A MATROID ANALOGUE OF A THEOREM OF BROOKS FOR GRAPHS

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Abstract. Brooks proved that the chromatic number of a loopless connected graph G is at most the maximum degree of G unless G is an odd cycle or a clique. This note proves an analogue of this theorem for $GF(p)$-representable matroids when p is prime, thereby verifying a natural generalization of a conjecture of Peter Nelson.

1. Introduction

The terminology and notation for matroids that is used here will follow [10]. For a matroid M having ground set E and rank function r, the *chromatic* or *characteristic polynomial* of M is defined by

$$p(M;\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(M)-r(X)}.$$

If M is the cycle matroid of a graph G and G has $\omega(G)$ components, then the chromatic polynomial $P_G(\lambda)$ of the graph G is linked to the chromatic polynomial of its cycle matroid $M(G)$ via the following equation:

$$P_G(\lambda) = \lambda^{\omega(G)} p(M(G);\lambda).$$

Of course, the chromatic number $\chi(G)$ of G is the smallest positive integer j for which $P_G(j)$ is positive unless G has a loop, in which case, the chromatic number is ∞. Let M be a rank-r simple matroid that is representable over $GF(q)$ and let T be a subset of $PG(r-1,q)$ such that $M \cong PG(r-1,q)|T$. Let Q be a flat of $PG(r-1,q)$ that avoids T and has maximum rank. The *critical exponent* $c(M;q)$ of M is $r - r(Q)$. If M is loopless but has parallel elements, we define $c(M;q) = c(\mathrm{si}(M);q)$. If M has a loop, $c(M;q) = \infty$. Ostensibly, $c(M;q)$ depends on the embedding of M in $PG(r-1,q)$ but the following fundamental result of Crapo and Rota [4] establishes that this is not the case.

Theorem 1.1. *Let M be a loopless matroid that is representable over $GF(q)$. Then*

$$c(M;q) = \min\{j : p(M;q^j) > 0\}.$$

Evidently, the critical exponent is an analogue of the chromatic number of a graph. Indeed, Geelen and Nelson [5] use the term 'critical number' rather than 'critical exponent' to highlight this analogy. For a loopless graph G, it is immediate that, for all prime powers q,

$$q^{c(M(G);q)-1} < \chi(G) \le q^{c(M(G);q)}.$$

Brooks [1] proved the following well-known result. For a graph G, let $\Delta(G)$ denote its maximum vertex degree.

Theorem 1.2. *Let G be a loopless connected graph. Then*

$$\chi(G) \le \Delta(G) + 1.$$

Indeed, $\chi(G) \le \Delta(G)$ unless G is an odd cycle or a complete graph.

The purpose of this note is to prove the following analogue of this result for $GF(q)$-representable matroids when q is prime. This new result was essentially conjectured by Peter Nelson [7]. An alternative analogue of Brooks's Theorem, one for regular matroids, was proved in [8, Theorem 2.12].

Theorem 1.3. *Let p be a prime and M be a loopless non-empty $GF(p)$-representable matroid whose largest cocircuit has c^* elements. Then*

$$c(M;p) \le \lceil \log_p(1 + c^*) \rceil.$$

Indeed, if M is connected, then $c(M;p) \le \lceil \log_p c^* \rceil$ unless M is a projective geometry or M is an odd circuit, where the latter only occurs when $p = 2$.

The requirement that M be connected appears in the last part of the theorem only to streamline the statement. It is not difficult to state a result in the absence of that requirement since the critical exponent of a loopless matroid M is the maximum of the critical exponents of its components while the maximum cocircuit size of M is the maximum of the maximum cocircuit sizes of its components.

We conjecture that Theorem 1.3 remains true if p is replaced by an arbitrary prime power q, but the proof technique used here only works when q is prime.

2. The proof

The proof of the main result will use three lemmas, the first of which is [8, Theorem 3.5]. For a matroid M, let $\mathcal{R}(M)$ be the set of simple restrictions of M, and let $\mathcal{C}^*(M)$ be the set of cocircuits of M.

Lemma 2.1. *Let M be a $GF(q)$-representable matroid having no loops. Then*

$$c(M;q) \le \lceil \log_q(1 + \max_{N \in \mathcal{R}(M)} (\min_{C^* \in \mathcal{C}^*(N)} |C^*|)) \rceil.$$

Murty [6] considered the class of matroids in which all circuits have the same cardinality. His main result, which can be stated as follows, determined all binary matroids with this property.

Lemma 2.2. *Let M be a connected binary matroid with at least two elements. Then every cocircuit of M has the same cardinality if and only if, for some positive integer t, the matroid M can be obtained by adding $t-1$ elements in parallel to each element of one of the following:*

(i) $U_{r,r+1}$ *for some* $r \geq 2$*;*
(ii) $PG(r-1,2)$ *for some* $r \geq 1$*; or*
(iii) $AG(r-1,2)$ *for some* $r \geq 2$*.*

The proof of this result relies heavily on a property that characterizes binary matroids, namely, that the symmetric difference of two circuits is a disjoint union of circuits. Although the last result has not been generalized to all $GF(q)$-representable matroids, the next result is a partial generalization of it that treats the case when q is prime and the cardinality of all cocircuits is a power of q. Recall that a *point* in a matroid is a rank-1 flat in the matroid.

