FORK-DECOMPOSITIONS OF MATROIDS

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Abstract. One of the central problems in matroid theory is Rota's conjecture that, for all prime powers q, the class of $GF(q)$–representable matroids has a finite set of excluded minors. This conjecture has been settled for $q \leq 4$ but remains open otherwise. Further progress towards this conjecture has been hindered by the fact that, for all $q > 5$, there are 3–connected $GF(q)$–representable matroids having arbitrarily many inequivalent $GF(q)$–representations. This fact refutes a 1988 conjecture of Kahn that 3–connectivity would be strong enough to ensure an absolute bound on the number of such inequivalent representations. This paper introduces fork-connectivity, a new type of self-dual 4–connectivity, which we conjecture is strong enough to guarantee the existence of such a bound but weak enough to allow for an analogue of Seymour's Splitter Theorem. We prove that every fork-connected matroid can be reduced to a vertically 4–connected matroid by a sequence of operations that generalize $\Delta - Y$ and $Y - \Delta$ exchanges. It follows from this that the analogue of Kahn's Conjecture holds for fork-connected matroids if and only if it holds for vertically 4–connected matroids. The class of fork-connected matroids includes the class of 3–connected forked matroids. By taking direct sums and 2–sums of matroids in the latter class, we get the class $\mathcal{M}$ of forked matroids, which is closed under duality and minors. The class $\mathcal{M}$ is a natural subclass of the class of matroids of branch-width at most 3 and includes the matroids of path-width at most 3. We give a constructive characterization of the members of $\mathcal{M}$ and prove that $\mathcal{M}$ has finitely many excluded minors.

1. Introduction

Historically, much of the emphasis in matroid structure theory has been placed on 3–connectivity, which has numerous attractive properties. In particular, the class of 3–connected matroids is closed under duality; every matroid that is not 3–connected can be built from 3–connected matroids by direct sums and 2–sums; and the class of 3–connected matroids has very powerful inductive tools in Tutte's Wheels and Whirls Theorem and its extension, Seymour's Splitter Theorem, which implies that if N is a 3–connected minor of a 3–connected matroid M and $|E(N)| \geq 4$, then there is a 3–connected minor M' of M that has a minor isomorphic to N such that $|E(M')| - |E(N)|$ is 1 or 2. Furthermore, 3–connectivity is an important tool in matroid representation theory: the 3–connected members of the classes of binary, ternary, and quaternary matroids are all uniquely representable over their

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respective fields; and this fact played a crucial role in the determination of the sets of excluded minors for each of these classes. In the paper that proved the unique representability of quaternary matroids, Kahn [15] conjectured that, for every finite field $GF(q)$, there is an integer μ_q such that every 3–connected matroid has at most μ_q inequivalent $GF(q)$–representations. Regrettably this conjecture, while true for $q = 5$, is false for all larger fields [21]. This failure has prompted the search for an appropriate strengthening of 3–connectivity that will not only regain control of the number of inequivalent representations but will also retain some of the useful properties of 3–connectivity noted above. This paper introduces a new type of 4–connectivity for matroids, fork-connectivity, which we hope will be the right definition to allow further progress in matroid representation theory.

Let M be a matroid with ground set E. For a positive integer k, a subset X of E is *k–separating* if $r(X) + r(E - X) - r(E) \leq k - 1$. When equality holds here, X is *exactly k–separating*. A partition $\{X, Y\}$ of E is a *k–separation* of M if X is k–separating and $|X|, |Y| \geq k$. For an integer n exceeding one, Tutte [29] defined M to be *n–connected* if, for all k in $\{1, 2, \ldots, n-1\}$, it has no k–separation. It is easily checked that a matroid is n–connected if and only if its dual is, and, when $n \leq 3$, this definition has been both predictable and serviceable. For example, if G is a simple connected graph with at least 4 vertices, then $M(G)$ is a 3–connected matroid if and only if G is a 3–connected graph. However, strict 4–connectivity is a restrictive notion. For instance, unless it is very small, a 4–connected matroid can have no triangles. Thus, such well-structured objects as the cycle matroids of complete graphs and projective geometries are generally not 4–connected. Hence we seek a weaker notion of 4–connectivity. Cunningham [3], Inukai and Weinberg [14], and Oxley [16] independently introduced a matroid generalization of vertex connectivity called vertical connectivity. Since we are concentrating here on strengthenings of the notion of 3–connectivity, we augment their definition by insisting on 3–connectivity. In particular, we shall call a matroid *vertically 4–connected* if it is 3–connected and has no 3–separations $\{X, Y\}$ such that $r(X), r(Y) \geq 3$. All projective geometries are vertically 4–connected and we believe that the analogue of Kahn's Conjecture holds for vertically 4–connected matroids.

Conjecture 1.1. *For every finite field $GF(q)$, there is an integer ν_q such that every vertically 4–connected $GF(q)$–representable matroid has at most ν_q inequivalent representations over $GF(q)$.*

Vertical 4–connectivity also has its limitations. For example, the class of vertically 4–connected matroids is not closed under duality. Moreover, Rajan [22] has shown that there are vertically 4–connected matroids M and N with $|E(M)| - |E(N)|$ arbitrarily large such that N is the only vertically 4–connected proper minor of M that has a minor isomorphic to N. Thus, no analogue of Seymour's Splitter Theorem holds for vertically 4–connected matroids. By contrast, Geelen and Whittle [7] proved an analogue of the Wheels and Whirls Theorem for the class of *sequentially 4–connected matroids*, a class that both contains the class of vertically 4–connected matroids and is closed under duality. It is straightforward to show that Geelen and Whittle's result extends to fork-connectivity in that if M is fork-connected and is neither a wheel nor a whirl, then M has an element e such that either $M\backslash e$ or M/e is fork-connected. But we also believe that stronger results

exist and we conjecture that an analogue of Seymour's Splitter Theorem holds for the class of fork-connected matroids.

Each time we weaken 4–connectivity, it is easier to produce chain theorems and hence easier to obtain leverage for inductive arguments. Given this, it is natural to look for the weakest version of 4–connectivity that does not lose the benefit of the extra structure that was obtained by considering 4–connectivity in the first place. The notion of 4–connectivity introduced in this paper, namely fork–connectivity, is weaker even than sequential 4–connectivity. However, for many purposes, it is just as strong as vertical 4–connectivity since we prove, in Corollary 10.7, that *Conjecture 1.1 holds for fork–connected matroids if and only if it holds for vertically* 4*–connected matroids.* Moreover, we believe that fork–connectivity could be the right notion of connectivity to use to tackle Conjecture 1.1.

In defining fork-connectivity, we are attempting to impose some control on the situation when we have a partition $\{X, Y, Z\}$ of the ground set of a 3–connected matroid M such that each of X, Y, and Z is exactly 3–separating. If this occurs and, for example, M is representable, then we want that, for some N in $\{M, M^*\}$, when N is viewed as a restriction of a projective space, there is a line L of the projective space such that the intersection of the spans of any two of X, Y, and Z is L. In this case, we think of $\{X, Y, Z\}$ as a fork. This dual pair of conditions can be expressed as the following single rank inequality: $r(X) + r(Y) + r(Z) - r(M) \neq 3$.

Branch-width is a basic parameter for graphs that was introduced by Robertson and Seymour [27] and is closely related to their better-known parameter tree-width [25, 26, 23]. Moreover, branch-width has recently proved to be a very important tool for matroids. A matroid M has *branch-width* at most n if it has a *width–*n *branch-decomposition*, that is, a tree T with all its internal vertices of degree 3 and a one-to-one labelling of the leaves of T by the elements of M such that, for every edge e of T, if $\{X_e, Y_e\}$ is the partition of $E(M)$ induced by e, then X_e is n–separating. The class $\mathcal{B}_n$ of matroids of branch-width at most n is closed under both duality and minors. Geelen, Gerards, and Whittle [9] have made progress towards extending the Graph Minors Project to $GF(q)$–representable matroids by proving that, for all positive integers n and all prime powers q, the intersection of $\mathcal{B}_n$ with the class of $GF(q)$–representable matroids contains no infinite antichains. In addition, Geelen and Whittle [7] have recently proved that the intersection of $\mathcal{B}_n$ with the set of excluded minors for the class of $GF(q)$–representable matroids is finite, thereby adding credibility to Rota's Conjecture [24] that there are finitely many excluded minors for the class of $GF(q)$–representable matroids. The class $\mathcal{B}_2$ coincides with the class of direct sums of series-parallel networks [27]. In [13, 11], the authors proved that $\mathcal{B}_3$ is characterized by a finite set of excluded minors by showing that such a minor has at most 14 elements. We call a matroid *forked* if it has a width–3 branch-decomposition T such that, for every partition $\{X, Y, Z\}$ of $E(M)$ induced by an internal vertex of T, one of the following dual pair of conditions holds: $r(X \cup Y) + r(Y \cup Z) + r(X \cup Z) - 2r(M) \leq 2$ or $r(X) + r(Y) + r(Z) - r(M) \leq 2$. The class of forked matroids is closed under duality and minors. Moreover, it includes the matroids of path-width at most 3, that is, the class of matroids M for which there is an ordering $x_1, x_2, \ldots, x_n$ of $E(M)$ such that $\{x_1, x_2, \ldots, x_k\}$ is 3–separating for all k in $\{1, 2, \ldots, n-1\}$. The purpose of Section 8 is to prove

that the number of excluded minors for the class of forked matroids is finite by showing that such excluded minors have at most 37 elements. We do not attempt to explicitly determine these excluded minors. To prove this bound, we need the material of Section 7 on minimal non-fans but, otherwise, the results of Section 7 are independent of the rest of the paper.

Oxley, Semple and Vertigan [20] introduced an operation on matroids, termed segment-cosegment exchange, that generalizes the familiar $\Delta - Y$ exchange. This operation has fundamental connections with the class of forked matroids. In Section 9, we show that a 3–connected forked matroid of size at least three can always be transformed, for some $n \geq 3$, to either $U_{2,n}$ or its dual via a sequence of operations each consisting of a segment-cosegment exchange followed by a cosimplification or the dual of this composite operation. The main result of that section, Theorem 9.10, extends the last result to give a constructive characterization of 3–connected forked matroids.

In each of the weakenings of 4–connectivity that have been discussed above, certain 3–separations $\{A, B\}$ of a matroid are allowed as long as one side, A or B, has a certain size or structure. For fork-connectivity, it is the structure of one side, say A, that we focus on. In describing and understanding this structure, we will find that the individual elements of B are largely irrelevant and potentially distracting. To overcome this inconvenience, we consider a new object which has ground set is $\{\{a_1\}, \{a_2\}, \ldots, \{a_n\}, B\}$, where $A = \{a_1, a_2, \ldots, a_n\}$, and which has a rank function that is induced by the rank function of M. This object is an example of a *partitioned matroid*, that is, a matroid together with a partition of its ground set and the rank function that is induced on this partition by the matroid rank function. Much of this paper is set at the level of partitioned matroids. In particular, we define when a partitioned matroid is forked by extending the definition of a forked matroid given above. The matroid M is *fork-connected* if it is 3–connected and, for every 3–separation $\{A, B\}$ of M, there is a pairing $\{X, Y\} = \{A, B\}$ such that the partitioned matroid induced by M on X and the set of singleton subsets of Y is forked. It will follow, in particular, that every 3–connected forked matroid is fork-connected.

The paper is structured as follows. Section 2 introduces partitioned matroids and describes their basic properties. Section 3 introduces fork-decompositions of partitioned matroids. A fork-decomposition is a width–3 branch-decomposition with extra structure; a partitioned matroid is forked if it has a fork-decomposition. As with branch-decompositions, fork-decompositions need not be unique. It turns out to be important to have fork-decompositions of a certain desirable form and to know which separations can be displayed in a fork-decomposition. Sections 5 and 6 are devoted to this issue. In particular, it follows from Theorem 6.2 that if $\{A, B\}$ is a 3–separation of a forked partitioned matroid with $|A|, |B| \geq 3$ and $\{A, B\}$ cannot be displayed in a fork-decomposition, then either A or B is a fan. When we finally come to consider forked matroids and fork-connected matroids in Sections 8, 9, and 10, we obtain most of their properties as corollaries of results on forked partitioned matroids.

The main result of Section 10, Theorem 10.1, is that every fork-connected matroid can be transformed to a vertically 4–connected matroid by a sequence of moves consisting of a segment-cosegment exchange followed by a cosimplification or a cosegment-segment exchange followed by a simplification. It is known [10] that representations of a matroid M are in one-to-one correspondence with representations of a matroid obtained from M via a segment-cosegment or cosegment-segment exchange. Thus we derive at the end of Section 10 that the number of inequivalent representations of a fork-connected matroid over a finite field is equal to that of an associated vertically 4–connected matroid.

We conclude the introduction by fixing some terminology. Throughout the paper, unless otherwise indicated, we shall allow a block in a partition to be empty. The terminology used here will follow Oxley [17] with the exception of the definition of vertical 4–connectivity noted above and the use of $\mathrm{si}(N)$ and $\mathrm{co}(N)$ for the simplification and cosimplification, respectively, of a matroid N.

The property that a circuit and a cocircuit cannot have exactly one common element will be referred to as *orthogonality*. A basic structure in the study of 3–connected matroids consists of an interlocking chain of triangles and triads. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid N such that, for all i in $\{1, 2, \ldots, k-1\}$,

(i) $|T_i \cap T_{i+1}| = 2$;
(ii) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \ldots \cup T_i)$ is empty; and
(iii) in $\{T_i, T_{i+1}\}$, exactly one set is a triangle and exactly one set is a triad.

We call the sequence $T_1, T_2, \ldots, T_k$ a *fan* of N of *length* k with *links* $T_1, T_2, \ldots, T_k$. When this occurs, it is straightforward to show that N has $k+2$ distinct elements $x_1, x_2, \ldots, x_{k+2}$ such that $T_i = \{x_i, x_{i+1}, x_{i+2}\}$ for all i in $\{1, 2, \ldots, k\}$. When $k \geq 2$, the elements x_1 and x_{k+2} are the only elements of the fan that are in exactly one link. We call them the *ends* of the fan and call $x_2, x_3, \ldots, x_{k+1}$ the *internal elements* of the fan. There are three types of fans: *type*-1 when both T_1 and T_k are triangles; *type*-2 when both T_1 and T_k are triads; and *type*-3 when one of T_1 and T_k is a triangle and the other is a triad. While, formally, a fan is a sequence of triangles and triads as described above, it will often be convenient to use the term "fan" for what is strictly the ground set $\{x_1, x_2, \ldots, x_{k+2}\}$ of the fan. The terminology just introduced differs from that used in [19] where the term "chain" is used for what has just been defined as a fan, and where "fan" is used for a maximal chain.

