A Characterization of a Class of Non-binary Matroids

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A well-known result of Tutte is that $U_{2,4}$, the 4-point line, is the only non-binary matroid M such that, for every element e, both $M \backslash e$ and M/e, the deletion and contraction of e from M, are binary. This paper characterizes those non-binary matroids M such that, for every element e, $M \backslash e$ *or* M/e is binary. © 1990 Academic Press, Inc.

1. INTRODUCTION

The class of binary matroids is one of the best-known and most frequently studied classes of matroids. In this paper, we characterize a class of non-binary matroids that are, in a certain natural sense, close to being binary. Tutte [11] proved that $U_{2,4}$ is the only non-binary matroid for which every single-element deletion and every single-element contraction is binary. Here we characterize a larger class of non-binary matroids: those such that, for every element e, the deletion *or* the contraction of e is binary.

Most of the matroid terminology used here will follow Welsh [13]. The ground set, corank, and rank function of the matroid M will be denoted by $E(M)$, cork M, and rk, respectively. If $T \subseteq E(M)$, we shall say that M *uses* T. We shall denote by $M \backslash T$ or $M \mid (E(M) - T)$ the deletion of T from M, and by M/T the contraction of T from M.

Let n be a positive integer. The matroid M is *n-connected* [12] if, for all positive integers $k < n$, there is no partition $\{S, T\}$ of $E(M)$ such that $|S|$, $|T| \geqslant k$ and rk $S + $ rk $T - $ rk $M = k - 1$. Thus a matroid is 2-connected precisely when it is connected [13, p. 69]. Moreover, it is easy to show that M is *n*-connected if and only if M^* is *n*-connected.

For matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$, we denote the series and parallel connections of M_1 and M_2 with respect to the basepoint p by $S(M_1, M_2)$ and $P(M_1, M_2)$. If both $E(M_1)$ and $E(M_2)$ have at least three elements and p is neither a loop nor a coloop of M_1 or M_2, then the 2-*sum* of M_1 and M_2 is $P(M_1, M_2) \backslash p$, or equivalently, $S(M_1, M_2)/p$. We call M_1 and M_2 the *parts* of this 2-sum. One attractive feature of this operation is that the dual of the 2-sum of M_1 and M_2 is the 2-sum of M_1^* and M_2^*. Seymour [8, (2.6)] (see also [1, 3]) proved the following basic link between 3-connectedness and 2-sums.

(1.1) LEMMA. *A connected matroid M is not 3-connected if and only if there is a pair of matroids such that M is their 2-sum.*

We shall assume familiarity with other basic properties of 2-sums. Those needed here are summarized in [5, p. 664].

If $\{x, y\}$ is a circuit of the matroid M, we say that x and y are *in parallel* in M. If $\{x, y\}$ is a cocircuit, then x and y are *in series*. A *parallel class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct members of A, then a and b are in parallel. *Series classes* are defined analogously. A series or parallel class is *non-trivial* if it contains more than one element. The matroid N is a *series extension* of M if $M = N/T$ and every element of T is in series with some element of M. We call N a *parallel extension* of M if N^* is a series extension of M^*.

The main results of this paper use the following basic construction. Let C be a circuit–hyperplane of the matroid M, that is, C is both a circuit and a hyperplane of M. Let $\mathscr{B} = \{B : B$ is a basis of $M\} \cup \{C\}$. Then it is well known (see, for example, [7, p. 164; 9, p. 77]) that $\mathscr{B}$ is the set of bases of a matroid M' on $E(M)$. Following Kahn [4], we call M' a *relaxation* of M. We shall also say that M' has been obtained from M by *relaxing* the circuit–hyperplane C. Thus, for example, the non-Fano and non-Pappus matroids are relaxations of the Fano and Pappus matroids, respectively. Moreover, the whirl $\mathscr{W}^r$ [13, p. 81] is a relaxation of $M(\mathscr{W}_r)$, the cycle matroid of the r-spoked wheel.

The next two theorems are the main results of the paper. Although the second is weaker than the first, we state both since the proof of the second is a major step in the proof of the first.

(1.2) THEOREM. *The following two statements are equivalent for a matroid M.*

    (i)   *M is non-binary and, for every element e, $M \backslash e$ or M/e is binary.*

    (ii)   (a) *Both* rk M *and* cork M *exceed two and M can be obtained from a connected binary matroid by relaxing a circuit–hyperplane; or*

(b) M is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geqslant 5$; or

(c) M is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geqslant 5$; or

(d) M can be obtained from $U_{2,4}$ by series extension of a subset S of $E(U_{2,4})$ and parallel extension of a disjoint subset T of $E(U_{2,4})$ where S or T may be empty.

