THE BINARY MATROIDS WHOSE ONLY ODD CIRCUITS ARE TRIANGLES

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ABSTRACT. This paper generalizes a graph-theoretical result of Maffray to binary matroids. In particular, we prove that a connected simple binary matroid M has no odd circuits other than triangles if and only if M is affine, M is $M(K_4)$ or F_7, or M is the cycle matroid of a graph consisting of a collection of triangles all of which share a common edge. This result implies that a 2-connected loopless graph G has no odd bonds of size at least five if and only if G is Eulerian or G is a subdivision of either K_4 or the graph that is obtained from a cycle of parallel pairs by deleting a single edge.

1. INTRODUCTION

For each $n \geq 1$, let $K'_{2,n}$ be the graph that is obtained from $K_{2,n}$ by adding an edge joining the vertices in the two-vertex class (see Figure 1). Maffray [5, Theorem 2] proved the following result.

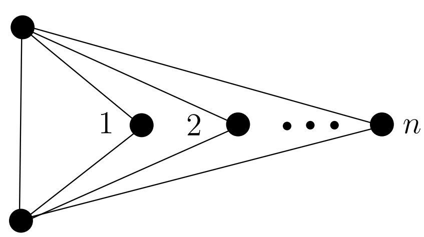

FIGURE 1. $K'_{2,n}$

Theorem 1.1. *A 2-connected simple graph G has no odd cycles of length exceeding three if and only if*

(i) *G is bipartite;*
(ii) *$G \cong K_4$; or*
(iii) *$G \cong K'_{2,n}$ for some $n \geq 1$.*

There is a long history of generalizing results for graphs to binary matroids (see, for example, [3, 7] or, more recently, [6, Section 15.4]). This paper

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continues this tradition by proving a generalization of Maffray's result. A circuit in a matroid is *even* if it has even cardinality; otherwise, it is *odd*. A *triangle* is a 3-element circuit. A binary matroid is *affine* if all of its circuits are even. Hence the cycle matroid, $M(G)$, of a graph G is affine if and only if G is bipartite. The following is the main result of the paper.

Theorem 1.2. *A connected simple binary matroid M has no odd circuits other than triangles if and only if*
- (i) *M is affine;*
- (ii) *$M \cong M(K_4)$ or F_7; or*
- (iii) *$M \cong M(K'_{2,n})$ for some $n \geq 1$.*

The terminology used here will follow Oxley [6]. Binary affine matroids have several attractive characterizations. Indeed, Welsh [8] proved that the link between bipartite and Eulerian graphs via duality extends to binary matroids. His result is the equivalence of the first two parts of the next theorem (see, for example, [6, Theorem 9.4.1]). The equivalence of the first and third parts was proved independently by Brylawski [2] and Heron [4].

Theorem 1.3. *The following are equivalent for a binary matroid M.*
- (i) *M is affine;*
- (ii) *M is loopless and its simplification is isomorphic to a restriction of $AG(r-1, 2)$ for some $r \geq 1$;*
- (iii) *$E(M)$ can be partitioned into cocircuits.*

Recall that a *bond* of a graph is a minimal edge cut. The next result follows immediately by applying our main result to the bond matroid of a graph, that is, to the dual of its cycle matroid.

Corollary 1.4. *A 2-connected loopless graph G has no odd bonds of size at least five if and only if*
- (i) *G is Eulerian; or*
- (ii) *G is a subdivision of either K_4 or the graph that is obtained from an n-edge cycle for some $n \geq 2$ by adding an edge in parallel to all but one of the edges.*

Another straightforward consequence of Theorems 1.2 and 1.3 is the following.

Corollary 1.5. *Let M be a connected cosimple binary matroid of rank at least four. Then M has no odd circuits of size exceeding three if and only if M is affine.*

2. The Proof of the Main Theorem

We shall use two lemmas.

Lemma 2.1. *A simple binary matroid having an even circuit meeting a triangle T in a single element has an odd circuit of size exceeding three.*

Proof. From among even circuits that meet T in a single element, choose C to have minimum cardinality. As M is binary, $C\Delta T$ is the disjoint union of k circuits for some $k \geq 1$. As $|C\Delta T| = |C| + 1$, if $k = 1$, then the lemma holds. Thus we may assume that $k \geq 2$. Since each circuit contained in $C\Delta T$ must contain an element of $T - C$, we deduce that $k \leq 2$, so $k = 2$. Thus, as $C\Delta T$ has odd cardinality, it is the disjoint union of an odd circuit and an even circuit, C_0, each of which meets T in a single element. As $|C_0| < |C|$, the choice of C is contradicted. □

Our second lemma is more general than we need to prove the theorem. For an integer n exceeding one, let $M_1, M_2, \ldots, M_n$ be matroids such that $E(M_i) \cap E(M_j) = \{p\}$ for all distinct i and j in $\{1, 2, \ldots, n\}$, and $\{p\}$ is not a component of any M_k. The *parallel connection* $P(M_1, M_2, \ldots, M_n)$ is the matroid with ground set $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)$ whose set of circuits consists of the union of the sets of circuits of $M_1, M_2, \ldots, M_n$ along with, for all distinct elements i and j of $\{1, 2, \ldots, n\}$, all sets of the form $(C_i - p) \cup (C_j - p)$ where C_i is a circuit of M_i containing p, and C_j is a circuit of M_j containing p (see, for example, [6, Proposition 7.1.18]). Thus if $M_k \cong U_{2,3}$ for all k, then $P(M_1, M_2, \ldots, M_n) \cong M(K'_{2,n})$. The element p is called the *basepoint* of the parallel connection.