An element e of a 3–connected matroid M is *essential* if neither $M\backslash e$ nor M/e is 3–connected. Tutte [29] showed that every essential element in a 3–connected matroid is in a triangle or a triad, so every essential element is in a fan. A 4–element set that is both a circuit and a cocircuit in a matroid is called a *quad*.

2. Partitioned Matroids

Let M be a matroid with rank function r_M and let P be a partition of $E(M)$ into non-empty sets. Let $E(P)$ be the set of blocks of P and, for all subsets X of $E(P)$, define r_P by $r_P(X) = r_M(\cup_{\xi\in X}\xi)$ for all subsets X of $E(P)$. We call P a *partitioned matroid* with *rank function* r_P and *underlying matroid* M. We shall also say that P is *induced by* M *on* $E(P)$. By $r(P)$, we shall mean $r_P(E(P))$. Evidently this equals $r(M)$. The matroid M can be viewed as a partitioned matroid by taking the partition of $E(M)$ consisting of singleton subsets. The reader may recognize a partition matroid as an example of a polymatroid. Moreover, every polymatroid is isomorphic to a partitioned matroid. But, whereas it can be problematic to define duality for arbitrary polymatroids, there are no such difficulties for partitioned matroids. Indeed, the *dual* P^* of the partitioned matroid P is the partitioned matroid with underlying matroid M^* and having the same partition of $E(M)$ as P. Thus $E(P^*) = E(P)$ and $(P^*)^* = P$. Several basic concepts from matroid theory extend to partitioned matroids. In particular, if P is a partitioned matroid and $X \subseteq E(P)$, we define the *closure* $\mathrm{cl}(X)$ to be $\{e \in E(P) : r_P(X \cup e) = r_P(X)\}$. A *matroid element* of a partitioned matroid P is an element e of the underlying matroid such that $\{e\}$ is block of the partition.

A *connectivity function on a finite set* S is a function λ defined on the set of subsets of S that is

(i) *integer-valued*: $\lambda(A)$ is an integer for all $A \subseteq S$;
(ii) *symmetric*: $\lambda(S - A) = \lambda(A)$ for all $A \subseteq S$; and
(iii) *submodular*: $\lambda(A) + \lambda(B) \geq \lambda(A \cup B) + \lambda(A \cap B)$ for all $A, B \subseteq S$.

If P is a partitioned matroid, and λ_P is defined, for all subsets A of $E(P)$ by $\lambda_P(A) = r_P(A) + r_P(E(P) - A) - r(P) + 1$, then λ_P is clearly integer-valued and symmetric, and it is not difficult to check that λ_P is submodular. Thus λ_P is a connectivity function. We call it the *connectivity function of* P. It is straightforward to prove that the connectivity function of a partitioned matroid and its dual are equal.

Lemma 2.1. *Let P be a partitioned matroid. Then, for all $A \subseteq E(P)$,*

$$\lambda_{P^*}(A) = \lambda_P(A).$$

Proof. Let M be the underlying matroid of P. Then M^* is the underlying matroid of P^*. By definition,

$$\begin{aligned}
\lambda_{P^*}(A) &= r_{P^*}(A) + r_{P^*}(E(P) - A) - r(P^*) + 1 \\
&= r_{M^*}(\cup_{a\in A} a) + r_{M^*}(\cup_{a\in E(P)-A} a) - r(M^*) + 1 \\
&= |\cup_{a\in A} a| - r(M) + r_M(E(M) - \cup_{a\in A} a) + |\cup_{a\in E(P)-A} a| - r(M) \\
&\quad + r_M(E(M) - \cup_{a\in E(P)-A} a) - r(M^*) + 1 \\
&= (|\cup_{a\in A} a| + |\cup_{a\in E(P)-A} a|) - (r(M) + r(M^*)) \\
&\quad + r_M(E(M) - \cup_{a\in E(P)-A} a) + r_M(E(M) - \cup_{a\in A} a) - r(M) + 1 \\
&= |E(M)| - |E(M)| + r_P(A) + r_P(E(P) - A) - r(P) + 1 \\
&= \lambda_P(A).
\end{aligned}$$

□

Let P be a partitioned matroid. A subset A of $E(P)$ is *k–separating* if $\lambda_P(A) \leq k$. The set A is *exactly k–separating* if $\lambda_P(A) = k$. We extend these definitions to partitions of $E(P)$ as follows. The partition $\{X_1, X_2, \ldots, X_n\}$ of $E(P)$ is *k–separating* if, for each i in $\{1, 2, \ldots, n\}$, the set X_i is k–separating. In addition, $\{X_1, X_2, \ldots, X_n\}$ is *exactly k–separating* if every X_i is exactly k–separating.

Let P be a partitioned matroid with underlying matroid M. We define P to be *2–connected* if M is 2–connected; and P to be *3–connected* if M is 3–connected. Thus P is 3–connected if and only if P is 2–connected and, whenever a subset A of $E(P)$ is 2–separating, either A or $E(P) - A$ is a matroid element. Note that, in a 3–connected polymatroid whose underlying matroid has at least two elements, a matroid element has rank one. Some words of caution seem appropriate here. In [13], we defined a connectivity function λ on a set S to be n–connected if, for all $k \in \{0, 1, \ldots, n-1\}$, whenever $\{A, B\}$ is a partition of S with $|A|, |B| \geq k$, then $\lambda(A) \geq k + 1$. It is tempting to think that, for example, a partitioned matroid P will be 3–connected if and only if its connectivity function is 3–connected. While this is true when P is a matroid, it is not true in general. For example, if M is the rank–3 matroid that is formed by taking the 2–sum with basepoint p of a 3–point line $\{p, a, e\}$ and a 4–point line $\{p, b, c, d\}$ (see Figure 1), then M is clearly not 3–connected. However, if P is the partitioned matroid induced by the partition $\{\{a, e\}, \{b\}, \{c\}, \{d\}\}$, then $\lambda_P(A) \geq 3$ whenever both A and $E(P) - A$ have size at least two.

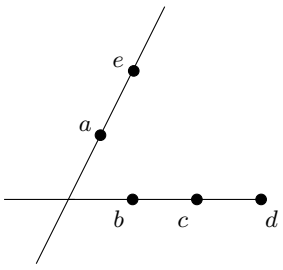


FIGURE 1. A non-trivial 2–separation of a partitioned matroid.

Let P be a partitioned matroid with underlying matroid M and let Z be a set of matroid elements in P. We call Z a *triangle*, a *triad*, or a *fan* of P if Z is, respectively, a *triangle*, a *triad*, or a *fan* of M. If every 3–element subset of Z is a triangle, then Z is a *segment* of P; if every 3–element subset of Z is a triad, then Z is a *cosegment* of P.

The next two results for partitioned matroids extend the corresponding results for matroids. Their straightforward proofs are omitted.

Lemma 2.2. *Let Z be a set of matroid elements in a 3–connected partitioned matroid P such that $|Z| \geq 3$. If there is an ordering $z_1, z_2, \ldots, z_n$ of the elements of Z such that, for all i in $\{1, 2, \ldots, n-2\}$, the set $\{z_i, z_{i+1}, z_{i+2}\}$ is a triangle or a triad, then Z is a segment, a cosegment, or a fan of P.*

Lemma 2.3. *Let X be a 3–element set of matroid elements in a 3–connected partitioned matroid P having at least four elements. If X is 3–separating, then X is a triangle or a triad.*

3. Branch-Decompositions and Fork-Decompositions

In this section, we introduce fork-decompositions. These are a special type of branch-decomposition, and the basic definitions associated with the latter will first be recalled from [9] and [13]. It would be quite straightforward to define fork-decompositions and fork-width for arbitrary k, but since we know of no applications for fork-width other than in the case $k = 3$, we confine our attention to this case.

Branch-decompositions are defined in terms of cubic trees, that is, trees in which every vertex has degree zero, one, or three. Such trees are sometimes called *ternary trees*. A *branch* of a cubic tree T is a subtree of T that is a component of $T\backslash e$ for some edge e of T. Equivalently, a branch is a component of $T\backslash v$ for some vertex v of T. We say that a branch is *displayed* by an edge e or a vertex v if it is one of the components of $T\backslash e$ or $T\backslash v$, respectively. Clearly, an edge displays two branches, while a vertex of degree three displays three branches.

Let P be a partitioned matroid. A 3–separating partition $\{X, Y, Z\}$ of $E(P)$ satisfies the *strong guts condition* if

$$r(X \cup Y) + r(X \cup Z) + r(Y \cup Z) - 2r(P) \leq 2.$$

On the other hand, $\{X, Y, Z\}$ satisfies the *strong coguts condition* if

$$r(X) + r(Y) + r(Z) - r(P) \leq 2.$$

The terminology here implies that the last two conditions are dual and this follows immediately from the next result, whose straightforward proof is omitted.

Lemma 3.1. *Let P be a partitioned matroid, and let $\{A, B, C\}$ be a partition of $E(P)$. Then*

$$r_P(A \cup B) + r_P(A \cup C) + r_P(B \cup C) - 2r(P) = r_{P^*}(A) + r_{P^*}(B) + r_{P^*}(C) - r(P^*).$$

In Figure 2, the partition $\{A, B, C\}$ satisfies the strong guts condition in the rank–5 partitioned matroid illustrated in (a) but does not satisfy the strong guts condition in the rank–6 partitioned matroid illustrated in (b). Both the strong guts and the strong coguts conditions can be formulated in equivalent ways, which we shall describe in the next section.

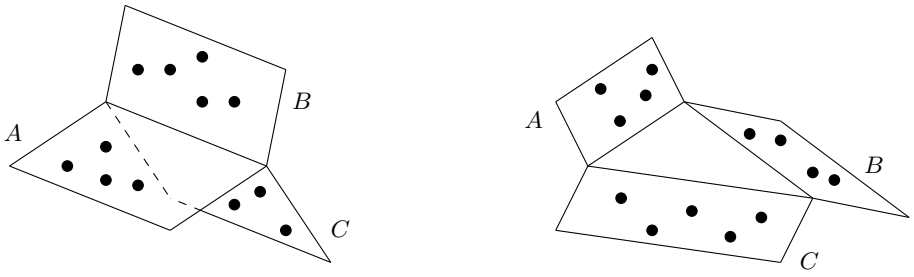


FIGURE 2

A *branch decomposition* of a partitioned matroid P is a cubic tree T together with a one-to-one labelling of a subset of the leaves of T by the elements of P. Each edge e of T induces a partition of $E(P)$ into two subsets, X_e and Y_e, and we say that the partition $\{X_e, Y_e\}$ is *displayed by* e. The *width of* e is $r(X_e) + r(Y_e) - r(P) + 1$, and the *width of* T is the maximum of the widths of the edges of T or is 1 if T has no edges. The *branch-width* of P is the minimum of the widths of its branch-decompositions. Each internal vertex v of a branch-decomposition of P induces a partition of $E(P)$ into three subsets. We call this the partition *displayed by* v.

At last, we are now in a position to define fork-decompositions. A branch-decomposition T of a partitioned matroid P is a *fork-decomposition* if every edge of T has width at most 3 and, for each internal vertex v of T, the partition displayed by v satisfies either the strong guts condition or the strong coguts condition. Moreover, a partitioned matroid is *forked* if it has a fork-decomposition. A fork-decomposition T of a partitioned matroid P is *reduced* if every leaf of T labels an element of P. Given a fork-decomposition T of a partitioned matroid P with $|E(P)| \geq 2$, we can obtain a reduced fork-decomposition by repeating the operation of deleting an unlabelled leaf and then contracting one of the edges incident with the resulting degree-two vertex.

If v is an internal vertex of a fork-decomposition of a partitioned matroid, then v is called a *guts vertex* if the 3–separating partition displayed by v satisfies the strong guts condition and v is a *coguts vertex* if this partition satisfies the strong coguts condition.

4. Basic Lemmas

The next lemma is an easy consequence of the fact that connectivity functions are submodular.

Lemma 4.1. *Let λ be a connectivity function on a finite set S. Let X and Y be 3–separating subsets of S.*

(i) *If $\lambda(X \cap Y) \geq 3$, then $X \cup Y$ is 3–separating.*
(ii) *If $\lambda(S - (X \cup Y)) \geq 3$, then $X \cap Y$ is 3–separating.*

The following consequence of the last lemma will be used frequently throughout the paper.

Corollary 4.2. *Let P be a 3–connected partitioned matroid, and let X and Y be 3–separating sets of P. If $r(E(P) - (X \cup Y)) \geq 2$, then $X \cap Y$ is 3–separating. In particular, if $X \cup Y$ avoids an exactly 3–separating set of P, then $X \cap Y$ is 3–separating.*

Proof. Suppose that $r(E(P) - (X \cup Y)) \geq 2$. If $r(X \cup Y) \geq 2$, then, as P is 3–connected, $\lambda_P(E(P) - (X \cup Y)) \geq 3$, so, by Lemma 4.1(ii), $X \cap Y$ is 3–separating. If $r(X \cup Y) \leq 1$, then

$$\lambda_P(X \cap Y) = r(X \cap Y) + [r(E(P) - (X \cap Y)) - r(P)] + 1 \leq r(X \cap Y) + 1 \leq 2,$$

and again $X \cap Y$ is 3–separating. We conclude that the first assertion holds. Now suppose that $X \cup Y$ avoids some exactly 3–separating set Z. Then $r(E(P) - (X \cup Y)) \geq r(Z) \geq 2$ and the second assertion follows from the first. □

Lemma 4.3. *Let P be a partitioned matroid, and let $\{X, Y, Z\}$ be a 3–separating partition of P.*

(i) *If X is 2–separating, then $\{X, Y, Z\}$ satisfies either the strong guts or the strong coguts condition.*
(ii) *If X and Y are both 2–separating, then $\{X, Y, Z\}$ satisfies the strong guts and the strong coguts conditions.*

Proof. To prove (i), suppose that $\{X, Y, Z\}$ does not satisfy the strong coguts condition. Then $r(X) + r(Y) + r(Z) - r(P) \geq 3$. Therefore, as X is 2–separating, and Y and Z are both 3–separating, we deduce that

$$\begin{aligned} r(X \cup Y) + r(X \cup Z) + r(Y \cup Z) - 2r(P) & \\ &\leq (r(P) + 2 - r(Z)) + (r(P) + 2 - r(Y)) + (r(P) + 1 - r(X)) - 2r(P) \\ &= 5 - (r(X) + r(Y) + r(Z) - r(P)) \\ &\leq 2. \end{aligned}$$

Hence $\{X, Y, Z\}$ satisfies the strong guts condition, thus proving (i).

