ON REMOVABLE CYCLES THROUGH EVERY EDGE

MANOEL LEMOS AND JAMES OXLEY

Abstract. Mader and Jackson independently proved that every 2-connected simple graph G with minimum degree at least four has a removable cycle, that is, a cycle C such that $G\backslash E(C)$ is 2-connected. This paper considers the problem of determining when every edge of a 2-connected graph G, simple or not, can be guaranteed to lie in some removable cycle. The main result establishes that if every deletion of two edges from G remains 2-connected, then, not only is every edge in a removable cycle but, for every two edges, there are edge-disjoint removable cycles such that each contains one of the distinguished edges.

1. Introduction

If G is a 2-connected simple graph in which the minimum degree $\delta(G)$ is at least four, then, as a special case of a result for k-connected graphs, Mader [7] showed that G has a cycle C such that $G\backslash E(C)$, the graph obtained by deleting the edges of C, is 2-connected. Independently, Jackson [5] proved that C can be chosen to avoid a nominated edge e and to have at least $\delta(G) - 1$ edges. We call a cycle D in a 2-connected graph H *removable* if $H\backslash E(D)$ is 2-connected. Lemos and Oxley [6] extended Jackson's theorem and obtained, as a corollary, that every simple 2-connected graph with $\delta(G) \geq 5$ has two removable edge-disjoint cycles each with at least $\delta(G) + 1$ edges.

Although every *simple* 2-connected graph G with minimum degree at least four has a removable cycle, the example shown in Figure 1(a), which was obtained independently by Robertson (in [5]) and Jackson [5], shows that the requirement that G be simple cannot be dropped. However, Fleischner and Jackson [2] proved that this requirement could be replaced by the condition that G is planar. Their result was extended by Goddyn, van den Heuvel, and McGuinness [3] who proved a strengthening of a conjecture of Jackson [5] by proving the following.

1.1. **Theorem.** *Let G be a 2-connected graph with minimum degree at least four. If G has no minor isomorphic to the Petersen graph, then G has two edge-disjoint removable cycles.*

In this paper, we consider when we can guarantee that every edge of a 2-connected graph G is in some removable cycle. When G is simple, the requirement that $\delta(G) \geq 4$ is not sufficient to ensure the existence of such a family of removable cycles for if G has an edge cut of size less than four, none of the edges in this cut will be in a removable cycle. But even if we strengthen the minimum-degree requirement to insist that every edge-cut has size at least four, we shall not be guaranteed that every edge is in a removable cycle. For example, every edge cut in the graph G in Figure 1(b) has size at least four, but the only edges that are in

1991 *Mathematics Subject Classification.* 05C40.

removable cycles are those edges that are in 2-cycles. Although G is not simple, a straightforward modification produces a simple graph G' that has no removable cycle meeting either the outer pentagon or the inner pentagram. Specifically, we replace each pair $\{x, y\}$ of parallel edges of G by a copy of K_5 whose vertex set meets $V(G)$ in the endpoints of x.

Figure 1.

In the following theorem, the main result of the paper, the hypothesis is strong enough to ensure not only that every edge is in a removable cycle but also that, for every two edges, there is a pair of edge-disjoint removable cycles one containing each of the distinguished edges. We observe here that, for the graph G in Figure 2, every edge is in a removable cycle, but the only removable cycle containing e also contains f. This example can be modified as above to produce a simple graph satisfying the same assertions.

Figure 2.

1.2**. Theorem.** *Let G be a 2-connected graph. Suppose that $G\backslash X$ is a 2-connected graph for every 2-subset X of $E(G)$, and let x and y be distinct edges of G. Then G has edge-disjoint cycles C_x and C_y containing x and y, respectively, such that both $G\backslash E(C_x)$ and $G\backslash E(C_y)$ are 2-connected.*

For comparison with Theorem 1.1, we note the following immediate consequence of Theorem 1.2.

