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On a matroid generalization of graph connectivity

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This paper relates the concept of n-connection for graphs to Tutte's theory of n-connection for matroids (12). In particular, we show how Tutte's definition may be modified to give a matroid concept directly generalizing the graph-theoretic notion of n-connection.

The terminology used here for matroids and graphs will in general follow Welsh (14) and Bondy and Murty (1) respectively. The ground set of a matroid M will be denoted by $E(M)$ and, if $T \subseteq E(M)$, we denote the rank of T by rkT.

If G is a graph and X is a subset of the edge-set $E(G)$ of G, then $G[X]$ will denote the subgraph of G *induced* by X. That is, $G[X]$ has X as its edge-set and $V(X)$, the set of endpoints of edges in X, as its vertex-set. If $U \subseteq V(G)$, then $G \backslash U$ denotes the subgraph of G obtained by deleting U and all those edges incident with a vertex of U. The *connectivity* $\kappa(G)$ of G is the least number of vertices that must be deleted from G in order to leave a disconnected or single-vertex graph. We shall say that G is *n-connected* if n is a non-negative integer not exceeding $\kappa(G)$. Thus a graph with at least two vertices is 1-connected if and only if it is connected.

The concept of n-connection for matroids was introduced by Tutte (12) in order to extend certain graph connectivity results to matroids. If M is a matroid and $\{S, T\}$ is a partition of its ground set $E(M)$, then let

$$\xi(M; S, T) = rkS + rkT - rkM + 1.$$

For a positive integer m, we say that M is *m-separated* if there is a partition $\{S, T\}$ of $E(M)$ such that $|S|, |T| \geqslant m$ and $\xi(M; S, T) = m$.

If there is a least positive integer j such that M is j-separated, it is called the *connectivity* $\lambda(M)$ of M. If there is no such integer, we say that $\lambda(M) = \infty$. Richardson ((8), lemma 2) and Inukai and Weinberg ((3), theorem 1) have independently shown that all matroids having infinite connectivity are uniform. Indeed ((3), p. 312),

$$\lambda(U_{r,m}) = \begin{cases} r+1, & \text{if } m \geqslant 2r+2; \\ m-r+1, & \text{if } m \leqslant 2r-2; \\ \infty & \text{otherwise.} \end{cases}$$

A matroid M will be said to be *n-connected* if n is a positive integer not exceeding $\lambda(M)$.

The above notion of matroid connectivity is a direct generalization of an alternative definition of graph connectivity given by Tutte ((11); (12), 3·5). The next result follows immediately from combining this fact with a result of Graver and Watkins ((2), proposition VIE7). The *girth* $g(G)$ of a graph G is defined to be ∞ if G has no circuits, and $\min\{|C| : C \text{ is a circuit of } G\}$ otherwise.

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(1) PROPOSITION. *If G is a graph without isolated vertices and G has at least three vertices and at least four edges, then*

$$\lambda(M(G)) = \min\{\kappa(G), g(G)\}.$$

We shall now indicate how Tutte's definition may be modified to yield a definition of connectivity for matroids which directly generalizes the notion of κ-connectivity for graphs. For a positive integer m, we shall say that the matroid M is *m-κ-separated* if there is a partition $\{S, T\}$ of $E(M)$ such that $rkS, rkT \geqslant m$ and $\xi(M; S, T) = m$. It is straightforward to check that the matroid M is m-κ-separated for some positive integer m if and only if M has a pair of disjoint cocircuits. Thus, provided M has a pair of disjoint cocircuits, we define the *κ-connectivity* $\kappa(M)$ of M to be the least positive integer j such that M is j-κ-separated. If M does not have a pair of disjoint cocircuits, we let $\kappa(M) = rkM$. Thus for the uniform matroid, $U_{r,m}$, it is straightforward to check that

$$\kappa(U_{r,m}) = \begin{cases} m-r+1, & \text{if} \quad m \leqslant 2r-2; \\ r & \text{otherwise.} \end{cases}$$

The main result of this paper is the following.

(2) THEOREM. *Let G be a connected graph. Then*

$$\kappa(M(G)) = \kappa(G).$$

Proof. It is straightforward to check that we may assume G is simple. Moreover, if G is complete, then

$$\kappa(G) = |V(G)| - 1 = rk(M(G)) = \kappa(M(G)).$$

Hence we shall suppose that G is not complete.

