2}{3} + \frac{2s}{3} + \delta_3 + \delta_4 < 0.$$

We obtain a contradiction provided each of $|Y_r| + |Y_b| - 1$, $|Y_r| + |Y_{sd}| + s - 4$, and $|Y_b| - |Y_0|$ is non-negative. The first is because G has a depth-1 edge by Lemma 12.2; the second is by (37); and the third is because $Y_0 \subseteq Y_b$. We may now assume that (α, β) equals $(\frac{5}{2}, 3)$. In that case, by Lemma 11.5,

$$\frac{|Y_r|}{2} + \frac{|Y_b|}{2} + \frac{|Y_0|}{2} + \frac{\kappa_2^{>}(G', X', Y') - 1}{2} + \frac{3\delta_2}{2} + s + \delta_3 + \delta_4 < 0,$$

so $\kappa_2^{>}(G', X', Y') = 0$. Substituting this value into the last inequality and rearranging it, we obtain

$$\frac{|Y_r| + |Y_b| - 1}{2} + \frac{|Y_0|}{2} + \frac{3\delta_2}{2} + s + \delta_3 + \delta_4 < 0.$$

Again we arrive at a contradiction because $Y_r \cup Y_b \neq \emptyset$ by Lemma 12.2. We conclude that Lemma 12.4 holds. □

We shall arrive at the final contradiction by proving the following:

12.5**. Lemma.** *G has no depth-*2 *edges.*

Proof. By Lemma 12.4, G has a depth-2 edge. Let e be such an edge and (G_1, G_2) be an admissible decomposition of G with respect to e such that $E(G_1) \cap (Y - e)$ contains only depth-0 or depth-1 edges. Then, as before, we construct the graphs J_2 and H_2'. Since G_1 has at least one depth-1 edge, it follows that $\kappa_2(H_2') < \kappa_2(H)$, so J_2 is smaller than G in our lexicographic order. Thus we can apply Lemma 11.4. Suppose first that $(\alpha, \beta) = (\frac{5}{2}, 3)$. Rearranging the second inequality in Lemma 11.4, we get

$$0 > \frac{|Y_r|}{2} + \frac{|Y_b|}{2} + \frac{|Y_0| + 1}{2} + \frac{\kappa_2^{>}(G', X', Y')}{2} + \frac{3\delta_2}{2} + s + \delta_3 + [3(\delta_1 - 1) + t + \delta_4].$$

By (21), we obtain, after rearranging terms, that

$$0 > \frac{|Y_r| + |Y_b| - 1}{2} + \frac{|Y_0|}{2} + \frac{\kappa_2^{>}(G', X', Y')}{2} + \frac{3\delta_2}{2} + s + \delta_3.$$

But this is a contradiction because $|Y_r| + |Y_b| - 1$ is non-negative since there is at least one depth-1 edge.

We may now assume that $(\alpha, \beta) = (\frac{5}{3}, \frac{10}{3})$. The rest of the proof will be divided explicitly into three cases depending on the type of (G_1, G_2).

Suppose that (G_1, G_2) has type-0. Then $t = 0$ and $\delta_1 = 1$. By Lemma 11.4, we have that

$$0 > |Y_r| + \frac{2|Y_b|}{3} - \frac{|Y_0| + 1}{3} + \frac{|Y_{sd}|}{3} + \frac{5\kappa_2^{>}(G', X', Y')}{3} + \frac{2\delta_2}{3} + s + \delta_3 + \delta_4.$$

As $|Y_b| \geq |Y_0|$, we obtain a contradiction unless $|Y_b| = |Y_0| = 0$. But in the exceptional case, $|Y_r| \geq 1$, because e is a depth-2 edge and so $Y_r \cup Y_b \neq \emptyset$. Again, we have a contradiction.

Suppose that (G_1, G_2) has type-1. Then $t = 2$, $\delta_1 = 0$, and $\delta_5 = 1$. Thus, by Lemma 11.4, we have, after rearranging terms, that

$$0 > |Y_r| + \frac{|Y_b|}{3} + \frac{|Y_b| - |Y_0|}{3} + \frac{|Y_{sd}|}{3} + \frac{5(\kappa_2^{>}(G', X', Y') - 1)}{3} + \frac{2\delta_2}{3} + s + \delta_3 + \delta_4. \quad (38)$$

Hence $\kappa_2^{>}(G', X', Y') = 0$ and so, by Lemma 11.2,

$$|Y_r| + |Y_{sd}| + s + 1 \geq 2\kappa_1(G' \backslash X'/Y') \geq 4. \quad (39)$$