1.3**. Corollary.** *Let G be a 2-connected graph such that $G\backslash X$ is 2-connected for every 2-subset X of $E(G)$. Then G has two edge-disjoint removable cycles.*

The proof of Theorem 1.2 will be given in the next section. Section 3 describes a counterexample to a natural extension of Theorem 1.2 to cographic matroids. The graph and matroid terminology that is used here will follow Bondy and Murty [1] and Oxley [8], respectively. A *block* is a connected graph with no cut-vertices. A graph with at least three vertices is a block if and only if it is 2-connected and loopless. For a path λ that is not a cycle, if u and v are vertices of λ, then $\lambda[u, v]$ denotes the subpath of λ having u and v as its ends.

2. Proof of the main result

The following result is the main tool in the proof of Theorem 1.2. Once it is proved, it will not be difficult to complete the proof of the theorem.

2.1. **Proposition.** *Suppose that G and H are blocks such that*

(i) *H is a subgraph of G; and*
(ii) *$G\backslash X$ is a block for every $X \subseteq E(G) - E(H)$ such that $|X| \leq 2$.*

Then, for each e in $E(G) - E(H)$, either

(iii) *there is no cycle of G that contains e and is edge-disjoint from H; or*
(iv) *there is a cycle C of G that contains e such that C is edge-disjoint from H and $G\backslash E(C)$ is a block.*

Proof. Suppose that the proposition fails and choose a counterexample (G, H, e) for which $(|E(G)|, -|E(H)|)$ is lexicographically minimal. Then there is a cycle C of G that contains e and is edge-disjoint from H. Moreover, $G\backslash E(C)$ is not a block. Let B be the block of $G\backslash E(C)$ that contains $E(H)$. By the choice of (G, H, e), we have that $B = H$. Choose C such that the connected component of $G\backslash E(C)$ that contains H as a block has the maximum number of edges.

In the rest of the proof, when λ is a subpath of C, we define its *complement* λ^c to be the other subpath of C having the same terminal vertices as λ. In particular, λ_i^c and λ'^c will denote the complements of λ_i and λ', respectively.

2.2. **Lemma.** *$G\backslash E(C)$ is connected.*

Proof. Let $K_0, K_1, K_2, \dots, K_n$ be the connected components of $G\backslash E(C)$ and let H be a block of K_0. We want to show that $n = 0$. Assume that $n \geq 1$. For each i in $\{1, 2, \dots, n\}$, let λ_i be the shortest subpath of C containing e such that its terminal vertices are in $V(K_i)$. If $\lambda_i = C$, then λ_i and K_i have a unique common vertex, say v_i. Since G is a block, v_i is not a cut-vertex of G so v_i is an isolated vertex of $G\backslash E(C)$. But then $G\backslash X$ is not a block when X are the two edges of G incident with v_i. This contradiction to (ii) implies that $\lambda_i \neq C$, so λ_i has different terminal vertices. Let λ'_i be a path in K_i joining the terminal vertices of λ_i. Note that $\lambda_i \cup \lambda'_i$ is a cycle C' of G that contains e and is edge-disjoint from H. Therefore

$$V(\lambda_i^c) \cap V(K_0) = \emptyset, \tag{1}$$

otherwise the connected component of $G\backslash E(C')$ that contains H contains $E(K_0) \cup E(\lambda_i^c)$, which is contrary to the choice of C.

Now, we shall define an auxiliary graph K such that $V(K) = \{1, 2, \dots, n\}$ and ij is an edge of K if and only if $i \neq j$ and $E(\lambda_i^c) \cap E(\lambda_j^c) \neq \emptyset$. Let K' be a connected component of K. Let λ be the union of all the paths λ_i^c for which i in $V(K')$. Note that λ is a subpath of C that does not contain e. We set

$$V' = \bigcup_{i \in V(K')} V(K_i).$$

Note that $V' \neq V(G)$ because $V' \cap V(K_0) = \emptyset$ by the definition of K'. Let X be the set of edges of G that join a vertex of V' to a vertex of $V(G) - V'$. By our construction, $X \subseteq E(C)$ and, since G is a block, $|X| \geq 2$. We shall prove that X is the set of terminal edges of λ^c.