Now let $\kappa(M(G)) = n$. We shall argue by contradiction to show that $\kappa(G) \leqslant n$. Hence suppose that $\kappa(G) > n$. As G is not complete, $M(G)$ has a pair of disjoint cocircuits, and so, since $\kappa(M(G)) = n$, there is a partition $\{S, T\}$ of $E(G)$ such that

$$rkS, rkT \geqslant n, \quad \text{and} \tag{3}$$

$$\xi(M(G); S, T) = n. \tag{4}$$

It follows easily from (4) (see ((12), 3·1)) that

$$|V(S) \cap V(T)| = \omega(S) + \omega(T) + n - 2, \tag{5}$$

where $\omega(S)$ and $\omega(T)$ denote the number of components of $G[S]$ and $G[T]$ respectively.

If $\omega(S) = \omega(T) = 1$, then $|V(S) \cap V(T)| = n$. Now $rkS \geqslant n$, so $|V(S)| \geqslant n+1$ and hence some edge of S has an endpoint v_1 which is not in $V(T)$. Similarly, as $rkT \geqslant n$, some edge of T has an endpoint v_2 which is not in $V(S)$. Thus in the graph

$$G \backslash (V(S) \cap V(T))$$

there is no path joining v_1 and v_2, and hence $\kappa(G) \leqslant |V(S) \cap V(T)| = n$; a contradiction. Therefore

$$\omega(S) + \omega(T) \geqslant 3. \tag{6}$$

Hence, we may suppose, without loss of generality, that

(7) $$\omega(S) \geqslant \omega(T) \quad \text{and} \quad \omega(S) \geqslant 2.$$

Assume, in addition, that among the sets S satisfying (3), (4) and (7) we have one of minimum cardinality.

Let $S_1, S_2, \ldots, S_p$ be the edge-sets of the components of $G[S]$ and $T_1, T_2, \ldots, T_q$ be the edge-sets of the components of $G[T]$. Then $p = \omega(S)$ and $q = \omega(T)$. Now G is connected so $V(S_i) \cap V(T) \neq \varnothing$ for all i in $\{1, 2, \ldots, p\}$. Moreover, as $|V(S_i)| \geqslant 2$ and $p \geqslant 2$, if $|V(S_i) \cap V(T)| = 1$, then $V(S_i) \cap V(T)$ is a vertex cut in G, so $\kappa(G) \leqslant 1$; a contradiction. Hence

(8) $$|V(S_i) \cap V(T)| \geqslant 2.$$

Next we note that, by (5), if $j \in \{1, 2, \ldots, p\}$, then

$$|V(S_j) \cap V(T)| = p+q+n-2-\Sigma\, |V(S_i) \cap V(T)|,$$

where the sum is taken from $i = 1$ to p, excepting $i = j$. But, by (7), $p \geqslant q$ and, by (8),

$$\Sigma\, |V(S_i) \cap V(T)| \geqslant 2(p-1).$$

Hence

(9) $$|V(S_j) \cap V(T)| \leqslant p+p+n-2-2(p-1) = n.$$

Now if $V(S_j) \neq V(S_j) \cap V(T)$, then $V(S_j) \cap V(T)$ is a vertex cut of G separating $V(S_j) \backslash V(T)$ from $V(S_k)$ where $k \neq j$. But, since $|V(S_j) \cap V(T)| \leqslant n$, this contradicts the fact that $\kappa(G) > n$. Therefore $V(S_j) \cap V(T) = V(S_j)$, and so, by (9),

(10) $$|V(S_j)| \leqslant n.$$

Moreover, $V(S) \subseteq V(T)$ and hence $V(T) = V(G)$ and $V(S) \cap V(T) = V(S)$. Therefore, by (5),

(11) $$|V(S)| = p+q+n-2.$$

Hence

(12) $$rkS = q+n-2,$$

and so, by (3),

(13) $$q \geqslant 2.$$

We show next that

(14) every edge of S joins vertices in different components of $G[T]$.

