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The 3-Connected Graphs with Exactly Three Non-Essential Edges*

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Abstract. An edge e of a simple 3-connected graph G is essential if neither the deletion $G\backslash e$ nor the contraction G/e is both simple and 3-connected. Tutte's Wheels Theorem proves that the only simple 3-connected graphs with no non-essential edges are the wheels. In earlier work, as a corollary of a matroid result, the authors determined all simple 3-connected graphs with at most two non-essential edges. This paper specifies all such graphs with exactly three non-essential edges. As a consequence, with the exception of the members of 10 classes of graphs, all 3-connected graphs have at least four non-essential edges.

1. Introduction

Let G be a simple 3-connected graph. An edge e is *deletable* if the deletion $G\backslash e$ is 3-connected; e is *simple-contractible* if G/e is both simple and 3-connected. Tutte [7] called an edge *essential* if it is neither deletable nor simple-contractible. This paper determines all simple 3-connected graphs with exactly three non-essential edges.

The graph terminology used here that is otherwise unexplained will follow Bondy and Murty [1]. A *triangle* in a graph is the edge set of a 3-cycle. A *triad* is the set of edges meeting a degree-3 vertex. An edge e of a graph G is *subdivided* if e is replaced by a path P that has length at least one, connects the ends of e, and has none of its internal vertices in G. This subdivision is *non-trivial* if P has length at least two.

The next result, Tutte's Wheels Theorem [7, (4.1)], characterizes all simple 3-connected graphs in which every edge is essential.

1.1. Theorem. *Let G be a simple 3-connected graph. Then every edge is essential if and only if G is a wheel.*

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The Wheels Theorem is of fundamental importance in the study of 3-connected graphs. It implies that all simple 3-connected graphs can be constructed starting from wheels by repeatedly adding edges or splitting vertices. The theorem ensures the existence of at least one non-essential edge in every simple 3-connected graph that is not a wheel. The existence of such edges has been a useful induction tool. In studying 3-connected graphs, it is sometimes important to know not only that non-essential edges exist but also to know the distribution of such edges. In [5], it is shown that each longest cycle in a minimally 3-connected graph contains at least two non-essential edges.

In this paper, we show that all simple 3-connected graphs with exactly three non-essential edges belong to one of seven infinite classes of graphs. It follows immediately that every edge in such a graph is in a triangle or an edge cut of size three. The problem of determining all simple 3-connected graphs with exactly k non-essential edges for some fixed k exceeding three seems more complex. However, we have completely determined all minimally 3-connected graphs with at most four non-essential edges. Reid and Wu [6] extended this by determining all minimally 3-connected graphs with at most five non-essential edges. This paper is structured as follows. In the next section, we introduce seven infinite classes of simple 3-connected graphs each with exactly three non-essential edges and state the main result of the paper, that these are the only such graphs. The proof of this theorem relies heavily on the main result of [3]. This is stated in Section 3 along with several preliminary lemmas. The main theorem is proved in Section 4 and that section also determines all minimally 3-connected graphs with exactly four non-essential edges.

2. The Main Result

In this section, we state the main result of the paper, a characterization of all simple 3-connected graphs with exactly three non-essential edges. We begin by describing all simple 3-connected graphs in which the set of non-essential edges consists of exactly two edges or exactly three edges that form a triangle. Three classes of graphs arise in this case. These are constructed from wheels in a way that we now describe. For all $k \geq 2$, a *triangle-sum of k wheels* (see Fig. 1(a)) is the graph that can be obtained from k disjoint wheels and a single triangle with edge set $\{x, y, z\}$ by identifying $\{x, y, z\}$ with a triangle in each of the k wheels. For all $k \geq 3$, a *k-dimensional wheel* (see Fig. 1(b)) is constructed as follows: begin with a clique with vertex set $\{v_1, v_2, v_3\}$; replace the edge v_1v_3 by k internally disjoint paths each having length at least two and avoiding v_2; and join each internal vertex of each of these paths to v_2. A *twisted wheel* (see Fig. 1(c)) is a graph that can be obtained from a clique with vertex set $\{v_1, v_2, v_3, v_4\}$ by subdividing each of the edges v_1v_2 and v_3v_4 non-trivially and then joining each of the new vertices obtained by subdividing v_1v_2 and v_3v_4 to v_3 and v_1, respectively. The following theorem [3, Corollary 5.4] was obtained as a consequence of a result for 3-connected matroids.

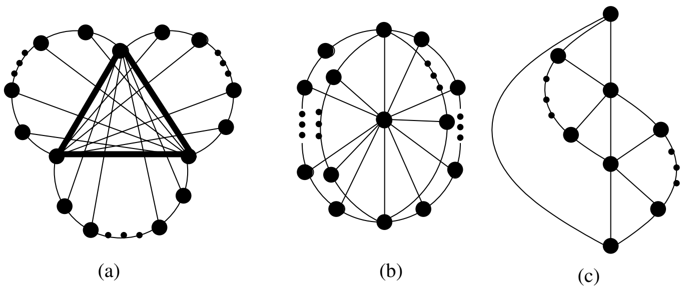


Fig. 1. The three classes of graphs in Theorem 2.1

2.1. Theorem. *Let G be a simple 3-connected graph other than a wheel. Then the set of non-essential edges contains at least two members. Moreover, this set contains either exactly two members or exactly three members forming a triangle if and only if G is a triangle-sum of n wheels for some* $n \geq 2$, *a k-dimensional wheel for some* $k \geq 3$, *or a twisted wheel.*