First we show that $X \cap E(\lambda) = \emptyset$. Suppose that $f \in X \cap E(\lambda)$. By definition of λ, the edge $f \in E(\lambda_i^c)$ for some $i \in V(K')$. Let $f = xy$ where $x \in V'$ and $y \in V(G) - V'$. By (1), $y \notin V(K_0)$. Thus $y \in V(K_j)$ for some j in $\{1, 2, \dots, n\}$. Hence $j \notin V(K')$. Let g be the edge of $C - \{f\}$ incident to y. By definition of λ_j, we have that $E(\lambda_j^c) \cap \{f, g\} \neq \emptyset$. The definition of K now implies that $E(\lambda_i^c) \cap E(\lambda_j^c) = \emptyset$. Hence $g \in E(\lambda_j^c) - E(\lambda_i^c)$ and $f \in E(\lambda_i^c) - E(\lambda_j^c)$. Thus f is a terminal edge and y a terminal vertex of λ_i^c and hence of λ_i. Therefore $y \in V(K_i)$. But $y \in V(K_j)$ so $j = i$. This contradiction implies that $X \cap E(\lambda) = \emptyset$.

We now know that $X \subseteq E(\lambda^c)$. Let $f \in X$. Then f is incident with some vertex x of V' and $x \in V(K_i)$, say. Since $f \in E(\lambda^c)$, we deduce that $f \in E(\lambda_i)$. Now no internal vertex of λ_i is in K_i, so x is a terminal vertex of λ_i. Thus the edge g of $C - f$ that is incident with x is in $E(\lambda_i^c)$ and so is in $E(\lambda)$. Hence f must be a terminal edge of λ^c. Thus $|X| \leq 2$. But $|X| \geq 2$, so $|X| = 2$ and we have a contradiction since $G \backslash X$ is disconnected. □

Let $w_1, w_2, \dots, w_n$ be the cut vertices of $G \backslash E(C)$ belonging to H. For each k in $\{1, 2, \dots, n\}$, let G_k be the connected component of $G \backslash (E(H) \cup E(C))$ that contains w_k. We set $V_k = V(G_k)$.

2.3. **Lemma.** *Suppose that $\{u, v\} \subseteq V_k \cap V(C)$ and that ζ is a uv-path of G_k. Let K be the connected component of $G_k \backslash E(\zeta)$ such that $w_k \in V(K)$. If λ is the path in C that joins u and v and does not contain e, then $V(\lambda) \subseteq V_k$ or $V(\lambda) \cap V(K) = \emptyset$.*

Proof. Observe that the union D of λ^c and ζ is an Eulerian graph since λ^c and ζ are edge-disjoint paths that have the same terminal vertices. Thus D is a union of cycles of G. One of these cycles, say C'', contains e. When $V(\lambda) \cap V(K) = \emptyset$, the lemma holds. Therefore we may suppose that there is a vertex x in $V(\lambda) \cap V(K)$. Let β be a $w_k x$-path in K. We choose x so that β has minimum length. Hence $V(\beta) \cap V(\lambda) = \{x\}$. In particular, the union of β with either of the paths $\lambda[u, x]$ or $\lambda[x, v]$ is also a path.

Now suppose that $V(\lambda) \not\subseteq V_k$ and choose a vertex w of $V(\lambda) - V_k$ such that its distance to x in the path λ is a minimum. Then w is a vertex of $\lambda[u, x]$ or $\lambda[x, v]$. Without loss of generality, we may assume the latter. It follows, by the choice of w, that $V(\lambda[x, w]) - V_k = \{w\}$. Consider the $w_k w$-path τ that is the union of β and $\lambda[x, w]$.

Suppose $w \in V(H)$. As τ is edge-disjoint from H and C'', it follows that $E(H) \cup E(\tau)$ is contained in some block H' of $G \backslash E(C'')$. This contradicts the choice of (G, H, e) since H' has more edges than H. Therefore $w \notin V(H)$.

