MATROID PACKING AND COVERING WITH CIRCUITS THROUGH AN ELEMENT

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Abstract. In 1981, Seymour proved a conjecture of Welsh that, in a connected matroid M, the sum of the maximum number of disjoint circuits and the minimum number of circuits needed to cover M is at most $r^*(M) + 1$. This paper considers the set $\mathcal{C}_e(M)$ of circuits through a fixed element e such that M/e is connected. Let $\nu_e(M)$ be the maximum size of a subset of $\mathcal{C}_e(M)$ in which any two distinct members meet only in $\{e\}$, and let $\theta_e(M)$ be the minimum size of a subset of $\mathcal{C}_e(M)$ that covers M. The main result proves that $\nu_e(M) + \theta_e(M) \leq r^*(M) + 2$ and that if M has no Fano-minor using e, then $\nu_e(M) + \theta_e(M) \leq r^*(M) + 1$. Seymour's result follows without difficulty from this theorem and there are also some interesting applications to graphs.

1. Introduction

For an element e of a matroid M, we denote by $\mathcal{C}_e(M)$ the set of circuits of M that contain e. For a subset X of $E(M)$, a set $\mathcal{D}$ of circuits *covers* X if every element of X is in some member of $\mathcal{D}$. Now suppose that M is connected but is not a coloop. Let $\nu_e(M)$ and $\theta_e(M)$ be, respectively, the maximum size of a subset of $\mathcal{C}_e(M)$ any two members of which meet in $\{e\}$ and the minimum size of a subset of $\mathcal{C}_e(M)$ that covers $E(M)$. The purpose of this paper is to prove the following result.

1.1. **Theorem.** *Let M be a connected matroid M other than a coloop and e be an element of M such that M/e is connected. Then*

$$\nu_e(M) + \theta_e(M) \leq r^*(M) + 2.$$

Moreover, when M has no F_7-minor using e,

$$\nu_e(M) + \theta_e(M) \leq r^*(M) + 1.$$

The bounds in this theorem are sharp with, for example, the first being attained by all odd-rank binary spikes having e as the tip, and the second by all free spikes where again e is the tip.

For a matroid M, let $\nu(M)$ be the maximum number of pairwise disjoint circuits of M, and $\theta(M)$ be the minimum number of circuits needed to cover $E(M)$. A consequence of our main result is the following theorem of Seymour [11], which verified a conjecture of Welsh and generalized a result of Oxley [8].

1.2. **Corollary.** *If M is a connected matroid other than a coloop, then*

$$\nu(M) + \theta(M) \leq r^*(M) + 1.$$

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The next two corollaries are obtained by applying the main result to the cycle and bond matroids of a graph. For distinct vertices u and v of a 2-connected loopless graph G, we denote by $\nu_{uv}(G)$ and $\theta_{uv}(G)$ the maximum number of edge-disjoint uv-paths in G and the minimum number of uv-paths needed to cover $E(G)$. We shall call a *minimal* set of edges whose removal from G puts u and v in separate components a *uv-cut*. Let $\nu^*_{uv}(G)$ and $\theta^*_{uv}(G)$ denote the maximum number of edge-disjoint uv-cuts in G and the minimum number of uv-cuts needed to cover $E(G)$.

1.3. **Corollary.** *Let u and v be distinct non-adjacent vertices of a 2-connected loopless graph G such that $G - \{u, v\}$ is connected. Then*

$$\nu_{uv}(G) + \theta_{uv}(G) \leq |E(G)| - |V(G)| + 3.$$

1.4. **Corollary.** *Let u and v be distinct non-adjacent vertices of a 2-connected loopless graph G. Then*

$$\nu^*_{uv}(G) + \theta^*_{uv}(G) \leq |V(G)|.$$

If e is a non-coloop element of a matroid M, let $g^*_e(M)$ and $c^*_e(M)$ be the minimum and maximum sizes of members of $\mathcal{C}_e(M)$. Evidently, $c^*_e(M) \leq r^*(M) + 1$. The authors [5, Theorem 2.4] proved that $\theta_e(M) \leq c^*_e(M) - 1$. Thus

$$\theta_e(M) \leq c^*_e(M) - 1 \leq r^*(M). \tag{1}$$

There is also a relation between $\nu_e(M)$ and $g^*_e(M)$. Let $C_1, C_2, \ldots, C_m$ be a maximum-sized set of circuits of M such that any two meet in $\{e\}$. If D^* is a cocircuit of M containing e, then, by orthogonality, D^* meets each C_i in an element other than e. Thus $m \leq |D^* - e|$. Hence

$$\nu_e(M) \leq g^*_e(M) - 1. \tag{2}$$

By the extension of Menger's Theorem to regular matroids [6] (see also [10, Theorem 11.3.14]), equality holds in this bound when M is regular. Thus, we have the following corollary of our main theorem.

1.5. **Corollary.** *Let M be a connected regular matroid M other than a coloop and e be an element of M such that M/e is connected. Then*

$$g^*_e(M) + \theta_e(M) \leq r^*(M) + 2.$$

The last corollary need not hold when M is non-regular. For example,

$$\theta_e(U_{r,n}) = \left\lceil \frac{n-1}{r} \right\rceil \text{ and } g^*_e(U_{r,n}) = n - r + 1 = r^*(U_{r,n}) + 1.$$

The matroid terminology used here will follow Oxley [10] except that the cosimplification of a matroid N will be denoted by $\mathrm{co}(N)$. In the next section, some preliminaries needed for the main proof are given. Two special classes of matroids that appear in this proof, Sylvester matroids and spikes, will be discussed in Sections 3 and 4. The proof of the main theorem appears in Section 5. Some consequences of this theorem will be given in Section 6 where the corollaries noted above will also be proved.

2. Preliminaries

In this section, we prove some lemmas that will be used in the proof of Theorem 1.1. Several of these concern extremal connectivity results. In addition, we recall Cunningham and Edmonds' tree decomposition of a connected matroid, which will also play an important role in the main proof.

2.1**. Lemma.** *Let $\{X, Y\}$ be a 2-separation of a connected cosimple matroid M and let C be a circuit of M that meets both X and Y. Then C has a 2-subset A such that $M\backslash A$ is connected.*

Proof. Suppose that the lemma fails. For each Z in $\{X, Y\}$, let M_Z be a matroid such that $E(M_Z) = Z \cup b$ and $M = M_X \oplus_2 M_Y$. If each M_Z has an element e_Z in $C \cap Z$ such that $M_Z\backslash e_Z$ is connected, then $M\backslash\{e_X, e_Y\}$, which equals $(M_X\backslash e_Y) \oplus_2 (M_Y\backslash e_Y)$, is connected; a contradiction. Thus, for some Z, the matroid $M_Z\backslash e_Z$ is disconnected for all e_Z in $C \cap Z$. Thus, as M_Z is connected, by a result of Oxley [9] (see also [10, Lemma 10.2.1]), $C \cap Z$ contains a 2-cocircuit of M_Z. This 2-cocircuit is also a 2-cocircuit of M, contradicting the fact that M is cosimple. □

The next lemma extends the following result of Akkari [1].

2.2**. Theorem.** *Let C be a circuit of a 3-connected matroid M satisfying $|E(M)| \geq 4$. Suppose that, when M is isomorphic to a wheel of rank at least four, C is not its rim. If $M\backslash A$ is disconnected for every 2-subset A of C, then every 2-subset of C is contained in a triad of M.*

2.3**. Lemma.** *Let C be a circuit of a 3-connected matroid M satisfying $|E(M)| \geq 4$. Suppose that, when M is isomorphic to a wheel of rank at least four, C is not its rim. If $M\backslash A$ is disconnected for every subset A of C such that $r^*(A) = 2$, then every 2-subset of C is contained in a triad of M whose third element is not in C.*

Proof. Suppose that the lemma is false and choose a counterexample M such that $|E(M)|$ is minimal. If $|E(M)| = 4$, then $M \cong U_{2,4}$. But the hypothesis fails for this matroid. Thus $|E(M)| \geq 5$. Since M is a counterexample, there is a 2-subset Y of C that is contained in no triad whose third element is in $E(M) - C$. By Theorem 2.2, Y is contained in a triad T^* of M. By assumption, we must have that $T^* \subseteq C$. Next we prove the following:

2.3.1**.** *For every e in T^*, the matroid M/e is not 3-connected.*

Suppose that M/e is 3-connected for some e in T^*. Let A be a 2-subset of $C - e$ such that $r^*(A) = 2$. If $M/e\backslash A$ is connected, then e is a coloop of $M\backslash A$ and so A spans e in M^*. Hence $r^*(A \cup e) = 2$ and $M\backslash(A \cup e)$, which equals $M/e\backslash A$, is connected. This contradiction implies that $M/e\backslash A$ is disconnected. By the choice of M, the result holds for M/e; that is, for each 2-subset X of $C - e$, there is a triad T^*_X of M/e such that $X \subseteq T^*_X$ and $T^*_X \not\subseteq C - e$. Evidently T^*_X is also a triad of M. Now $M^*|(T^* \cup T^*_{T^*-e}) \cong U_{2,4}$ and $T^* \cup T^*_{T^*-e}$ contains Y and the element f of $T^*_{T^*-e}$ not in C. Thus $Y \cup f$ is a triad of M; a contradiction. We conclude that (2.3.1) holds.

By the dual of Tutte's Triangle Lemma [13] (see also [10, Corollary 8.4.9]), the elements x_1, x_2, and x_3 of T^* can be ordered so that $E(M) - T^*$ contains elements x_0 and x_4 such that $\{x_0, x_1, x_2\}$ and $\{x_2, x_3, x_4\}$ are triangles of M. We arrive at a contradiction because, as is easily checked, $M\backslash T^*$ is connected, $T^* \subseteq C$, and $r^*(T^*) = 2$. □

An important tool in the proof of the main theorem, which will also be used in the next result, is the following idea of decomposing a connected matroid M. Assume $|E(M)| \geq 3$. A *tree decomposition* of M is a tree T with edges labelled $e_1, e_2, \dots, e_{k-1}$ and vertices labelled by matroids $M_1, M_2, \dots, M_k$ such that

(i) each M_i is 3-connected having at least four elements or is a circuit or cocircuit with at least three elements;
(ii) $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k) = E(M) \cup \{e_1, e_2, \dots, e_{k-1}\}$;
(iii) if the edge e_i joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$;
(iv) if no edge joins the vertices M_{j_1} and M_{j_2}, then $E(M_{j_1}) \cap E(M_{j_i})$ is empty;
(v) M is the matroid that labels the single vertex of the tree $T/e_1, e_2, \dots, e_{k-1}$ at the conclusion of the following process: contract the edges $e_1, e_2, \dots, e_{k-1}$ of T one by one in order; when e_i is contracted, its ends are identified and the vertex formed by this identification is labelled by the 2-sum of the matroids that previously labelled the ends of e_i.

Cunningham and Edmonds [3] proved the following result.

2.4. **Theorem.** *Every connected matroid M has a tree decomposition $T(M)$ in which no two adjacent vertices are both labelled by circuits or are both labelled by cocircuits. Furthermore, the tree $T(M)$ is unique to within relabelling of its edges.*

We shall call $T(M)$ the *canonical tree decomposition of* M and let $\Lambda_2^u(M)$ be the set of matroids that label vertices of $T(M)$.

Next we extend Lemma 2.3 from 3-connected matroids to cosimple connected matroids.