Lemma 2.3. *Let p be a prime exceeding two and M be a loopless non-empty $GF(p)$-representable matroid in which all cocircuits have p^k elements for some non-negative integer k. If M' is a component of M, then* $\mathrm{si}(M')$ *is a projective geometry and every parallel class of M' has the same size, this size being a power of p.*

Proof. For each e in $E(M)$, we write $d(e)$ for $|\mathrm{cl}(\{e\})|$. We argue by induction on $r(M)$ noting that the result is immediate if $r(M) = 1$. If $r(M) = 2$, then either $M \cong U_{1,p^k} \oplus U_{1,p^k}$, or M is a line with n points, for some $n \geq 3$. As the lemma holds in the former case, we consider the latter case, letting $\{e_1, e_2, \ldots, e_n\}$ be a transversal of the set of points of M. Then $|E(M)| - d(e_i) = p^k$ for all i. Thus $d(e_i) = d(e_j)$ for all distinct i and j. Now $(n-1)d(e_1) = p^k$. As p is prime, we deduce that $n = p+1$ and $d(e_1) = p^{k-1}$. Hence M is obtained from $PG(1,p)$ by replacing each element by p^{k-1} elements in parallel. Thus the lemma holds when $r(M) = 2$.

Assume the lemma holds when $r(M) < r$ and let $r(M) = r \geq 3$. If M is disconnected, the result follows by the induction assumption. Thus we may assume that M is connected.

We show next that

2.3.1. $M/\mathrm{cl}(\{e\})$ *is connected for all e in $E(M)$.*

Assume that $M/\mathrm{cl}(\{e\})$ is disconnected for some e in $E(M)$. Then every cocircuit of $M/\mathrm{cl}(\{e\})$ has p^k elements and, by the induction assumption, each component is a projective geometry in which every parallel class has the same size, this size being a power of p. As $M/\mathrm{cl}(\{e\})$ is disconnected, by a result of Brylawski [2], M can be written as the parallel connection, with basepoint e, of some set, S, of connected matroids. Let $M_1, M_2, \ldots, M_s$ be the matroids in S that have rank at least two. The only other possible member of S is $M|\mathrm{cl}(\{e\})$ and it is present if and only if $d(e) > 1$. For each i in $\{1, 2, \ldots, s\}$, take $N_i = M|[E(M_i) \cup \mathrm{cl}(\{e\})]$. Then N_i is certainly connected.

Consider $\operatorname{si}(N_i)$ for some i. By [9, Lemma 2.3] (see also [10, Lemma 4.3.10]), each cocircuit of $\operatorname{si}(N_i)$ has an element f_i such that $N_i/\operatorname{cl}(\{f_i\})$ is connected. Choose a cocircuit of N_i avoiding e. Then $M/\operatorname{cl}(\{f_i\})$ is connected and so its simplification is a projective geometry and all the parallel classes of $M/\operatorname{cl}(\{f_i\})$ have the same size. It follows that $s = 2$ and $r(N_1) = r(N_2) = 2$ otherwise $M/\operatorname{cl}(\{f_1\})$ and $M/\operatorname{cl}(\{f_2\})$ cannot be matroids whose simplifications are projective geometries. We deduce that $r(M) = 3$. As $M/\operatorname{cl}(\{f_i\})$ is connected, $\operatorname{si}(N_i) \cong U_{2,p+1}$ for each i. Thus $|\operatorname{cl}_M(\{f\})| = p^{k-1}$ for all $f \neq e$. But, in $M/\operatorname{cl}(\{f_i\})$, the parallel class containing e has more than $(p-1)p^{k-1}$ elements whereas every other parallel class has exactly p^{k-1} elements; a contradiction. Thus (2.3.1) holds.

By the induction assumption, $M/\operatorname{cl}(\{e\})$ is obtained from $PG(r-2,p)$ by replacing each element by the same number of parallel elements. Every cocircuit of $M/\operatorname{cl}(\{e\})$ contains p^k elements and contains p^{r-2} parallel classes. Thus, in $M/\operatorname{cl}(\{e\})$, each parallel class has size p^{k-r+2}. This number will be the same irrespective of the choice of e.

Next we show the following.