One of the families of simple 3-connected graphs with exactly three non-essential edges is the class of triangle-sums of wheels. The remaining six classes are constructed from a wheel with 3 or 4 spokes or from a prism graph, $(K_5 - e)^*$, in a way that we now describe. The first of these classes is constructed as follows from K_4 labelled as in Fig. 2(a). Add m, n, and p edges parallel to a, b, and c, respectively, such that $m \geq n \geq 1$ and $p \geq 0$. If $p > 0$, we non-trivially subdivide every edge in each non-trivial parallel class and then join each newly created vertex to the hub h. We follow the same procedure if $p = 0$ except that we might or might not subdivide the edge c. The class of graphs constructed in this way is denoted by $\mathscr{A}$.

The members of the second and third classes are constructed from the prism graph labelled as in Fig. 2(b) or 2(c) by subdividing each of a and b non-trivially and joining each newly created vertex to the vertices h and k, respectively. We denote by $\mathscr{B}$ and $\mathscr{C}$, respectively, the classes of graphs formed in this way and call the members of $\mathscr{B}$ and $\mathscr{C}$ *split-wheels* and *crossed split-wheels*.

The graphs in the fourth class $\mathscr{D}$ are constructed from the graph K_4 labelled as in Fig. 2(d). First we subdivide the edge b non-trivially and then join each newly created vertex to the vertex k. Next we add some non-empty set of new edges parallel to a. We non-trivially subdivide each edge in the resulting parallel class and join each newly created vertex to the vertex h.

The graphs in the fifth class $\mathscr{E}$ are constructed from the wheel $\mathscr{W}_4$ drawn as in Fig. 2(e). First, for some $m \geq 0$, we add m new edges parallel to a. If $m \geq 1$, we non-trivially subdivide each edge in the parallel class containing a and join each newly created vertex to the vertex h. Then we subdivide the edge b non-trivially and join each newly created vertex to the vertex k. Finally, we subdivide the edge c, possibly trivially, and, when there are newly created vertices, join each to the vertex j. If $m = 0$, we subdivide both b and c non-trivially and then join each newly created vertex to the vertices k and j, respectively. Then we either leave a

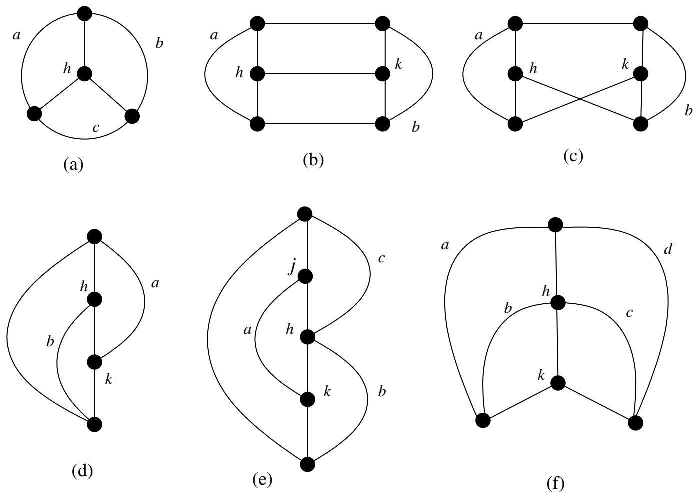


Fig. 2. The starting graphs for the classes $\mathscr{A}$–$\mathscr{F}$

alone or subdivide it non-trivially and join each newly created vertex to the vertex h.

The graphs in the sixth class $\mathscr{F}$ are again constructed from the wheel $\mathscr{W}_4$, this time labelled as in Fig. 2(f). First, we non-trivially subdivide each of the edges b and c and join each newly created vertex to the vertex k. Next, for each of a and d, we either leave the edge unchanged, or we subdivide it non-trivially and join each newly created vertex to the vertex h.

Fig. 3(a)–(f) show some examples of graphs in the classes $\mathscr{A}$–$\mathscr{F}$. The following is the main result of the paper.

2.2. Theorem. *Let G be a simple 3-connected graph other than a wheel. Then G has exactly three non-essential edges if and only if G is a triangle-sum of n wheels for some $n \geq 2$ or G is a graph in one of the six classes $\mathscr{A}$–$\mathscr{F}$.*

On combining this theorem with Theorems 1.1 and 2.1, we obtain the following description of all simple 3-connected graphs with at most three non-essential edges.

2.3. Corollary. *Let G be a 3-connected graph with at most three non-essential edges. Then G is a wheel, a twisted wheel, a triangle-sum of n wheels for some $n \geq 2$, a k-dimensional wheel for some $k \geq 2$, or one of a graphs in the classes $\mathscr{A}$–$\mathscr{F}$.*

A consequence of Theorem 2.1 is that a minimally 3-connected graph other than a wheel has at least three non-essential edges. The next corollary identifies the graphs that attain equality here.