Observe that (38) can be rewritten as

$$0 > \frac{|Y_r| + |Y_{sd}| + s - 3}{3} + \frac{|Y_r| + |Y_b| - 1}{3} + \frac{|Y_b| - |Y_0|}{3} + \frac{|Y_r|}{3} + \frac{2\delta_2}{3} + \frac{2s}{3} + \delta_3 + \delta_4 - \frac{1}{3}.$$

Since all of $|Y_r| + |Y_{sd}| + s - 3$, $|Y_r| + |Y_b| - 1$, $|Y_b| - |Y_0|$, $|Y_r|$, and s are non-negative, we get a contradiction unless all of these are zero. Thus $|Y_b| = 1$, $|Y_{sd}| = 3$, and $|Y_0| = 1$. Hence, by (8), (9), and (10),

$$1 = |Y_0| = \kappa_1(G' \backslash X'/Y') + \kappa_2^{>}(G', X', Y') - 2 - \delta_2 = \kappa_1(G' \backslash X'/Y') - 2.$$

Thus $\kappa_1(G' \backslash X'/Y') = 3$. But this contradicts (39) since $|Y_r| + |Y_{sd}| + s - 3 = 0$.

Finally, suppose that (G_1, G_2) has type-2. Then $t = 1$, $\delta_4 \geq 1$, and $\delta_5 = 0$. Thus, it follows by Lemma 11.4 that

$$0 > |Y_r| + \frac{|Y_b|}{3} + \frac{|Y_b| - |Y_0|}{3} + \frac{|Y_{sd}|}{3} + \frac{5(\kappa_2^{>}(G', X', Y') - 1)}{3} + \frac{2\delta_2}{3} + s + \delta_3 + (\delta_4 - 1). \quad (40)$$

Hence $\kappa_2^{>}(G', X', Y') = 0$ so, by Lemma 11.2, $|Y_r| + |Y_{sd}| + s \geq 2\kappa_2(H') \geq 4$. Rewriting (40), we get

$$0 > \frac{|Y_r| + |Y_{sd}| + s - 4}{3} + \frac{|Y_r| + |Y_b| - 1}{3} + \frac{|Y_b| - |Y_0|}{3} + \frac{|Y_r|}{3} + \frac{2\delta_2}{3} + \frac{2s}{3} + \delta_3 + (\delta_4 - 1).$$

This contradiction completes the proof of Lemma 12.5. □

Theorem 8.1 follows by combining the last two lemmas. □

We may rewrite the bound in Theorem 8.1 for some special values of α and β.

12.6. **Corollary.** *If $G \in \mathcal{G}_L(H)$, then*

(i) $|E(G)| - |E(H)| \leq \kappa_1(H) + 4\kappa_2(H) - 5$;
(ii) $|E(G)| - |E(H)| \leq 5\kappa_2(H) - 5$; *and*
(iii) $|E(G)| - |E(H)| \leq 3\kappa_1(H) + 3\kappa_2(H) - 6$.

Part (i) and (iii) of this corollary are two best-possible linear bounds on $|E(G)| - |E(H)|$ in which $\kappa_1(H)$ and $\kappa_2(H)$ have integer coefficients. The bound in (ii) is interesting, since we can compare it to the bound obtained for the corresponding matroid problem. When M is a minor-minimal matroid with respect to being 2-connected and having a non-empty matroid N as a minor, Theorem 1.4 gives that $|E(M)| - |E(N)| \leq 2k - 2$, where k is the number of 2-connected components of N. If $M = M(G)$ for some graph G, then $N = M(H)$ for a minor H of G having no isolated vertices, and $2k - 2$ equals $2\kappa_2(H) - 2$. Thus the matroid bound is exactly $\frac{2}{5}$ of the bound obtained in the graph case. This strange situation occurs because the cycle matroids of two graphs are equal provided the sets of blocks with at least one edge in these two graphs coincide [9].

13. A sharper bound

For all (α,β) on the boundary of A, one of the examples constructed in Section 5 attains the bound

$$|E(G)| - |E(H)| \leq \lfloor \alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1)\rfloor \tag{41}$$

unless (α,β) is on the oblique half-line $\alpha+\beta=5$ and $\alpha \leq \frac{5}{3}$. In the exceptional case, provided $\kappa_2(H)$ is not too small, the bound in (41) can be improved so that it is also attained by an appropriate example from Section 5. This improvement is contained in the next theorem. Corollary 13.2 is a straightforward consequence of this theorem that sharpens the bound in Corollary 12.6(ii) when $\kappa_2(H) \geq 3$.