2.3.2. *Every parallel class of M has the same size, n_1; and every line of M contains the same number, n_2, of points. Moreover, (n_1, n_2) is $(p^{k-r+2}, 2)$ or $(p^{k-r+1}, p+1)$.*

Consider a line L of M. Let $\{e_1, e_2, \dots, e_n\}$ be a transversal of the set of points of L. Then, in $M/\operatorname{cl}(\{e_i\})$, we have a parallel class having exactly $[d(e_1) + d(e_2) + \cdots + d(e_n)] - d(e_i)$ elements. Thus, for all i,

$$p^{k-r+2} + d(e_i) = \sum_{j=1}^{n} d(e_j).$$

Therefore $d(e_1) = d(e_2) = \cdots = d(e_n)$. Hence

$$(n-1)d(e_1) = p^{k-r+2}. \tag{2.1}$$

Now let f be an element not in $\operatorname{cl}(\{e_1\})$. Since L was chosen arbitrarily, every point on the line $\operatorname{cl}(\{e_1, f\})$ has the same size, so $d(f) = d(e_1)$. Thus every parallel class of M has the same size. Morever, if some line containing f has n' points, then

$$(n'-1)d(f) = p^{k-r+2}. \tag{2.2}$$

From (2.1) and (2.2), we see that $n' = n$, so every line of M contains the same number of points. It follows since p is prime that (n_1, n_2) is $(p^{k-r+2}, 2)$ or $(p^{k-r+1}, p+1)$, that is, (2.3.2) holds.

Suppose every line of M contains exactly two points. Take a 3-element independent set, $\{e, f, g\}$, in $\operatorname{si}(M)$. Then, in $\operatorname{si}(M)/e$, the line through f and g must have $p+1$ elements. Thus, the plane in $\operatorname{si}(M)$ spanned by $\{e, f, g\}$ has $p+2$ elements. As p is odd, M does not have $U_{3,p+2}$ as a minor (see, for example, [10, Table 6.1]). This contradiction implies that every line of M contains exactly $p+1$ points. Hence M is a projective geometry in which every point has p^{k-r+1} elements. Thus the lemma follows by induction. □

We are now ready to prove the main result of the paper.

Proof of Theorem 1.3. By Lemma 2.1, M has a simple restriction N such that

$$c(M;p) \leq \lceil \log_p(1 + \min_{C^* \in \mathcal{C}^*(N)} |C^*|) \rceil.$$

Every cocircuit of N is a subset of a cocircuit of M, so

$$c(M;p) \leq \lceil \log_p(1 + \max_{C^* \in \mathcal{C}^*(M)} |C^*|) \rceil. \tag{2.3}$$

Hence the first part of the theorem holds.

Now suppose that M is connected and that

$$c(M;p) > \lceil \log_p(\max_{C^* \in \mathcal{C}^*(M)} |C^*|) \rceil. \tag{2.4}$$

By combining (2.3) and (2.4), we deduce that $\max_{C^* \in \mathcal{C}^*(M)} |C^*| = p^k$ for some positive integer k. Then $c(M;p) = k+1$, so $k+1 \leq \lceil \log_p(1 + \min_{C^* \in \mathcal{C}^*(N)} |C^*|) \rceil$. Hence $\min_{C^* \in \mathcal{C}^*(N)} |C^*| \geq p^k$. Thus every cocircuit of N has exactly p^k elements.

Let $N = M \backslash T$. The cocircuits of N are the minimal non-empty sets in $\{C^* - T : C^* \in \mathcal{C}^*(M)\}$. If M has a cocircuit D^* that meets both T and $E - T$, then $|D^*| > p^k$; a contradiction. Thus T is a (possibly empty) union of components of M. But M is connected, so $T = \emptyset$, and $N = M$. By Lemmas 2.2 and 2.3, M is isomorphic to one of $PG(r-1,p)$, $U_{r,r+1}$, and $AG(r-1,2)$ with the last two possibilities only arising when $p = 2$. But $c(U_{r,r+1};2) = 1$ when r is odd, while $c(AG(r-1,2);2) = 1$ for all choices of r, so the theorem follows. □

One may hope to be able to eliminate the ceiling function in Theorem 1.3 but this is not possible. To see this, observe that $c(M(K_5);2) = 3$ but $\max_{C^* \in \mathcal{C}^*(M)} |C^*| = 6$. This example is far from the only exception one would need to add. To show this, we shall use the following result of Brylawski [3, Theorem 7.8].

Lemma 2.4. *Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = X$ and $M_1|X = M_2|X$. Let X be a modular flat in M_1 and let M be the generalized parallel connection of M_1 and M_2 across X. Then*

$$p(M;\lambda) = \frac{p(M_1;\lambda)p(M_2;\lambda)}{p(M_1|X;\lambda)}.$$

As is well known and follows easily from Theorem 1.1,

$$p(PG(r-1,q);\lambda) = (\lambda-1)(\lambda-q)(\lambda-q^2)\dots(\lambda-q^{r-1}).$$

Combining this with the last lemma, it is straightforward to check that, for $s \leq r$, the generalized parallel connection M of $PG(r-1,q)$ and $PG(s-1,q)$ across a $PG(t-1,q)$ for $1 \leq t < s < r$ has critical exponent r. Now a largest cocircuit of $PG(r-1,2)$ has q^{r-1} elements, and one easily checks that a largest cocircuit of M has fewer than q^r elements. Hence $c(M;q) >$

$\log_q(\max_{C^* \in \mathcal{C}^*(M)} |C^*|)$. One can delete elements from $PG(s-1, q)$ that are not in the common $PG(t-1, q)$ to obtain numerous other examples that exhibit this behaviour.

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