Now consider (ii). Then, as X and Y are both 2–separating, and Z is 3–separating,

$$\begin{aligned} r(X \cup Y) + r(X \cup Z) + r(Y \cup Z) - 2r(P) &\leq r(X \cup Y) + (r(P) + 1 - r(Y)) \\ &\quad + (r(P) + 1 - r(X)) - 2r(P) \\ &= 2 + r(X \cup Y) - (r(X) + r(Y)) \\ &\leq 2, \qquad \text{by submodularity.} \end{aligned}$$

Thus, in this case, $\{X, Y, Z\}$ satisfies the strong guts condition in P. By Lemma 2.1, X and Y are both 2–separating sets of P^*, and Z is a 3–separating set of P^*. It follows from above that $\{X, Y, Z\}$ satisfies the strong guts condition in P^*, and therefore, by Lemma 3.1, $\{X, Y, Z\}$ satisfies the strong coguts condition in P. □

Lemma 4.4. *Let P be a partitioned matroid, and let $\{X, Y, Z\}$ be a 3–separating partition of P.*

(i) *If P is 3–connected and Z is not exactly 3–separating, then either $Z = \emptyset$ or $Z = \{z\}$ for some matroid element z.*
(ii) *Assume that $Z = \{z\}$ for some matroid element z. If Z is exactly 2–separating, and both X and Y are exactly 3–separating, then*
   (a) *$\{X, Y, Z\}$ satisfies the strong guts condition if and only if $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$.*
   (b) *$\{X, Y, Z\}$ satisfies the strong coguts condition if and only if $r(X \cup z) = r(X) + 1$ and $r(Y \cup z) = r(Y) + 1$.*

Proof. To prove (i), assume that P is 3–connected. Since Z is not exactly 3–separating, it follows from the definition of 3–connectivity that, provided Z is non-empty, $Z = \{z\}$ for some matroid element z. Thus (i) holds.

Now suppose that $Z = \{z\}$ for some matroid element z, that Z is exactly 2–separating, and that both X and Y are exactly 3–separating. Then both X and $X \cup z$ are exactly 3–separating and so

$$r(X \cup z) - r(X) = (r(P) + 2 - r(Y)) - (r(P) + 2 - r(Y \cup z)) = r(Y \cup z) - r(Y).$$

Thus $z \in \mathrm{cl}(X)$ if and only if $z \in \mathrm{cl}(Y)$, and $r(X \cup z) = r(X) + 1$ if and only if $r(Y \cup z) = r(Y) + 1$. We freely use these observations in the rest of the proof.

To prove (ii)(a), first assume that $z \in \mathrm{cl}(X)$. Then

$$r(X \cup Y) + r(X \cup z) + r(Y \cup z) = r(X \cup Y) + r(X) + r(Y) \leq 2r(P) + 2$$

and the strong guts condition holds. Now assume that $\{X, Y, Z\}$ satisfies the strong guts condition. As Z and X are exactly 2– and exactly 3–separating, respectively, we deduce that $r(X \cup Y) = r(P)$ and $r(Y \cup z) = r(P) + 2 - r(X)$. Therefore, as

$$r(X \cup Y) + r(X \cup z) + r(Y \cup z) - 2r(P) \leq 2,$$

it follows that $r(X \cup z) \leq r(X)$. Hence $z \in \mathrm{cl}(X)$, and so (ii)(a) holds.

For the proof of (ii)(b), first assume that $r(X \cup z) = r(X) + 1$. Then

$$r(X) + r(z) + r(Y) = r(X \cup z) + r(Y) = r(P) + 2$$

and the strong coguts condition holds. Now assume that $\{X, Y, Z\}$ satisfies the strong coguts condition. As Y is exactly 3–separating, $r(Y) = r(P) + 2 - r(X \cup z)$. Therefore, as

$$r(X) + r(z) + r(Y) - r(P) \leq 2,$$

we deduce that $r(X) + 1 \leq r(X \cup z)$, and so $r(X \cup z) = r(X) + 1$. This completes the proof of the lemma. □

While the strong guts and strong coguts conditions are defined as inequalities, it turns out, for 3–connected partitioned matroids, that, when they hold, they hold with equality.

Lemma 4.5. *Let P be a partitioned matroid, and let $\{X, Y, Z\}$ be a 3–separating partition of $E(P)$ such that X, Y, and Z are all non-empty.*

(i) *If P is 3–connected and not isomorphic to $U_{1,3}$ or $U_{2,3}$, then at least one of X, Y, and Z is exactly 3–separating.*
(ii) *Assume that exactly one of X, Y, and Z is exactly 3–separating. Then*
  (a) *The strong guts condition holds for $\{X, Y, Z\}$ if and only if*
$$r(X \cup Y) + r(X \cup Z) + r(Y \cup Z) - 2r(P) = 2.$$
  (b) *The strong coguts condition holds for $\{X, Y, Z\}$ if and only if*
$$r(X) + r(Y) + r(Z) - r(P) = 2.$$

Proof. To prove (i), assume that X, Y, and Z are all 2–separating. Then each of these sets consists of a single matroid element and it is easily seen that P is isomorphic to either $U_{1,3}$ or $U_{2,3}$. Thus one of X, Y, and Z is exactly 3–separating.

Now consider (ii). Without loss of generality, we may assume that Z is exactly 3–separating. Assume that the strong coguts condition holds for $\{X, Y, Z\}$. Since Z is exactly 3–separating, $r(Z) = r(P) + 2 - r(X \cup Y)$. Thus
$$r(X) + r(Y) + r(Z) - r(P) = r(X) + r(Y) - r(X \cup Y) + 2.$$
Using submodularity, we deduce that $r(X) + r(Y) + r(Z) - r(P) \geq 2$. But, since the strong coguts condition holds, we also have $r(X) + r(Y) + r(Z) - r(P) \leq 2$ and we conclude that $r(X) + r(Y) + r(Z) - r(P) = 2$. Thus (b) holds and (a) follows immediately by Lemma 3.1. □

The next lemma enables us to quickly test the strong guts and strong coguts conditions.

Lemma 4.6. *Let $\{X, Y, Z\}$ be an exactly 3–separating partition of a partitioned matroid P. Then, in each of* (i) *and* (ii)*, statements* (a), (b), *and* (c) *are equivalent.*

(i) (a) *$\{X, Y, Z\}$ satisfies the strong guts condition.*
  (b) *$r(X \cup Y) \leq r(X) + r(Y) - 2$.*
  (c) *$r(X \cup Y) = r(X) + r(Y) - 2$.*
(ii) (a) *$\{X, Y, Z\}$ satisfies the strong coguts condition.*
  (b) *$r(X \cup Y) \geq r(X) + r(Y)$.*
  (c) *$r(X \cup Y) = r(X) + r(Y)$.*

Proof. We shall prove (i) and omit the similar proof of (ii). Using the fact that X and Y are exactly 3–separating, we see that
$$\begin{aligned} r(X \cup Y) + r(Y \cup Z) + r(X \cup Z) - 2r(P) - 2 \\ &= r(X \cup Y) + [r(P) + 2 - r(X)] + [r(P) + 2 - r(Y)] - 2r(P) - 2 \\ &= r(X \cup Y) - r(X) - r(Y) + 2. \end{aligned}$$

The equivalence of (a) and (b) follows immediately from the above equation, while the equivalence of (a) and (c) follows from the above equation using Lemma 4.5. □

Lemma 4.7. *Let $\{W, X, Y \cup Z\}$ and $\{W, X \cup Z, Y\}$ be exactly 3–separating partitions of a partitioned matroid P. Then*

(i) *$\{W, X, Y \cup Z\}$ satisfies the strong guts condition if and only if $\{W, X \cup Z, Y\}$ satisfies the strong guts condition; and*
(ii) *$\{W, X, Y \cup Z\}$ satisfies the strong coguts condition if and only if $\{W, X \cup Z, Y\}$ satisfies the strong coguts condition.*

Proof. Suppose that $\{W, X, Y \cup Z\}$ satisfies the strong guts condition. Then, by Lemma 4.6,

$$r(W \cup X) \leq r(W) + r(X) - 2.$$

By submodularity,

$$r(W \cup X \cup Z) \leq r(W \cup X) + r(X \cup Z) - r(X).$$

Thus,

$$r(W \cup X \cup Z) \leq r(W) + r(X \cup Z) - 2.$$

It now follows from Lemma 4.6 that $\{W, X \cup Z, Y\}$ satisfies the strong guts condition and, by symmetry, (i) holds. Part (ii) follows by duality. □

Lemma 4.8. *Let P be a 3–connected partitioned matroid, and let $\{X, Y, Z\}$ and $\{W, B\}$ be 3–separating partitions of P where the first is exact. If $W \cap X$ is exactly 3–separating, then $\{W \cup X, Y \cap B, Z \cap B\}$ is a 3–separating partition of P. Moreover,*

(i) *if $\{W \cup X, Y \cap B, Z \cap B\}$ is not exactly 3–separating, then it satisfies the strong guts or the strong coguts condition; and*
(ii) *if $\{W \cup X, Y \cap B, Z \cap B\}$ is exactly 3–separating, then*
   (a) *$\{W \cup X, Y \cap B, Z \cap B\}$ satisfies the strong guts condition if and only if $\{X, Y, Z\}$ satisfies the strong guts condition; and*
   (b) *$\{W \cup X, Y \cap B, Z \cap B\}$ satisfies the strong coguts condition if and only if $\{X, Y, Z\}$ satisfies the strong coguts condition.*

Proof. Since W and X are 3–separating and $W \cap X$ is exactly 3–separating, it follows by Lemma 4.1 that $W \cup X$ is 3–separating. Again, since Y and B are 3–separating and their union avoids $W \cap X$, which is exactly 3–separating, it follows by Corollary 4.2 that $Y \cap B$ is 3–separating. By symmetry, $Z \cap B$ is also 3–separating. Thus $\{W \cup X, Y \cap B, Z \cap B\}$ is a 3–separating partition of P. Part (i) is immediate from Lemma 4.3(i). Now suppose that $\{W \cup X, Y \cap B, Z \cap B\}$ is exactly 3–separating. Then, as $\{X, Y, Z\}$ is also exactly 3–separating, we deduce that each of $Y \cap B$ and Z is exactly 3–separating. Now $W \cup X$ and $X \cup Y$ are 3–separating and their union avoids $Z \cap B$ which is exactly 3–separating. Thus, by Corollary 4.2, $(W \cup X) \cap (X \cup Y)$ is 3–separating, that is, $X \cup (Y \cap W)$ is 3–separating. Moreover, $X \cup (Y \cap W)$ and its complement contain the exactly 3–separating sets $X \cap W$ and $Y \cap B$, respectively. Thus $r(X \cup (Y \cap W)) \geq r(X \cap W) \geq 2$ and $r(E(P) - [X \cup (Y \cap W)]) \geq r(Y \cap B) \geq 2$. Hence $\lambda_P(X \cup (Y \cap W)) \geq 3$ and so $X \cup (Y \cap W)$ is exactly 3–separating. We conclude that $\{X \cup (Y \cap W), Y \cap B, Z\}$ is exactly 3–separating. By applying the above argument with the last partition

replacing $\{X, Y, Z\}$, we deduce that $\{X \cup (Y \cap W) \cup (Z \cap W), Y \cap B, Z \cap B\}$ is exactly 3–separating. Then, by successive applications of Lemma 4.7, we deduce that the following statements are equivalent, where condition gc is either the strong guts or the strong coguts condition:

1. $\{X, Y, Z\}$ satisfies condition gc;
2. $\{X \cup (Y \cap W), Y \cap B, Z\}$ satisfies condition gc;
3. $\{X \cup (Y \cap W) \cup (Z \cap W), Y \cap B, Z \cap B\}$ satisfies condition gc.

Since $X \cup (Y \cap W) \cup (Z \cap W) = W \cup X$, parts (a) and (b) of (ii) follow immediately. □

5. Sorting Lemmas

Just as with branch-decompositions, fork-decompositions of partitioned matroids are generally not unique. A key technique is to move from a given fork-decomposition to one of a more desirable form. The lemmas in this section consider operations that can be performed on fork-decompositions to produce new fork-decompositions.

Lemma 5.1. *Let T be a fork-decomposition of a partitioned matroid P. Let e be an edge of T, and let A and $B \cup x$ be the sets displayed by e, where x is a matroid element not in B. Let $\hat{T}$ be obtained from T by subdividing e; inserting a new vertex v, adding a new leaf adjacent to v; and then moving the label x from its original leaf in T to the new leaf. If either*

(i) $r(A \cup x) = r(A)$, *or*
(ii) $r(B \cup x) = r(B) + r(x)$,

then $\hat{T}$ is a fork-decomposition of P where v is a guts vertex in case (i) *and a coguts vertex in case* (ii).

Proof. We first show that, in each case, $\hat{T}$ is a branch-decomposition of P of width at most 3. Let f be an edge of $\hat{T}$. Then either f displays a partition $\{X, Y\}$ that was also displayed in T, in which case $w(f) \leq 3$; or f displays a partition $\{X - x, Y \cup x\}$ where $\{X, Y\}$ is displayed in T and $x \in X$. In the latter case, $A \cup x \subseteq Y \cup x$. Thus, if (i) holds, then $r(Y \cup x) = r(Y)$, so

$$w(f) = r(X - x) + r(Y \cup x) - r(P) + 1 \leq r(X) + r(Y) - r(P) + 1 \leq 3.$$

If (ii) holds, then, since $B \supseteq X - x$, it follows that $r(X) = r(X - x) + r(x)$ and so

$$\begin{aligned} w(f) &= r(X) - r(x) + r(Y \cup x) - r(P) + 1 \\ &= [r(X) + r(Y) - r(P) + 1] + [r(Y \cup x) - r(Y) - r(x)] \\ &\leq 3, \end{aligned}$$

where the last inequality holds since $r(Y \cup x) \leq r(Y) + r(x)$. We conclude that if either (i) or (ii) holds, then $\hat{T}$ is indeed a branch-decomposition of P of width at most 3.

Next we need to show that $\hat{T}$ is a fork-decomposition of P. Consider the vertex v of $\hat{T}$. Certainly $\{x\}$ is 2–separating. Assume that $\{x\}$ is exactly 2–separating, and that both A and B are exactly 3–separating. Then it follows by Lemmas 4.3(i) and 4.4(ii) that, if (i) holds, the strong guts condition holds at v, while, if (ii) holds, then the strong coguts condition holds at v. We may now assume that either A or B is 2–separating. Then, two of the sets displayed by v are 2–separating and so, by Lemma 4.3(ii), both the strong guts and strong coguts conditions hold at v.

Let u be an internal vertex of $\hat{T}$ different from v. Let $\{Z_1 \cup x, Z_2, Z_3\}$ be the partition displayed by u. We need to show that this partition satisfies the strong guts or the strong coguts condition. This certainly holds if the partition is displayed by a vertex of T. Thus we may assume that it is not. Then $A \subseteq Z_1$, and, without loss of generality, $\{Z_1, Z_2 \cup x, Z_3\}$ is displayed by a vertex of T.