If not, then S has an edge e joining two vertices in the same component of $G[T]$. Evidently $rk(T \cup e) = rkT$. Moreover, $rk(S \backslash e) = rkS$, for otherwise $rk(S\backslash e) = rkS - 1$ and so $\xi(M(G); S\backslash e, T \cup e) = n-1$ and $rk(S\backslash e), rk(T \cup e) \geqslant n-1$; a contradiction to the fact that $\kappa(M(G)) = n$. It follows that $\omega(S\backslash e) = \omega(S)$ and $\omega(T \cup e) = \omega(T)$. Thus the partition $\{S\backslash e, T \cup e\}$ of $E(G)$ satisfies (3), (4) and (7) and so the choice of S is contradicted. We conclude that (14) holds.

Since $\kappa(G) > n$ and G is not complete, every vertex of G is adjacent to at least $n+1$ other vertices. We now show that

$$\text{(15)} \qquad \text{if} \quad |V(S) \cap V(T_i)| \leqslant n \quad \text{for some } i, \text{ then} \quad V(T_i) = V(S) \cap V(T_i) \quad \text{and} \quad rkS \geqslant 2n+2-|V(T_i)| \geqslant n+2.$$

If $V(T_i) \neq V(S) \cap V(T_i)$, then $V(S) \cap V(T_i)$ is a vertex cut of G having at most n members, contrary to the fact that $\kappa(G) > n$. Therefore $V(T_i) = V(S) \cap V(T_i)$ and so $|V(T_i)| \leqslant n$. Now if $v \in V(T_i)$, then v meets at least $n+2-|V(T_i)|$ edges of S. Moreover, by (10), $|V(S_j)| \leqslant n$ for all j, so T_i meets at least two different components of $G[S]$. Thus we may choose vertices v_1 and v_2 in $V(T_i)$ which are in different components of S. Consider the subset A of S formed by taking one edge of S meeting each vertex of $V(T_i)\backslash\{v_1, v_2\}$, together with all those edges of S which meet v_1 or v_2. Evidently

$$|A| \geqslant 2(n+2-|V(T_i)|)+(|V(T_i)|-2) = 2n+2-|V(T_i)|.$$

Moreover, since v_1 and v_2 are in different components of S, the set A is independent in $M(G)$. Hence $rkS \geqslant 2n+2-|V(T_i)|$. But $|V(T_i)| \leqslant n$ and so (15) holds.

Following (13) we now distinguish two cases, (I) $q = 2$ and (II) $q > 2$.

In case I, by (12), $rkS = n$. Now consider $|V(S) \cap V(T_1)|$. If this number exceeds n, then choose one edge of S incident with each vertex in $V(S) \cap V(T_1)$. Evidently the set of edges so chosen is independent in $M(G)$ and so we have a contradiction to the fact that $rkS = n$. Therefore $|V(S) \cap V(T_1)| \leqslant n$ and so, by (15), $rkS \geqslant n+2$ and again we have a contradiction.

Now consider case II. By (12), $rkS > n$. We shall show that, for all i in $\{1, 2, \ldots, p\}$,

$$\text{(16)} \qquad |V(S_i)| \geqslant 3.$$

As $|S_i| \geqslant 1$, $|V(S_i)| \geqslant 2$, so assume that $|V(S_i)| = 2$. Then S_i contains a single edge e. Clearly $rk(S\backslash e) = rkS-1$ and $\omega(S\backslash e) = \omega(S)-1$. In addition, by (14), $rk(T \cup e) = rkT+1$ and $\omega(T \cup e) = \omega(T)-1$. Thus, as $\omega(S) \geqslant \omega(T)$, we have $\omega(S\backslash e) \geqslant \omega(T \cup e)$. But the partition $\{S\backslash e, T \cup e\}$ of $E(G)$ satisfies (3) and (4) and so, by (6), $\omega(S\backslash e)+\omega(T \cup e) \geqslant 3$. Since $\omega(S\backslash e) \geqslant \omega(T \cup e)$, it follows that $\omega(S\backslash e) \geqslant 2$, and so $\{S\backslash e, T \cup e\}$ satisfies (3), (4) and (7), contrary to choice of S. We conclude that (16) is satisfied.