2.5. **Lemma.** *Let C be a circuit of a cosimple connected matroid M such that $|C| \geq 3$. If $M\backslash A$ is disconnected for every subset A of C such that $r^*(A) = 2$, then there is a 3-connected matroid H in $\Lambda_2^u(M)$ such that H has at least four elements, C is a circuit of H, and*

(i) *H is isomorphic to a wheel having C as its rim; or*
(ii) *every 2-subset of C is contained in a triad of H not contained in C.*

Moreover, there is a subset W of $E(H) - C$ and a set $\mathcal{F}$ of connected matroids $\{N_b : b \in W\}$ such that M is the 2-sum of H with all the matroids in $\mathcal{F}$.

Proof. First, we observe the following immediate consequence of Lemma 2.1.

2.5.1. *For every 2-separation $\{X, Y\}$ of M, either $C \subseteq X$ or $C \subseteq Y$.*

From (2.5.1), there is a matroid H in $\Lambda_2^u(M)$ such that C is a circuit of H. If H is a circuit, then $E(H) = C$ and so $E(M) = C$, a contradiction to the fact that M is cosimple. If H is a cocircuit, then $|C| = 2$, a contradiction to the hypothesis. Thus H is a 3-connected matroid having at least four elements. Now let X be a subset of C such that $r_{H^*}(X) = 2$. Then, as H^* is a vertex of $T(M^*)$, it follows that $H^*|X = M^*|X$ so $r_{M^*}(X) = 2$. Thus $M\backslash X$ is disconnected and so $H\backslash X$ is disconnected.

We may now apply Lemma 2.3 to H. Thus, either H is isomorphic to a wheel having C as its rim, or every 2-subset of C is in a triad of H that is not contained in C. Each element b of $E(H) - E(M)$ labels an edge of $T(M)$ and it follows from the structure of $T(M)$ that there is a connected matroid N_b such that $E(N_b) \cap E(H) = \{b\}$, and M is the 2-sum of H and all these matroids N_b. $\square$

If we weaken the hypothesis of the last lemma to require only that $M\backslash A$ is disconnected for every 2-subset A of C, then the lemma remains true if we omit the requirement that the triads of H in (ii) meet $E(H) - C$.

The following consequence of Lemma 2.3 will also be used in the main proof.

2.6. **Lemma.** *Let C be a circuit of a 3-connected matroid M such that $|E(M)| \geq 4$. Suppose that $M\backslash A$ is disconnected for every subset A of C such that $r^*(A) = 2$. Let $Z = \{e \in E(M) - C : A \cup e$ is a triad of M for some 2-subset A of $C\}$. Then either*

(i) *$|Z| = 1$ and $M \cong U_{|C|-1,|C|+1}$; or*
(ii) *M has no circuit D such that $|D \cap Z| = 1$.*

Proof. By Lemma 2.3, either M is a wheel of rank at least four having C as its rim, or every two elements of C are in some triad of M with an element not in C. In the former case, Z is the set of spokes of the wheel and, by orthogonality, (ii) holds. Thus we may assume that every two elements a and b of C are in a triad $T^*_{a,b}$ of M that contains an element of Z. Then $M\backslash Z$ has C as a circuit and a series class and hence as a component. Thus C is a circuit of $M\backslash Z/[E(M) - (Z \cup C)]$, so C is a circuit of $M/[E(M) - (Z \cup C)]$. Let $N = M/[E(M) - (Z \cup C)]$. Every triad of M contained in $Z \cup C$ is a triad of N, so N is connected.

Suppose that (ii) does not hold and let D be a circuit of M such that $|D \cap Z| = 1$. Let e be the unique element of $D \cap Z$. Now $D \cap E(N)$ is a union of circuits of N and so it contains a circuit D' such that $D' - C = \{e\}$. Since N is connected, $D' \cap C \neq \emptyset$. Now choose a in $D' - e$. Then, for all b in $C - D'$, it follows by orthogonality that $e \in T^*_{a,b}$. Hence $\{e, a\}$ spans $C - D'$ in N^*. Thus $\{e, a\} \cup (C - D')$ is contained in a line L^* of N^*. Evidently, for each b in $C - D'$, the set $\{e, b\}$ spans L^* in N^* so $L^* \supseteq D' - e$. Thus $L^* \supseteq C \cup e$, so $C \cup e$ has rank 2 in N^* and hence in M^*. Therefore

$$r_M(C \cup e) + r_{M^*}(C \cup e) - |C \cup e| \leq |C| + 2 - (|C| + 1) = 1.$$

But M is 3-connected, so $|E(M) - (C \cup e)| \leq 1$. As $r_{M^*}(C \cup e) = 2$, it follows that $r(M^*) = 2$ so every 3-subset of $E(M)$ is in a triad of M. We conclude, by orthogonality, that $|Z| = 1$ and $M \cong U_{|C|-1,|C|+1}$. □

2.7. **Lemma.** *Suppose that C_1 and C_2 are circuits of a cosimple connected matroid M such that $C_1 \cap C_2 = \{e\}, E(M) - (C_1 \cup C_2) = \{f\}$, and $\min\{|C_1|, |C_2|\} \geq 3$. If $M\backslash f/e$ has two components whose ground sets are $C_1 - e$ and $C_2 - e$, then there are circuits D_1 and D_2 of M such that $\{e, f\} \subseteq D_1 \cup D_2$ and $E(M) = D_1 \cup D_2$.*

Proof. The matroid $M^*/f\backslash e$ has rank 2. Thus the simple matroid $M^*\backslash e$ is the parallel connection of two lines with ground sets $(C_1 - e) \cup f$ and $(C_2 - e) \cup f$. For each i in $\{1, 2\}$, let a_i and b_i be distinct elements of $C_i - e$. Consider the four lines of M^* spanned by $\{a_1, a_2\}, \{a_2, b_1\}, \{b_1, b_2\}$, and $\{b_2, a_1\}$. The fact that M^* is simple implies that e does not lie on two lines that are consecutive in the specified cyclic order. It follows that there are two such lines that are non-consecutive in this cyclic order such that e avoids both. The complements of these lines are circuits D_1 and D_2 of M satisfying the required conditions. □

3. Sylvester matroids

Murty [7] has called a matroid a *Sylvester matroid* if every pair of distinct elements is in a triangle. Such matroids will arise naturally in the proof of our main theorem and we shall need some covering properties of them. The following characterization of Sylvester matroids extends a similar characterization of Akkari and Oxley [2].

3.1. **Lemma.** *Let N be a matroid with at least four elements. Then N is the dual of a Sylvester matroid if and only if N is cosimple and connected, and $N\backslash A$ is disconnected for every 2-subset A of $E(N)$.*

Proof. If N is the dual of a Sylvester matroid with at least four elements, then it is clear that N is cosimple and connected and that $N\backslash A$ is disconnected for every 2-subset A of $E(N)$. Now assume that the latter conditions on N hold. As $|E(N)| \geq 4$, it follows, by a result of Akkari and Oxley [2] (see also [10, Proposition 10.2.5]), that it suffices to show that N is 3-connected. But this follows immediately from Lemma 2.1. □

A set $\mathcal{D}$ of circuits of a matroid M *double covers* a subset X of $E(M)$ if every element of X is in at least two members of $\mathcal{D}$.

3.2. **Lemma.** *Suppose that N^* is a 3-connected Sylvester matroid with at least four elements or that $N \cong U_{1,m}$ for some $m \geq 3$. Then, for all circuits C_1 of N and all elements g of C_1, there are circuits $C_2, C_3, \ldots, C_{n+1}$ of N such that*

(a) *$C_1, C_2, \ldots, C_{n+1}$ are distinct;*
(b) *$\{C_1, C_2, \ldots, C_{n+1}\}$ double covers $E(N)$;*
(c) *$C_i - (C_{i-1} \cup C_{i-2} \cup \cdots \cup C_1) \neq \emptyset$ for all i in $\{2, 3, \ldots, n\}$;*
(d) *$g \notin C_2 \cup C_3 \cup \cdots \cup C_n$; and*
(e) *$n = r^*(N)$.*

Proof. Suppose first that $N \cong U_{1,m}$. Let $C_1 = \{a_1, a_2\}$ and $g = a_1$. Let $E(N) = \{a_1, a_2, \ldots, a_m\}$, let $C_i = \{a_i, a_{i+1}\}$ for all i in $\{2, 3, \ldots, m-1\}$, and let $C_m = \{a_1, a_m\}$. Then $m = r^*(N) + 1$, so the lemma holds with $n = m - 1$.

Next assume that N^* is a 3-connected Sylvester matroid having at least four elements. Then C_1 is a cocircuit of N^* containing g. Let $g = b_1$ and let $\{b_2, b_3, \ldots, b_n\}$ be a basis for the hyperplane $E(N^*) - C_1$ of N^*. Then $\{b_1, b_2, \ldots, b_n\}$ is a basis for N^*. For each i in $\{2, 3, \ldots, n\}$, let C_i be the fundamental cocircuit of b_i in N^* with respect to $E(N^*) - \{b_1, b_2, \ldots, b_n\}$. Note that C_1 is the fundamental cocircuit of b_1 with respect to $E(N^*) - \{b_1, b_2, \ldots, b_n\}$. As N^* is a Sylvester matroid, for each i in $\{2, 3, \ldots, n\}$, there is an element b'_i on the line of N^* spanned by b_1 and b_i that is different from both b_1 and b_i. Then, in N^*/b_1, each b'_i is parallel to b_i. Thus $\{b'_2, b'_3, \ldots, b'_n\}$ is a basis of N^*/b_1. Hence $\{b'_2, b'_3, \ldots, b'_n\}$ spans a hyperplane of N^* that avoids $\{b_1, b_2, \ldots, b_n\}$. Let C_{n+1} be the complement of this hyperplane. Then $n = r^*(N)$ and $b_1 \notin C_2 \cup C_3 \cup \cdots \cup C_n$. Moreover, $b_i \in C_i - (C_{i-1} \cup C_{i-2} \cup \cdots \cup C_1)$ for all i in $\{2, 3, \ldots, n\}$. Since $C_1, C_2, \ldots, C_n$ is the set of fundamental cocircuits of N^* with respect to $E(N^*) - \{b_1, b_2, \ldots, b_n\}$, this set of fundamental cocircuits covers $E(N^*)$ because N^* has no loops. If there is an element x of $E(N^*) - \{b_1, b_2, \ldots, b_n\}$ that is in exactly one of $C_1, C_2, \ldots, C_n$, say C_i, then, by orthogonality, the fundamental circuit of N^* with respect to $\{b_1, b_2, \ldots, b_n\}$ is $\{x, b_i\}$. This contradicts the fact that N^* is 3-connected having at least four elements. Therefore every element of $E(N^*) - \{b_1, b_2, \ldots, b_n\}$ is in at least two of $C_1, C_2, \ldots, C_n$. Finally, since $C_{n+1} \supseteq \{b_1, b_2, \ldots, b_n\}$, we deduce that every element of N^* is in at least two of $C_1, C_2, \ldots, C_{n+1}$. □

4. Spikes

In this section, we prove some results for spikes that will be used in the proof of the main theorem. For $r \geq 3$, a rank-r matroid M is a *spike* with *tip* p and *legs* $L_1, L_2, \dots, L_r$ if $\{L_1, L_2, \dots, L_r\}$ is a subset of $\mathcal{C}_p$ covering $E(M)$; each L_i is a triangle; and, for all k in $\{1, 2, \dots, r-1\}$, the union of any k of $L_1, L_2, \dots, L_r$ has rank $k+1$. Thus, for example, both the Fano and non-Fano matroids are rank-3 spikes although the tips of these spikes are not unique. It follows from (ii) below that spikes of rank at least four have unique tips. In general, if M is a rank-r spike with tip p, then

(i) $(L_i \cup L_j) - \{p\}$ is a circuit and a cocircuit of M for all distinct i and j;
(ii) apart from $L_1, L_2, \dots, L_r$ and those sets listed in (i), every non-spanning circuit of M avoids p, is a circuit-hyperplane, and contains a unique element from each of $L_1 - p, L_2 - p, \dots, L_r - p$;
(iii) M/p can be obtained from an r-element circuit by replacing each element by two elements in parallel; and
(iv) if $\{x, y\} = L_i - p$ for some i, then each of $M\backslash p/x$ and $(M\backslash p\backslash x)^*$ is a rank-$(r-1)$ spike with tip y.