(1.3) THEOREM. *The following two statements are equivalent for a matroid M.*

(i)   *M is non-binary, 3-connected, and, for every element e, $M \backslash e$ or M/e is binary.*

(ii)   (a) *M is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geqslant 4$; or*

(b)   *both the rank and corank of M exceed two and M can be obtained from a 3-connected binary matroid by relaxing a circuit–hyperplane.*

The proofs of these theorems will be given in Section 2. In the remainder of this section, we note some preliminaries that will be needed in the proofs. We begin with a number of properties of relaxation that were noted by Kahn [4, p. 320].

(1.4) LEMMA. *Suppose that M_2 is obtained from M_1 by relaxing the circuit–hyperplane C. Then*

(i)   *M_2^* is obtained from M_1^* by relaxing the circuit–hyperplane $E(M_1) - C$;*

(ii)   *if $a \in C$ and $b \in E(M) - C$, then $M_2 \backslash a = M_1 \backslash a$ and $M_2/b = M_1/b$;*

(iii)   *if M_1 is n-connected, then so is M_2; and*

(iv)   *if M_1 is connected, then M_2 is non-binary.*

The following result enables one to recognize when a matroid is a relaxation of another matroid. The straightforward proof is omitted.

(1.5) LEMMA. *Let M be a matroid having rank at least one and Y be a basis of M. Suppose that, for all e in $E(M) - Y$, the fundamental circuit of e with respect to Y is $Y \cup e$. Then*

$$(\{Z: Z \text{ is a circuit of } M\} - \{Y \cup e: e \in E(M) - Y\}) \cup \{Y\}$$

is the set of circuits of a matroid N on $E(M)$. Moreover, Y is a hyperplane of N and M is obtained from N by relaxing Y.

The next two lemmas are structural results for non-binary 3-connected

matroids. Figure 1 gives Euclidean representations for the matroids P_6 and Q_6 that appear in the second of these.

(1.6) LEMMA [9, (3.1)]. *If x and y are elements of a non-binary 3-connected matroid M, then M has a $U_{2,4}$-minor using $\{x, y\}$.*

(1.7) LEMMA [5, Theorem 3.1]. *Let M be a non-binary 3-connected matroid having rank and corank at least three. Then M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathscr{W}^3$.*

Theorem 1.2 contains one generalization of Tutte's excluded-minor characterization of binary matroids [11]. The following is an alternative generalization of that result.

(1.8) LEMMA. *Let M be a non-binary matroid such that, for some element e, both $M \backslash e$ and M/e are binary. Then M is obtained from a 4-point line having ground set $\{e, e_1, e_2, e_3\}$ by a sequence of at most three 2-sums where the basepoints of these 2-sums are e_1, e_2, and e_3, the other part of each 2-sum is binary, and each of e_1, e_2, and e_3 is the basepoint of at most one of these 2-sums.*

Proof. Evidently we may assume that M is connected. The lemma is now immediate from [5, Theorem 3.8]. ▮

As an immediate consequence of the last result, we have the following:

(1.9) COROLLARY [5, Corollary 3.9]. *If M is 3-connected, non-binary and, for some element e, both $M \backslash e$ and M/e are binary, then $M \cong U_{2,4}$.*

2. THE PROOFS

In this section, we prove Theorems 1.2 and 1.3, beginning with the latter.

Proof of Theorem 1.3. Assume that (ii) holds. If $M \cong U_{2,n}$ or $U_{n-2,n}$ for some $n \geqslant 4$, then (i) holds. Now suppose that both the rank and corank of M exceed two and that M is obtained from a binary 3-connected matroid

FIGURE 1

N by relaxing a circuit–hyperplane C. Then, by Lemma 1.4(iii) and (iv), M is 3-connected and non-binary. Moreover, by Lemma 1.4(ii), if $e \in C$, then $M \backslash e$ is binary, while if $e \in E(M) - C$, then M/e is binary. We conclude that (ii) implies (i).

Now suppose that (i) holds. Evidently, if rk $M = 2$ or cork $M = 2$, then, as M is 3-connected, it is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geqslant 4$. Thus we may assume that both the rank and corank of M exceed two.

