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Note

On the Matroids Representable over $GF(4)$

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The purpose of this note is to present a counterexample to a conjecture of Kahn and Seymour on the minor-minimal matroids not representable over $GF(4)$.

Kahn and Seymour [2] have conjectured that a matroid is representable over $GF(4)$ if and only if it has no minor isomorphic to any of the matroids $U_{2,6}$, $U_{4,6}$, F_7^-, $(F_7^-)^*$, and P_6, where F_7^- is the non-Fano matroid and P_6 is the 6-element rank-3 self-dual matroid for which a Euclidean representation is shown in Fig. 1.

Let P_8 be the matroid induced by linear independence on the set of columns of the following matrix over $GF(3)$.

$$\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & -1 \\ & 1 & 0 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 \\ & -1 & 1 & 1 & 0 \end{array}\right].$$

It is not difficult to check, using, for example, a list of its 4-circuits, that P_8 has a transitive automorphism group. Thus every single-element contraction of P_8 is isomorphic to $P_8/1$. The latter is isomorphic to P_7, the matroid for which a Euclidean representation is shown in Fig. 2. Now P_7 is easily shown to be representable over every field other than $GF(2)$ [3]. Using this and the fact that P_8 is self-dual, we deduce that no proper minor

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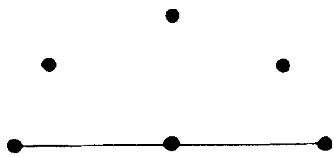

FIGURE 1

of P_8 is a counterexample to Kahn and Seymour's conjecture. The next result completes the argument that P_8 itself is a counterexample.

THEOREM. *P_8 is representable over a field F if and only if the characteristic of F is not two.*

Proof. Suppose that A is a 4×8 matrix representing P_8 over a field F. Then, using Brylawski and Lucas's theory of coordinatizing paths [1], we can assume that

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$A = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & d \\ I_4 & 1 & 0 & 1 & 1 \\ & a & 1 & 0 & e \\ & b & 1 & c & 0 \end{array}\right]$$

where a, b, c, d, and e are non-zero elements of F. It is straightforward to show, by considering P_8/i for each i in $\{4, 3, 2\}$, that

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$A = \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 2 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \\ & 2 & 1 & 1 & 0 \end{array}\right].$$

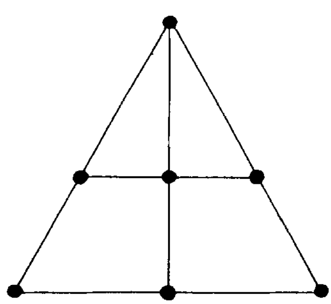

FIGURE 2

Since every non-zero entry in this matrix must be non-zero modulo F, we deduce that F does not have characteristic two. It is now routine to check that A represents P_8 over all fields of characteristic other than two. ▮

REFERENCES

1. T. H. BRYLAWSKI AND D. LUCAS, Uniquely representable combinatorial geometries, *in* "Teorie Combinatorie, Proc. 1973 Internat. Colloq.," Accademia Nazionale dei Lincei, Roma, 1976, pp. 83–104.
2. J. KAHN AND P. D. SEYMOUR, private communication, February 1984.
3. G. P. WHITTLE, Some Aspects of the Critical Problem for Matroids, Ph.D. thesis, University of Tasmania, 1985.