Sometimes spikes are considered with the tips removed. The rank-r *free spike* has no non-spanning circuits except the legs and those sets listed in (i). There is a unique rank-r binary spike. It is represented by the matrix $[I_r|J_r - I_r|\mathbf{1}]$ where J_r is the $r \times r$ matrix of all ones and $\mathbf{1}$ is the vector of all ones. This vector corresponds to the tip of the spike.

Let C and D be circuits of a matroid N where $D = \{e, a, b\}$ and $C \cap D = \{e, a\}$. We say that C is *indifferent with respect to $D - e$ in N* if $(C - a) \cup b$ is also a circuit of N.

4.1. **Lemma.** *For $r \geq 3$, let M be a spike with legs $L_1, L_2, \dots, L_r$ where $L_i = \{e, a_i, b_i\}$ for all i. Then M has a circuit D of the form $\{e, d_1, d_2, \dots, d_r\}$ where $d_i \in \{a_i, b_i\}$ for all i. Moreover, if M has a spike minor on $L_1 \cup L_2 \cup L_3$ that is not isomorphic to F_7, then D can be chosen so that it is indifferent with respect to $L_1 - e, L_2 - e$, or $L_3 - e$ in M.*

Proof. Let M_1 be a spike minor of M on $L_1 \cup L_2 \cup L_3$. Then M_1 has a 4-circuit C of the form $\{e, d_1, d_2, d_3\}$ where $d_i \in \{a_i, b_i\}$ for all i in $\{1, 2, 3\}$. By relabelling we may assume that $C = \{e, a_1, a_2, a_3\}$. Since M_1 is a spike minor of M, it follows that $M_1 = M\backslash X/Y$ where $X \cup Y = (L_4 \cup L_5 \cup \dots \cup L_r) - e$ and $|Y \cap L_i| = 1$ for all $i \geq 4$. Since $(L_i \cup L_j) - e$ is a cocircuit of M for all distinct i and j, it follows by orthogonality that $C \cup Y$ is a circuit D of M. We may assume that D is indifferent with respect to none of $C_1 - e, C_2 - e$, and $C_3 - e$. Then M_1 has $\{a_i, a_j, b_k\}$ as a circuit for all $\{i, j, k\} = \{1, 2, 3\}$. Now suppose that $M_1 \not\cong F_7$. Then $\{b_1, b_2, b_3\}$ is not a circuit of M_1. Thus $\{e, b_1, b_2, b_3\}$ is a circuit of M_1 that is indifferent with respect to $L_1 - e$. Then, by orthogonality again, $\{e, b_1, b_2, b_3\} \cup Y$ is a circuit of M and this circuit is indifferent with respect to $L_1 - e$. The lemma follows. □

4.2. **Lemma.** *Let M be a spike of rank at least three having legs $L_1, L_2, \dots, L_r$ and tip e. Then M has circuits D_1 and D_2 each containing e such that $L_2 \cup L_3 \cup \dots \cup L_r \subseteq D_1 \cup D_2$. Furthermore, unless M is a binary spike of odd rank, D_1 and D_2 can be chosen so that, in addition, $L_1 \subseteq D_1 \cup D_2$.*

Proof. Let $L_i = \{e, a_i, b_i\}$ for all i. Suppose first that M is a binary spike and view M as a restriction of the r-dimensional vector space over $GF(2)$ letting $b_1, b_2, \dots, b_r$ be the natural basis vectors and e be the vector of all ones. If $r(M)$ is even, then $\{e, b_1, b_2, \dots, b_r\}$ and $\{e, a_1, a_2, \dots, a_r\}$ are circuits of M that cover $E(M)$. If $r(M)$ is odd, then $\{e, b_1, b_2, \dots, b_r\}$ and $\{e, b_1, a_2, \dots, a_r\}$ are circuits of M that cover $E(M) - a_1$. Hence the lemma holds if M is binary.

We may now assume that M is non-binary. Then, by a result of Seymour [12], M has a $U_{2,4}$ minor using $\{e, a_1\}$ and hence has such a minor M_1 using L_1. Without loss of generality, we may assume that $E(M_1) - L_1 = \{a_2\}$. Let $M_1 = M\backslash X/Y$. Then we may assume that $|Y| = r - 2$. For all i, both $M\backslash\{a_i, b_i\}$ and $M/\{a_i, b_i\}$ are binary. Thus, for all $i \geq 3$, one of a_i and b_i is in X and the other is in Y. By relabelling if necessary, we may assume that each such a_i is in Y. Since $|Y| = r - 2$, it follows that $b_2 \in X$. Thus $Y \cup a_2$ is a series class of $M/e\backslash X$ and hence of $M\backslash X$. Therefore both $Y \cup \{e, a_1, a_2\}$ and $Y \cup \{e, b_1, a_2\}$ are circuits of M, so $\{e, a_1, a_2, \dots, a_r\}$ is a circuit D_1 of M that is indifferent with respect to $L_1 - e$. Since $\{b_1, b_2, \dots, b_r\}$ is a circuit of M/e, it is straightforward to show that $\{e, b_1, b_2, \dots, b_r\}$ or $\{e, a_1, b_2, b_3, \dots, b_r\}$ is a circuit of M and we take this circuit to be D_2. Clearly, $L_2 \cup L_3 \cup \dots \cup L_r \subseteq D_1 \cup D_2$. Moreover, since D_1 is indifferent with respect to $L_1 - e$, we can replace D_1 by whichever of D_1 and $(D_1 - a_1) \cup b_1$ contains $D_2 - \{a_1, b_1\}$ to obtain that $L_1 \subseteq D_1 \cup D_2$. □

5. The proof of the main result

In this section, we prove the main result of the paper.

Proof of Theorem 1.1. Suppose the theorem is false and choose a counterexample M that minimizes $|E(M)|$. First we note that

5.0.1. *M is not a spike with tip e.*

Assume the contrary. Clearly $\nu_e(M) = r(M) = r^*(M) - 1$. Moreover, by Lemma 4.2, provided M is not a binary spike of odd rank, $\theta_e(M) = 2$. In the exceptional case, M has an F_7-minor using e and $\theta_e(M) \leq 3$. Thus, in both cases, M satisfies the theorem. This contradiction establishes (5.0.1).

For a connected minor M' of M using e such that M'/e is connected and $M' \not\cong U_{1,1}$, define $s(M') = 1$ if M' has no F_7-minor using e, and $s(M') = 2$ otherwise. Evidently, if $s(M') = 2$, then $s(M) = 2$, so

$$s(M') \leq s(M). \tag{3}$$

As M is a counterexample to the theorem,

$$\nu_e(M) + \theta_e(M) \geq r^*(M) + s(M) + 1.$$

Observe that M is not a circuit and so $r^*(M) > 1$. Let $C_1, C_2, \dots, C_m$ be a maximum-sized subset of $\mathcal{C}_e(M)$ such that the intersection of any two of them equals $\{e\}$. By definition, $m = \nu_e(M)$. First, we prove that

5.1. $m \geq 2$.

If $m = 1$, then

$$1 + \theta_e(M) = \nu_e(M) + \theta_e(M) \geq r^*(M) + s(M) + 1.$$

Thus $\theta_e(M) \geq r^*(M) + s(M) \geq r^*(M) + 1$, which contradicts (1). Hence (5.1) holds.

Next we show the following:

5.2. *M has no cocircuit D^* containing e that is contained in some C_i.*

Suppose that such a cocircuit D^* exists. By orthogonality, $C_j \cap D^* \supsetneqq \{e\}$ for all j in $\{1, 2, \ldots, m\}$. But $D^* \subseteq C_i$ and $C_j \cap C_i = \{e\}$ when $j \neq i$. Hence $m = 1$; a contradiction to (5.1). Thus (5.2) holds.

Observe that

5.3. *M is cosimple.*

If not, then M has a non-trivial series class S. By (5.2), $e \notin S$. Choose f in S. Clearly M/f contradicts the choice of M provided that $M/f/e$ is connected. Thus assume that $M/f/e$ is disconnected. Then, as M/e is connected, $M/e\backslash f$ is connected. Since $M\backslash f$ is disconnected, it follows that $\{e, f\}$ is a cocircuit of M; a contradiction. Hence (5.3) holds.

Next, we prove the following:

5.4. *For every f in $E(M) - (C_1 \cup C_2 \cup \cdots \cup C_m)$, the matroid $M/e\backslash f$ is disconnected.*

Suppose that, for some such element f, the matroid $M/e\backslash f$ is connected. Then $M\backslash f$ is connected because $\{e, f\}$ is not a cocircuit of M since M is cosimple. By the choice of M,

$$\nu_e(M\backslash f) + \theta_e(M\backslash f) \leq r^*(M\backslash f) + s(M\backslash f).$$

Evidently, $r^*(M\backslash f) = r^*(M) - 1$ and, since $f \notin C_1 \cup C_2 \cup \cdots \cup C_m$, we have $\nu_e(M\backslash f) = \nu_e(M)$. Moreover, $\theta_e(M\backslash f) \geq \theta_e(M) - 1$ because a set of circuits in $\mathcal{C}_e(M\backslash f)$ that covers $E(M\backslash f)$ can be completed to a set of circuits in $\mathcal{C}_e(M)$ that covers $E(M)$ by adding a circuit that contains $\{e, f\}$. Since, by (3), $s(M\backslash f) \leq s(M)$, it follows that $\nu_e(M) + \theta_e(M) \leq r^*(M) + s(M)$. This contradiction to the fact that M is a counterexample to the theorem completes the proof of (5.4).

We show next that:

5.5. **Lemma.** *If i in $\{1, 2, \ldots, m\}$, then $M/e\backslash A$ is disconnected for every subset A of $C_i - e$ having at least two elements.*

Proof. Suppose that $M/e\backslash A$ is connected for some subset A of $C_i - e$ such that $|A| \geq 2$. If $M\backslash A$ is also connected, then, by the choice of M, the theorem holds for $M\backslash A$ and so

$$\nu_e(M\backslash A) + \theta_e(M\backslash A) \leq r^*(M\backslash A) + s(M\backslash A).$$

But $r^*(M\backslash A) \leq r^*(M) - 2$ because M is cosimple; $\nu_e(M\backslash A) \geq \nu_e(M) - 1$ because $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_m$ are circuits of $M\backslash A$; and $\theta_e(M\backslash A) \geq \theta_e(M) - 1$ because a set of circuits in $\mathcal{C}_e(M\backslash A)$ that covers $E(M\backslash A)$ can be completed to a set of circuits in $\mathcal{C}_e(M)$ that covers $E(M)$ by adding C_i. Hence, as $s(M\backslash A) \leq s(M)$,

$$\begin{aligned} \nu_e(M) + \theta_e(M) &\leq [\nu_e(M\backslash A) + 1] + [\theta_e(M\backslash A) + 1] \\ &\leq r^*(M\backslash A) + s(M\backslash A) + 2 \leq r^*(M) + s(M). \end{aligned}$$

This contradiction implies that $M\backslash A$ is disconnected. As $M/e\backslash A$ is connected, it follows e is a coloop of $M\backslash A$ and so $A \cup e$ contains a cocircuit D^* of M such that $e \in D^*$. Since this contradicts (5.2), we deduce that the lemma holds. □

5.6. **Lemma.** $E(M) = C_1 \cup C_2 \cup \cdots \cup C_m$.