Suppose that rk $M = 3$. Then, by Lemma 1.7, M has a minor isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$. In the first three cases, if the element e is as marked in Fig. 2, then both $M \backslash e$ and M/e are non-binary. We conclude that M has a $\mathcal{W}^3$-minor, but has no minor isomorphic to $U_{3,6}$, P_6, or Q_6. Using this and the fact that, for all elements e of M, $M \backslash e$ or M/e is binary, it is not difficult to check that M is isomorphic to $\mathcal{W}^3$ or the non-Fano matroid. As these matroids are relaxations of $M(\mathcal{W}_3)$ and the Fano matroid, respectively, the theorem holds if rk $M = 3$. By duality, it also holds if cork $M = 3$.

We now assume that both rk M and cork M exceed three. Let $X = \{x \in E(M) : M \backslash x \text{ is non-binary}\}$ and $Y = \{y \in E(M) : M/y \text{ is non-binary}\}$. By Corollary 1.9, $X \cup Y = E(M)$ and, by hypothesis, $X \cap Y = \varnothing$.

(2.1) LEMMA. *Y is a basis and X is a cobasis of M.*

Proof. As $\{X, Y\}$ is a partition of $E(M)$, it suffices to show that Y is a basis. Since M is non-binary, for some pair, A and D, of disjoint subsets of $E(M)$, $M \backslash A/D \cong U_{2,4}$. Evidently $D \subseteq Y$ and $A \subseteq X$. Moreover, as $X \cap Y = \varnothing$, we may assume that D is independent and A is coindependent in M.

Since rk M and cork M exceed three, we can choose 2-element subsets $\{d_1, d_2\}$ and $\{a_1, a_2\}$ of D and A, respectively. By Lemma 1.6, M has $U_{2,4}$-minors $M \backslash A'/D'$ and $M \backslash A''/D''$ that use $\{d_1, d_2\}$ and $\{a_1, a_2\}$, respectively. Since $X \cap Y = \varnothing$, both D' and A'' contain at least two elements of $E(M) - (A \cup D)$, and $D' \cap A'' = \varnothing$. Thus exactly two elements, say e_1 and e_2, of $E(M) - (A \cup D)$ are in Y and the other two elements are in X. Since D and $\{e_1, e_2\}$ are bases of $M \mid D$ and M/D, respectively, $D \cup \{e_1, e_2\}$ is a basis of M. Since $D \cup \{e_1, e_2\} = Y$, the lemma holds. ∎
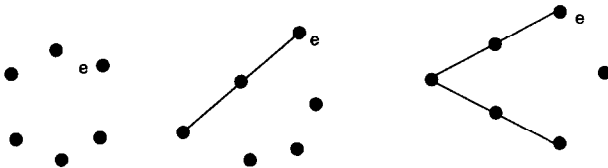

FIGURE 2

We shall show next that it is the set Y whose relaxation produces the matroid M.

(2.2) LEMMA. *For all e in X, the fundamental circuit $C(e, Y)$ is $Y \cup e$.*

Proof. Suppose that, for some element e of X, the fundamental circuit $C(e, Y)$ does not contain Y. Then we can choose an element y_1 from $Y - C(e, Y)$. Let y_2 be an element of $Y - y_1$. By Lemma 1.6, M has a $U_{2,4}$-minor $M \backslash Z_1 / Z_2$ using $\{ y_1, y_2 \}$. Evidently $Z_2 \subseteq Y$ and $|Z_2| = \text{rk } M - 2 = |Y| - 2$. Therefore $Z_2 = Y - \{ y_1, y_2 \}$. Thus, in M/Z_2, there are two possibilities: either (I) e is a loop, or (II) e is parallel to y_2. In the first case, $M/Z_2/e$, and hence, M/e is non-binary, contrary to the fact that $e \in X$. In case II, $M/Z_2 \backslash y_2$ has a $U_{2,4}$-minor, contrary to the fact that $y_2 \in Y$. We conclude that the lemma holds. ∎

Now define the collection $\mathscr{C}$ to be

$$(\{ Z : Z \text{ is a circuit of } M \} - \{ Y \cup e : e \in X \}) \cup \{ Y \}.$$

Then, by Lemma 1.5, $\mathscr{C}$ is the set of circuits of a matroid N on $E(M)$ and Y is a hyperplane of N. Moreover, M is obtained from N by relaxing the circuit–hyperplane Y. By Lemma 1.4(i), M^* is obtained from N^* by relaxing the circuit–hyperplane X of N^*. Thus the set of cocircuits of N is

$$(\{ Z^* : Z^* \text{ is a cocircuit of } M \} - \{ X \cup f : f \in Y \}) \cup \{ X \}.$$

The next two lemmas complete the proof that (i) implies (ii) by showing that N is 3-connected and binary.