2.4. Corollary. *Let G be a minimally 3-connected graph other than a wheel. Then G has at least three non-essential edges. Moreover, G has exactly three non-essential edges if and only if G is a split-wheel or a crossed split-wheel.*

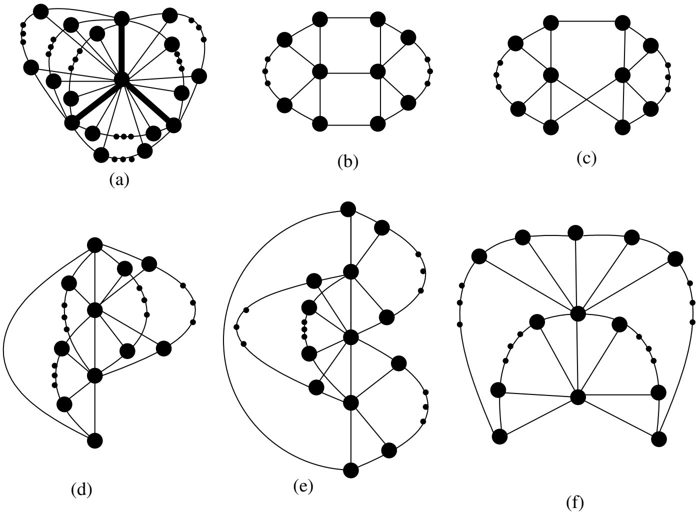


Fig. 3. Some graphs in the classes $\mathscr{A}$–$\mathscr{F}$

Theorem 2.2 will be proved in Section 4, where we will also determine all minimally 3-connected graphs with exactly four non-essential edges.

3. Preliminaries

Let G be a simple 3-connected graph other than a wheel and F be a subset of $E(G)$ containing $k+2$ elements for some $k \geq 1$. Then F is a *fan* if F is a maximal subset of $E(G)$ for which the edges can be ordered $a_1, a_2, \ldots, a_{k+2}$ such that, for all i in $\{1, 2, \ldots, k\}$, either

(a) $\{a_i, a_{i+1}, a_{i+2}\}$ is a triangle when i is odd and a triad when i is even; or
(b) $\{a_i, a_{i+1}, a_{i+2}\}$ is a triad when i is odd and a triangle when i is even.

A fan is *trivial* if $k < 3$. If F is a non-trivial fan, then it follows from [3, Lemma 3.4] that the ordering of F for which (a) or (b) holds is unique up to a complete reversal that interchanges a_j and $a_{(k+2)-j}$ for all j. We call this ordering the *canonical ordering* of F. Its existence implies that the *ends* a_1 and a_{k+2} of F are well-defined. Moreover, the canonical ordering of F satisfies exactly one of the following:

(i) k is odd and $\{a_1, a_2, a_3\}$ is a triangle;
(ii) k is odd and $\{a_1, a_2, a_3\}$ is a triad; or
(iii) k is even and $\{a_1, a_2, a_3\}$ is a triangle.

A trivial fan can be ordered in more than one way so that (a) or (b) holds, but each of these orderings satisfies the same one of (i)–(iii). We say that F is of *type*-1, *type*-2, or *type*-3 depending on which of (i), (ii), or (iii), respectively, holds.

Figure 4 illustrates the three types of fans, where each circled vertex has degree 3 in G. In each case, the vertex v in the *hub* of the fan while the vertices u, w, and z in Fig. 4(b)–(c) are called *vertex-ends* of the fan. We observe that, when $|F| \leq 4$, the choice of the hub and the vertex-ends depends on the ordering of F so we arbitrarily fix such an ordering that obeys (a) or (b). We also use this ordering to determine the *ends* of F as the first and last elements on the ordering. The set of vertices incident with some edge of a fan F will be denoted by $V(F)$. A fan with ends x and y will be called an *xy-fan*.

Fans were originally defined in [3] for matroids using the definition above. However, in a matroid, a "triad" is a 3-element cocircuit. Thus a triad in the cycle matroid $M(G)$ of a graph G is a minimal edge cut of G of size three. Now let G be simple and 3-connected and let T be a triad in $M(G)$ that meets a triangle. Then T must consist of the three edges meeting a degree-3 vertex. Hence every triad in the matroid $M(G)$ that is involved in a non-trivial fan in $M(G)$ must be a triad of the graph G. However, a minimal edge cut in G of size three that does not consist of the three edges meeting a degree-3 vertex is a fan in $M(G)$ but not in G. Although a crucial tool here will be the next theorem, a specialization to graphs of a matroid result [3], this minor difference between fans in $M(G)$ and fans in G will not create any difficulties. This is because Tutte [7] proved that every essential edge of G is in a triangle or a triad of G and hence is in a fan of G. It is easy to see that every edge that is in both a triangle and a triad of G is essential. The next