Suppose that (i) holds. Then, as $A \subseteq Z_1$, we have $r(Z_1 \cup x) = r(Z_1)$. Thus

$$r(Z_1 \cup x) + r(Z_2) + r(Z_3) - r(P) \leq r(Z_1) + r(Z_2 \cup x) + r(Z_3) - r(P),$$

and

$$\begin{aligned} r(Z_1 \cup x \cup Z_2) + r(Z_1 \cup x \cup Z_3) + r(Z_2 \cup Z_3) - 2r(P) \\ \leq r(Z_1 \cup Z_2 \cup x) + r(Z_1 \cup Z_3) + r(Z_2 \cup x \cup Z_3) - 2r(P). \end{aligned}$$

Hence, as $\{Z_1, Z_2 \cup x, Z_3\}$ satisfies the strong guts or the strong coguts condition, so does $\{Z_1 \cup x, Z_2, Z_3\}$ in case (i). The same conclusion holds in case (ii) for then $r(Z_2 \cup x) = r(Z_2) + r(x)$ and $r(Z_2 \cup Z_3 \cup x) = r(Z_2 \cup Z_3) + r(x)$, and hence

$$\begin{aligned} r(Z_1 \cup x) + r(Z_2) + r(Z_3) - r(P) &\leq r(Z_1) + r(x) + r(Z_2) + r(Z_3) - r(P) \\ &= r(Z_1) + r(Z_2 \cup x) + r(Z_3) - r(P), \end{aligned}$$

and

$$\begin{aligned} r(Z_1 \cup x \cup Z_2) + r(Z_1 \cup x \cup Z_3) + r(Z_2 \cup Z_3) - 2r(P) \\ \leq r(Z_1 \cup Z_2 \cup x) + r(Z_1 \cup Z_3) + r(x) + r(Z_2 \cup Z_3) - 2r(P) \\ = r(Z_1 \cup Z_2 \cup x) + r(Z_1 \cup Z_3) + r(Z_2 \cup x \cup Z_3) - 2r(P). \end{aligned}$$

□

The next lemma is an extension of [13, Lemma 4.2]. Indeed, the construction used at the start of the proof is identical to that used in the earlier paper. For completeness here, this part of the argument is repeated.

Lemma 5.2. *Let T be a fork-decomposition of a 3–connected partitioned matroid P. Let $\{W, B\}$ be a 3–separating partition of P, and let h and j be edges of T having the following properties:*

(i) *the label set H of the branch T_H of h that does not contain j is a subset of W and $\lambda_P(H) = 3$; and*
(ii) *the label set J of the branch T_J of j that does not contain h is a subset of B and $\lambda_P(J) = 3$.*

Then there is a fork-decomposition $\hat{T}$ of P that displays W. Indeed, $\hat{T}$ can be obtained as follows: let T^+ and T^- be copies of the branches of $T\backslash j$ and $T\backslash h$ that

contain h and j, respectively, such that all leaf labels in B are removed in T^+ and all leaf labels in W are removed in T^-; finally, connect T^+ with T^- by a new edge e joining the vertex corresponding to v in T^+ to the vertex corresponding to u in T^-.

Proof. Since $\lambda_P(H) = 3 = \lambda_P(J)$, both H and J are non-empty. If either $|W| = 1$ or $|B| = 1$, then T displays W. Therefore we may assume that $|W|, |B| \geq 2$.

Let u and v be the end-vertices of h and j, respectively, such that the path that joins u and v in T does not contain h or j. Clearly, u and v need not be distinct. The construction of $\hat{T}$ is illustrated in Figure 3 for the case $u \neq v$. Since

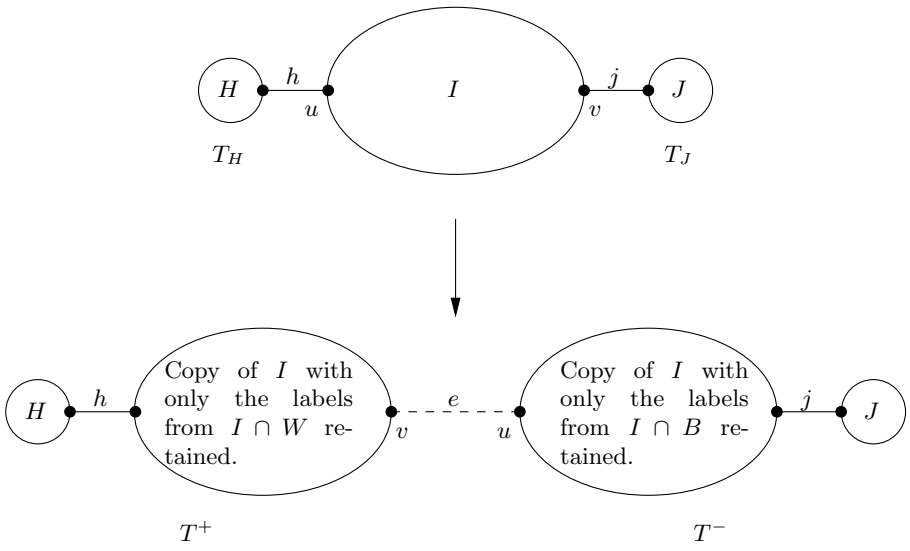


FIGURE 3

the connectivity function of a 3–connected partitioned matroid is 3–connected, the proof that $\hat{T}$ is a width–3 branch decomposition is identical to that of [13, Lemma 4.2] and we omit it here. Evidently, W is displayed in $\hat{T}$ by the edge e.

It remains to show that $\hat{T}$ is a fork-decomposition. To do this, we need to show that, at each vertex $\hat{t}$ of $\hat{T}$, the partition displayed by $\hat{t}$ satisfies the strong guts or the strong coguts condition. Without loss of generality, we may assume that $\hat{t} \in V(T^-)$. Now $\hat{t}$ is a copy of a vertex t of T. Let $\{X, Y, Z\}$ be the partition of $E(P)$ displayed by t in T. If t is a vertex of T_J, then the partition displayed by $\hat{t}$ in $\hat{T}$ is also $\{X, Y, Z\}$, so the strong guts or the strong coguts condition holds at $\hat{t}$. Thus we may assume that t is not a vertex of T_J. Then we may also assume that $X \supseteq H$ and that either $Y \supseteq J$ or $X \supseteq J$. Since $\hat{t} \in V(T^-)$, it follows that the partition displayed by $\hat{t}$ is $\{X \cup W, Y \cap B, Z \cap B\}$. If $\{X, Y, Z\}$ is exactly 3–separating, then, by Lemma 4.8, since the strong guts or the strong coguts condition holds for $\{X, Y, Z\}$, one of these conditions holds for $\{X \cup W, Y \cap B, Z \cap B\}$. If $\{X, Y, Z\}$ is not exactly 3–separating, then, by Lemma 4.4(i), X, Y, or Z is empty

or consists of a single matroid element. Since $X \supseteq H$ and $\lambda_P(H) = 3$, we deduce that Y or Z is empty or consists of a single matroid element. Thus $Y \cap B$ or $Z \cap B$ is empty or consists of a single matroid element. Hence, by Lemma 4.3(i), $\{X \cup W, Y \cap B, Z \cap B\}$ satisfies the strong guts or the strong coguts condition and the lemma is proved. □

Lemma 5.3. *Let T be a fork-decomposition of a 3–connected partitioned matroid P. Assume that T has a path $v_0v_1v_2v_3$ such that both the sets displayed by each of v_0v_1 and v_2v_3 are non-empty, v_1 and v_2 have the same label a, where $a \in \{g, c\}$, and P has matroid elements x_1 and x_2 that label leaves of T that are adjacent to v_1 and v_2 respectively. Let $\hat{T}$ be constructed from T by contracting the edge v_1v_2, forming a new vertex v_{12}; splitting the vertex v_{12} into two adjacent vertices v'_{12} and v''_{12}, where the other vertices adjacent to v'_{12} are v_0 and v_3, and the other vertices adjacent to v''_{12} are x_1 and x_2; and v'_{12} and v''_{12} are both labelled a while all other vertices of $\hat{T}$ retain their labels from T. Then $\hat{T}$ is a fork-decomposition of P in which $v'_{12}v''_{12}$ has width three.*

Proof. The construction of $\hat{T}$ is illustrated in Figure 4. Each edge of $\hat{T}$ other

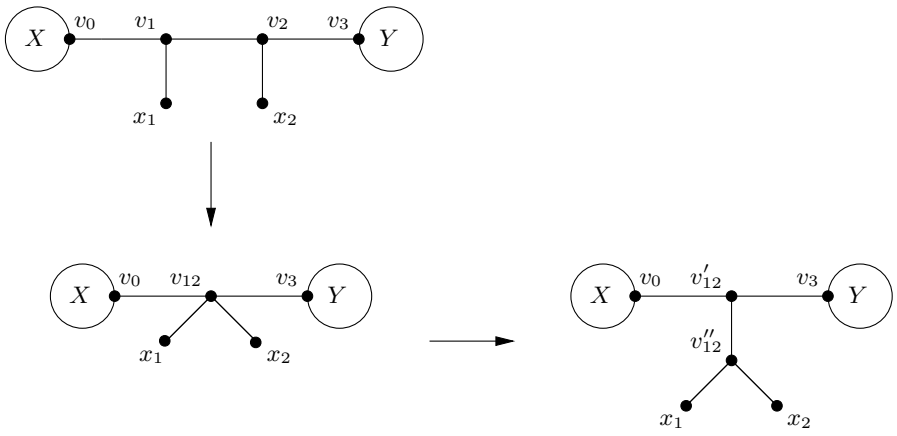


FIGURE 4

than $v'_{12}v''_{12}$ has the same width in $\hat{T}$ as in T. Since x_1 and x_2 are matroid elements, $\lambda_P(\{x_1, x_2\}) \leq 3$, and, since P is 3–connected with at least four elements, $\lambda_P(\{x_1, x_2\}) \geq 3$. Thus the width of $v'_{12}v''_{12}$ is 3. Hence $\hat{T}$ is a width–3 branch-decomposition of P. To check that $\hat{T}$ is a fork-decomposition of P, we need only check that the strong guts or the strong coguts condition holds at each of v'_{12} and v''_{12}. By Lemma 4.3(ii), both conditions hold at v''_{12}, so consider v'_{12}. Let $\{X, Y \cup \{x_1, x_2\}\}$ be the partition displayed by the edge v_0v_1 of T, where $Y \cap \{x_1, x_2\} = \emptyset$. Then the partition displayed by v'_{12} is $\{X, Y, \{x_1, x_2\}\}$. Assume that each of X and Y is exactly 3–separating. Then, from T, we deduce that $Y \cup x_2$ and $X \cup x_1$ are exactly 3–separating. If $a = g$, then, by applying Lemma 4.4(ii)(a) to $\{X, Y \cup x_2, x_1\}$ and $\{X \cup x_1, Y, x_2\}$, we obtain that $x_1 \in \mathrm{cl}(X)$ and $x_2 \in \mathrm{cl}(X \cup x_1)$. Thus $r(X \cup \{x_1, x_2\}) = r(X) = r(X) + r(\{x_1, x_2\}) - 2$, and it follows from Lemma 4.6 that the strong guts condition holds at v'_{12}. If $a = c$, then,

by Lemma 4.4(ii)(b), $r(X \cup x_1) = r(X) + 1$ and $r((X \cup x_1) \cup x_2) = r(X \cup x_1) + 1$. Thus $r(X \cup \{x_1, x_2\}) = r(X) + 2$, that is, $r(X \cup \{x_1, x_2\}) = r(X) + r(\{x_1, x_2\})$ and it follows from Lemma 4.6 that the strong coguts condition holds at v'_{12}.

We may now assume that X or Y, say X, is not exactly 3–separating. Then, by Lemma 4.4(i), X consists of a single matroid element. We may also assume that Y does not consist of a single matroid element otherwise, by Lemma 4.3(ii), $\{X, Y, \{x_1, x_2\}\}$ satisfies both the strong guts and strong coguts conditions. If $a = g$, then, from v_2, we deduce that

$$r(Y \cup x_2) + r(Y \cup X \cup x_1) + r(X \cup \{x_1, x_2\}) - 2r(P) \leq 2.$$

But $r(Y \cup x_2) = r(P) + 2 - r(X \cup x_1) = r(P) + 2 - r(\{x_1, x_2\}) = r(Y \cup X)$, and $r(Y \cup X \cup x_1) = r(P) = r(Y \cup \{x_1, x_2\})$. Thus

$$r(Y \cup X) + r(Y \cup \{x_1, x_2\}) + r(X \cup \{x_1, x_2\}) - 2r(P) \leq 2,$$

so $\{X, Y, \{x_1, x_2\}\}$ satisfies the strong guts condition. If $a = c$, then, from v_2,

$$2 \geq r(Y) + r(x_2) + r(X \cup x_1) - r(P) = r(Y) + r(X) + r(\{x_1, x_2\}) - r(P),$$

so $\{X, Y, \{x_1, x_2\}\}$ satisfies the strong coguts condition. □

Lemma 5.4. *Let T be a fork-decomposition of a 3–connected partitioned matroid P. For some $n \geq 2$, let $v_0v_1v_2 \dots v_{n+1}$ be a path in T such that both v_0v_1 and v_nv_{n+1} have width three; for each i in $\{1, 2, \dots, n\}$, the vertex v_i is adjacent to a leaf that labels a matroid element w_i, and the set $W = \{w_1, w_2, \dots, w_n\}$ is 3–separating. Then every consecutive 3–element subset of W is 3–separating. Moreover, if, for some j in $\{1, 2, \dots, n-1\}$, both v_j and v_{j+1} have the same label a where $a \in \{g, c\}$, then P has a fork-decomposition that displays W.*

Proof. Let v_0v_1 display the sets X_1 and $W \cup X_2$ where $W \cap X_2 = \emptyset$. For all i in $\{3, 4, \dots, n\}$, since W and $X_1 \cup \{w_1, w_2, \dots, w_i\}$ are 3–separating and their union avoids X_2, it follows by Corollary 4.2 that their intersection, $\{w_1, w_2, \dots, w_i\}$, is 3–separating. Similarly, $\{w_{i-2}, w_{i-1}, \dots, w_n\}$ is 3–separating. Thus, by Corollary 4.2 again, $\{w_1, w_2, \dots, w_i\} \cap \{w_{i-2}, w_{i-1}, \dots, w_n\}$ is 3–separating, that is, every consecutive 3–element subset of W is 3–separating. Now suppose that v_j and v_{j+1} are both labelled g or are both labelled c. Then, by Lemma 5.3, P has a fork-decomposition as shown in Figure 5. Furthermore, by Lemma 2.3, every consecutive 3–element subset of W is a triangle or a triad. Thus, by $j-1$ applications