Proof. Suppose that the lemma fails and choose f in $E(M) - (C_1 \cup C_2 \cup \cdots \cup C_m)$. By (5.4), $M/e\backslash f$ is disconnected. As M/e is connected by hypothesis, we deduce that $M/e/f$ is connected. Thus M/f is connected and so the theorem holds for this matroid. Hence

$$\nu_e(M/f) + \theta_e(M/f) \leq r^*(M/f) + s(M/f).$$

Now $r^*(M/f) = r^*(M)$ and $\nu_e(M/f) \geq \nu_e(M)$ because each of $C_1, C_2, \ldots, C_m$ contains a circuit of M/f containing e. As $s(M/f) \leq s(M)$, it follows that

$$\nu_e(M) + \theta_e(M/f) \leq r^*(M) + s(M).$$

Since the theorem fails for M, we deduce that $\theta_e(M) > \theta_e(M/f)$.

Let $D_1, D_2, \ldots, D_n$ be a minimum-sized subset of $\mathcal{C}_e(M/f)$ that covers $E(M/f)$. For each i, either D_i or $D_i \cup f$ is a circuit of M containing e. Since $\theta_e(M) > \theta_e(M/f)$, it follows that each D_i is a circuit of M. Thus $D_1, D_2, \ldots, D_n$ are in $\mathcal{C}_e(M\backslash f)$ and cover $E(M\backslash f)$. Hence $D_1 - e, D_2 - e, \ldots, D_n - e$ are circuits of $M\backslash f/e$ that cover $E(M\backslash f/e)$. As $M\backslash f/e$ is disconnected, we may assume, by relabelling if necessary, that $D_1 - e$ and $D_2 - e$ are in different components of $M\backslash f/e$. Then $(M/e)|[(D_1 \cup D_2) - e] = (M/e)|(D_1 - e) \oplus (M/e)|(D_2 - e)$. Let C' be a circuit of M/e that meets both $D_1 - e$ and $D_2 - e$ such that $C' - (D_1 \cup D_2)$ is minimal. As $M/e\backslash f$ has $D_1 - e$ and $D_2 - e$ in different components, $f \in C'$. We show next that

5.6.1. *$C' - (D_1 \cup D_2)$ is a series class of $(M/e)|[(D_1 \cup D_2 \cup C') - e]$.*

If not, then $(M/e)|[(D_1 \cup D_2 \cup C') - e]$ has a circuit C'' that contains some but not all of $C' - (D_1 \cup D_2)$. By the choice of C', we may assume that C'' meets D_2 but avoids D_1. Take d_1 in $D_1 \cap C'$ and $c \in (C'' \cap C') - (D_1 \cup D_2)$. Then $(M/e)|[(D_1 \cup D_2 \cup C') - e]$ has a circuit C''' such that $d_1 \in C''' \subseteq (C' \cup C'') - c$. Then C''' must contain an element of $C'' - C'$ and so C''' meets D_2 and contradicts the choice of C'. Hence (5.6.1) holds.

Now $M|(D_1 \cup D_2 \cup C')$ is connected and has $C' - (D_1 \cup D_2)$ as a series class. Consider the cosimplification of this matroid labelled so that f is an element of it. If, in $\text{co}(M|(D_1 \cup D_2 \cup C'))$, only two elements of D_1 remain, then $D_1 - e$ is a series class of $M|(D_1 \cup D_2 \cup C')$, and hence is a series class of $(M/e)|[(D_1 \cup D_2 \cup C') - e]$. But the last matroid is connected and has $D_1 - e$ as a circuit; a contradiction. Thus, in $\text{co}(M|(D_1 \cup D_2 \cup C'))$, at least three elements of D_1 remain and, similarly, at least three elements of D_2 remain. Let $D'_i = D_i \cap E(\text{co}(M|(D_1 \cup D_2 \cup C')))$ for each i in $\{1, 2\}$. Then, by applying Lemma 2.7 to the circuits D'_1 and D'_2 of $\text{co}(M|(D_1 \cup D_2 \cup C'))$, we get that the last matroid has circuits D''_1 and D''_2 that both contain $\{e, f\}$ and that cover $E(\text{co}(M|(D_1 \cup D_2 \cup C')))$. Hence $M|(D_1 \cup D_2 \cup C')$ has circuits D'''_1 and D'''_2 that both contain $\{e, f\}$ and that cover $D_1 \cup D_2 \cup C'$. Hence $D'''_1, D'''_2, D_3, D_4, \ldots, D_n$ covers $E(M)$, so $\theta_e(M) \leq \theta_e(M/f)$; a contradiction. □

Without loss of generality, we may assume that there is a non-negative integer l such that $|C_i| \geq 4$ if $1 \leq i \leq l$, and $|C_i| = 3$ if $l + 1 \leq i \leq m$.

By hypothesis, M/e is a connected matroid. For all i in $\{1, 2, \ldots, l\}$, the set $C_i - e$ is a circuit of M/e and $|C_i - e| \geq 3$. Now M/e is cosimple and, by Lemma 5.5, $M/e\backslash A$ is disconnected for every subset A of $C_i - e$ such that $|A| \geq 2$. Thus, by Lemma 2.5, there is a 3-connected matroid H_i in $\Lambda^u_2(M/e)$ with at least four elements such that $C_i - e$ is a circuit of H_i and

(i) H_i is isomorphic to a wheel having $C_i - e$ as its rim; or
(ii) every 2-subset of $C_i - e$ is contained in a triad of H_i not contained in $C_i - e$.

Moreover, there is a subset W_i of $E(H_i) - (C_i - e)$ and a set $\mathcal{F}_i$ of connected matroids $\{N_b : b \in W_i\}$ such that M/e is the 2-sum of H_i with all the matroids in $\mathcal{F}_i$. We also define

$$Z_i = \{f \in E(H_i) - C_i : A \cup f \text{ is a triad of } H_i \text{ for some 2-subset } A \text{ of } C_i - e\}.$$

5.7. **Lemma.** *If $i \in \{1, 2, \ldots, l\}$, then $Z_i \subseteq W_i$.*

Proof. Suppose that $f \in Z_i - W_i$. Let T^* be a triad of M/e and so of M such that $f \in T^*$ and $T^* - f \subseteq C_i - e$. By Lemma 5.6, $f \in C_j$ for some j in $\{1, 2, \ldots, m\}$. By orthogonality, $T^* \cap C_j \neq \{f\}$, say $g \in (T^* \cap C_j) - f$. As $T^* \subseteq E(M/e)$, it follows that $g \neq e$ and so $g \in (C_i - e) \cap (C_j - e)$. Hence $i = j$; a contradiction because $f \notin C_i$. Thus $Z_i \subseteq W_i$. □

5.8. **Lemma.** *If $i \in \{1, 2, \ldots, l\}$ and $z \in Z_i$, then $r^*(H_i.[(C_i - e) \cup z]) > 2$.*

Proof. Suppose that $r^*(H_i.[(C_i - e) \cup z]) \leq 2$. As H_i is cosimple and $|C_i - e| \geq 3$, it follows that $r^*(H_i.[(C_i - e) \cup z]) = 2$ and that $H_i.[(C_i - e) \cup z]$ is cosimple. Thus every 3-subset of $(C_i - e) \cup z$ is a triad of H_i. Since $|(C_i - e) \cup z| \geq 4$, it follows that if $f \in C_i - e$, then H_i/f is connected. Therefore, from the remarks preceding Lemma 5.7, we deduce that $M/e/f$ is connected. If M/f is disconnected, then $\{e, f\}$ is a circuit of M contradicting the fact that M/e is connected. Thus M/f is connected. By the choice of M, we have that

$$\nu_e(M/f) + \theta_e(M/f) \leq r^*(M/f) + s(M/f) \leq r^*(M/f) + s(M).$$

As each of $C_1 - f, C_2 - f, \ldots, C_m - f$ contains a circuit of M/f having e as one of its elements, it follows that $\nu_e(M) \leq \nu_e(M/f)$. Since $r^*(M/f) = r^*(M)$ and M is a counterexample to the theorem, we deduce that $\theta_e(M/f) < \theta_e(M)$. Let $D_1, D_2, \ldots, D_n$ be a minimum-sized subset of $\mathcal{C}_e(M/f)$ that covers $E(M/f)$. For each i in $\{1, 2, \ldots, n\}$, either D_i or $D_i \cup f$ is a circuit of M. As $\theta_e(M/f) < \theta_e(M)$, it follows that each D_i is a circuit of M. In particular, none of $D_1, D_2, \ldots, D_n$ contains $C_i - f$. Now, either $|C_i - \{e, f\}| = 2$, or every 3-subset of $C_i - \{e, f\}$ is a triad of H_i and hence is a triad of M and so of M/f. As $D_1 \cup D_2 \cup \cdots \cup D_n \supseteq C_i - \{e, f\}$, we may assume that $D_1 \cap (C_i - \{e, f\}) \neq \emptyset$. Since $D_1 \not\supseteq C_i - f$, it follows by orthogonality in M or from the size of $C_i - \{e, f\}$ that there is a unique element x_2 of $C_i - \{e, f\}$ that is not in D_1. Without loss of generality, we may assume that $x_2 \in D_2$. Again, there is a unique element x_1 of $C_i - \{e, f\}$ that is not in D_2. Now $D_1 \cap (C_i - e)$ and $D_2 \cap (C_i - e)$ are both unions of circuits of $M.(C_i - e)$ and both sets avoid f. Furthermore, x_1 is in the first set but not the second, while x_2 is in the second but not the first. Thus $\{f, x_1, x_2\}$ is coindependent in $M.(C_i - e)$. Hence $3 \leq r^*(M.(C_i - e)) = r^*_M(C_i - e) = 2$; a contradiction. □

5.9. **Lemma.** *If $i \in \{1, 2, \ldots, l\}$ and C is a circuit of H_i, then $|C \cap Z_i| \neq 1$ and $|Z_i| \geq 2$.*

Proof. Suppose that $|C \cap Z_i| = 1$ or $|Z_i| = 1$. In the latter case, the connected matroid H_i has a circuit D such that $|D \cap Z_i| = 1$. Thus, in both cases, by Lemma 2.6, $H_i \cong U_{|C_i|-2,|C_i|}$. Therefore, $r^*(H_i) \leq 2$; a contradiction to Lemma 5.8. □