(2.3) LEMMA. *N is 3-connected.*

Proof. Suppose that N is not 3-connected. Then, as $E(N) = E(M)$, for some k in $\{ 1, 2 \}$, there is a partition $\{ S, T \}$ of $E(M)$ such that $|S|, |T| \geqslant k$ and

$$\text{rk}_N(S) + \text{rk}_N(T) - \text{rk } N = k - 1. \qquad (2.4)$$

Now, all subsets of $E(M)$ except Y have the same rank in N as they do in M, while $\text{rk}_N(Y) = \text{rk } N - 1$. Thus, as M is 3-connected, (2.4) implies that S or T, say T, equals Y. Hence, $S = X$ and $\text{rk}_M X = k$. As $|X| \geqslant 4$ and M is 3-connected, $k \neq 1$. Hence $k = 2$. Thus, in M, the set X is a cobasis contained in a line L that has at least four points. Now, since $\text{rk } M \geqslant 4$, there is an element y of M not in L. As $M \backslash y$ has a $U_{2,4}$-minor, it is non-binary. But $y \notin X$ and so we have a contradiction that completes the proof of the lemma. ∎

(2.5) LEMMA. *N is binary.*

Proof. Assume that N is non-binary. Then, by a well-known result of Seymour [6, p. 360], N has a circuit C and a cocircuit C^* such that $|C \cap C^*| = 3$. Now C is a circuit of M unless $C = Y$, and C^* is a cocircuit of M unless $C^* = X$. As X and Y are disjoint, we cannot have both $C = Y$ and $C^* = X$.

Suppose that $C = Y$. Then C^* is a cocircuit of M and is not equal to X. Hence $C^* \not\supseteq X$, so we can choose an element x from $X - C^*$. Now $Y \cup x$ is a circuit of M meeting C^* in exactly three elements. Thus M/x is non-binary because Y and C^* are a circuit and a cocircuit of it that meet in exactly three elements. But, since $x \in X$, this is a contradiction. Therefore $C \neq Y$ and, by duality, $C^* \neq X$. Hence C is a circuit of M and C^* is a cocircuit of M.

Suppose $y \in C - C^*$. Then $C - y$ is a circuit and C^* is a cocircuit of M/y. Thus M/y is non-binary and so $y \in Y$. Hence $C - C^* \subseteq Y$, and, by duality, $C^* - C \subseteq X$. Consider $C \cap C^*$. As $C \not\subseteq Y$ and $C^* \not\subseteq X$, neither $C \cap C^* \cap X$ nor $C \cap C^* \cap Y$ is empty. Since $|C \cap C^*| = 3$, it follows that $|C \cap C^* \cap X|$ or $|C \cap C^* \cap Y|$ is 1. By duality, we may assume the former. Let $C \cap C^* \cap X = \{z\}$. Then $C \subseteq Y \cup z$. But $Y \cup z$ is a circuit of M. Hence $C = Y \cup z$ and so C is not a circuit of N. This contradiction completes the proof of Lemma 2.5 and thereby that of Theorem 1.3. ∎

Proof of Theorem 1.2. We omit the straightforward argument showing that if (ii) holds, then so does (i). Now assume that (i) holds. We argue by induction on $|E(M)|$ to show that (ii) holds. If M is 3-connected, then the result follows easily from Theorem 1.3. Assume the result is true for all matroids having fewer elements than M. It is straightforward to check that M must be connected. Hence, as M is connected but not 3-connected, Lemma 1.1 implies that, for some matroids M_1 and M_2 with $E(M_1) \cap E(M_2) = \{p\}$, $M = P(M_1, M_2) \backslash p$ where $|E(M_1)|$, $|E(M_2)| \geq 3$. Now M_1 or M_2 is non-binary. Without loss of generality, assume the former.

(2.6) LEMMA. *M_2 is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geq 3$.*

Proof. Since M is connected, so is M_2. For each x in $E(M_2) - p$, let C_x and C_x^* be a maximum-sized circuit and a maximum-sized cocircuit of M_2 containing $\{p, x\}$. If both $|C_x|$ and $|C_x^*|$ exceed two, then both $M \backslash x$ and M/x have M_1 as a minor, so both are non-binary, a contradiction. Thus, for all x in $E(M_2) - p$, $|C_x| = 2$ or $|C_x^*| = 2$. If x and y are distinct elements of $E(M_2) - p$ and $|C_x| = 2 = |C_y^*|$, then $C_x \cap C_y^* = \{p\}$, a contradiction. Thus either $|C_x| = 2$ for all x in $E(M_2) - p$, or $|C_x^*| = 2$ for all such x. The lemma follows immediately. ∎

By the last lemma, we may assume that $M_2 \cong U_{1,n}$ for some $n \geq 3$, otherwise we replace M by M^* in the argument that follows. We may also

suppose that M_1 has no elements in parallel with p, since any such element can be taken to be in M_2 rather than in M_1. Thus M is obtained from M_1 by replacing p by $n-1$ elements in parallel.