13.1. **Theorem.** *Suppose that $\alpha+\beta=5$ and $\alpha \leq \frac{5}{3}$. If $G \in \mathcal{G}_L(H)$ and $\kappa_2(H) \geq \beta - \frac{7}{3}$, then $|E(G)| - |E(H)| \leq \lfloor \alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - (\beta - \frac{10}{3})\rfloor$.*

Proof. It suffices to prove that

$$\beta - \tfrac{10}{3} \leq \alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - (|E(G)| - |E(H)|). \tag{42}$$

By taking $(\alpha,\beta) = (\frac{5}{3}, \frac{10}{3})$, it follows from Theorem 8.1 that if

$$h = \tfrac{5}{3}(\kappa_1(H)-1) + \tfrac{10}{3}(\kappa_2(H)-1) - [|E(G)| - |E(H)|],$$

then h is non-negative. Now, since $\alpha+\beta=5$, it follows that $(\beta-\frac{10}{3})(\kappa_2(H) - \kappa_1(H)) = (\alpha - \frac{5}{3})(\kappa_1(H)-1) + (\beta - \frac{10}{3})(\kappa_2(H)-1)$. Thus

$$\alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - [|E(G)| - |E(H)|] = h + (\beta - \tfrac{10}{3})(\kappa_2(H) - \kappa_1(H)).$$

Suppose that $\kappa_2(H) \geq \kappa_1(H)+1$. Then, as $h \geq 0$, it follows that

$$\alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - [|E(G)| - |E(H)|] \geq \beta - \tfrac{10}{3},$$

that is, (42) holds. We may now assume that $\kappa_2(H) \leq \kappa_1(H)$. Thus $\kappa_2(H) = \kappa_1(H)$. Then, by Theorem 3.5,

$$\begin{aligned} |E(G)| - |E(H)| &\leq 4\kappa_2(H) - 4 \\ &= \alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - (\kappa_2(H)-1). \end{aligned}$$

But, by assumption, $\kappa_2(H) \geq \beta - \frac{7}{3}$, so $\kappa_2(H) - 1 \geq \beta - \frac{10}{3}$. Hence

$$|E(G)| - |E(H)| \leq \alpha(\kappa_1(H)-1) + \beta(\kappa_2(H)-1) - (\beta - \tfrac{10}{3})$$

so (42) holds. □

13.2. **Corollary.** *If $G \in \mathcal{G}_L(H)$ and $\kappa_2(H) \geq 3$, then*

$$|E(G)| - |E(H)| \leq 5\kappa_2(H) - 7.$$

14. Some consequences

We conclude the paper by using Corollary 3.3 to generalize some results of Dirac [2] and Lemos and Oxley [5] for minimally 2-connected graphs, where a graph G is minimally 2-connected if, for all e in $E(G)$, the graph $G\backslash e$ is not 2-connected.

14.1. **Corollary.** *Let M be a matching in a 2-connected graph G and assume that no proper 2-connected subgraph of G has M as a matching. Then*

$$|E(G)| \leq 2|V(G)| - |E(M)| - 2.$$

Proof. The corollary follows by applying Corollary 3.3 to the graph with vertex-set $V(G)$ and edge-set $E(M)$. □

To see that the last result is sharp, let G be the graph that is constructed by joining two vertices u and v by k internally disjoint paths where $k \geq 2$ and two of the paths P_1 and P_2 have length two while the rest have length three. Let v_1 and v_2 be the internal vertices of P_1 and P_2. Let F be the set of edges of G that are incident with neither u nor v. Then $\{uv_1, vv_2\} \cup F$ is the edge-set of a matching in G and no 2-connected proper subgraph of G has M as a matching. Moreover, $|E(G)| = 2|V(G)| - |E(M)| - 2$.

The next result, due to Dirac [2], is obtained by applying the last corollary to a 2-edge matching.

14.2**. Corollary.** *A minimally* 2*-connected graph* G *with at least four vertices has at most* $2|V(G)| - 4$ *edges.*

14.3**. Corollary.** *Let* $C_1, C_2, \ldots, C_k$ *be vertex-disjoint cycles in a* 2*-connected graph* G*. Assume that no proper* 2*-connected subgraph of* G *has all of* $C_1, C_2, \ldots, C_k$ *as cycles. Then*

$$|E(G)| \leq 2|V(G)| + 2(k-1) - \sum_{i=1}^{k} |E(C_i)|.$$

Proof. The corollary follows by applying Corollary 3.3, taking H to be the subgraph of G with vertex-set $V(G)$ and edge-set $\cup_{i=1}^{k} E(C_i)$. □

By taking $k = 1$ in the last corollary and letting C_1 be a maximum-sized cycle in G, we obtain the following result of Oxley and Lemos [5] that was originally derived from the corresponding result for matroids.

14.4**. Corollary.** *Let* G *be a minimally* 2*-connected graph with circumference* c*. Then* $|E(G)| \leq 2|V(G)| - c$*.*

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