For each i in $\{l+1, l+2, \ldots, m\}$, let $C_i = \{e, a_i, b_i\}$. Now $C_1-e, C_2-e, \ldots, C_m-e$ is a set of disjoint circuits of M/e that covers $E(M/e)$. By (5.3), M is cosimple so M/e is cosimple. Thus every non-trivial series class of $M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}$ contains at most one element not in $\{b_{l+1}, b_{l+2}, \ldots, b_m\}$. Hence, by orthogonality with each of $C_1 - e, C_2 - e, \ldots, C_l - e$, every such series class is contained in $\{b_{l+1}, b_{l+2}, \ldots, b_m\}$. Let $N = \text{co}(M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\})$. Then, clearly, N is connected and has all of $C_1 - e, C_2 - e, \ldots, C_l - e$ among its circuits. For the remainder of the proof of the theorem, we take

$$X = (C_1 \cup C_2 \cup \cdots \cup C_l) - e.$$

Thus

$$E(N) - X \subseteq \{b_{l+1}, b_{l+2}, \ldots, b_m\}.$$

Moreover, since M is not a spike with tip e,

$$r(N) > 0.$$

Now consider the canonical tree decomposition $T(M/e)$ of M/e. For each i in $\{1, 2, \ldots, l\}$, the matroid H_i is in $\Lambda_2^u(M/e)$. Thus, by possibly relabelling some elements in the set W_i, we may assume that each H_i labels a vertex of $T(M/e)$, and W_i labels the edges of $T(M/e)$ incident with this vertex. We observe that the vertices $H_1, H_2, \ldots, H_l$ need not be distinct. Now contract every edge of $T(M/e)$ that is not labelled by a member of $W_1 \cup W_2 \cup \cdots \cup W_l$ and, after each such contraction, label the new composite vertex by the 2-sum of the two matroids that previously labelled the ends of the edge. At the conclusion of this process, we obtain a tree $T'(M/e)$ with edge-set $W_1 \cup W_2 \cup \cdots \cup W_l$ such that if $l > 0$, then $\{H_1, H_2, \ldots, H_l\}$ is a dominating set of vertices of the tree. Moreover, since $Z_i \subseteq W_i$ and $|Z_i| \geq 2$ for all i, it follows that no H_i is a terminal vertex of $T'(M/e)$, and $|E(H_i)| \geq |C_i - e| + |Z_i| \geq 3 + 2 = 5$. Note that if $l = 0$, then we take $T'(M/e)$ to consist of a single vertex labelled by M/e. For each matroid H that labels a vertex of $T'(M/e)$ other than $H_1, H_2, \ldots, H_l$, the set $E(H) - (W_1 \cup W_2 \cup \cdots \cup W_l)$ is a disjoint union of 2-circuits from $C_{l+1} - e, C_{l+2} - e, \ldots, C_m - e$.

From $T'(M/e)$, we construct a tree $T'(N)$ for N by first replacing each matroid H labelling a vertex of $T'(M/e)$ other than $H_1, H_2, \ldots, H_l$ by the matroid obtained from it by deleting $E(H) \cap \{a_{l+1}, a_{l+2}, \ldots, a_m\}$ and contracting $(E(H) \cap \{b_{l+1}, b_{l+2}, \ldots, b_m\}) - E(N)$. After this, if some vertex is labelled by a 2-element matroid H, then H must contain at least one b_i for $l + 1 \leq i \leq m$. Hence H must be a terminal vertex of the current tree with its second element being an element w_j of some W_j. When this occurs, we contract the edge w_j of the tree and relabel the element w_j of H_j by b_i. At the conclusion of this process, we obtain the tree $T'(N)$ which will be important throughout the rest of the argument. Evidently, for each i in $\{1, 2, \ldots, l\}$, there is a vertex H'_i of $T'(N)$ that is labelled by a matroid that is obtained from H_i by possibly relabelling some members of W_i by elements of $\{b_{l+1}, b_{l+2}, \ldots, b_m\}$. Let Z'_i be the set Z_i after this relabelling.

5.10. **Lemma.** *If C is a circuit of N, then $|C - X| \neq 1$.*

Proof. Since $r(N) > 0$ and N is connected, the result holds if $l = 0$. Thus suppose that $l > 0$ and $|C - X| = 1$. To each subtree T' of $T'(N)$, we can associate a connected matroid $M(T')$ formed by taking the 2-sum of the matroids that label the vertices of T' using, as basepoints, the labels of the edges of T'. Choose such a subtree T' of T for which $M(T')$ contains a circuit C' such that $|C' - X| = 1$

and $|V(T')|$ is a minimum. As $|C'| > 1$, it follows that $C' \cap (C_j - e) \neq \emptyset$ for some j in $\{1, 2, \ldots, l\}$, say $j = 1$. Thus $C_1 - e$ meets $E(M(T'))$, and the construction of T' implies that $C_1 - e \subseteq E(M(T'))$. Suppose that $M(T')$ is 3-connected. Then $M(T') = H'_1$. By orthogonality, either $C' \supseteq C_1 - e$, or $C' \cap Z'_1 \neq \emptyset$. In the first case, $C' \supsetneqq C_1 - e$; a contradiction. Thus $C' \cap Z'_1 \neq \emptyset$. Since $C' \cap Z'_1 \subseteq C' - X$ and the last set has exactly one element, say f, we deduce that $C' \cap Z'_1 = C' - X = \{f\}$. Therefore, by Lemma 2.6, $|Z_1| = 1$; a contradiction to Lemma 5.9. We conclude that $M(T')$ is not 3-connected. In particular, T' does not consist of a single vertex and so has an edge b. Moreover, H'_1 must label a vertex of T'. Let T_1 and T_2 be the connected components of $T' - b$. Then $M(T')$ is the 2-sum with basepoint b of the matroids $M(T_1)$ and $M(T_2)$. By the choice of T', neither $M(T_1)$ nor $M(T_2)$ has C' as a circuit. Hence, for each i in $\{1, 2\}$, there is a circuit D_i of $M(T_i)$ such that $b \in D_i$ and $C' = (D_1 \cup D_2) - b$. Without loss of generality, we may assume that $f \in D_1$. Thus $D_2 - b \subseteq X$, that is, $\{b\} = D_2 - X$. This contradicts the choice of T' and the lemma follows. □

Now let $B = \{b_{l+1}, b_{l+2}, \ldots, b_m\} \cap E(N)$.

5.11. **Lemma.** *If A is a 2-subset of B and $N\backslash A$ is connected, then A is a circuit of N.*

Proof. Suppose that A is not a circuit of N. Then there is a circuit C of N such that $|C| \geq 3$ and $A \subseteq C$. By the definition of N, there is a circuit D of $M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}$ such that $C = D \cap E(N)$. Let $A = \{s_1, s_2\}$. As $|C| \geq 3$, the circuit D meets at least three series classes of $M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}$ including S_1 and S_2 that contain s_1 and s_2, respectively. As $N\backslash A$ is connected, $[M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}]\backslash(S_1 \cup S_2)$ is connected. Let $S' = \{a_i, b_i : b_i \in S_1 \cup S_2\}$. Then $M/e\backslash S'$ is obtained from $[M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}]\backslash(S_1 \cup S_2)$ by, for each j in $\{l+1, l+2, \ldots, m\}$ such that $a_j \notin S'$, adding a_j in parallel to b_j. Hence $M/e\backslash S'$ is connected. Moreover, $M\backslash S'$ is connected, otherwise e is a coloop of $M\backslash S'$ contradicting the fact that $E(M\backslash S')$ is the union of the circuts in $\{C_1, C_2, \ldots, C_m\} - \{C_i : b_i \in S_1 \cup S_2\}$. Thus $M\backslash S'$ satisfies the hypotheses of the theorem, so

$$\nu_e(M\backslash S') + \theta_e(M\backslash S') \leq r^*(M\backslash S') + s(M\backslash S'). \tag{4}$$

Evidently, $\nu_e(M\backslash S') = \nu_e(M) - (|S_1| + |S_2|)$. Moreover, $r^*(M\backslash S') = r^*(M/e\backslash S')$, and, in M/e, the elements of S' consist of 2 distinct series classes in which each elements has been replaced by two parallel elements. Thus $r^*(M/e\backslash S') = r^*(M/e) - (|S_1|+1) - (|S_2|+1)$, so $r^*(M\backslash S') = r^*(M) - (|S_1|+|S_2|+2)$. Since $s(M\backslash S') \leq s(M)$, we obtain, by substituting into (4), that

$$\nu_e(M) - (|S_1| + |S_2|) + \theta_e(M\backslash S') \leq r^*(M) - (|S_1| + |S_2| + 2) + s(M),$$

that is,

$$\nu_e(M) + (\theta_e(M\backslash S') + 2) \leq r^*(M) + s(M).$$

We shall complete the proof of the lemma by showing the following:

5.11.1. *M has two circuits both containing e whose union contains S'.*

This will show that

$$\theta_e(M) \leq \theta_e(M\backslash S') + 2$$

and thereby establish the contradiction that M satisfies the theorem.

To prove (5.11.1), it suffices to show that:

5.11.2. *M has a spike-minor M' with tip e whose legs include all the sets $\{e, a_i, b_i\}$ such that $b_i \in S_1 \cup S_2$ together with at least one other set.*

This is because, by Lemma 4.2, if (5.11.2) holds, then M' has two circuits both containing e whose union contains S' and therefore (5.11.1) holds.

We now prove (5.11.2). There are two possibilities for the circuit C:

(i) $C \cap (B - A) \neq \emptyset$; and

(ii) $C \cap B = A$.

Suppose that (i) holds and let s_3 be an element of $C \cap (B - A)$. Then the circuit D of $M/e\backslash\{a_{l+1}, a_{l+2}, \dots, a_m\}$ contains S_1, S_2, and the series class S_3 containing s_3. Now consider the restriction of M/e to the set D' that is obtained from D by adding all a_i such that $b_i \in S_1 \cup S_2 \cup S_3$. Then it is not difficult to check that by contracting from $M|(D' \cup e)$ all the elements of D that are not in $S_1 \cup S_2 \cup S_3$, we obtain a spike with tip e and legs all the sets $\{e, a_i, b_i\}$ such that $b_i \in S_1 \cup S_2 \cup S_3$. Thus, in case (i), (5.11.2) holds.

We may now assume that (ii) holds. Then we have that $\{C_1, C_2, \dots, C_m\}$ covers $E(M)$, that $|C| \geq 3$, and that $C - X = C \cap B = \{s_1, s_2\}$. It follows that C meets $C_j - e$ for some j in $\{1, 2, \dots, l\}$. This circuit $C_j - e$ will be used to manufacture the leg of the spike minor M' that is different from all $\{e, a_i, b_i\}$ such that $b_i \in S_1 \cup S_2$.

We show next that $S_1 \cup S_2$ is contained in a series class of $(M/e)|[D \cup (C_j - e)]$. Suppose not. Then, since $C \cap B = \{s_1, s_2\}$, it follows that

$$N|[C \cup (C_j - e)] = [M/[e \cup (S_1 - s_1) \cup (S_2 - s_2)]]|[C \cup (C_j - e)].$$

Thus s_1 and s_2 are not in series in $N|[C \cup (C_j - e)]$. Therefore N has a circuit C' that contains exactly one of s_1 and s_2, and this circuit must meet $C_j - e$. As $C - X = \{s_1, s_2\}$, we deduce that $|C' - X| = 1$; a contradiction to Lemma 5.10. We conclude that $S_1 \cup S_2$ is contained in a series class of $(M/e)|[D \cup (C_j - e)]$.

Now let D_1 be a circuit of $(M/e)|[D \cup (C_j - e)]$ such that $D_1 \supseteq S_1 \cup S_2$ and $D_1 - (C_j - e)$ is a circuit of $((M/e)|[D \cup (C_j - e)])/(C_j - e)$. Then $D_1 - (C_j - e)$ is a series class of $(M/e)|[D_1 \cup (C_j - e)]$. Clearly there is a 2-element subset $\{c_j, d_j\}$ of $C_j - e$ such that

$$((M/e)|[D_1 \cup (C_j - e)])/(C_j - \{e, c_j, d_j\})$$

consists of a circuit with ground set $(D_1 - (C_j - e)) \cup c_j$ and the element d_j in parallel with c_j.