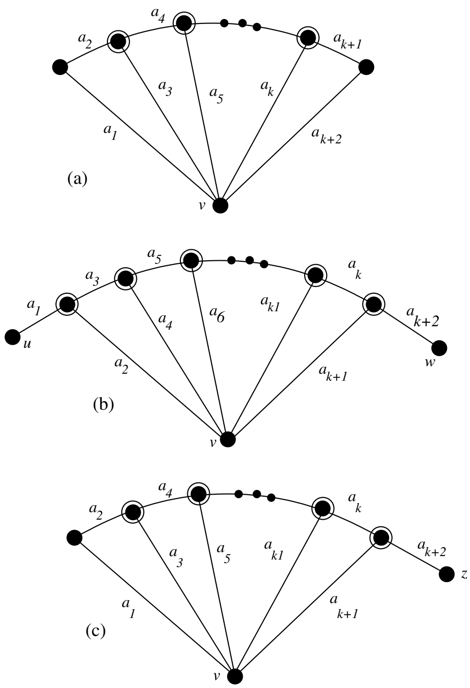


Fig. 4. The three types of fans

theorem, which will be used frequently in the paper, shows that the essential edges in a simple
3-connected graph other than a wheel can be partitioned into subsets so that two edges are in the same subset if and only if there is a fan containing them both. We remark that, except where otherwise indicated, all fans and triads from now on will be fans and triads in graphs rather than in their cycle matroids.

3.1. Theorem. *Let G be a simple 3-connected graph other than a wheel. Suppose that e is an essential edge of G. Then e is in a fan, both ends of which are non-essential. Moreover, this fan is unique unless*

(*i*) *every fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$;*
(*ii*) *e is in exactly two triads and each such triad has exactly two non-essential edges;*
(*iii*) *e is in exactly three fans; these three fans are of the same type, each has five edges, together they contain a total of six edges; and, depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of G to this set of six edges is isomorphic to K_4.*

The rest of this section presents five lemmas that will be used in the proof of the main result. The first of these was proved in [6, Lemma 7].

3.2. Lemma. *Let G be a simple 3-connected graph other than a wheel and F be a non-trivial type-2 fan with two ends x and y. Then x and y are not adjacent.*

3.3. Lemma. *Let G be a simple 3-connected graph other than a wheel. Suppose G has two type-2 fans N_1 and N_2 with the same ends, x and y. Then G is a split-wheel.*

Proof. The maximality of N_1 and N_2 implies that the hubs, v_1 and v_2, respectively, of these two fans are distinct. Moreover $\{v_1, v_2\}$ is not a vertex cut of G, so $V(G) = V(N_1) \cup V(N_2)$. As $\{x, y\}$ is not an edge cut of G, it follows that $v_1v_2 \in E(G)$ and so G is indeed a split-wheel. □

3.4. Lemma. *Let G be a simple 3-connected graph. Suppose that G has two type-3 fans, F_1 and F_2, with the same ends, x and y. Then G is a twisted wheel.*

Proof. Without loss of generality, we may assume that that x is deletable and y is simple-contractible. Let $x = uv$. Clearly the subgraph of G induced by $V(F_1) \cup V(F_2)$ is either a wheel or a twisted wheel. But y is simple-contractible, so it is not in any triangle. Thus neither of the fans is trivial and $V(F_1) \cup V(F_2)$ induces a twisted wheel as a subgraph of G. Next we show that $V(G) = V(F_1) \cup V(F_2)$. Assume that G has an edge f that is not in F_1 or F_2 but is adjacent to exactly one vertex in $V(F_1) \cup V(F_2)$. Since both F_1 and F_2 are type-3 fans, f must be adjacent to u or v. Therefore, $\{u, v\}$ is a vertex cut of G; a contradiction. Hence $V(G) = V(F_1) \cup V(F_2)$ and it follows that G is a twisted wheel. $\square$

3.5. Lemma. *Let G be a simple 3-connected graph other than a wheel. Suppose that G has a fan F. Then G has at least two edges not in the fan.*

Proof. Since G is simple and 3-connected, $|E(G)| \geq 6$, we may assume that F is non-trivial having ends x and y. Now $F \neq E(G)$ otherwise G is a wheel. Suppose that G has a unique edge z not in F. If F is of type-1, then $\{x, y, z\}$ is a triangle and G is a wheel; a contradiction. If F is of type-2, then, since each vertex has degree at least three, x, y, and z must meet at a common vertex; a contradiction to Lemma 3.2. If F is type-3, then G must have a vertex of degree at most two. This contradiction completes the proof of the lemma. $\square$

3.6. Lemma. *Let G be a simple 3-connected graph with exactly three non-essential edges x, y, and z. Then none of these three edges is both deletable and simple-contractible.*

Proof. Suppose that x is both deletable and simple-contractible. Since G has exactly three non-essential edges, by Theorem 3.1, every edge not in $\{x, y, z\}$ must be in a fan with two non-essential edges as its ends. Since x is both deletable and simple-contractible, it is not in any triangle or triad. Thus x is not in any fan. Therefore every essential edge e is in a yz-fan F. Now it is not difficult to see that none of (i)–(iii) in Theorem 3.1 could hold for e. Thus every essential edge is in a unique fan. Let $F_1, F_2, \ldots, F_k$ be all the fans of G. If $k = 1$, then $E(G) = F_1 \cup x$; a contradiction to Lemma 3.5. Hence $k \geq 2$. All of $F_1, F_2, \ldots, F_k$ must be of the same type as each has y and z as its ends. Thus $G\backslash x$ can be constructed by sticking together some yz-fans of the same type. If these fans are of type-1, then $G\backslash x$ can be constructed by sticking the fans $F_1, F_2, \ldots, F_k$ together along y and z. Clearly, x can only be adjacent to the end-vertices of y and z. It follows that $\{x, y, z\}$ must be a triangle in G. This is a contradiction as x is simple-contractible. Now suppose all of $F_1, F_2, \ldots, F_k$ are of type-2. By Lemma 3.3, G is a split-wheel; a contradiction as x is not deletable. Finally, assume that all of the fans are of type-3. Since $k \geq 2$, by Lemma 3.4, G must be a twisted wheel and so G has exactly two non-essential edges. This contradiction completes the proof of the lemma. $\square$

4. A Proof of The Main Theorem

Proof of Theorem 2.2. If G is a triangle-sum of n wheels for some $n \geq 2$ or a graph in one of the six classes $\mathscr{A}$–$\mathscr{F}$, then it is straightforward to check that G has exactly three non-essential edges.