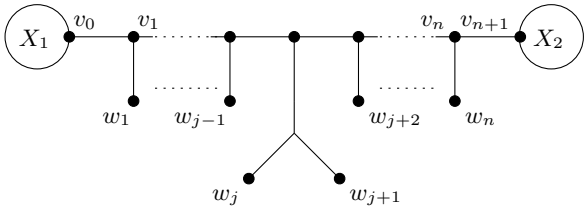


FIGURE 5

of Lemma 5.1, we deduce that P has a fork-decomposition as shown in Figure 6. A

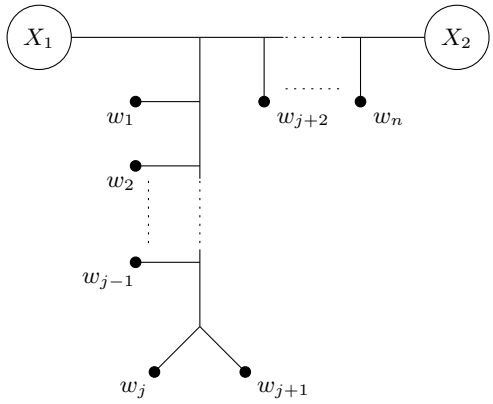


FIGURE 6

further $n - j + 1$ applications of Lemma 5.1 gives a fork-decomposition of P that displays W. □

Lemma 5.5. *Let T be a fork-decomposition of a 3–connected partitioned matroid P. For some $n \geq 3$, let $v_0v_1v_2 \dots v_{n+1}$ be a path in T such that each edge has width three and P has elements $b_1, b_2, \dots, b_n$ that label leaves of T that are adjacent to $v_1, v_2, \dots, v_n$, respectively. Let $B = \{b_1, b_2, \dots, b_n\}$. If B is 3–separating and cannot be displayed in a fork-decomposition of P, then either $E(P) - B$ consists of exactly two matroid elements, or B is a fan and the vertices $v_1, v_2, \dots, v_n$ are alternately guts and coguts vertices.*

Proof. Let the partition displayed by the edge v_0v_1 of T be $\{A, B \cup C\}$, where B and C are disjoint. If either A or C is empty, then B is displayed, so both these sets are non-empty. Assume that $\lambda_P(A) = 2$. Then, A consists of a single matroid element a. If C consists of a single matroid element, then the lemma holds. So we may assume that $\lambda_P(C) = 3$. Now both B and $B \cup a$ are exactly 3–separating, so, by [13, Lemma 2.6] either $a \in \mathrm{cl}(C)$ or $r(B) \cup a = r(B) + 1 = r(B) + r(a)$. In either case, it follows from Lemma 5.1 that we can display B. Thus we may assume that both A and C are exactly 3–separating.

We show first that B consists entirely of matroid elements. Suppose that there is some element b_i of B that is not a matroid element. Then $\lambda_P(b_i) = 3$. Now, since $\lambda_P(A) = 3$, we may apply Lemma 5.2 taking $(H, J) = (A, \{b_i\})$ to obtain a fork-decomposition of P that displays $A \cup C$ and hence displays B. This contradiction implies that B must consist entirely of matroid elements.

It follows immediately from Lemma 5.4 that $v_1, v_2, \dots, v_n$ are alternately guts and coguts vertices, and that, for each i in $\{1, 2, \dots, n-2\}$, the set $\{b_i, b_{i+1}, b_{i+2}\}$ is 3–separating and hence, by Lemma 2.3, is either a triangle or a triad.

If $n = 3$, then B is certainly a fan. Now suppose that $n > 3$ and that B is not a fan. Then, by Lemma 2.2, B is either a segment or a cosegment. Assume the former. Then both b_3 and b_4 are in $\mathrm{cl}(\{b_1, b_2\})$. Thus $b_3 \in \mathrm{cl}(A \cup \{b_1, b_2\})$

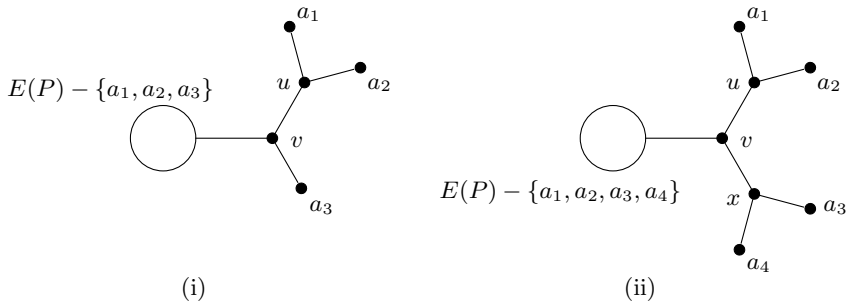


FIGURE 7

and $b_4 \in \mathrm{cl}(A \cup \{b_1, b_2, b_3\})$, so, by Lemmas 4.3(i) and 4.4(ii)(a), both v_3 and v_4 are guts vertices; a contradiction. Now assume that B is a cosegment. Then $r(E(P) - \{b_2, b_3\}) = r(P) = r(E(P) - \{b_1, b_2, b_3\}) + 1$, so $r(A \cup b_1) = r(A) + 1$. Similarly, $r(A \cup \{b_1, b_2\}) = r(A \cup b_1) + 1$. Thus, by Lemmas 4.3(i) and 4.4(ii)(b), both v_1 and v_2 are coguts vertices; a contradiction. □

6. Displaying Separations

In this section, we characterize precisely which 3–separating partitions cannot be guaranteed to be displayed in some fork-decomposition of a 3–connected forked partitioned matroid. We begin with a lemma that extends [13, Lemma 5.3].

Lemma 6.1. *Let P be a 3–connected forked partitioned matroid,x and let T be a reduced fork-decomposition of P. If, for some $n \in \{3, 4\}$, there are matroid elements $a_1, a_2, \ldots, a_n$ such that T has a vertex v that displays $\{a_1, a_2\}$, $\{a_3, a_n\}$, and $E(P) - \{a_1, a_2, \ldots, a_n\}$, then every permutation of $\{a_1, a_2, \ldots, a_n\}$ in T produces another fork-decomposition of P.*

Proof. Evidently T is as shown in Figure 7(i) or (ii), where exactly two of the branches at v are shown completely. Since every set of one or two matroid elements is 3–separating, it follows that every permutation of $\{a_1, a_2, \ldots, a_n\}$ in T produces another width–3 branch-decomposition of P. To check that we retain a fork-decomposition, we observe that this is immediate in (i) since each of u and v is incident with an edge of width 2. For the same reason, we need only check the vertex v in case (ii). Then, at v, symmetry implies that the strong guts or strong coguts condition holds after relabelling unless two of the sets displayed by v are $\{a_1, a_3\}$ and $\{a_2, a_4\}$. Assume that the exceptional case arises. Then we may suppose that each of the sets displayed by v is exactly 3–separating. Now

$$r(\{a_1, a_3\}) + r(\{a_2, a_4\}) - r(\{a_1, a_2, a_3, a_4\}) \\ = r(\{a_1, a_2\}) + r(\{a_3, a_4\}) - r(\{a_1, a_2, a_3, a_4\}).$$

It now follows from Lemma 4.6 that the strong guts or strong coguts condition holds at v after relabelling, since it holds before relabelling. □

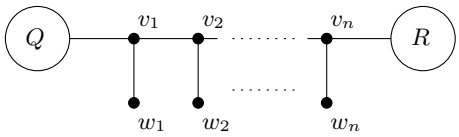


FIGURE 8

Theorem 6.2. *Let P be a 3–connected forked partitioned matroid. Let $\{W, B\}$ be a 3–separating partition of the ground set of P that cannot be displayed in any fork-decomposition of P. Then, up to interchanging the sets W and B,*

(i) *W consists entirely of matroid elements, and*
(ii) *either $|W| \leq 2$, or W is the ground set of a fan. Moreover, if $|W| > 3$, then P has a fork-decomposition T of the form shown in Figure 8, where $W = \{w_1, w_2, \ldots, w_n\}$ and the vertices $v_1, v_2, \ldots, v_n$ are alternately guts and coguts vertices.*

Proof. Let T be a reduced fork-decomposition of P. If T has edges h and j of width three such that h displays a subset H of W and j displays a subset J of B, then, by Lemma 5.2, $\{W, B\}$ can be displayed in a fork-decomposition of P; a contradiction. Thus we may assume that no fork-decomposition of P has such edges h and j.

Next we establish

6.2.1. *P has a reduced fork-decomposition T_1 that has an edge f of width three such that one of the sets displayed by f is a subset of W or B.*

Proof. This is immediate if P has an element of rank exceeding one. Thus we may assume that P is a matroid. Take a longest path in T, letting one end of this path be a_1, and letting v be the vertex on this path whose distance from a_1 is two. Evidently T is as shown in Figure 7(i) or (ii), where exactly two of the branches at v are shown completely. Then, by Lemma 6.1, we can relabel T such that $\{a_1, a_2\}$ is a subset of W or B, and we conclude that (6.2.1) holds. □

Without loss of generality, assume that one of the sets displayed by f in T_1 is a subset of B. Then, by the first paragraph of the proof of the theorem, W consists entirely of matroid elements. Assume that the partitioned matroid P is a counterexample to the theorem that is chosen to minimize $|W|$. Then $|W| \geq 3$. Moreover, if $|W| = 3$, then, by Lemma 2.3, W is a triangle or a triad, so W is certainly a fan. Thus we may assume that $|W| \geq 4$.

Let Z be the subset of B displayed by f. It is easily seen that there is a vertex v_1 in T_1 that displays a partition $\{Y_1, Y_2, X \cup Z\}$ as shown in Figure 9 such that $|Y_1 \cap W| = 1$ and $|Y_2 \cap W| = 1$. Here $X \cap Z = \emptyset$.

We next construct from T_1 a reduced fork-decomposition T_2 of P as shown in Figure 10, where Y_1 is the disjoint union of Y_1' and Y_1'', and f' has width three and displays a subset of B. This construction is done in one of two ways depending

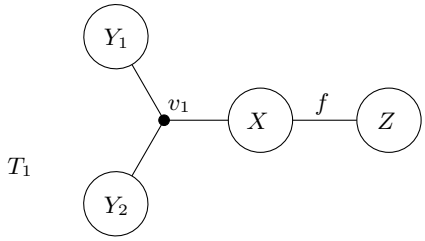


FIGURE 9

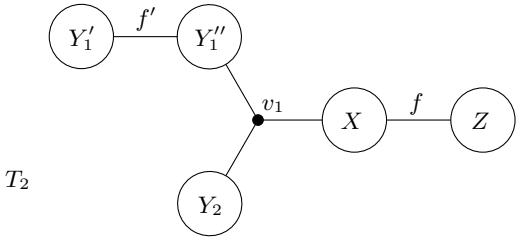


FIGURE 10

upon whether

(i) there is an element y of either Y_1 or Y_2 that is not a matroid element, or
(ii) every element of Y_1 and Y_2 is a matroid element.

Consider (i). Without loss of generality, we may assume that y is in Y_1. In this case, choose T_2 to be T_1, let f' denote the pendant edge of T_2 that displays this element, and let Y_1' denote the set consisting of this element.

Now consider (ii). Then either $|Y_1| \geq 3$ or $|Y_2| \geq 3$, for otherwise, using Lemma 6.1, we can obtain a fork-decomposition of P that has edges h and j as described in the first paragraph of the proof of this theorem. Without loss of generality, assume that $|Y_1| \geq 3$. Take a longest path in T_1 that starts at v_1 and whose first edge is the edge of T_1 displaying Y_1. Let a_1 denote the terminal vertex of this path and let v denote the vertex of this path whose distance from a_1 is two. Evidently, the local neighbourhood of v is as shown in Figure 7(i) or (ii). In either case, since at most one of a_1, a_2, and a_3 is an element of W, it follows by Lemma 6.1 that we can relabel T_1 so that $\{a_1, a_2\}$ is a subset of B. Choose T_2 to be the resulting fork-decomposition of P. As P is 3–connected, $\{a_1, a_2\}$ is exactly 3–separating. In this case, let $Y_1' = \{a_1, a_2\}$ and let f' denote the edge of T_2 that displays Y_1'.

Having constructed T_2, let $Y = Y_1 \cup Y_2$. Clearly, Y is exactly 3–separating as P is 3–connected. Since $Y \cap W$ consists of two matroid elements, $Y \cap W$ is exactly 3–separating. Therefore, by Lemma 4.1, $Y \cup W$ is 3–separating. Since $\{Y \cup W, (X \cap B) \cup Z\}$ is a 3–separating partition of $E(P)$, and both Y and Z are

exactly 3–separating with $Y \subseteq Y \cup W$ and $Z \subseteq (X \cap B) \cup Z$, it follows by Lemma 5.2 that P has a reduced fork-decomposition T_3 as shown in Figure 11.

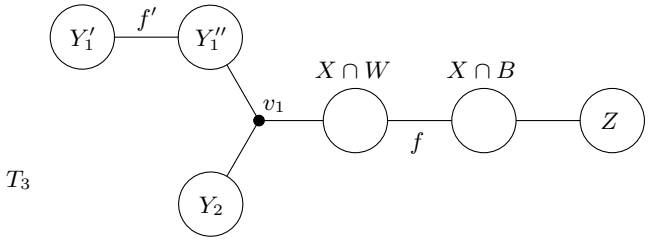


FIGURE 11

Since $|W| \geq 4$ and $|Y \cap W| = 2$, we have $|(X \cup Z) \cap W| \geq 2$, and so, as P is 3–connected, $\lambda_P((X \cup Z) \cap W) \geq 3$. Therefore, by Lemma 4.1, $(X \cup Z) \cup W$ is 3–separating. Since $\{Y \cap B, (X \cup Z) \cup W\}$ is a 3–separating partition of $E(P)$,and both Y_1' and Z are exactly 3–separating with $Y_1' \subseteq Y \cap B$ and $Z \subseteq (X \cup Z) \cup W$, it follows by Lemma 5.2 again that P has a reduced fork-decomposition T_4 as shown in Figure 12.

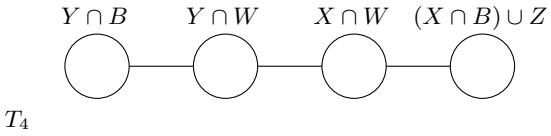


FIGURE 12

By the first paragraph of the proof of this theorem, T_4 does not display an exactly 3–separating subset of W, so T_4 must be as shown in Figure 13, where $W = \{w_1, w_2, \ldots, w_n\}$. By Lemma 5.5, W is a fan and the vertices $v_1, v_2, \ldots, v_n$ are alternately guts and coguts vertices. This completes the proof of the theorem.

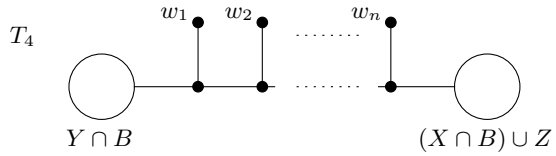


FIGURE 13

□

7. Minimal Non-Fans

In this section, we bound the size of a fully closed set A when $\{A, B\}$ is a 3–separation of a 3–connected matroid M such that A is not a fan but A' is a fan for

every proper subset A' of A for which $A' \cup B$ is the ground set of a 3–connected minor of M.