Now recall that $S' = \{a_i, b_i : b_i \in S_1 \cup S_2\}$ and let

$$M'' = (M|[D_1 \cup S' \cup C_j])/(C_j - \{e, c_j, d_j\}).$$

Observe that:

5.11.3. *If T is a triangle of M or of M'' such that $e \in T$ and $T - e$ is a circuit of $M''/(V \cup e)$ for some V avoiding e, then either e is a loop of M''/V, or T is a triangle of M''/V.*

Clearly M'' has $\{e, c_j, d_j\}$ as a circuit. Thus, by (5.11.3), for each b_i in $S_1 \cup S_2$, the matroid M'' has $\{e, a_i, b_i\}$ as a circuit. Observe that M''/e has $(D_1 - (C_j - e)) \cup c_j$ as a circuit. Let $Y = (D_1 - (C_j - e)) - (S_1 \cup S_2)$. Then $M''/e/Y$ has, among its circuits, the sets $S_1 \cup S_2 \cup c_j$, $\{c_j, d_j\}$, and all $\{a_i, b_i\}$ with b_i in $S_1 \cup S_2$. In order to show that M''/Y is the desired spike minor of M, we shall show next that e is not a loop of M''/Y. But $(D_1 - (C_j - e)) \cup c_j$ is a circuit of M''/e so either $(D_1 - (C_j - e)) \cup c_j$ or $(D_1 - (C_j - e)) \cup c_j \cup e$ is a circuit of M''. In the former case, choose some i such that $b_i \in S_1 \cup S_2$. Then $\{e, a_i, b_i\}$ is a circuit of M''. As $(D_1 - (C_j - e)) \cup c_j$ is a circuit of M'' containing b_i and not spanning e, it follows that $((D_1 - b_i) - (C_j - e)) \cup c_j \cup \{a_i, e\}$ is a circuit of M''. Thus, by interchanging the labels on this a_i and b_i, we may assume that $(D_1 - (C_j - e)) \cup c_j \cup e$ is a circuit of M''. As Y is a subset of the last set, we conclude that e is not a loop of M''/Y. Thus, by (5.11.3), $\{e, c_j, d_j\}$ and all $\{e, a_i, b_i\}$ with b_i in $S_1 \cup S_2$ are triangles of M''/Y so this matroid is, indeed, the desired spike minor of M. Hence (5.11.2) holds and the lemma is proved.

□

5.12**. Lemma.** *Either $r(N) = 1$, or N is 3-connected having at least four elements. In the latter case, $H'_1 = H'_2 = \cdots = H'_l = N$.*

Proof. Suppose that $r(N) \geq 2$. Since N is cosimple and connected, N has at least four elements. To prove that N is 3-connected, it is enough to prove that $T'(N)$ has just one vertex. Suppose that $T'(N)$ has at least two vertices. Let K_1 and K_2 be terminal vertices of $T'(N)$ and, for each i let k_i be the element of K_i that labels an edge of $T'(N)$. We prove next that:

5.12.1. *For each i in $\{1, 2\}$, there is an element e_i in $E(K_i) \cap B$ such that $K_i \backslash e_i$ is connected and, when $r(K_i) \neq 1$, the set $\{e_i, k_i\}$ is not a circuit of K_i.*

Suppose first that $K_i = H'_j$ for some j in $\{1, 2, \ldots, l\}$. Then $Z'_j - k_i$ contains an element e_i since, by Lemma 5.9, $|Z_j| \geq 2$ and $|Z'_j| = |Z_j|$. Since K_i is a terminal vertex of $T'(N)$, the element e_i must be in B. As K_i is 3-connected having at least four elements, $K_i \backslash e_i$ is connected and $\{e_i, k_i\}$ is not a circuit of K_i. Hence (5.12.1) holds if $K_i = H'_j$.

Now suppose that $K_i \notin \{H'_1, H'_2, \ldots, H'_l\}$. From the construction of $T'(N)$, it follows that $E(K_i) \subseteq B \cup k_i$. Moreover, since N is cosimple, if K_i has a 2-cocircuit, then this 2-cocircuit is unique and must cointain k_i. Choose a circuit of K_i that contains k_i. Then, provided $r(K_i) \neq 1$, we can choose this circuit to have at least three elements. By a result of Oxley [9] (see also [10, Lemma 10.2.1]), this circuit must contain an element e_i such that $K_i \backslash e_i$ is connected. Moreover, $\{e_i, k_i\}$ is not a circuit unless $r(K_i) = 1$. We conclude that (5.12.1) holds in this case and therefore holds in general.

Now $N = K_1 \oplus_2 K_2$ or $N = N' \oplus_2 K_1 \oplus_2 K_2$ for some connected matroid N'. As $K_1 \backslash e_1$ and $K_2 \backslash e_2$ are connected, it follows that, in each case, $N \backslash \{e_1, e_2\}$ is connected because it is a 2-sum of connected matroids. Thus, by Lemma 5.11, $\{e_1, e_2\}$ is a 2-circuit of N. Therefore, $(E(K_1) - k_1) \cup (E(K_2) - k_2)$ is contained in a parallel class of N. But K_1 and K_2 were arbitrarily chosen terminal vertices of $T'(N)$. Hence $r(N) = 1$; a contradiction.

Finally, we note that it is an immediate consequence of the construction of $T'(N)$ that, when N is 3-connected having at least four elements, $H'_1 = H'_2 = \cdots = H'_l = N$. □

Recall that $(C_1 \cup C_2 \cup \cdots \cup C_l) - e = X$.

5.13**. Lemma.** *If C is a circuit of N and $C \notin \{C_1 - e, C_2 - e, \ldots, C_l - e\}$, then $C - X$ is a circuit of N/X. Moreover, $N|X$ is the direct sum of the l circuits $C_1 - e, C_2 - e, \ldots, C_l - e$. In particular, $l < m$.*

Proof. Clearly, we may assume that $l \neq 0$. As $C_1 - e$ is a circuit of N having at least 3 elements, it follows from the last lemma that N is 3-connected having at least four elements. Moreover, $H'_1 = H'_2 = \cdots = H'_l = N$.

First, we prove that $C \not\subseteq X$. Suppose that $C \subseteq X$. Hence $C \cap (C_i - e) \neq \emptyset$ for some i in $\{1, 2, \ldots, l\}$, say $i = 1$. As $C \neq C_1 - e$ by hypothesis, $(C_1 - e) - C$ and $C \cap C_1$ contain elements a and b, respectively. Since $N = H'_1$, there is a triad T^* of N containing $\{a, b\}$ whose third element, c say, is not in C_1. By the orthogonality of the circuit C and the triad T^*, it follows that $c \in C$. As $C \subseteq X$, it follows that $c \in C_i$ for some $i \in \{2, 3, \ldots, l\}$; a contradiction to orthogonality because $C_i \cap T^* = \{c\}$. Hence $C - X \neq \emptyset$.

To complete the proof of the first part of the lemma, it suffices to show that:

5.13.1. *N has no circuit C' such that $C' - X$ is a non-empty proper subset of $C - X$.*

For each i in $\{1, 2, \ldots, l\}$, choose g_i in $(C_i - e) - C$. Assume that (5.13.1) fails and let be a circuit C' of N that demonstrates this failure and minimizes $|C' \cap \{g_1, g_2, \ldots, g_l\}|$. Suppose that $C' \cap \{g_1, g_2, \ldots, g_l\}$ is non-empty and choose g_i in this set. Since $C' - X$ is non-empty, it contains an element c. By circuit elimination, N has a circuit C'' such that $c \in C'' \subseteq [C' \cup (C_i - e)] - g_i$. Clearly $C'' - X$ is a non-empty subset of $C' - X$. Moreover, $C'' \cap \{g_1, g_2, \ldots, g_l\} \subseteq (C' \cap \{g_1, g_2, \ldots, g_l\}) - g_i$; a contradiction to the choice of C'. Hence $C' \cap \{g_1, g_2, \ldots, g_l\} = \emptyset$.

If $C' \cap (C_j - e) \subseteq C \cap (C_j - e)$ for all j in $\{1, 2, \ldots, l\}$, then $C' \cap X \subseteq C \cap X$ and so $C' \subsetneqq C$; a contradiction. Thus $C' \cap (C_j - e) \not\subseteq C \cap (C_j - e)$ for some j in $\{1, 2, \ldots, l\}$. Now choose $h_j \in (C' - C) \cap (C_j - e)$. As $N = H'_j$, there is a triad T^*_j of N such that $T^*_j \cap (C_j - e) = \{g_j, h_j\}$ and $T^*_j - (C_j - e) = \{f_j\}$, say. As $g_j \notin C'$, it follows by the orthogonality of C' and T^*_j that $f_j \in C'$. Thus $f_j \in C$. But $C \cap \{g_j, h_j\} = \emptyset$ and this contradiction to orthogonality completes the proof of (5.13.1) and thereby proves the first part of the lemma. The second assertion of the lemma follows from the fact that, by the first part, $N|X$ has no circuits except $C_1 - e, C_2 - e, \ldots, C_l - e$.

To verify the last assertion, assume that $l = m$. Then $X = E(M/e)$ so $N|X = M/e$. By assumption, the matroid on the right-hand side is connected, whereas by the second part and the fact that $m \geq 2$, the matroid on the left-hand side is disconnected. This contradiction implies that $l < m$. □

5.14. **Lemma.** *N/X is cosimple and connected.*

Proof. Observe that N/X is cosimple because N is cosimple. Let a and b be elements of N/X. As N is connected, there is a circuit C of N such that $\{a, b\} \subseteq C$. By Lemma 5.13, $C - X$ is a circuit of N/X that contains both a and b. Thus N/X is connected. □

5.15. **Lemma.** *Suppose that $r(N/X) \geq 2$. If A is a 2-subset of $E(N) - X$, then $N/X \backslash A$ is disconnected.*

Proof. Since $r(N/X) \geq 2$, it follows by Lemma 5.12 that N is 3-connected having at least four elements, and $H'_1 = H'_2 = \cdots = H'_l = N$. Suppose that $N/X \backslash A$ is connected. Since N is simple, it does not have A as a circuit. As $A \subseteq E(N) - X \subseteq B$, it follows by Lemma 5.11, that $N \backslash A$ is disconnected. Since $N/X \backslash A$ is connected, we deduce that $N \backslash A$ has a component H such that $E(H) \subseteq X$. If $E(H) = \{h\}$, then, as N is connected, h must be a coloop of $N \backslash A$. Since N is also cosimple, $h \cup A$ is a triad of it. But this triad meets some $C_i - e$ in a single element, namely h. This contradiction to orthogonality implies that $|E(H)| \geq 2$, so $E(H)$ contains a circuit of N. Since every circuit of H is contained in X, by Lemma 5.13, the only circuits of H are members of $\{C_1 - e, C_2 - e, \ldots, C_l - e\}$. But the members of the last set are disjoint and H is connected, so $E(H) = C_i - e$ for some i, say $i = 1$. Thus $C_1 - e$ is the ground set of a component of $N \backslash A$. As $N = H_1'$, every 2-subset of $C_1 - e$ is contained in a triad of N whose third element is in Z_1'. Since every 2-subset of $C_1 - e$ is a cocircuit of $N \backslash A$, every element of $Z_1' - A$ is a coloop of $N \backslash A$ and so is a coloop of $N/X \backslash A$. This is a contradiction as the last matroid is connected having at least two elements. Hence $Z_1' \subseteq A$. By Lemma 5.9, $|Z_1| \geq 2$. As $|Z_1'| = |Z_1|$, it follows that $Z_1' = A$. We now apply Lemma 2.6 to N to deduce that (ii) of that lemma holds for N. Therefore A contains a minimal non-empty subset of $E(N)$ that does not meet any circuit in exactly one element. Hence A contains a cocircuit of N (see, for example, [10, Proposition 2.1.20]); a contradiction. □

5.16. **Lemma.** *$r(N/X) < 2$ or N/X is the dual of a 3-connected Sylvester matroid with at least four elements.*

Proof. Suppose that $r(N/X) \geq 2$. Since, by Lemma 5.14, N/X is cosimple and connected, it follows that this matroid has at least four elements. The lemma follows from Lemmas 5.15 and 3.1. □

In the next lemma, we construct a special cover of N by circuits. This cover will be used in the subsequent lemma to construct a cover of M.