By (11), $\Sigma_{i=1}^{p} |V(S_i)| = p+q+n-2$. Therefore, for some k in $\{1, 2, \ldots, p\}$,

$$|V(S_k)| \leqslant \frac{1}{p}(p+q+n-2) = 1+\frac{q}{p}+\frac{n-2}{p}.$$

But $p \geqslant q$ and $|V(S_k)| \geqslant 3$, hence

$$\text{(17)} \qquad p \leqslant n-2.$$

We now apply (11) again together with the fact that $V(T) = V(G)$ to get that

$$\sum_{i=1}^{q} |V(S) \cap V(T_i)| = p+q+n-2.$$

Therefore, for some j in $\{1, 2, \ldots, q\}$,

$$|V(S) \cap V(T_j)| \leqslant \frac{p+q+n-2}{q}$$

$$= \frac{p+n-2}{q} + 1.$$

Hence, by (17),

$$|V(S) \cap V(T_j)| \leqslant \frac{2n-4}{q} + 1.$$

But, by (13), $q \geqslant 2$, so

$$|V(S) \cap V(T_j)| \leqslant n-1.$$

Therefore, by (15),

$$rkS \geqslant 2n+2-|V(T_j)|$$

and

$$|V(T_j)| \leqslant \frac{2n-4}{q} + 1.$$

Hence $rkS \geqslant 2n+2-(2n-4)/q-1$. But, by (12), $rkS = q+n-2$, so

$$q+n-2 \geqslant 2n+1-(2n-4)/q.$$

It follows easily from this that

$$q(q-3) \geqslant n(q-2)+4.$$

But $q(q-2) > q(q-3)$, hence $q(q-2) > n(q-2)+4$ and so $q > n$. Therefore, by (7), $p > n$. This contradiction to (17) completes the proof of the fact that

$$(18) \qquad \kappa(G) \leqslant \kappa(M(G)).$$

To establish the reverse inequality, suppose that $\kappa(G) = k$. Then, as G is not complete, it has a k-vertex cut U. Partition the set of components of $G \backslash U$ into two non-empty subsets A_1 and A_2 and, for $i = 1, 2$, let $S_i = \{e \in E(G) : e = uv \text{ where } u \in U \text{ and } v \in V(A_i)\} \cup E(A_i)$. Then, for each u in U and each i in $\{1, 2\}$, there is an edge in S_i having u as an endpoint, otherwise $U \backslash u$ is a vertex cut of G. Thus $rkS_i \geqslant k$ and $V(S_i) \supseteq U$. Therefore $V(S_1) \cap V(S_2) \supseteq U$. But $V(S_1) \cap V(S_2) \subseteq U$; hence $V(S_1) \cap V(S_2)$ equals U. Now let $S = S_1 \cup E(G[U])$ and $T = S_2$. Then $\{S, T\}$ is a partition of $E(G)$ and $rkS, rkT \geqslant k$. Moreover, both $G[S]$ and $G[T]$ are connected. Hence, by (5),

$$\xi(M(G); S, T) = |V(S) \cap V(T)|.$$

But $|V(S) \cap V(T)| = |V(S_1) \cap V(S_2)| = k$ and so $M(G)$ is k-κ-separated. Hence $\kappa(M(G)) \leqslant \kappa(G)$. The theorem follows on combining this with (18).

The next result generalizes Proposition 1 to matroids. By analogy with graphs we define the *girth* $g(M)$ of a matroid M to be ∞ if M is free, and $\min\{|C| : C \text{ is a circuit of } M\}$ otherwise.

(19) Proposition. *If M is a non-uniform matroid, then*

$$\lambda(M) = \min\{\kappa(M), g(M)\}.$$

Proof. As M is non-uniform, $g(M) \leqslant rkM$. Now, if C is a circuit of M having $g(M)$ elements, then

$$\xi(M; C, E(M)\backslash C) = g(M) + rk(E(M)\backslash C) - rkM.$$

Hence if $t = \min\{g(M), rk(E(M)\backslash C)\}$, then $\xi(M; C, E(M)\backslash C) \leqslant t$ and $|C|, |E(M)\backslash C| \geqslant t$. Thus M is m-separated for some $m \leqslant t \leqslant g(M)$ and so

$$\lambda(M) \leqslant g(M) \leqslant rkM. \tag{20}$$

It is clear that, if M is m-κ-separated, then M is m-separated. Moreover, $\lambda(M) \leqslant rkM$; hence

$$\lambda(M) \leqslant \kappa(M). \tag{21}$$

Combining this with (20), we get

$$\lambda(M) \leqslant \min\{\kappa(M), g(M)\}. \tag{22}$$

To establish equality here, we first note that, as M is $\lambda(M)$-separated, there is a partition $\{S, T\}$ of $E(M)$ such that $|S|, |T| \geqslant \lambda(M)$ and $\xi(M; S, T) = \lambda(M)$. Now suppose that $\lambda(M) < \min\{\kappa(M), g(M)\}$. Then $\lambda(M) < g(M)$, so $rkS, rkT \geqslant \lambda(M)$ and hence M is $\lambda(M)$-κ-separated. Thus $\kappa(M) \leqslant \lambda(M)$; a contradiction.