We may now suppose that $w \in V_l$ for some $l \neq k$. Let ζ_l be a ww_l-path in G_l. Observe that $V(\zeta_l) \cap V(\tau) = \{w\}$ because $V(\beta) \subseteq V(K) \subseteq V_k$ and $V(\lambda[x, w]) - V_k = \{w\}$. Thus the union of τ and ζ_l is a $w_k w_l$-path ε that is edge-disjoint from H and C''. Therefore $E(H) \cup E(\varepsilon)$ is contained in a block H'' of $G \backslash E(C'')$. This contradicts the choice of (G, H, e) since H'' has more edges than H. We deduce that w does not exist, so $V(\lambda) \subseteq V_k$. □

Let λ_k be the longest path in C that has both ends in V_k and does not contain e. Next we show that

2.4. $V(\lambda_k) - V_k \neq \emptyset$.

Let X be the set of edges of G that join a vertex of $V_k - \{w_k\}$ to a vertex of $V(G) - V_k$. Suppose that $V(\lambda_k) \subseteq V_k$. Then $X \subseteq E(\lambda_k^c)$. But the choice of λ_k means that only the terminal vertices of λ_k^c are in V_k. Hence each edge of X is terminal in λ_k^c. Thus $|X| \leq 2$. Since $G \backslash X$ has w_k as a cut-vertex, it is not difficult to obtain a contradiction. We conclude that (2.4) holds.

Let a_k and b_k be the ends of λ_k. Traverse λ_k beginning at a_k and let v_k be the first vertex that is not in V_k. Let λ'_k be the shortest subpath of λ_k that contains v_k and has both ends in V_k. Suppose that λ'_k joins the vertices a'_k and b'_k and that $a_k, a'_k, v_k, b'_k, b_k$ appear in this order in λ_k. Note that a'_k, v_k, and b'_k are distinct, but a_k may equal a'_k and b_k may equal b'_k. We define

$$A_k = V(\lambda_k[a_k, a'_k]) \cap V_k \text{ and } B_k = V(\lambda_k[b'_k, b_k]) \cap V_k.$$

By the choice of v_k, we have that $A_k = V(\lambda_k[a_k, a'_k])$.

We now define a simple auxiliary graph M with vertex set A_k. If a and a' are distinct members of A_k, then $aa' \in E(M)$ if and only if there is no b in B_k such that G_k has two edge-disjoint paths joining $\{a, a'\}$ and $\{b, w_k\}$.

When f is an isthmus of G_k, we denote by $G_k(f)$ the connected component of $G_k \backslash f$ whose vertex set does not contain w_k.

2.5. Lemma. *Let $\{a, a'\}$ be a subset of A_k such that if $a \neq a'$, then $aa' \in E(M)$. Then*

- (i) *G_k has an isthmus that separates $\{a, a'\}$ from w_k.*
- (ii) *If f is an isthmus of G_k that separates $\{a, a'\}$ from w_k such that $|V(G_k(f))|$ is a minimum, then*
  - (a) *$V(G_k(f)) \cap B_k = \emptyset$; and*
  - (b) *if N is the connected component of M such that $\{a, a'\} \subseteq V(N)$, then $V(G_k(f)) \cap A_k \subseteq V(N)$.*

Proof. We shall prove (i) and (ii)(a) simultaneously. To do this, we take $J = G_k$ and $w = w_k$ when there is no isthmus separating $\{a, a'\}$ from w_k; and, when the hypothesis of (ii)(a) holds, we take J to be $G_k(f)$ and w to be the unique vertex in $V(G_k(f)) \cap V(\{f\})$.

Next we observe that there is no isthmus in J that separates $\{a, a'\}$ from w. This is true by definition when $J = G_k$. Moreover, if $J = G_k(f)$ and g is such an isthmus, then g is also an isthmus in G_k that separates $\{a, a'\}$ from w_k. Since $G_k(g)$ is clearly a subgraph of $G_k(f) - w$, the choice of f is contradicted. Hence g does not exist.

By the result of the last paragraph, Menger's Theorem implies that J has an aw-path α and an $a'w$-path α' such that $E(\alpha) \cap E(\alpha') = \emptyset$.