5.17. **Lemma.** *Let X_1 be a circuit of N that is not in $\{C_1 - e, C_2 - e, \ldots, C_l - e\}$. Let g be an element of $X_1 - X$. Then N has circuits $X_2, X_3, \ldots, X_{n+1}$ such that:*

(i) *$X_1, X_2, \ldots, X_{n+1}$ are distinct;*
(ii) *$\{X_1, X_2, \ldots, X_{n+1}\}$ covers $E(N)$;*
(iii) *$\{X_1, X_2, \ldots, X_{n+1}\}$ double covers $E(N) - X$;*
(iv) *$X_i - (X_1 \cup X_2 \cup \cdots \cup X_{i-1} \cup X) \neq \emptyset$ for all i in $\{2, 3, \ldots, n\}$;*
(v) *$g \in X_1 - (X_2 \cup \cdots \cup X_n \cup X)$; and*
(vi) *$n = r^*(N/X)$.*

Proof. By Lemma 5.13, $X_1 - X$ is a circuit D_1' of N/X. By Lemmas 5.14, 5.15, and 5.16, N/X is a loop, a uniform matroid of rank one, or the dual of a 3-connected Sylvester matroid with at least four elements. It cannot be a loop by Lemma 5.10. By Lemma 3.2, there are circuits $X_2', X_3', \ldots, X_{n+1}'$ of N/X such that

(a) $X_1', X_2', \ldots, X_{n+1}'$ are distinct;
(b) $\{X_1', X_2', \ldots, X_{n+1}'\}$ double covers $E(N/X)$;
(c) $X_i' - (X_1' \cup X_2' \cup \cdots \cup X_{i-1}') \neq \emptyset$ for all i in $\{2, 3, \ldots, n\}$;
(d) $g \in X_1' - (X_2' \cup X_3' \cup \cdots \cup X_n')$; and
(e) $n = r^*(N/X)$.

For each i in $\{2, 3, \ldots, n + 1\}$, let X_i be a circuit of N such that $X_i' = X_i - X$. Choose $X_2, X_3, \ldots, X_{n+1}$ such that $|E(N) - (X_1 \cup X_2 \cup \cdots \cup X_{n+1})|$ is minimized. Then (i) and (iii)–(vi) follow from (a)–(e), respectively. Hence we need only show that (ii) holds. Assume it does not. Then X contains an element x that is not in $X_1 \cup X_2 \cup \cdots \cup X_{n+1}$. Without loss of generality, we may assume that $x \in C_1 - e$. Thus N has a circuit with at least three elements so $r(N) \geq 2$. Therefore, by Lemma 5.12, N is 3-connected and $N = H_1'$. Let $L_1^*, L_2^*, \ldots, L_k^*$ be the non-trivial lines of N^* that contain x. As $H_1' = N$, each 2-subset of $C_1 - e$ containing x is in a triad of N whose third element is not in $C_1 - e$. Thus each element of $C_1 - e$ is in some L_t^* for $1 \le t \le k$. Moreover, for all i in $\{1, 2, \ldots, k\}$, there is an element e_i in $L_i^* - C_1$ and, by orthogonality, e_i is unique. If $e_i \in X$, then $e_i \in C_j - e$ for some j in $\{2, 3, \ldots, l\}$, so $C_j - e$ meets L_i^* in a single element, contradicting orthogonality. Thus $e_i \notin X$. Now $L_1^* - x, L_2^* - x, \ldots, L_k^* - x$ are non-trivial series classes of $N \backslash x$. Moreover, $\{X_1, X_2, \ldots, X_{n+1}\}$ is a set of circuits of $N \backslash x$ that double covers $E(N) - X$. Thus each of $e_1, e_2, \ldots, e_r$ is in at least two of $X_1, X_2, \ldots, X_{n+1}$. As every element of $C_1 - \{e, x\}$ is in a series class of $N \backslash x$ with some e_i, it follows that $\{X_1, X_2, \ldots, X_{n+1}\}$ double covers $C_1 - \{e, x\}$. Thus, for some $i \ge 2$, say $i = 2$, the circuit X_i meets $C_1 - \{e, x\}$. By Lemma 5.13, $N|X$ is the direct sum of l circuits. Moreover, $X_2 - X$ is a circuit of N/X. An elementary rank calculation using these observations shows that $r(X_2 \cup X) = |X_2 \cup X| - (l + 1)$. Now, in $N|(X_2 \cup X)$, if we delete an element of each $(C_i - e) - X_2$ with $2 \le i \le l$, we do not alter the rank of the matroid. Thus the last matroid has corank 2 and has $C_1 - e$ and X_2 as intersecting circuits. Hence $N|[X_2 \cup (C_1 - e)]$ is connected, has corank 2, and has both $X_2 - (C_1 - e)$ and $(C_1 - e) - X_2$ as series classes. Therefore, $N|[X_2 \cup (C_1 - e)]$, and hence N, has a circuit X_2'' that contains both of these series classes and so contains x. Clearly $X_2'' - X = X_2 - X$. Now $X_2 - X_2'' \subseteq C_1 - e \subseteq X$ and $\{X_1, X_2, \ldots, X_{n+1}\}$ double covers $C_1 - \{e, x\}$, so $\{X_1, X_2'', X_3, \ldots, X_{n+1}\}$ double covers $E(N) - X$ and covers $X_1 \cup X_2 \cup \cdots \cup X_{n+1}$. Since $x \in X_2''$, it follows that $|E(N) - (X_1 \cup X_2'' \cup \cdots \cup X_{n+1})| < |E(N) - (X_1 \cup X_2 \cup \cdots \cup X_{n+1})|$ and so the choice of $\{X_1, X_2, \ldots, X_{n+1}\}$ is contradicted and the result follows. □

5.18. **Lemma.**

$$\theta_e(M) \le r^*(M) + s(M) = \begin{cases} r^*(N/X) + 1 & \textit{if } M \textit{ has no } F_7\textit{-minor using } e; \\ r^*(N/X) + 2 & \textit{otherwise.} \end{cases}$$

Proof. First we show that:

5.18.1. *For all i in $\{l + 1, l + 2, \ldots, m\}$, the element b_i is in a non-trivial series class of $M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\}$.*

Suppose that b_i is in a trivial series class of $M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\}$ for some i in $\{l + 1, l + 2, \ldots, m\}$. Then, since $N = \mathrm{co}(M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\})$, it follows that b_i is an element of N. By Lemma 5.12, either $r(N) = 1$, or N is 3-connected having at least four elements. Thus $N \backslash b_i$ is connected. However, by Lemma 5.5, $[M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\}] \backslash b_i$ is disconnected. Therefore, b_i is in a non-trivial series class of $M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\}$. This contradiction establishes (5.18.1).

From (5.18.1) and the construction of N, we deduce that M is obtained from N by:

(i) replacing each element f of $E(N) - X$ by a non-trivial series class S_f to give $M/e \backslash \{a_{l+1}, a_{l+2}, \ldots, a_m\}$, which we denote by N_1;
(ii) adding an element in parallel to each element of each S_f to give M/e;
(iii) coextending by e to give M.

By Lemma 5.12, $|E(N) - X| \ge 2$ so, by (5.18.1), there are at least two non-trivial series classes S_f in N_1. Let D be a circuit of N_1 that contains two such non-trivial series classes. Then D is a circuit of M/e and D contains at least four members of $\{b_{l+1}, b_{l+2}, \ldots, b_m\}$ including, say, $b_{m-3}, b_{m-2}, b_{m-1}, b_m$. Now D or $D \cup e$ is a circuit of M. In the former case, $(D - b_{m-3}) \cup \{a_{m-3}, e\}$ is a circuit of M and we interchange the labels on b_{m-3} and a_{m-3} so that $D \cup e$ is again a circuit of M. Then it is straightforward to check that $[M|(D \cup \{e, a_{m-2}, a_{m-1}, a_m\})]/(D -$

$\{b_{m-2}, b_{m-1}, b_m\}$) is a rank-3 spike. If this spike is isomorphic to F_7, then we leave $D \cup e$ unchanged. In the other case, by Lemma 4.1 and relabelling if necessary, we can choose D so that D is a circuit of N_1 containing $\{b_{m-2}, b_{m-1}, b_m\}$ and $D \cup e$ is a circuit of M that is indifferent with respect to $\{a_m, b_m\}$. In both cases, we take $g = b_m$. Since N is the cosimplification of N_1, we may assume that $g \in E(N)$. Let X_1 be the circuit $D \cap E(N)$ of N. We observe that, in particular, $g \in X_1$ and if M has no F_7-minor using e, then $D \cup e$ is indifferent with respect to $\{a_m, b_m\}$ in M.

Now let $X_2, X_3, \ldots, X_{n+1}$ be circuits of N such that (i)-(vi) of (5.17) hold. For each i in $\{1, 2, \ldots, n+1\}$, let $X'_i = \cup\{S_h : h \in X_i\}$. By (ii) and (iii), $\{X'_1, X'_2, \ldots, X'_{n+1}\}$ covers $E(N_1)$ and double covers $E(N_1) - X$. Let $X'_1 = D_1$. Next we construct circuits $D_2, D_3, \ldots, D_{n+1}$ of M/e as follows. Each series class S_h for $h \in E(N) - X$ is contained in at least two of $X'_1, X'_2, \ldots, X'_{n+1}$. Proceed through the list $X'_1, X'_2, \ldots, X'_{n+1}$ in order and, the second time each S_h is contained in some X'_i, replace each element of S_h in that X'_i by the element of M/e that is parallel to it. Clearly $\{D_1, D_2, \ldots, D_{n+1}\}$, the resulting set of circuits of M/e, covers $E(M/e)$. For each i in $\{1, 2, \ldots, n+1\}$, let D'_i be the circuit of M that is in $\{D_i, D_i \cup e\}$.

Next we describe an inductive construction of a subset $\{D''_1, D''_2, \ldots, D''_{n+1}\}$ of $\mathcal{C}_e(M)$ that covers $E(M) - a_m$. This set of circuits also covers M provided M has no F_7-minor using e.

Suppose that $D''_1, D''_2, \ldots, D''_{i-1}$ have been constructed in $\mathcal{C}_e(M)$ such that $\{D''_1 - e, D''_2 - e, \ldots, D''_{i-1} - e, D_i, D_{i+1}, \ldots, D_{n+1}\}$ is a set of circuits of M/e that covers $E(M/e)$. If $e \in D'_i$, then we take D''_i to be D'_i. Now assume that $e \notin D'_i$. The definition of D''_i in this case will depend on the value of i. We observe that, since $e \in D'_1$, we must have $i > 1$.