Now suppose that G is a simple 3-connected graph with exactly three non-essential edges x, y, and z. By Lemma 3.6, none of these three edges is both deletable and simple-contractible. Therefore exactly one of the following cases occurs:

(a) all of x, y, and z are deletable but not simple-contractible;
(b) all of x, y, and z are simple-contractible but not deletable;
(c) one edge is simple-contractible but not deletable, and the other two edges are deletable but not simple-contractible; and
(d) two edges are simple-contractible but not deletable, and the other edge is deletable but not simple-contractible.

In case (a), if $\{x, y, z\}$ forms a triangle, then it follows by Theorem 2.1 that G is a triangle-sum of n wheels for some $n \geq 2$. Therefore we may assume that $\{x, y, z\}$ is not a triangle. Since none of the edges in G is simple-contractible, G has only type-1 fans. If G has only trivial fans, then, as G is simple, for each 2-subset $\{j, k\}$ of $\{x, y, z\}$, there is at most one jk-fan. As each essential edge is in a fan, G has at most six edges. As G is 3-connected, $G \cong K_4$; a contradiction. Thus we may assume that G has a fan, say an xy-fan N, having at least five edges.

Next assume that every edge except z is in some xy-fan. Then, by Theorem 3.1, any two distinct xy-fans have $\{x, y\}$ as their intersection. Thus $G \backslash z$ can be constructed by sticking type-1 fans together along the edges x and y. Hence z can only be adjacent to the end-vertices of x or y. As G is simple, it follows that $\{x, y, z\}$ is a triangle; a contradiction.

We may now suppose that G has an edge e that is not in $\{x, y, z\}$ and is not in any xy-fan. Then, without loss of generality, e is in a yz-fan. Next we show that x, y, and z must meet a common vertex. Suppose not. Since x and y are adjacent and y and z are adjacent but $\{x, y, z\}$ is not a triangle, it follows that x and z are not adjacent. Therefore there is no xz-fan, so every edge is in some xy- or yz-fan. Since any xy-fan and any yz-fan will have the end-vertices of y as their only common vertices, it follows that these end-vertices form a vertex cut; a contradiction. We conclude that x, y, and z do indeed meet a common vertex. We also know that G has both an xy-fan and a yz-fan. Moreover, G must also have an xz-fan, otherwise the end-vertices of y will again form a vertex cut. Since G is not a wheel and has exactly three non-essential edges, it is not difficult to see now that G is in $\mathscr{A}$.

In case (b), since none of the edges of G is deletable, G is a minimally 3-connected graph. By Theorem 3.1, every essential edge is in a type-2 fan. If G has at least two fans with identical ends, then, by Lemma 3.3, G is a split-wheel and is in $\mathscr{B}$. Thus we may assume that G has at most one xy-fan, at most one yz-fan, and at most one xz-fan. Without loss of generality, suppose that G has an xy-fan F_0. By Lemma 3.5, there is at least one essential edge e_1 not in F_0. By Theorem 3.1, G has a type-2 fan F_1 containing e_1. Without loss of generality, we may assume that F_1 is an xz-fan.

Suppose next that there is an edge e_2 that is not in F_0 or F_1. Then e_2 is in a yz-fan F_2, which must be of type-2. Since G has at most one xy-fan, at most one yz-fan, and at most one xz-fan, every edge of G is in one of F_0, F_1, and F_2. Since G is 3-connected, we conclude that the hubs of these three fans are equal. Thus G is a wheel. This contradiction implies that every edge in G is in F_0 or F_1. Since G is 3-connected, we conclude that y meets the hub of the xz-fan F_1 and z meets the hub of the xy-fan F_0. Moreover, as G is 3-connected but not a wheel, neither fan is trivial. Thus G is a crossed split-wheel and is in $\mathscr{C}$. This finishes case (b).

The next lemma will be used in cases (c) and (d).

4.1. Lemma. *Let G be a simple 3-connected graph having exactly three non-essential edges and suppose these edges satisfy (c) or (d). Then every essential edge is in a unique fan. Moreover, every simple-contractible edge is in at most two fans.*

Proof. Suppose that some essential edge e is in more than one fan. Then, as G has only three non-essential edges, we deduce that Theorem 3.1(iii) occurs. Thus all three non-essential edges are deletable or all three are simple-contractible; a contradiction to the assumption that (c) or (d) holds. Hence every essential edge is in a unique fan. Since G has at most two simple-contractible edges, it follows by Theorem 3.1 that no essential edge of G is in more than one fan. As every edge of G meets at at most two degree-3 vertices, we deduce that every simple-contractible edge is in at most two fans. $\square$

In case (c), let x and y be the deletable edges of G and z be the simple-contractible edge.