Next we show that

2.5.1. $V(J) \cap B_k = \emptyset$.

If $b \in V(J) \cap B_k$, then choose a path β of minimum length joining b to some vertex of α or α', say α'. Let b' be the common vertex of α' and β and let γ be the union of $\alpha'[a', b']$ and β. Then α and γ are edge-disjoint paths of J joining a to w and a' to b, respectively. The path α can be completed to an aw_k-path ε of G_k that is edge-disjoint from γ by taking its union with a ww_k-path in $G_k \backslash E(J)$. Hence $aa' \notin E(M)$ and so $a = a'$. Now apply Lemma 2.3 taking (u, v, ζ) to be (a', b, γ). The path α implies that $a' \in V(\lambda_k[a', b]) \cap V(K)$, where K is the connected component of $G_k \backslash E(\gamma)$ that contains w_k. Thus $V(\lambda_k[a', b]) \cap V(K) \neq \emptyset$. Moreover, the definitions of A_k and B_k imply that $V(\lambda_k[a', b]) \not\subseteq V_k$. Thus Lemma 2.3 is contradicted. Therefore b does not exist and so (2.5.1) holds.

As $\emptyset \neq B_k \subseteq V(G_k)$, it follows that $J \neq G_k$. Thus (i) holds. Moreover, $J = G_k(f)$ and (ii)(a) holds by (2.5.1). Observe that (ii)(b) follows because any path of G_k joining a vertex of $V(G_k(f))$ to a vertex of $\{w_k\} \cup B_k$ must contain f as an edge by (ii)(a). It follows by the definition of M that any two distinct elements of $A_k \cap V(G_k(f))$ are adjacent in M. □

2.6. **Lemma.** *If N is a connected component of M, then there is an isthmus f of G_k such that*

$$V(G_k(f)) \cap A_k = V(N) \text{ and } V(G_k(f)) \cap B_k = \emptyset.$$

Proof. We shall prove the following assertion by induction on $|V|$.

2.6.1. *If $V \subseteq V(N)$ and $N[V]$ is connected, then there is an isthmus f_V of G_k such*

$$V \subseteq V(G_k(f_V)) \cap A_k \subseteq V(N) \text{ and } V(G_k(f_V)) \cap B_k = \emptyset.$$

The lemma follows by taking $V = V(N)$ in (2.6.1).

Suppose that $|V| = 1$, say $V = \{a\}$. Now taking $a' = a$ in Lemma 2.5(i), we get that there is an isthmus f of G_k that separates $\{a, a'\}$ from w_k. Choose f so that $|V(G_k(f))|$ is a minimum. Then (2.6.1) follows from Lemma 2.5(ii) by taking $f_{\{a\}} = f$. Now assume that (2.6.1) holds for $|V| < n$ and let $|V| = n \geq 2$. Since $N[V]$ is connected, there is an a in V such that $N[V - \{a\}]$ is connected. By induction, the isthmus $f_{V-\{a\}}$ exists. If $a \in V(G_k(f_{V-\{a\}}))$, then we take $f_V = f_{V-\{a\}}$ and the result follows. We may now suppose that $a \notin V(G_k(f_{V-\{a\}}))$. Choose a' in $V - \{a\}$ such that $aa' \in E(N)$. By Lemma 2.5(i), there is an isthmus f of G_k that separates $\{a, a'\}$ from w_k because $aa' \in E(M)$. Choose such an f for which $|V(G_k(f))|$ is a minimum.