Suppose that $i \leq n$. We now choose an element h. Since $i > 1$, by (iii), we may choose h in $X_i - (X_{i-1} \cup X_{i-2} \cup \cdots \cup X_1 \cup X)$. Let P_h be the parallel class of M/e that meets S_h in $\{h\}$. Then $|P_h| = 2$ and P_h is a parallel class of $[M|(D'_i \cup P_h \cup e)]/e$ and so $D'_i - (P_h \cup e)$ is a series class of this matroid and hence is a series class of $M|(D'_i \cup P_h \cup e)$. Also $(P_h \cup e) - D'_i$ is a series class of this matroid. As $|D'_i \cap (P_h \cup e)| = 1$, it follows that $D'_i \triangle (P_h \cup e)$ is a circuit of M. We take this circuit to be D''_i. Note that the element belonging to $P_h \cap D'_i$, which is not in D''_i, may be in none of $D_{i+1}, D_{i+2}, \ldots, D_{n+1}$. As $i \leq n$, it follows by the choice of h and (ii) that, for some $j > i$, we have $h \in X_j$, so $P_h \cap D_j \neq \emptyset$. If $P_h \cap D_j = P_h \cap D'_i$, then $\{D''_1 - e, D''_2 - e, \ldots, D''_i - e, D_{i+1}, \ldots, D_{n+1}\}$ is a set of circuits of M/e that covers $E(M/e)$. If $P_h \cap D_j \neq P_h \cap D'_i$, then we replace D_j by $D_j \triangle P_h$, another circuit of M/e. Again, $\{D''_1 - e, D''_2 - e, \ldots, D''_i - e, D_{i+1}, \ldots, D_{n+1}\}$ is a set of circuits of M/e that covers $E(M/e)$.

Now suppose that $i = n + 1$. Then, since we are in the case when $e \notin D_i$, we have that $e \notin D_{n+1}$. By (v), $\{a_m, b_m\} \cap D_{n+1} \neq \emptyset$. In this case, D_{n+1} is a circuit of M and, therefore, so is $D_{n+1} \triangle \{a_m, b_m, e\}$. We take the last circuit to be D''_{n+1} and let $\mathcal{D}$ be $\{D''_1, D''_2, \ldots, D''_{n+1}\}$. Clearly $\mathcal{D}$ covers $E(M) - a_m$ and, if $a_m \in D''_{n+1}$, then $\mathcal{D}$ covers $E(M)$. We now assume that $a_m \notin D''_{n+1}$. Then $b_m \in D''_{n+1}$. If M has no F_7-minor using e, then $D \cup e$, which equals D''_1, is indifferent with respect to $\{a_m, b_m\}$ in M. Thus $(D''_1 - b_m) \cup a_m$ is a circuit of M. Replacing D''_1 by this circuit, we get that $\mathcal{D}$ covers $E(M)$.

We conclude that either $\mathcal{D}$ covers $E(M)$, or $\mathcal{D}$ covers $E(M) - a_m$ with the former holding if M has no F_7-minor using e. In the former case, $\theta_e(M) \leq n + 1 =$

$r^*(N/X)+1$. In the latter case, let D''_{n+2} be a circuit of M containing $\{a_m, e\}$, then $\mathcal{D} \cup \{D''_{n+2}\}$ covers $E(M)$ and so $\theta_e(M) \leq n+2 = r^*(N/X)+2$. □

We now complete the proof of the theorem. We know that $\nu_e(M) = m$. Moreover, by Lemma 5.13, $r^*(N|X) = l$, that is, $r(N^*.X) = l$, so $r(N^*) - r(N^*\backslash X) = l$. Hence

$$r^*(N) - r^*(N/X) = l. \tag{5}$$

Since $N = \mathrm{co}(M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\})$ and a_i is parallel to b_i in M/e for each i in $\{l+1, l+2, \ldots, m\}$, we have

$$\begin{aligned} r^*(N) &= r^*(M/e\backslash\{a_{l+1}, a_{l+2}, \ldots, a_m\}) \\ &= r^*(M/e) - (m-l) \\ &= r^*(M) - (m-l). \end{aligned}$$

Substituting into (5), we get $r^*(M) - (m-l) - r^*(N/X) = l$, so $r^*(M) - r^*(N/X) = m$. Thus

$$\nu_e(M) = r^*(M) - r^*(N/X).$$

On combining this with the last lemma, we get

$$\begin{aligned} \nu_e(M) + \theta_e(M) &\leq (r^*(M) - r^*(N/X)) + (r^*(N/X) + s(M)) \\ &= r^*(M) + s(M). \end{aligned}$$

This contradicts the fact that M is a counterexample to the theorem and thereby completes the proof. □

6. Consequences

In this section, we prove several consequences of the main theorem including the corollaries that were stated in the introduction.

Proof of Corollary 1.2. Let N^* be the matroid obtained from M^* by freely adding an element e. Note that N^* does not have an F_7^*-minor using e, because every minor of N^* has e as a free element and F_7^* has no free elements. Now $N^*\backslash e = M^*$, so $M = N/e$. In particular, $r^*(N) = r^*(M)$. We also have that

$$\mathcal{C}_e(N) = \{e \cup C : C \in \mathcal{C}(M)\}.$$

In particular, $\theta(M) = \theta_e(N)$ and $\nu(M) = \nu_e(N)$. The result follows from Theorem 1.1 because N is a connected matroid without an F_7-minor using e and N/e is connected. □

Observe that when M attains the bound in Corollary 1.2, the matroid N constructed in the last proof attains the bound in Theorem 1.1. Lemos [4] characterized the binary matroids that attain the bound in Corollary 1.2 but the characterization in general remains open. A characterization of the matroids attaining the bounds in Theorem 1.1 seems to be more difficult, since there are matroids attaining the bounds other than those described at the beginning of this paragraph. One such extremal example is given after Theorem 1.3 and we now describe some others. It is not difficult to check that, for all $q > 2$, the dual of the projective geometry $PG(r-1, q)$ attains the second bound in the theorem. Lest the reader suspect that binary spikes are the only matroids attaining the first bound, we now construct another class of matroids attaining that bound. Begin with $U_{1,n}$ for some odd $n \geq 3$ and replace each element by m elements in series for some even m. Then, in the resulting matroid, add an element in parallel to each element. Finally, construct the simple binary coextension of this matroid by the element e. Let the resulting matroid be M. Then $r^*(M) = r^*(M/e) = mn + n - 1$ and $\nu_e(M) = mn$. If $\{C_1, C_2, \dots, C_k\}$ is a minimum-sized subset of $\mathcal{C}_e(M)$ covering $E(M)$, then the first bound implies that $k \leq n+1$. Suppose that $k = n$. Then k is odd. Moreover, each $C_i - e$ is a circuit of M/e and so has at most $2m$ elements. Since $\{C_1, C_2, \dots, C_k\}$ covers $E(M)$, it follows that $|C_i - e| = 2m$ for all i and $C_i \cap C_j = \{e\}$ for all distinct i and j. Now think of M as being represented by a matrix D with $r(M)$ rows and let e correspond to the last natural basis vector. Then, in each $C_i - e$, there must be an odd number of ones in the last row. Because the sets $C_1 - e, C_2 - e, \dots, C_k - e$ are disjoint and k is odd, it follows that there are an odd number of ones altogether in the last row of D, not counting the one in the column corresponding to e. But the last row of D has exactly $mn + 1$ ones; a contradiction since m is even. We conclude that $k \neq n$ and so M does, indeed, attain the first bound in the theorem.

The next result extends the main theorem by allowing M/e to be disconnected.

6.1. **Corollary.** *Let e be an element of a connected matroid M where M is not a coloop, and let $E_1, E_2, \dots, E_n$ be the ground sets of the connected components of M/e. Suppose that $M|(E_i \cup e)$ has an F_7-minor using e if and only if i is in $\{1, 2, \dots, k\}$. Then*

$$\nu_e(M) + \theta_e(M) \leq r^*(M) + n + k.$$

Proof. The matroid M is the parallel connection of the n matroids $M|(E_1 \cup e)$, $M|(E_2 \cup e), \dots, M|(E_n \cup e)$, each of which is connected. Because each of ν_e, θ_e, and r^* is additive under the operation of parallel connection along the element e, it follows by Theorem 1.1 that $\nu_e(M) + \theta_e(M) \leq r^*(M) + 2k + (n-k)$, as required. □

Equality is attained in the bound in the last corollary by assuming that $M|(E_i \cup e)$ is an odd-rank binary spike with tip e if $1 \leq i \leq k$, and otherwise is a free spike with tip e.

To prove Corollaries 1.3 and 1.4, we construct the graph G' from G by adding an edge e joining u and v and then apply Theorem 1.1 to, respectively, the cycle and bond matroids of G'.

Next we describe a graph G_n for which equality is attained in Corollary 1.3. Let G_n be the graph obtained from a path P_n of length n by adding two non-adjacent vertices u and v both adjacent to every vertex in P_n. In this case, $\nu_{uv}(G_n) = \theta_{uv}(G_n) = n+1$ and $|E(G_n)| - |V(G_n)| + 3 = (3n+2) - (n+3) + 3 = 2n+2$. Let G'_n be the graph that is obtained from G_n by adding a new edge e joining u and v. Then, from above, G'_n attains the bound in Theorem 1.1. However, it does not attain the bound in Corollary 1.2, since $\nu(G'_n) = n$ and $\theta(G'_n) = \lceil \frac{n+3}{2} \rceil$.

The next result is the natural extension of Corollary 1.3 to the case when $G - \{u, v\}$ need not be connected. It is not difficult to give examples that attain equality in this bound.

6.2. **Corollary.** *If u and v are distinct non-adjacent vertices of a 2-connected graph G and $G - \{u, v\}$ has k components, then*

$$\nu_{uv}(G) + \theta_{uv}(G) \leq |E(G)| - |V(G)| + k + 2.$$

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References

[1] Akkari, S., A minimal connectedness result for matroids, *Discrete Math.* **103** (1992), 221–232.
[2] Akkari, S. and Oxley, J., Some extremal connectivity results for matroids, *J. Combin. Theory Ser. B* **52** (1991), 301–320.
[3] Cunningham, W.H., *A combinatorial decomposition theory*, Ph. D. thesis, University of Waterloo, 1973.
[4] Lemos, M., On Seymour's question about packing and covering with matroid circuits, *Ars Combin.* **20** (1985), 27–34.
[5] Lemos, M. and Oxley, J., A sharp bound on the size of a connected matroid, *Trans. Amer. Math. Soc.* **353** (2001), 4039–4056.
[6] Minty, G.J., On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network programming, *J. Math. Mech.* **15** (1966), 485–520.
[7] Murty, U.S.R., Matroids with the Sylvester property, *Aequationes Math.* **4** (1970), 44–50.
[8] Oxley, J.G., Cocircuit coverings and packings for binary matroids, *Math. Proc. Camb. Phil. Soc.* **83** (1978), 347–351.
[9] Oxley, J.G., On connectivity in matroids and graphs, *Trans. Amer. Math. Soc.* **265** (1981), 47–58.
[10] Oxley, J.G., *Matroid Theory,* Oxford University Press, New York, 1992.
[11] Seymour, P.D., Packing and covering with matroid circuits, *J. Combin. Theory Ser. B* **28** (1980), 237–242.
[12] Seymour, P.D., On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.
[13] Tutte, W.T., Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

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