Lemma 2.2. *Let M be a simple connected matroid. Then M has an element p such that the only circuits of M that contain p are triangles if and only if M is isomorphic to $U_{1,1}$ or to $U_{2,k}$ for some $k \geq 3$, or M is the parallel connection with basepoint p of some collection of simple rank-2 matroids each of which contains at least three points.*

Proof. It is straightforward to check that, for each of the matroids listed, the only circuits containing p are triangles. Now assume that the only circuits of M containing p are triangles. We may assume that $r(M) \geq 3$ otherwise the result certainly holds. As M is connected, each of its elements is in some circuit with p. By hypothesis, this circuit must be a triangle. Thus, in M/p, every element is in a non-trivial parallel class. If every component of M/p has rank one, then it follows by a result of Brylawski [1] (see also [6, Theorem 7.1.16]) that M is a parallel connection as asserted. Therefore we may assume that M/p has a component of rank exceeding one. Thus M/p has a circuit D of size exceeding two and, as $D \cup p$ is not a circuit of M, we deduce that D is a circuit of M. Similarly, $(D - d) \cup d'$ is a circuit of M where d is some element of D, and d' is parallel to d in M/p. Thus $\mathrm{cl}_M(D - d)$ contains $\{d, d'\}$ and so contains p. Then $r_{M/p}(D - d) < |D - d|$; a contradiction. □

We are now ready to prove the main result.

Proof of Theorem 1.2. It is easily checked that $M(K_4)$, F_7, and each $M(K'_{2,n})$ are binary having no odd circuits of size greater than three. For the converse, assume that M has no odd circuits of size greater than three. Suppose M is not affine. If $r(M) = 3$, then clearly M is isomorphic to $M(K'_{2,2})$, $M(K_4)$, or F_7. Thus we may assume that $r(M) \geq 4$. First we show the following.

2.3.1. *If T_0 is a triangle of M and C is a circuit that meets but is not equal to T_0, then $|C| \leq 4$ and $M|(T_0 \cup C) \cong M(K'_{2,2})$.*

This is certainly true if C is a triangle, so we assume that $|C| \geq 4$. By Lemma 2.1, $|C \cap T_0| = 2$. Then $C\Delta T_0$ is a circuit of M of cardinality $|C| - 1$. Thus $|C| = 4$ and $C\Delta T_0$ is a triangle T_1 meeting T_0 in a single element. Hence $M|(T_0 \cup C) = M|(T_0 \cup T_1) \cong M(K'_{2,2})$, and (2.3.1) holds.

As M is not affine, it contains a triangle T. As M is connected, it follows by (2.3.1) that M has a triangle T' that meets T in a single element, say f.

2.3.2. *For each g not in $\mathrm{cl}(T \cup T')$, there is a triangle that contains $\{g, f\}$.*

As M is connected, it has a circuit D that contains g and meets $T \cup T'$. Without loss of generality, we may assume that D meets T. By (2.3.1), $M|(D \cup T) \cong M(K'_{2,2})$. Thus M has a triangle T'' that contains g and meets T in a single element, h. We may assume that $h \neq f$ otherwise (2.3.2) holds. Then T'' meets the 4-element circuit $(T \cup T') - f$ in a single element; a contradiction to Lemma 2.1. We deduce that (2.3.2) holds.

We may assume that M has a circuit C' that contains f and is not a triangle otherwise the result follows by Lemma 2.2. By Lemma 2.1, C' meets each triangle containing f in two elements. Moreover, by (2.3.1), $|C'| = 4$. Hence M has at most three triangles containing f. But, as $r(M) \geq 4$, it follows that $r(M) = 4$, and M has exactly two elements not in $\mathrm{cl}(T \cup T')$, these elements being contained in a common triangle with f.

If $T \cup T'$ is a flat of M, then $M \cong M(K'_{2,3})$. Thus we may assume that $\mathrm{cl}(T \cup T') - (T \cup T')$ contains an element h. Then $M|(T \cup T' \cup h) \cong M(K_4)$, so $T \cup T' \cup h$ contains a 4-circuit D' containing $\{f, h\}$. By (2.3.2), M has a triangle that meets D' in $\{f\}$. This contradiction to Lemma 2.1 completes the proof of the theorem. □

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