The following property of fans [19, Lemma 3.4] will be used repeatedly.

Lemma 7.1. *Let e_1, e_2, e_3, e_4, e_5 be distinct elements of a 3–connected matroid M that is not isomorphic to a rank–3 wheel. Suppose that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles and $\{e_2, e_3, e_4\}$ is a triad of M. Then these two triangles and this one triad are the only triangles and triads of M that contain e_3.*

Lemma 7.2. *Let M be a 3–connected matroid that is not a wheel or a whirl and let $\mathcal{F}$ be a fan of M. If $\{x, y, z\}$ is a triangle or a triad of M and each of x, y, and z is in $E(\mathcal{F})$, then $\{x, y, z\}$ is a link of $\mathcal{F}$.*

Proof. By duality, we may assume that $\{x, y, z\}$ is a triangle of M. Let $\mathcal{F}$ be

$$\{x_0, x_1, x_2\}, \{x_1, x_2, x_3\}, \ldots, \{x_{k-2}, x_{k-1}, x_k\}$$

and assume that $\{x, y, z\}$ is not a link of $\mathcal{F}$. Suppose first that $\mathcal{F}$ has type–1 so that $\{x_0, x_1, x_2\}$ and $\{x_{k-2}, x_{k-1}, x_k\}$ are triangles. Then $k \geq 4$ and, by Lemma 7.1, the only triangles and triads of M containing any of $x_2, x_3, \ldots, x_{k-2}$ are those in $\mathcal{F}$. Thus $\{x, y, z\} \subseteq \{x_0, x_1, x_{k-1}, x_k\}$ so, without loss of generality, $x_1 = x$. This contradicts orthogonality unless $k = 4$. In the exceptional case it follows by orthogonality that, without loss of generality, we may assume that $y = x_3$. But then $z \neq x_2$ and $z \notin \{x_0, x_4\}$, otherwise $\{x_0, x_1, x_2, x_3, x_4\}$ has rank two; a contradiction. We conclude that $\mathcal{F}$ does not have type–1.

Suppose next that $\mathcal{F}$ has type–3 where $\{x_0, x_1, x_2\}$ is a triangle. Then, by Lemma 7.1 again, $\{x, y, z\} \subseteq \{x_0, x_1, x_{k-1}, x_k\}$. By orthogonality, we may assume that $\{x, y\} = \{x_{k-1}, x_k\}$ and that either $z = x_0$, or $k = 3$ and $z = x_1$. In the latter case, $\{x_1, x_2, x_3\}$ is a triangle and a triad of M, so M is isomorphic to $U_{2,4}$, which is the rank–2 whirl; a contradiction. In the former case, by [19, Lemma 2.4], M is a wheel or a whirl; a contradiction.

Finally, suppose that $\mathcal{F}$ has type–2. Then $\{x, y, z\} \subseteq \{x_0, x_1, x_{k-1}, x_k\}$ and orthogonality is contradicted. □

We show next that the links in a fan with at least five elements induce a unique ordering on the ground set of the fan.

Lemma 7.3. *Let $\mathcal{F}$ be a fan in a 3–connected matroid M. Suppose that $|E(\mathcal{F})| = n \geq 5$ and that $\mathcal{F}'$ is another fan with $E(\mathcal{F}) = E(\mathcal{F}')$. Then either $\mathcal{F}' = \mathcal{F}$, or $\mathcal{F}'$ is obtained from $\mathcal{F}$ by reversing the order of the links.*

Proof. Suppose that $\mathcal{F}$ has as its links

$$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\}.$$

It follows using Lemma 7.2 that these links are the only triangles and triads contained in $E(\mathcal{F})$. Now a_1 and a_n are the only members of $E(\mathcal{F})$ that are in unique links. Once the links $\{a_1, a_2, a_3\}$ and $\{a_{n-2}, a_{n-1}, a_n\}$ are removed from $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\}$, the elements a_2 and a_{n-1} are the

only elements of $E(\mathcal{F}) - \{a_1, a_n\}$ that are in exactly one of the remaining links. The lemma follows by repeating this process. □

Because a fan $\mathcal{F}$ can be thought of as a partial wheel, when the fan has at least five elements, it inherits some terminology from wheels. Thus, if, in the canonical order $a_1, a_2, \ldots, a_n$ determined by the links of $\mathcal{F}$, the set $\{a_1, a_2, a_3\}$ is a triangle, then the *spokes* of $\mathcal{F}$ are $a_1, a_3, \ldots$, while the *rim* of $\mathcal{F}$ consists of the elements of $\mathcal{F}$ that are not spokes. If, instead, $\{a_1, a_2, a_3\}$ is a triad, then the *spokes* of $\mathcal{F}$ are $a_2, a_4, \ldots$, and the *rim* is again the set of non-spokes.

The next lemma shows how a fan in a 3–connected matroid can be shrunk to a fan in a smaller 3–connected matroid by deleting a spoke and contracting an adjacent rim element.

Lemma 7.4. *Let $\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\}$ be a fan $\mathcal{F}$ in a 3–connected matroid M. If $3 \leq i \leq n-2$ and a_i is a spoke of $\mathcal{F}$, then $M\backslash a_i/a_{i+1}$ is 3–connected unless a_{i+1} is in a triangle of M that is not in $\mathcal{F}$. In particular, if $n \geq 5$, then $M\backslash a_i/a_{i+1}$ is 3–connected unless M is a wheel of rank three.*

Proof. If M is a rank–3 wheel, then $i \in \{3, 4\}$ and it is easily checked that a_{i+1} is in a triangle that is not in $\mathcal{F}$. Thus we may assume that M is not a rank–3 wheel.

Now suppose that $i \leq n-3$. Evidently M/a_i has $\{a_{i+1}, a_{i+2}\}$ as a circuit and $M/a_i\backslash a_{i+1}$ has $\{a_{i+2}, a_{i+3}\}$ as a cocircuit. Thus $\mathrm{si}(M/a_i)$ is not 3–connected unless it is isomorphic to $U_{1,1}$ or $U_{2,3}$. Consider the exceptional cases. Then $r(M) = 2$ or $r(M) = 3$. But $3 \leq i \leq n-3$, so $n \geq 6$. As $r(M) \leq 3$, it follows that $n = 6$ and $r(M) = 3$, so $i = 3$ and $\mathrm{si}(M/a_i) \cong U_{2,3}$. Thus $\{a_1, a_5, a_6\}$ is a circuit of M/a_3 and so, by orthogonality, is a circuit of M. Therefore, by Lemma 7.1, M is isomorphic to a rank–3 wheel; a contradiction.

We may now assume that when $i \leq n-3$ or, by symmetry, when $i \geq 4$, the matroid $\mathrm{si}(M/a_i)$ is not 3–connected. Then, by Bixby's Lemma [1], $\mathrm{co}(M\backslash a_i)$ is 3–connected. Now $M\backslash a_i$ has $\{a_{i-1}, a_{i+1}\}$ as a cocircuit. Moreover, by Lemma 7.1, $M\backslash a_i$ has no other 2–cocircuits and so $M\backslash a_i/a_{i+1}$ is 3–connected.

It remains to consider the case when $3 = i = n-2$, so $n = 5$. Then, by Lemma 7.1, M/a_3 has $\{a_1, a_2\}$ and $\{a_4, a_5\}$ as its only 2–circuits, so

$$\mathrm{si}(M/a_3) \cong M/a_3\backslash a_2, a_4 = M\backslash a_2, a_3, a_4 = M/a_4\backslash a_3, a_2.$$

Moreover, $\mathrm{co}(M\backslash a_3) \cong M\backslash a_3/a_4 = M/a_4\backslash a_3$. Now one of $M/a_4\backslash a_3, a_2$ and $M/a_4\backslash a_3$ is 3–connected. If $M/a_4\backslash a_3, a_2$ is 3–connected, then so is $M/a_4\backslash a_3$ unless a_2 is in a 2–circuit in $M/a_4\backslash a_3$. In the exceptional case, $\{a_2, a_4\}$ is in a triangle of M. But this has been excluded by hypothesis. We deduce that $M\backslash a_3/a_4$ is 3–connected. □

In the next result, the graph C_3^2 is obtained from a triangle by adding an edge in parallel to each original edge.

Theorem 7.5. *Let $\{A, B\}$ be a 3–separation of a 3–connected matroid M in which A is a fully closed set that is not the ground set of a fan. Assume that if $\{A', B\}$ is*

a 3–separation of a 3–connected proper minor M' of M, then A' is the ground set of a fan. Then $|A| \leq 6$. Moreover, for some N in $\{M, M^*\}$, one of the following occurs:

(i) *A is a 4–point line of N;*
(ii) *A is a quad of N;*
(iii) *A is a 4–cocircuit of N that contains a triangle of N;*
(iv) *$N|A \cong M(K_4)$ and one of the triads of $N|A$ is a triad of N;*
(v) *$N|A$ is the direct sum of two triangles and $N.A$ is isomorphic to the cycle matroid of C_3^2;*
(vi) *$A = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ where N has $\{e_3, e_2, e_1\}$ and $\{e_3, e_4, e_5\}$ as circuits, and $\{e_3, e_2, e_4\}$ and $\{e_3, e_1, e_5, e_6\}$ as cocircuits*

Proof. We first establish the following useful result.

Lemma 7.6. *Let M' be a 3–connected minor of M with ground set $A' \cup B$ where A' is a proper subset of A having at least three elements. Then A' is the ground set of a fan $\mathcal{F}$ that is a maximal fan in M'. In particular, the ends of $\mathcal{F}$ are non-essential in M'.*

Proof. By the hypothesis of the theorem, A' is certainly the ground set of a fan $\mathcal{F}$. Suppose that $\mathcal{F}$ is not a maximal fan in M'. Then M' has a triangle or a triad X that can be adjoined to $\mathcal{F}$ to give a longer fan. By duality, we may assume that X is a triangle. Evidently X has exactly two elements in common with A'. Thus $X \subseteq \mathrm{cl}_{M'}(A')$, so $X \subseteq \mathrm{cl}_M(A)$; a contradiction to the fact that A is fully closed. We conclude that $\mathcal{F}$ is a maximal fan in M'. The fact that the ends of $\mathcal{F}$ are non-essential follows by [19, Lemma 1.5]. □

Since A is fully closed, $|A| > 3$ otherwise A is a triangle or a triad; a contradiction. Suppose that, for all e in A, the element e is essential. Then, by [18], M has a fan whose internal elements are in A and whose ends are in B. Thus A spans or cospans some element of B; a contradiction. We conclude, by duality, that we may assume that A has an element e such that $M\backslash e$ is 3–connected. Then $\{A - e, B\}$ is a 3–separation of $M\backslash e$, so $r(A - e) = r(A)$ and $A - e$ is the ground set of a fan $\mathcal{F}$ in $M\backslash e$.

We shall distinguish cases 1, 2, and 3 depending on whether $\mathcal{F}$ is a type–1, a type–2, or a type–3 fan, respectively.

First consider case 1, that is, $\mathcal{F}$ is a type–1 fan. Since $|A - e|$ is odd and $|A| > 3$, we have $|A - e| \geq 3$. Suppose first that $|A - e| = 3$. Then $A - e$ is a triangle in $M\backslash e$. As $r(A) = r(A - e)$, it follows that A is a 4–point line of M. Now suppose that $|A - e| \geq 5$. Let $\mathcal{F}$ be (see Figure 14)

$$\{a_0, a_1, a_2\}, \{a_1, a_2, a_3\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}.$$

Let $i \in \{1, 2, \ldots, n\}$. Clearly $M\backslash e\backslash a_{2i-2}/a_{2i-1} \cong M\backslash e\backslash a_{2i}/a_{2i-1}$. Then, since $|E(M\backslash e)| \geq 8$, it follows by Lemma 7.4 that $M\backslash e\backslash a_{2i-2}/a_{2i-1}$ is 3–connected. Assume that $M\backslash a_{2i-2}/a_{2i-1}$ is 3–connected. Then $A - \{a_{2i-2}, a_{2i-1}\}$ is the ground set of a fan in this matroid. Since e is not in a triad of $M\backslash a_{2i-2}/a_{2i-1}$, it follows

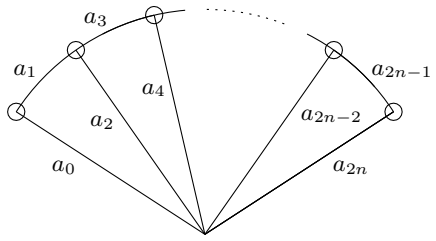


FIGURE 14

that e is in a triangle and e is an end of a maximal fan in $M\backslash a_{2i-2}/a_{2i-1}$. Thus, since $|A-\{a_{2i-2}, a_{2i-1}\}|$ is even, $M\backslash a_{2i-2}/a_{2i-1}$ has a type–3 fan $\mathcal{F}'$ with ground set $A-\{a_{2i-2}, a_{2i-1}\}$ and first link a triangle containing e. Now delete e from $M\backslash a_{2i-2}/a_{2i-1}$. Since $M\backslash a_{2i-2}/a_{2i-1}\backslash e$ is 3–connected, all the triangles and triads of $\mathcal{F}'$ except for the triangle containing e remain intact when e is deleted. Thus, in $M\backslash a_{2i-2}/a_{2i-1}\backslash e$, we deduce from considering $\mathcal{F}'$ with e deleted that $r^*(A-\{e, a_{2i-2}, a_{2i-1}\}) \leq n$; and, from considering $\mathcal{F}$, we have $r(A-\{e, a_{2i-2}, a_{2i-1}\}) \leq n$. But $|A-\{e, a_{2i-2}, a_{2i-1}\}| = 2n-1$. Thus, in $M\backslash a_{2i-2}/a_{2i-1}\backslash e$,

$$r(A-\{e, a_{2i-2}, a_{2i-1}\}) + r^*(A-\{e, a_{2i-2}, a_{2i-1}\}) - |A-\{e, a_{2i-2}, a_{2i-1}\}| \leq 1.$$

This contradicts the fact that $M\backslash a_{2i-2}/a_{2i-1}\backslash e$ is 3–connected. We conclude that $M\backslash a_{2i-2}/a_{2i-1}$ is not 3–connected. Since $M\backslash a_{2i-2}/a_{2i-1}\backslash e$ is 3–connected and A is closed, we deduce that:

7.7. *For all i in $\{1, 2, \ldots, n\}$, the matroid M has a triangle that contains $\{e, a_{2i-1}\}$, avoids a_{2i-2}, and is contained in A.*

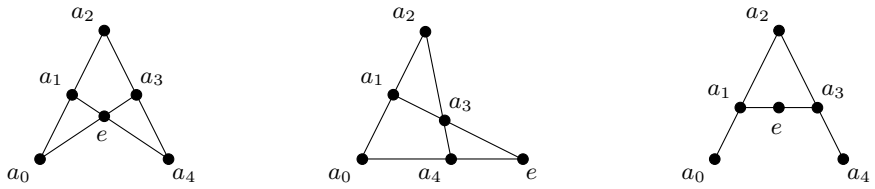
By Lemma 7.6, since a_0 is an end of $\mathcal{F}$ that is in a triangle, $M\backslash e\backslash a_0$ is 3–connected. Hence $M\backslash a_0$ is also 3–connected. Thus $M\backslash a_0$ has a fan $\mathcal{F}_0$ with ground set $A-a_0$. By Lemma 7.2, we deduce that $\mathcal{F}_0$ has all of $\{a_2, a_3, a_4\}, \{a_4, a_5, a_6\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}$ as triangles. Now e is not in a triad of $M\backslash a_0$. Thus e must be an end of $\mathcal{F}_0$, and $\mathcal{F}_0$ must be of type–1. Therefore $\mathcal{F}_0$ has exactly n triangles, $n-1$ of which are listed above. Since the union of the n triangles of $\mathcal{F}_0$ is $A-a_0$, the unique triangle T containing e must also contain a_1. Now T cannot meet $\{a_4, a_6, \ldots, a_{2n-2}\}$ as each of these elements is already in two triangles. Moreover, T cannot meet $\{a_5, a_7, \ldots, a_{2n-3}\}$ since no triangle of a fan has each of its elements in another triangle of the fan. Thus the third element of T is in $\{a_2, a_3, a_{2n-1}, a_{2n}\}$. We now separate into two subcases:

(I) $n > 2$; and
(II) $n = 2$.