Note that, since $a' \in V - \{a\} \subseteq V(G_k(f_{V-\{a\}}))$, any $a'w_k$-path in G_k contains $f_{V-\{a\}}$ and f as edges. Moreover, since $f_{V-\{a\}}$ separates a from a' in G_k, it follows that, in each $a'w_k$-path in G_k, the edge $f_{V-\{a\}}$ is closer to a' than f is. Thus $G_k(f)$ has $G_k(f_{V-\{a\}})$ as a subgraph. Hence $V \subseteq V(G_k(f))$. The remaining parts of (2.6.1) follow from Lemma 2.5(ii) by taking $f_V = f$. □

Now let N be a connected component of M. By the previous lemma, there is an isthmus f of G_k such that

$$V(G_k(f)) \cap A_k = V(N) \text{ and } V(G_k(f)) \cap B_k = \emptyset. \tag{2}$$

Let $\{x, y\}$ be the subset of $V(G_k(f)) \cap A_k$ such that a_k, x, y, a'_k appear in this order in λ_k and all the other vertices of $V(G_k(f)) \cap A_k$ lie between x and y on λ_k. Next we prove the following:

2.6.2. *Every vertex of $\lambda_k[x, y]$ is in $V(G_k(f)) \cap A_k$.*

Suppose that there is a vertex z of G such that $z \in V(\lambda_k[x,y])$ but $z \notin V(G_k(f))$. Since $V(\lambda_k[a_k,a'_k]) = A_k$ and $\{x,y\} \subseteq V(\lambda_k[a_k,a'_k])$, it follows that $z \in A_k$. By the definition of M, the vertices x and z are in different components of M. Thus there are edge-disjoint paths, ζ and η, joining $\{x,z\}$ to $\{w_k,b\}$ for some $b \in B_k$. First suppose that ζ and η join x to b and z to w_k, respectively. We know that v_k is in the path in C that joins x and b and does not contain e. Since $v_k \notin V_k$, Lemma 2.3 implies that z and w_k are in different connected components of $G_k \backslash E(\zeta)$. This contradiction to the fact that η joins z to w_k implies that we may assume that ζ joins z to b and η joins x to w_k.

Let $V(\{f\}) \cap V(G_k(f)) = \{w\}$. Since $x \in V(G_k(f))$, the isthmus f of G_k separates w_k from x. Thus f is an edge of η. Hence w is a vertex of η, and f is not an edge of ζ. As $y \in V(G_k(f))$, there is a yw-path γ of $G_k(f)$. The union ε of γ and $\eta[w,w_k]$ is a yw_k-path of G_k. Moreover, ε is edge-disjoint from ζ because ζ and η are edge-disjoint, and $V(\gamma) \subseteq V(G_k(f))$ while $V(\zeta)$ avoids $V(G_k(f))$ since f is not an edge of ζ and $z \notin V(G_k(f))$. Now apply Lemma 2.3, letting $(u,v,\zeta) = (z,b,\zeta)$. Since $\lambda_k[z,b]$ contains v_k, it is clear that $V(\lambda_k[z,b]) \not\subseteq V_k$. Therefore $V(\lambda_k[z,b]) \cap V(K) = \emptyset$, where K is the connected component of $G_k \backslash E(\zeta)$ that contains w_k. But $y \in V(\lambda_k[z,b])$, and the path ε joins y to w_k so $y \in V(K)$. Therefore $y \in V(\lambda_k[z,b]) \cap V(K)$. This contradiction implies that (2.6.2) holds.

We now complete the proof of Proposition 2.1. From (2.6.2), (2), and the choice of x and y, we have that $V(G_k(f)) \cap V(C) = V(\lambda_k[x,y])$. Thus there are just two edges of C that join a vertex of $G_k(f)$ to a vertex outside $G_k(f)$. Delete these edges. The resulting graph has f as an isthmus and has at least three vertices. Therefore it is not a block. This contradiction completes the proof of Proposition 2.1. □

Proof of Theorem 1.2. The graph obtained from G by deleting all the loops is a block. If we can prove the theorem in the case that G is a block, then it will follow in general because each loop is itself a removable cycle. Thus assume that G is a block. First, observe that $G\backslash e$ is a block for every edge e of G. Let x and y be distinct edges of G. Since $G\backslash y$ is a block, G has a cycle C that contains x but not y. Now applying Proposition 2.1 taking H to be the block with edge set $\{y\}$, we obtain a cycle C_x of G such that $G\backslash E(C_x)$ is a block, $x \in E(C_x)$, and $y \notin E(C_x)$. Since $G\backslash E(C_x)$ is a block, there is a cycle C' of this graph such that $y \in E(C')$. Applying Proposition 2.1 again, this time taking H to be the cycle C_x, we obtain a cycle C_y containing y such that $E(C_x) \cap E(C_y) = \emptyset$ and $G\backslash E(C_y)$ is a block. □

