Note

On the Matroids Representable over $GF(4)$

JAMES G. OXLEY*

Mathematics Department, Louisiana State University,
Baton Rouge, Louisiana 70803

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The purpose of this note is to present a counterexample to a conjecture of Kahn and Seymour on the minor-minimal matroids not representable over $GF(4)$.

Kahn and Seymour [2] have conjectured that a matroid is representable over $GF(4)$ if and only if it has no minor isomorphic to any of the matroids $U_{2,6}$, $U_{4,6}$, $F_7$, $(F_7)^*$, and $P_6$, where $F_7$ is the non-Fano matroid and $P_6$ is the 6-element rank-3 self-dual matroid for which a Euclidean representation is shown in Fig. 1.

Let $P_8$ be the matroid induced by linear independence on the set of columns of the following matrix over $GF(3)$.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
I_4 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\
& 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
& & 1 & 1 & 0 & 1 & & & & \\
& & & -1 & 1 & 1 & 0 & & & \\
\end{bmatrix}
\]

It is not difficult to check, using, for example, a list of its 4-circuits, that $P_8$ has a transitive automorphism group. Thus every single-element contraction of $P_8$ is isomorphic to $P_8/1$. The latter is isomorphic to $P_7$, the matroid for which a Euclidean representation is shown in Fig. 2. Now $P_7$ is easily shown to be representable over every field other than $GF(2)$ [3]. Using this and the fact that $P_8$ is self-dual, we deduce that no proper minor

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of $P_8$ is a counterexample to Kahn and Seymour's conjecture. The next result completes the argument that $P_8$ itself is a counterexample.

**Theorem.** $P_8$ is representable over a field $F$ if and only if the characteristic of $F$ is not two.

**Proof.** Suppose that $A$ is a $4 \times 8$ matrix representing $P_8$ over a field $F$. Then, using Brylawski and Lucas's theory of coordinatizing paths [1], we can assume that

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 1 & d \\
1 & 0 & 1 & 1 \\
a & 1 & 0 & e \\
b & 1 & c & 0
\end{bmatrix}$$

where $a, b, c, d,$ and $e$ are non-zero elements of $F$. It is straightforward to show, by considering $P_8/i$ for each $i$ in \{4, 3, 2\}, that

$$A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
2 & 1 & 1 & 0
\end{bmatrix}.$$
Since every non-zero entry in this matrix must be non-zero modulo $F$, we deduce that $F$ does not have characteristic two. It is now routine to check that $A$ represents $P_8$ over all fields of characteristic other than two. ■

REFERENCES