Suppose first that (I) holds. By (7.7), each of $\{e, a_3\}$ and $\{e, a_{2n-1}\}$ is contained in a triangle of $M|A$ but T is the only triangle of $M|(A-a_0)$ containing e. If T avoids $\{a_3, a_{2n-1}\}$, then $\{e, a_3, a_0\}$ and $\{e, a_{2n-1}, a_0\}$ are triangles of M, so $\{a_0, a_3, a_{2n-1}\}$ is a triangle of M. But, by Lemma 7.2, since $\{a_0, a_3, a_{2n-1}\}$ is not a triangle of $\mathcal{F}$, we have a contradiction. Thus T contains a_3 or a_{2n-1}. Hence either both $\{e, a_1, a_3\}$ and $\{e, a_0, a_{2n-1}\}$ are triangles of M, or both $\{e, a_1, a_{2n-1}\}$

and $\{e, a_0, a_3\}$ are triangles of M. In each case, by elimination and Lemma 7.2, $\{a_0, a_1, a_3, a_{2n-1}\}$ is a circuit of M. This contradicts orthogonality unless $n = 3$. In the exceptional case, A is spanned by $\{a_0, a_1, a_3\}$ so $r(A) \leq 3$. As $r^*(A) \leq |A| - 2$, we get a contradiction. We conclude that (I) cannot hold. Thus (II) holds.

By (7.7), each of $\{e, a_1\}$ and $\{e, a_3\}$ is in a triangle of $M|A$, so there are the three possibilities for $M|A$ shown in Figure 15. The unique triad T^* of $\mathcal{F}_0$ contains the



(a) (b) (c)

FIGURE 15

element that is in both triangles of $\mathcal{F}_0$, so T^* contains a_3 in cases (b) and (c), and contains a_4 in case (a). Now T^* or $T^* \cup a_0$ is a cocircuit of M. Also, as $\{a_1, a_2, a_3\}$ is a cocircuit of $M\backslash e$, either $\{a_1, a_2, a_3\}$ or $\{a_1, a_2, a_3, e\}$ is a cocircuit of M. Since M is 3–connected and $r(A) = 3$ while $|A| = 6$, we must have that $r^*(A) \geq 5$. Hence A contains at most one cocircuit of M. The only way for this to occur is for T^* to be equal to $\{a_1, a_2, a_3\}$ and for this set to be a cocircuit of M. We conclude that (b) holds and so (iv) of the theorem holds.

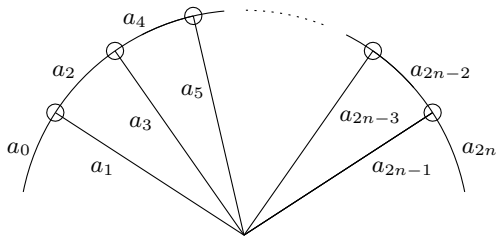


FIGURE 16

Next consider case 2, that is, suppose that $A - e$ is the ground set of a type–2 fan $\mathcal{F}$. Then $|A - e|$ is odd and exceeds two. Suppose that $|A - e| = 3$. Then $A - e$ is a triad of $M\backslash e$ and $A - e$ spans e. Thus either A is a quad, or A is a 4–element cocircuit that contains a triangle. In each case, the theorem holds. We may now suppose that $|A - e| \geq 5$. Let $\mathcal{F}$ be (see Figure 16)

$$\{a_0, a_1, a_2\}, \{a_1, a_2, a_3\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}.$$

By Lemma 7.6, both $M\backslash e/a_0$ and $M\backslash e/a_{2n}$ are 3–connected.

Suppose that M/a_0 is 3–connected. Then $A - a_0$ is the ground set of a fan with an odd number of elements. If this fan has type–2, then $A - a_0$ is the ground set

of a type–1 fan of $M^*\backslash a_0$ and we can deduce the result from case 1. Thus we may assume that $A - a_0$ is the ground set of a type–1 fan $\mathcal{F}_0$ of M/a_0. Now every triad of M/a_0 is a triad of M. Thus e is not in a triad of M/a_0, so e is an end of $\mathcal{F}_0$. Moreover, each of $\{a_1, a_2, a_3\}, \{a_3, a_4, a_5\}, \ldots, \{a_{2n-3}, a_{2n-2}, a_{2n-1}\}$ is a triangle of $\mathcal{F}_0$. Since $\mathcal{F}_0$ has exactly n triangles and their union is $A - a_0$, it follows that $\{e, a_{2n}\}$ is contained in a triangle T_{2n} of M/a_0.

Every triad of $\mathcal{F}_0$ contains and so equals a triad of $M\backslash e/a_0$. Now $A - \{a_0, e\}$ is the ground set of a fan $\mathcal{F}_0\backslash e$ of $M\backslash e/a_0$. Since $\mathcal{F}_0$ and $\mathcal{F}_0\backslash e$ have the same number of triads, it follows that the triads of M/a_0 in $\mathcal{F}_0$ coincide with the triads of $M\backslash e/a_0$ in $\mathcal{F}_0\backslash e$. All the triads of M/a_0 are triads of M. Thus all the triads of $M\backslash e/a_0$ in $\mathcal{F}_0\backslash e$ are triads of M. Moreover, all the triangles of $M\backslash e/a_0$ in $\mathcal{F}_0\backslash e$ are triangles of M. Hence

$$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}$$

is a type–3 fan in M and hence in the 3–connected matroid M/a_0, and the first link of this fan is a triangle.

By orthogonality, the triangle T_{2n} of M/a_0 that contains $\{e, a_{2n}\}$ also contains a_{2n-1} or a_{2n-2}, where the latter can only occur if $n = 2$. Now $\{a_0, a_1, a_2\}$ is a triad of $M\backslash e$. Suppose $T_{2n} = \{e, a_{2n}, a_{2n-1}\}$. By orthogonality, $\{e, a_{2n}, a_{2n-1}\}$ is a triangle of M if and only if $\{a_0, a_1, a_2\}$ is a triad of M. It follows that if T_{2n} is a triangle of M, then A is the ground set of a fan in M; a contradiction. Thus if $T_{2n} = \{e, a_{2n}, a_{2n-1}\}$, then

7.8. *$\{e, a_{2n}, a_{2n-1}\}$ is not a circuit of M and $T_{2n} \cup a_0$ is a circuit of M while $\{e, a_0, a_1, a_2\}$ is a cocircuit of M.*

Now let $T_{2n} = \{e, a_{2n}, a_{2n-2}\}$. Then $n = 2$. If $\{e, a_4, a_3\}$ is a circuit of M, then, by exchange, $\{a_4, a_3, a_2\}$ is a circuit of M/a_0. Since the last set is also a cocircuit of the 3–connected matroid M/a_0, we deduce that $|E(M/a_0)| = 4$; a contradiction. Thus $\{e, a_4, a_3\}$ is not a circuit of M. Now $\{e, a_4, a_2\}$ is a circuit of M/a_0 and $\{a_0, a_1, a_2\}$ is a cocircuit of $M\backslash e$. Thus, by orthogonality, one of the following holds:

(I) $\{e, a_4, a_2, a_0\}$ is a circuit of M and $\{e, a_0, a_1, a_2\}$ is a cocircuit of M;
(II) $\{e, a_4, a_2\}$ is a circuit of M and $\{e, a_0, a_1, a_2\}$ is a cocircuit of M; and
(III) $\{e, a_4, a_2, a_0\}$ is a circuit of M and $\{a_0, a_1, a_2\}$ is a cocircuit of M.

If (II) holds, then M has $\{a_2, a_4, e\}$ and $\{a_2, a_3, a_1\}$ as circuits, and has $\{a_2, a_4, a_3\}$ and $\{a_2, e, a_1, a_0\}$ as cocircuits, so (vi) of the theorem holds for $N = M$. If (III) holds, then M has $\{a_2, a_1, a_0\}$ and $\{a_2, a_3, a_4\}$ as cocircuits, and has $\{a_2, a_1, a_3\}$ and $\{a_2, a_0, a_4, e\}$ as circuits, so (vi) of the theorem holds for $N = M^*$. Finally, if (I) holds, then so does (7.8). We conclude that if M/a_0 is 3–connected, then either (vi) of the theorem holds, or (7.8) holds. Thus we may assume the latter.

Now suppose that M/a_{2n}, as well as M/a_0, is 3–connected. Then, by symmetry, either (vi) of the theorem holds, or $\{e, a_{2n}, a_{2n-1}, a_{2n-2}\}$ is a cocircuit of M. But $\{a_{2n-2}, a_{2n-1}, a_{2n}\}$ is a triad of M, so the latter does not occur.

We may now assume that M/a_0 or M/a_{2n}, say M/a_{2n}, is not 3–connected. As $M\backslash e/a_{2n}$ is 3–connected, it follows that $\{e, a_{2n}\}$ is in a triangle T'_{2n} of M. Suppose that M/a_0 is 3–connected. From (7.8), $T_{2n} \cup a_0$ is a circuit of M and $\{e, a_0, a_1, a_2\}$ is a cocircuit of M. Moreover, all of $\{a_2, a_3, a_4\}, \{a_4, a_5, a_6\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}$ are triads of M. By orthogonality and (7.8), $T'_{2n} = \{e, a_{2n}, a_{2n-2}\}$ and $n = 2$. But then A is spanned by $\{a_2, a_3, a_4\}$ and cospanned by $\{a_0, a_1, a_2, a_3\}$, so $r(A) + r^*(A) - |A| \leq 1$; a contradiction. We conclude that M/a_0 is not 3–connected. Thus $\{e, a_0\}$ is in a triangle, say $\{e, a_0, x_0\}$, of M.

Let $T'_{2n} = \{e, a_{2n}, x_{2n}\}$. As A is closed, $\{x_0, x_{2n}\} \subseteq A$. Suppose that $\{e, a_0, a_{2n}\}$ is not a circuit of M. Then $\{a_1, a_3, \ldots, a_{2n-1}, e\}$ spans $A - \{a_0, a_{2n}\}$ and hence spans $\{x_0, x_{2n}\}$. Therefore it also spans a_0 and a_{2n}, so $r(A) \leq n + 1$. Also $\{e, a_0, a_2, \ldots, a_{2n}\}$ cospans A, so $r^*(A) \leq n+2$; a contradiction. Thus $\{e, a_0, a_{2n}\}$ is a triangle of M. It follows by orthogonality using the triads of $\mathcal{F}$ that $\{e, a_0, a_1, a_2\}$, $\{a_2, a_3, a_4\}, \{a_4, a_5, a_6\}, \ldots, \{a_{2n-4}, a_{2n-3}, a_{2n-2}\}$, and $\{a_{2n-2}, a_{2n-1}, a_{2n}, e\}$ are cocircuits of M.

Now, by the dual of Lemma 7.4, $M\backslash e\backslash a_1/a_2$ is 3–connected otherwise a_2 is in a triad of $M\backslash e$ with an element of $\{a_1, a_3\}$ and some element of B, so A is not coclosed in M; a contradiction. Suppose that $M\backslash a_1/a_2$ is not 3–connected. Then M has a triangle T_2 containing $\{e, a_2\}$ and avoiding a_1. If $n = 2$, then, since $\{e, a_0, a_4\}$, $\{a_1, a_2, a_3\}$, and T_2 are circuits of M, it follows that $r(A) \leq 3$. Since $\{e, a_0, a_1, a_2\}$ and $\{a_2, a_3, a_4, e\}$ are cocircuits of M, we have $r^*(A) \leq 4$ and so we obtain a contradiction. Thus we may assume that $n \geq 3$. By orthogonality with the cocircuit $\{a_2, a_3, a_4\}$, we deduce that the third element of T_2 is a_3 or a_4. The cocircuit $\{a_{2n-2}, a_{2n-1}, a_{2n}, e\}$ gives a contradiction in the first case. Thus $T_2 = \{e, a_2, a_4\}$ and the cocircuit $\{a_{2n-2}, a_{2n-1}, a_{2n}, e\}$ implies that $n = 3$. Then $|A| = 8$ and A contains the cocircuits $\{e, a_0, a_1, a_2\}, \{a_2, a_3, a_4\}, \{a_4, a_5, a_6, e\}$, so $r^*(A) \leq 5$. Also $\{a_1, a_3, a_5\}$ spans $\{a_2, a_4\}$ and hence also spans e. The circuit $\{e, a_0, a_6\}$ now implies that $r(A) \leq 4$; a contradiction to the fact that M is 3–connected.

We may now suppose that $M\backslash a_1/a_2$ is 3–connected. Thus, as $A - \{a_1, a_2\}$ has an even number of elements, it is the ground set of a type–3 fan $\mathcal{F}_{12}$ in $M\backslash a_1/a_2$. Now e is not in a triad of this matroid so e is an end of $\mathcal{F}_{12}$ that is in a triangle. M has all of $\{a_3, a_4, a_5\}, \{a_5, a_6, a_7\}, \ldots, \{a_{2n-3}, a_{2n-2}, a_{2n-1}\}$ as triangles. It also has $\{e, a_0, a_{2n}\}$ as a triangle. Thus if $n \geq 3$, then the ground set of $\mathcal{F}_{12}$ is a union of triangles in $M\backslash a_1/a_2$. This is a contradiction, since the restriction of $M\backslash a_1/a_2$ to the ground set of $\mathcal{F}_{12}$ has a coloop. We deduce that $n = 2$. Thus $\mathcal{F}_{12}$ has a unique triad T^*, which contains a_3 and exactly two elements of $\{e, a_0, a_4\}$. Since $M\backslash e\backslash a_1/a_2$ is 3–connected, $e \notin T^*$. Thus $T^* = \{a_0, a_3, a_4\}$ and, by orthogonality with the circuit $\{a_1, a_2, a_3\}$ of M, it follows that $\{a_0, a_1, a_3, a_4\}$ is a cocircuit of M. The two triangles contained in A imply that $r(A) \leq 4$. Moreover, the 4–cocircuits contained in A imply that $r^*(A) \leq 4$. Since $|A| = 6$ and M is 3–connected, we deduce that $r(A) = 4 = r^*(A)$. It follows that $M|A$ is the direct sum of two triangles, while $M.A$ is isomorphic to $M(C_3^2)$.