3. Matroid (non)-extensions

Mader's theorem [7], with which we began this paper, implies that every 2-connected simple graph in which every bond has at least four edges has a removable cycle. In particular:

3.1. *Every 2-connected simple graphic matroid M in which every cocircuit has at least four elements has a removable circuit, that is, a circuit C such that $M\backslash C$ is 2-connected.*

The matroid $U_{3,7}$ shows that one cannot delete the word "graphic" from the last result. Oxley [8, Problem 14.4.8] asked whether one could replace "graphic" with "binary" in (3.1) and, specializing this, Goddyn, van den Heuvel, and McGuinness (in an earlier version of [3]) conjectured that (3.1) holds when we replace "graphic" with "cographic". Lemos and Oxley [6] gave a counterexample to this conjecture.

We describe this example next for it provides a counterexample to a natural extension of Theorem 1.2 to cographic matroids. We begin with a copy of $K_{5,5}$ having vertex classes V_1 and V_2. Then, for every 3-element set Z that is a subset of V_1 or of V_2, adjoin two new degree-3 vertices v_Z and w_Z making each of them adjacent to all the members of Z. The resulting graph G has 50 vertices and 145 edges. Evidently G is bipartite and has girth four. Moreover, G is 3-connected and 3-edge-connected. It is shown in [6, Proposition 2.1] that G has no bond C^* for which the contraction G/C^* is 2-connected. Thus $M^*(G)$ has no removable circuit. However, $M^*(G)\backslash X$ is 2-connected for all 2-subsets X of $E(G)$ because every single-edge contraction of G is 3-connected. Thus, by contrast with the graphic case dealt with in Theorem 1.2, in the cographic case, the condition that $M\backslash X$ is 2-connected for all 2-subsets X of $E(M)$ is not strong enough to guarantee the existence of even one removable circuit.

Although Mader's theorem does not extend to binary or even cographic matroids, by strengthening the hypothesis slightly, Goddyn and Jackson [4] were able to extend the theorem to a class of binary matroids that includes cographic matroids. In particular, they proved that if e is an element of a connected binary matroid M such that M does not have both the Fano matroid and its dual as minors and $|X| \geq 5$ for all cocircuits X of M not containing e, then M has a removable circuit C that avoids e such that $r(M\backslash C) = r(M)$.

Acknowledgements

The first author was partially supported by CNPq (Grant No. 522910/96-3), ProNEx/CNPq (Grant No. 664107/97-4), and CAPES (Grant No. AEX1307/00-2). The second author was partially supported by the National Security Agency (Grant Nos. MDA904-99-1-0030 and MDA904-01-1-0026).

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1976.

[2] H. Fleischner and B. Jackson, Removable cycles in planar graphs, *J. London Math. Soc. (2)* **31** (1985), 193–199.

[3] L.A. Goddyn, J. van den Heuvel, and S. McGuinness, Removable circuits in multigraphs, *J. Combin. Theory Ser. B* **71** (1997), 130–143.

[4] L.A. Goddyn and B. Jackson, Removable circuits inbinary matroids, *Combin. Probab. Comput.* **8** (1999), 539–545.

[5] B. Jackson, Removable cycles in 2-connected graphs of minimum degree at least four, *J. London Math. Soc. (2)*, **21** (1980), 385–392.

[6] M. Lemos and J. Oxley, On removable circuits in graphs and matroids, *J. Graph Theory* **30** (1999), 51–66.

[7] W. Mader, Kreuzungfreie a, b-Wege in endlichen Graphe, *Abh. Math. Sem. Univ. Hamburg* **42** (1974), 187–204.

[8] J.G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Pernambuco 50740-540, Brazil

E-mail address: manoel@dmat.ufpe.br

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803–4918, USA

E-mail address: oxley@math.lsu.edu