4.2. Lemma. *There is at least one xy-fan.*

Proof. Suppose that G has no xy-fans. Then, by Lemma 4.1, every essential edge is in an xz- or yz-fan. Since z is simple-contractible and x and y are deletable, these fans must all be of type-3. Without loss of generality, we may suppose that G has an xz-fan. If there is more than one xz-fan, then, by Lemma 3.4, G is a twisted wheel and has exactly two non-essential edges; a contradiction. Thus G has exactly one xz-fan N_1. By Lemma 3.5, G has at least one essential edge not in N_1. Thus G has a yz-fan N_2. By Lemma 3.4 again, we deduce that G has exactly one yz-fan. Since G has no xy-fans, it follows that $E(G) = N_1 \cup N_2$. Moreover, by Lemma 4.1, $N_1 \cap N_2 = \{z\}$. Since G is simple and 3-connected, we conclude that $x = y$ and G is either a wheel or a twisted wheel. This contradiction completes the proof of the lemma. $\square$

We now know that G has an xy-fan F. If every essential edge is in an xy-fan, then $G\backslash z$ can be constructed by sticking some type-1 fans together along x and y. It follows that $\{x, y, z\}$ is a triangle, contradicting the fact that z is simple-contractible. Thus there is an xz-fan or a yz-fan. By Lemma 4.1, z is in at most two type-3 fans.

Suppose that z is in exactly one type-3 fan, say an xz-fan N. If every edge is in $F \cup N$, then, as G is 3-connected, y and z must be adjacent and so G is either a wheel or a twisted wheel; a contradiction. Hence $E(G) - (F \cup N)$ is non-empty. Since z is in only one type-3 fan N, each edge in $E(G) - (F \cup N)$ must be in an xy-fan. As x, y, and z are the only non-essential edges of G, each of these xy-fans is non-trivial. We now know that G has at least two xy-fans, exactly one xz-fan N, and no yz-fans. Let $y = uv$ where v is the common hub of the xy-fans. Clearly, z can be only incident to u or v. Since G is 3-connected, z must be incident to u. If N

is trivial, then z is in a triangle; a contradiction as it is simple-contractible. The hub of N cannot be v otherwise z is again in a triangle. We conclude that G is in $\mathscr{D}$.

We may now assume that z is in exactly two type-3 fans. By Lemma 3.4, these two fans cannot have identical ends. Thus one of these fans is the xz-fan N and the other is a yz-fan, say T. If there is an edge in none of the fans F, N, and T, then it must be in some xy-fan. Thus there is an integer $t \geq 1$ such that G has t xy-fans, one xz-fan, and one yz-fan. Note that if $t = 1$, then neither N nor T is trivial, otherwise G is a wheel or a twisted wheel; a contradiction. Thus, when $t = 1$, the graph G is in $\mathscr{E}$. Suppose $t > 1$. Then none of the xy-fans is trivial otherwise the three edges of this fan are all non-essential. If both the xz-fan N and the yz-fan T are trivial, then z is in a triangle; a contradiction as z is simple-contractible. We conclude that N or T is non-trivial and so, when $t > 1$, the graph G is again in $\mathscr{E}$.

In case (d), let x be the deletable edge of G and y and z be its simple-contractible edges. Then G has no type-1 fans. Thus, by Lemma 4.1, every essential edge of G is in a unique type-2 or type-3 fan. Suppose first that G has no type-3 fans. Then every essential edge is in a type-2 fan, which must have ends y and z. By Lemma 3.5 and Theorem 3.1, G has at least two such fans. By Lemma 3.3, G is a split-wheel so x is not deletable; a contradiction.

We may now assume that G has a type-3 fan P_1 and, without loss of generality, we may assume that P_1 is an xy-fan. By Lemma 3.4, P_1 is the unique xy-fan otherwise G is a twisted wheel; a contradiction. We show next that G has an xz-fan. Suppose not. By Lemma 3.5 and Theorem 3.1, G has a fan P_0 different from P_1. From above, P_0 is not an xy-fan and, by assumption, P_0 is not an xz-fan. Thus P_0 is a yz-fan. Since y is in both P_1 and P_0, Lemma 4.1 implies that G has no other fans. Thus $E(G) = P_1 \cup P_0$. Let $x = h_1k_1$ where h_1 is the hub of P_1. Let h_0 be the hub of P_0. Then $h_0 \neq h_1$ since y is not in any triangles. As G is 3-connected, we deduce that $h_0 = k_1$ and z meets h_1. Hence z is in a triangle, contradicting the fact that z is simple-contractible.