(2.7) LEMMA. *For all e in $E(M_1)$, $M_1 \backslash e$ or M_1/e is binary.*

Proof. If $e \neq p$, then, by [2, Propositions 4.7 and 5.6], $M \backslash e = P(M_1 \backslash e, M_2) \backslash p$ and $M/e = P(M_1/e, M_2) \backslash p$. Since $M \backslash e$ or M/e is binary and M_2 is also binary, it follows that $M_1 \backslash e$ or M_1/e is binary. Now suppose that $e = p$. Pick an element q of $E(M_2) - p$. Then $M \backslash q$ is non-binary, so M/q is binary. But $M/q \cong M_1/p \oplus U_{0,n-2}$. Hence

$$M_1/p \text{ is binary.} \qquad (2.8)$$

We conclude that Lemma 2.7 holds. ∎

By the induction assumption, one of (ii)(a)–(d) holds for M_1. Suppose (ii)(a) holds, that is, both rk M_1 and cork M_1 exceed two, and M_1 can be obtained from a connected binary matroid N_1 by relaxing a circuit–hyperplane C. If $p \notin C$, then M is isomorphic to the matroid obtained from M_1 by adjoining $n-2$ elements in parallel with p. If we adjoin $n-2$ elements in parallel with p in N_1, we get a connected binary matroid N_2 that still has C as a circuit–hyperplane. It is not difficult to check that M is isomorphic to the matroid obtained from N_2 by relaxing C. Thus if $p \notin C$, then (ii)(a) holds for M. We may therefore assume that $p \in C$. Then, by Lemma 1.4(ii), $M_1 \backslash p = N_1 \backslash p$. Hence $M_1 \backslash p$ is binary. In addition, by (2.8), M_1/p is binary. Therefore, by Lemma 1.8, M_1 can be obtained from $U_{2,4}$ by a sequence of at most three 2-sums. By Lemma 2.6, one part of each of these 2-sums is either a rank-one uniform matroid or a corank-one uniform matroid. We conclude that if M_1 satisfies (ii)(a), then M satisfies (ii)(d).

If M_1 satisfies (ii)(b), then, clearly, so does M. We may now suppose that M_1 satisfies (ii)(c) or (ii)(d). If p is in a non-trivial series class, then, since M_1 is non-binary, so is M_1/p, a contradiction to (2.8). Thus we may assume that p is not in a non-trivial series class. It follows that if M_1 satisfies (ii)(c), then M_1/p is isomorphic to a series extension of $U_{n-3,n-1}$ and again (2.8) is contradicted. Hence we may suppose that M_1 satisfies (ii)(d). Then, by the choice of M_2, the element p is not in a non-trivial parallel class of M_1. Thus p is an element of $U_{2,4}$ that is not involved in any of the series and parallel extensions used to form M_1. We conclude that M satisfies (ii)(d), thereby completing the proof of Theorem 1.2. ∎

References

1. R. E. BIXBY, "Composition and Decomposition of Matroids and Related Topics," Ph. D. thesis, Cornell University, 1972.
2. T. BRYLAWSKI, A combinatorial model for series–parallel networks, *Trans. Amer. Math. Soc.* **154** (1971), 1–22.
3. W. H. CUNNINGHAM, "A Combinatorial Decomposition Theory," Ph. D. thesis, University of Waterloo, 1973.
4. J. KAHN, A problem of P. Seymour on nonbinary matroids, *Combinatorica* **5** (1985), 319–323.
5. J. G. OXLEY, On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.
6. P. D. SEYMOUR, The forbidden minors of binary clutters, *J. London Math. Soc. (2)* **12** (1976), 356–360.
7. P. D. SEYMOUR, Matroid representation over *GF*(3), *J. Combin. Theory Ser. B* **26** (1979), 159–173.
8. P. D. SEYMOUR, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.
9. P. D. SEYMOUR, Recognizing graphic matroids, *Combinatorica* **1** (1981), 75–78.
10. P. D. SEYMOUR, On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.
11. W. T. TUTTE, A homotopy theorem for matroids, I, II, *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
12. W. T. TUTTE, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.
13. D. J. A. WELSH, "Matroid Theory", Academic Press, London, 1976.