Although Tutte's definition of connectivity for matroids does not directly generalize the graph-theoretic notion of κ-connectivity, it does incorporate matroid duality. Indeed, for all matroids M, we have ((12), (12))

$$\lambda(M) = \lambda(M^*). \tag{23}$$

The next result is a consequence of Proposition 19.

(24) COROLLARY. *Let M be a non-uniform matroid. Then*

$$\lambda(M) = \min\{\kappa(M), \kappa(M^*)\}.$$

Proof. By Proposition 19, $\lambda(M) \leqslant \kappa(M)$ and $\lambda(M^*) \leqslant \kappa(M^*)$. But, by (23),

$$\lambda(M) = \lambda(M^*);$$

hence

$$\lambda(M) \leqslant \min\{\kappa(M), \kappa(M^*)\}. \tag{25}$$

Therefore, if $\lambda(M) = \kappa(M)$ or $\kappa(M^*)$ the required result is immediate. Hence, we may suppose that $\lambda(M) < \kappa(M)$ and $\lambda(M) < \kappa(M^*)$. Thus, by Proposition 19 and (23)

$$g(M) = \lambda(M) = \lambda(M^*) = g(M^*).$$

Hence M is $g(M)$-separated and so $|E(M)| \geqslant 2g(M)$. Now let C^* be a cocircuit of M having $g(M^*)$ elements. Then

$$\xi(M; C^*, E(M)\backslash C^*) = rk(C^*) \leqslant g(M^*) = g(M).$$

Moreover, $|E(M)\backslash C^*| \geqslant g(M)$. Hence, as M is not m-separated for $m < g(M)$, we have $rk(C^*) = g(M)$. Now $rk(E(M)\backslash C^*) = rkM - 1$ and so, if $rkM - 1 \geqslant g(M)$, then M is $g(M)$-κ-separated; that is, $\kappa(M) \leqslant g(M)$. But $g(M) = \lambda(M)$ and $\lambda(M) < \kappa(M)$; hence we have a contradiction. It follows that we may assume that $rkM - 1 < g(M)$; that is, $rkM \leqslant g(M)$. But then $\lambda(M) = g(M) \geqslant rkM \geqslant \kappa(M)$ and we again have a contradiction.

It is straightforward to check that deleting an element other than a coloop from a matroid cannot increase its κ-connectivity, although as Welsh observes ((14), ex. 5·6·2), the deletion may increase the connectivity. Thus, just as loops do not affect the connectivity of a graph, they do not affect the κ-connectivity of a matroid. It is not difficult to show that if M is a matroid of rank at least two and L is its set of loops then M is 2-κ-connected if and only if $M\backslash L$ is non-separable.

Although the concepts of n-κ-connection and n-connection do not coincide in general, when $n = 2$ and $n = 3$ they are very closely related. Minimally 2- and 3-connected matroids have been studied in detail in (4, 5, 6, 7, 9, 10) and we note that the classes of minimally 2-connected matroids and minimally 2-κ-connected matroids coincide. Moreover, the only difference between the classes of minimally 3-connected matroids and minimally 3-κ-connected matroids is that $U_{2,3}$ is in the first but not the second. Most of the known results for n-connected matroids have been restricted to the cases $n = 2$ and $n = 3$ and can be restated in terms of n-κ-connection.

Tutte's stated aim (13) in introducing a theory of n-connection for matroids was to allow his 'Wheels Theorem' for graphs ((11), (4·1)) to be extended to matroids. The resulting 'Wheels and Whirls Theorem'((12), 8·3) can be reformulated in terms of 3-κ-connection as follows.

(26) Theorem. *Let M be a simple 3-κ-connected matroid such that for all elements e of M neither $M\backslash e$ nor M/e is both simple and 3-κ-connected. Then M is isomorphic to a whirl, the cycle matroid of a wheel or $U_{3,5}$.*

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