We now know that G has an xz-fan P_2. Moreover, by Lemma 3.4, it is the unique such fan. If P_1 and P_2 are the only fans of G, then $E(G) = P_1 \cup P_2$ and it is not difficult to see that G cannot be 3-connected. Hence G has a third fan P_3, which must be a yz-fan. By Lemma 4.1, since P_1 and P_3 contain y, there are no other yz-fans in G. Therefore $E(G) = P_1 \cup P_2 \cup P_3$. Let h_3 be the hub of P_3 and recall that $x = h_1k_1$ where h_1 is the hub of P_1. As G is 3-connected, $h_3 = h_1$ or $h_3 = k_1$. If h_1 is not the hub of P_2, then k_1 is the hub of P_2 and it is not difficult to check that y or z must be in a triangle, contradicting the fact that both edges are simple-contractible. Therefore h_1 is the hub of both P_2 and P_1. Since G is not a wheel, it follows that $h_3 \neq h_1$, so $h_3 = k_1$. Moreover, neither P_1 nor P_2 is trivial, otherwise y or z, respectively, is in a triangle. We conclude that G is in $\mathscr{F}$. Note that P_3 can be trivial. This completes case (d) and thereby finishes the proof of the theorem. $\square$

Proof of Corollary 2.4. Let G be a minimally 3-connected graph other than a wheel. By Theorem 2.1, G has at least two non-essential edges. Moreover, G has exactly two such edges if and only if G is a twisted wheel or a k-dimensional wheel for some $k \geq 3$. But neither a twisted wheel nor a k-dimensional wheel with $k \geq 3$

is minimally 3-connected. Thus G has at least three non-essential edges. If G has exactly three non-essential edges, then G is in one of the seven classes of graphs specified in Theorem 2.2. But, among these graphs, only split-wheels and crossed split-wheels are minimally 3-connected. Hence G is a split-wheel or a crossed split-wheel. □

Next we determine all minimally 3-connected graphs with exactly four non-essential edges. We begin by constructing two such families of graphs. Take a cycle C and partition its vertex set into four non-empty subsets V_1, V_2, V_3, and V_4 such that vertices in each of these sets, as well as those in each of $V_1 \cup V_2, V_2 \cup V_3, V_3 \cup V_4$, and $V_4 \cup V_1$, induce paths in C. Now add two new vertices v_{13} and v_{24} joining the first to each vertex in $V_1 \cup V_3$ and the second to each vertex in $V_2 \cup V_4$. Let the resulting graph be G. When $|V_i| \geq 2$ for all i, the graph G is a *doubly interlocked wheel* (see Fig. 5(b)). Now suppose that $|V_2|, |V_4| \geq 2$ and $V_1 = \{u_1\}$. The graph G/u_1v_{13} is a *three-fan* (see Fig. 5(a)).

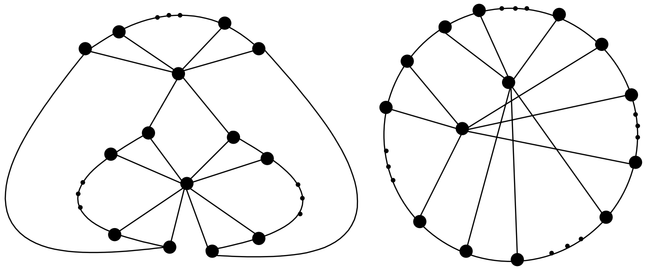
4.3. Theorem. *A graph G is minimally 3-connected having exactly four non-essential edges if and only if G is a three-fan or a doubly interlocked wheel.*

Proof. If G is a three-fan or a doubly interlocked wheel, then it is straightforward to check that G is a minimally 3-connected graph with exactly four non-essential edges.

Now suppose that G is a minimally 3-connected graph with exactly four non-essential edges a, b, c, and d. Then G has no deletable edges. Hence, by Theorem 3.1, every essential edge of G is in a type-2 fan with two non-essential edges as its ends. Moreover, there is a partition of the essential edges such that two are in the same class if and only if there is a fan that contains both or, equivalently, the two edges belong to precisely the same fans. Let k denote the number of such equivalence classes. Since G is 3-connected, it is straightforward to check that $k \geq 2$. Next we bound k above.

4.4. Lemma. $k \leq 4$. *Moreover,* $k = 4$ *if and only if G is a doubly interlocked wheel.*

Proof. Consider the number of pairs (X, f) such that X is an equivalence class and f is a non-essential edge for which $X \cup f$ is contained in a fan. For each non-



(a) (b)

Fig. 5. (a) A three-fan, (b) A doubly interlocked wheel

essential edge f, there are at most two such pairs as f meets at most two degree-3 vertices. Hence the total number of these pairs is at most $4 \cdot 2$. As each fan has two ends, the number of such pairs is at least $2k$. Thus $2k \leq 8$, so $k \leq 4$. Furthermore, $k = 4$ if and only if each non-essential edge is an end of exactly two fans and each equivalence class is contained in a unique fan.

Suppose that $k = 4$. For each 2-subset $\{x, y\}$ of $\{a, b, c, d\}$, let F_{xy} be the unique fan with ends x and y, and let v_{xy} be its hub. As none of a, b, c, or d is in a triangle, $\{v_{ab}, v_{cd}\} \cap \{v_{bc}, v_{ad}\} = \emptyset$. Since G is 3-connected, it follows that $v_{ab} = v_{cd}$ and $v_{bc} = v_{ad}$. Now both v_{ab} and v_{bc} have degree at least three. Thus at most one member of each of $\{F_{ab}, F_{cd}\}$ and $\{F_{bc}, F_{ad}\}$ is trivial. If one of F_{ab}, F_{bc}, F_{cd}, or F_{ad} is trivial, then it is straightforward to check that G has at least five non-essential edges; a contradiction. We conclude that all of F_{ab}, F_{bc}, F_{cd}, and F_{ad} are non-trivial. Hence G is a doubly interlocked wheel. □

By the last lemma, we may assume that $2 \leq k \leq 3$. First suppose that $k = 2$. Let F_1 and F_2 be two fans, one containing each of the classes of essential edges. Let the hubs of F_1 and F_2 be v_1 and v_2, respectively. Without loss of generality, assume that F_1 has a and b as it ends. Next we show that one of a and b is an end of F_2. Assume the contrary. Then c and d are the ends of F_2. Suppose that b and d meet in a common vertex v. Then clearly $v \neq v_1, v_2$. By Lemma 3.2, a and b are not adjacent and c and d are also not adjacent. Thus the vertex v has degree two in G; a contradiction. Hence b and d are not adjacent. By symmetry and Lemma 3.2, no two edges in the set $\{a, b, c, d\}$ are adjacent. Let $a = st$ and $b = uv$ where t and v are the vertex-ends of F_1. As t has degree at least three, it follows that $t = v_2$. We conclude that v has degree one; a contradiction.

We may now assume that a or b, say b, is an end of F_2. Then, as b is not in a triangle, $v_1 \neq v_2$. Moreover, by Lemma 3.3, a is not an end of F_2. Without loss of generality, assume the other end of F_2 is c. Now $E(G) = F_1 \cup F_2 \cup \{a, b, c, d\}$. Suppose that a and c are not adjacent. Then, as every vertex has degree at least three, it follows that a is incident to v_2, and c is incident to v_1. Moreover, $d = v_1v_2$. Thus G is a crossed split-wheel and has exactly three non-essential edges; a contradiction. We may now assume that a and c meet in a common vertex v_3. Then $v_3 \neq v_1, v_2$. As v_3 has degree at least three, d is also incident with v_3. The other end-vertex of d is either v_1 or v_2 and, in either case, d is in a triangle. This contradiction implies that $k \neq 2$.

It remains to consider the case when $k = 3$. Let F_1, F_2, and F_3 be three fans, one containing each class of essential edges, and let v_i be the hub of F_i for all i. As G has only four non-essential edges and each fan has two ends, at least one non-essential edge is an end of two fans. By Lemma 3.3, no two fans have the same set of ends. Therefore, without loss of generality, we may assume that F_1 has a and b as its ends, and F_2 has b and c as its ends. Suppose that F_3 has ends a and c. Then v_1, v_2, and v_3 are distinct as none of a, b, and c is in a triangle. It follows that $d \in \{v_1v_2, v_2v_3, v_1v_3\}$ and hence that G is not 3-connected; a contradiction. We conclude that either a or c is not an end of F_3. Note that b is also not an end of F_3 as it is already an end of the two fans F_1 and F_2. Therefore, without loss of generality, c and d are the ends of F_3. Suppose that $v_1 \neq v_3$. Then v_1, v_2, and v_3 are distinct as neither b nor c is in any triangle. As each vertex has degree at least three, a is incident to v_3 or v_2, and d is incident to v_1 or v_2. In each case, there are two non-essential edges that form an edge cut of G; a contradiction. We conclude that $v_1 = v_3$. As each vertex has degree at least three, both a and d are incident to v_2. Since a is not in a triangle, F_1 is non-trivial. Similarly, F_3 is non-trivial. By contrast, the fan F_2 can be trivial. We conclude that G is a three-fan. This completes the proof of the theorem. □

On combining the last theorem with Corollary 2.4, we immediately obtain the following result.

4.5. Corollary. *Let G be a minimally 3-connected with at most four non-essential edges. Then G is a wheel, a split-wheel, a crossed split-wheel, a three-fan, or a doubly interlocked wheel.*

To conclude, we note an immediate consequence of the main theorem that emphasizes the importance of fans.

4.6. Corollary. *Let G be a simple 3-connected graph having at most three non-essential edges. Then G has at most one edge that is not in a fan. Moreover, G has exactly one such edge if and only if G is a split-wheel.*

Observe that, when G is a split-wheel, the edge that is in neither a triangle nor a triad of G is in a triad and hence in a fan of the matroid $M(G)$.

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References

1. Bondy, J. A., Murty, U.S.R.: Graph Theory with Applications, London: Macmillan, New York: North-Holland 1976
2. Oxley, J. G.: Matroid Theory, New York: Oxford University Press 1992
3. Oxley, J. G., Wu, H.: On the structure of 3-connected matroids and graphs. Eur. J. Comb. **21**, 667–688 (2000)
4. Oxley, J. G., Wu, H.: Matroids and graphs with few non-essential edges. Graphs Comb. **16**, 199–229 (2000)
5. Reid, T. J., Wu, H.: A longest cycle version of Tutte's Wheels Theorem. J. Comb. Theory, Ser. B **70**, 202–215 (1997)
6. Reid, T. J., Wu, H.: A non-planar version of Tutte's wheels theorem. Australas. J. Comb. **20**, 3–12 (1999)
7. Tutte, W. T.: A theory of 3-connected graphs, Nederl. Wetensch. Proc. Ser. A **64**, 441–455 (1961)

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