Finally, consider case 3, that is, suppose that $A - e$ is the ground set of a type–3 fan $\mathcal{F}$. Then $|A - e|$ is even. Since $|A - e| \geq 3$, we deduce that $|A - e| \geq 4$. Let $\mathcal{F}$

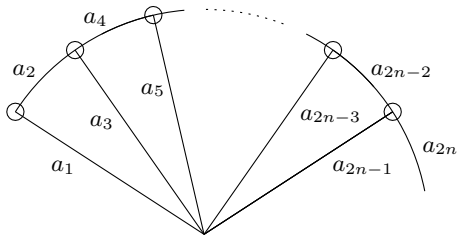


FIGURE 17

be (see Figure 17)

$$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}$$

where the first link is a triangle. By Lemma 7.6, both $M\backslash e/a_{2n}$ and $M\backslash e\backslash a_1$ are 3–connected. The latter implies that $M\backslash a_1$ is 3–connected. Thus $A - a_1$ is the ground set of a type–3 fan $\mathcal{F}_1$. This fan includes all of the triangles of $\mathcal{F}$ except $\{a_1, a_2, a_3\}$. The only elements of $A - a_1$ that are in none of these triangles are e, a_2, and a_{2n}. Now $\mathcal{F}_1$ has exactly one more triangle T apart from those already noted, and T contains exactly two of e, a_2, and a_{2n} since the restriction of $M\backslash a_1$ to $A - a_1$ has a unique coloop. In $M\backslash a_1$, the element e must be in a triangle or a triad that is contained in $A - a_1$. Since $M\backslash e\backslash a_1$ is 3–connected, e is not in a triad of $M\backslash a_1$. Therefore e is in exactly one triangle of $(M\backslash a_1)|(A-a_1)$ and this triangle must be T.

We know that the triangle T contains exactly one of a_2 and a_{2n}. Suppose that $a_2 \in T$. Then $a_{2n} \notin T$. Therefore, since $M\backslash e/a_{2n}$ is 3–connected, M/a_{2n} is 3–connected unless $\{e, a_{2n}\}$ is contained in a triangle T' of M. Consider the exceptional case. As T' is contained in A but not in $A - a_1$, it follows that $T' = \{e, a_{2n}, a_1\}$. Therefore, as $T - e \subseteq A - \{e, a_{2n}\}$, the set $\{a_1, a_3, \ldots, a_{2n-1}\}$ spans $A - a_{2n}$ and so, because of T', spans A. Thus $r(A) \leq n$. Since $\{a_2, a_4, \ldots, a_{2n}\} \cup \{e, a_1\}$ cospans A, we deduce that $r^*(A) \leq n + 2$. This is a contradiction since $|A| = 2n + 1$ and M is 3–connected. We conclude that M/a_{2n} is 3–connected. The last matroid has $A - a_{2n}$ as the ground set of a type–3 fan $\mathcal{F}_{2n}$ and has no triad containing e. Every triangle of $\mathcal{F}$ is a triangle of M/a_{2n} and so is a triangle of $\mathcal{F}_{2n}$. Therefore $\mathcal{F}_{2n}$ has no more triangles and so has no triangle containing e. Thus e is an element of the fan $\mathcal{F}_{2n}$ that is in neither a triangle nor a triad. This contradiction implies that $a_2 \notin T$, so $a_{2n} \in T$. Thus $M\backslash a_1$ has $\{e, a_{2n}\}$ in a triangle and has no triangle containing a_2.

Since $M\backslash e\backslash a_1$ is 3–connected, we observe that, by removing the first link from $\mathcal{F}$, we obtain a fan $\mathcal{F}\backslash a_1$ in $M\backslash e\backslash a_1$ having $A - \{e, a_1\}$ as its ground set. The triads $\{a_2, a_3, a_4\}, \{a_4, a_5, a_6\}, \ldots, \{a_{2n-2}, a_{2n-1}, a_{2n}\}$ of $\mathcal{F}$ remain triads in $M\backslash e\backslash a_1$. Also, since $M\backslash a_1\backslash e$ is 3–connected, all of the triads of $\mathcal{F}_1$ are triads of $M\backslash a_1\backslash e$. Since each of $\mathcal{F}_1$ and $\mathcal{F}$ has exactly $n - 1$ triads, we deduce that $\mathcal{F}_1$ has as its triads all of the triads of $\mathcal{F}$. By orthogonality, the triangle T of $M\backslash a_1$ that contains $\{e, a_{2n}\}$ must contain a_{2n-1}. Thus, in M, we have all of the triangles of $\mathcal{F}$ together with $\{a_{2n-1}, a_{2n}, e\}$. We also know that each of the triads of $\mathcal{F}_1$ is a triad of $M\backslash e$. By orthogonality with $\{a_{2n-1}, a_{2n}, e\}$, we deduce that each of

$\{a_2, a_3, a_4\}, \{a_4, a_5, a_6\}, \ldots, \{a_{2n-4}, a_{2n-3}, a_{2n-2}\}$ is a triad of M. By orthogonality with $\{a_1, a_2, a_3\}$, we deduce that either $n = 2$, or $\{a_{2n-2}, a_{2n-1}, a_{2n}\}$ is a triad of M. In the latter case, A is the ground set of a fan in M; a contradiction. In the former case, $\{a_2, a_3, a_4\}$ is a triad of each of $M\backslash a_1$ and $M\backslash e$ but not of M. Thus $\{a_1, a_2, a_3, a_4\}$ and $\{e, a_2, a_3, a_4\}$ are cocircuits of M, so $r^*(A) \leq 3$. The circuits $\{a_1, a_2, a_3\}$ and $\{a_3, a_4, e\}$ imply that $r(A) \leq 3$. Thus we have a contradiction. We conclude that $A - e$ is not the ground set of a type–3 fan. □

8. Bounding the Size of an Excluded Minor

In this section, we bound the size of an excluded minor for the class $\mathcal{M}$ of forked matroids using the results of earlier sections. Recall that a matroid M on a set E is forked if the partitioned matroid induced on the set of singleton subsets of E is forked. In Theorem 8.12, we establish that all excluded minors for $\mathcal{M}$ have at most 37 elements.

We begin by showing that $\mathcal{M}$ has several attractive properties including being closed under minors.

Lemma 8.1. *The class $\mathcal{M}$ of forked matroids is closed under duality, minors, direct sums, and 2–sums.*

Proof. Let M be a member of $\mathcal{M}$, and let T be a fork-decomposition of M. Let X be a subset of $E(M)$. Then, by Lemma 3.1 and the fact that $\lambda_M(X) = \lambda_{M^*}(X)$, it follows that the tree T^* obtained from T by interchanging the labels g and c on the internal vertices of T is a fork-decomposition of M^*. Hence $\mathcal{M}$ is closed under duality. To show that $\mathcal{M}$ is closed under minors, let x be an element of $E(M)$. It is straightforward to check that by deleting the leaf label x from T, we obtain a fork-decomposition for both $M\backslash x$ and M/x.

To show that $\mathcal{M}$ is closed under direct sums and 2–sums, let M_1 and M_2 be members of $\mathcal{M}$. Let T_1 and T_2 be fork-decompositions of M_1 and M_2, respectively. First consider the direct sum. Subdivide an edge of T_1 and an edge of T_2. Join the new vertices with an edge e. The width of e is 1. Arbitrarily label the end-vertices of e either g or c. It is easily checked that the new tree is a fork-decomposition of $M_1 \oplus M_2$.

Finally, consider the 2–sum of M_1 and M_2 with respect to the basepoints p_1 and p_2. We may assume that each p_i is neither a loop nor a coloop of M_i, for otherwise the 2–sum is a direct sum. Now identify the vertices of T_1 and T_2 labelled by p_1 and p_2 and suppress the resulting degree–2 vertex, letting f be the resulting edge. Then f has width 2. The routine check that the resulting tree is a fork-decomposition of the 2–sum is omitted. □

Let M be a matroid, and let $\{a_1, a_2, \ldots, a_n\}$ be a 3–separating set A of M. We say that A is *forked* if the partitioned matroid P induced on

$$\{E(M) - A, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$$

by M is forked.

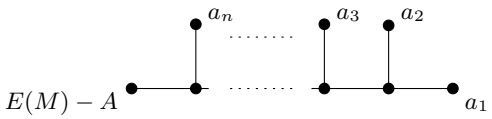


FIGURE 18

Lemma 8.2. *Let A be the ground set of a fan in a 3–connected matroid M. Then A is forked.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$, where

$$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \ldots, \{a_{n-2}, a_{n-1}, a_n\}$$

are the links of a fan. Then it is straightforward to check that, for each i in $\{1, 2, \ldots, n\}$ and each of the three types of fan,

$$r(\{a_1, a_2, \ldots, a_i\}) + r^*(\{a_1, a_2, \ldots, a_i\}) - |\{a_1, a_2, \ldots, a_i\}| \leq 2.$$

Thus each $\{a_1, a_2, \ldots, a_i\}$ is 3–separating in M. It follows that the tree T shown in Figure 18 is a width–3 branch-decomposition of the induced partitioned matroid on $\{E(M) - A, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$. Since every internal vertex v of T meets an edge of width two, the strong guts or strong coguts condition holds at v. We conclude that A is forked. □

The following is a useful consequence of Theorem 6.2.

Lemma 8.3. *Let $\{W, B\}$ be a 3–separation of a 3–connected forked partitioned matroid P. Then W or B is forked.*

Proof. If W or B is the ground set of a fan, then it is forked by Lemma 8.2. Thus we may assume that neither W nor B is the ground set of a fan. Since $|W|, |B| \geq 3$, it follows by Theorem 6.2 that $\{W, B\}$ can be displayed in a fork-decomposition of P. In this case, both W and B are forked. □

Lemma 8.4. *Let $\{A, B\}$ be a 3–separating partition of a matroid M. If both A and B are forked, then M is forked and there is a fork-decomposition of M that displays $\{A, B\}$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ and let $B = \{b_1, b_2, \ldots, b_m\}$. Let P_A be the partitioned matroid induced by M on $\{A, \{b_1\}, \{b_2\}, \ldots, \{b_m\}\}$ and let P_B be the partitioned matroid induced by M on $\{B, \{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$. Let T_A and T_B be fork-decompositions of P_A and P_B, respectively, and let $\hat{T}$ be the tree that is obtained by identifying the leaf of T_A labelled by A with the leaf of T_B labelled by B and then suppressing the resulting degree–2 vertex. It is easily seen that $\hat{T}$ is a fork-decomposition of M as every edge and every vertex of $\hat{T}$ corresponds to an edge or vertex of T_A or T_B. □

A set A of elements of a matroid M is *coclosed* if it is closed in M^*. We say that A is *fully closed* if A is both closed and coclosed. Since the intersection of closed sets

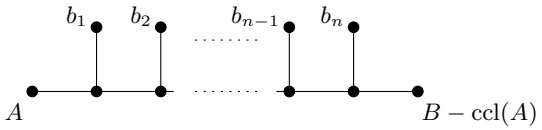


FIGURE 19

is closed, it follows that the intersection of fully closed sets is fully closed. Thus, for a given set A, there is a unique minimal fully closed set containing A. Denote this set by $\mathrm{ccl}(A)$. One way to find $\mathrm{ccl}(A)$ is to first take $\mathrm{cl}(A)$, then the coclosure of $\mathrm{cl}(A)$, then the closure of the result, and so on until, after some finite number of steps, no new elements are added; when this occurs, we have found $\mathrm{ccl}(A)$. We use the notation $x \in \mathrm{cl}^{(*)}(X)$ to denote that $x \in \mathrm{cl}(X)$ or $x \in \mathrm{cl}^*(X)$.

The closure operators of a matroid and its dual are linked through the following well-known result.

Lemma 8.5. *Let X, Y, and $\{x\}$ be disjoint sets whose union is the ground set of a matroid. Then $x \in \mathrm{cl}^*(X)$ if and only if $x \notin \mathrm{cl}(Y)$.*

The next lemma was proved in [13, Lemma 2.3].

Lemma 8.6. *If X is a subset of the ground set of a matroid M, and $x \in \mathrm{cl}^{(*)}(X)$, then $\lambda_M(X \cup x) \leq \lambda_M(X)$.*

Lemma 8.7. *Let $\{A, B\}$ be a 3–separating partition of a 3–connected matroid M. Then $\mathrm{ccl}(A)$ is 3–separating. Moreover,*

(i) *If A is a forked, then $\mathrm{ccl}(A)$ is forked; and*
(ii) *if $B - \mathrm{ccl}(A)$ is forked, then B is forked.*

Proof. To form $\mathrm{ccl}(A)$ from A, we add a sequence of elements $b_1, b_2, \ldots, b_n$ to A, where $b_i \in \mathrm{cl}^{(*)}(A \cup \{b_1, b_2, \ldots, b_{i-1})$ for all i in $\{1, 2, \ldots, n\}$. Since $\lambda_M(A) \leq 3$, it follows by Lemma 8.6 that, for each i, we have $\lambda_M(A \cup \{b_1, b_2, \ldots, b_i\}) \leq 3$, so $\mathrm{ccl}(A)$ is 3–separating in M.

Let P be the partitioned matroid induced by M on $\{A, \{b_1\}, \{b_2\}, \ldots, \{b_n\}, B - \mathrm{ccl}(A)\}$. As M is 3–connected, P is 3–connected. Consider the tree T shown in Figure 19. By Lemma 8.5, T is a width–3 branch-decomposition of P. Furthermore, T is a fork-decomposition of P since T can be obtained from the single-edge tree whose leaves are labelled A and B by repeatedly applying Lemma 5.1. In particular, v_i is a guts vertex of T if $b_i \in \mathrm{cl}(A \cup \{b_1, b_2, \ldots, b_{i-1}\})$ and v_i is a coguts vertex if $b_i \in \mathrm{cl}^*(A \cup \{b_1, b_2, \ldots, b_{i-1}\})$. It follows immediately that if A is forked, then $\mathrm{ccl}(A)$ is forked, and if $B - \mathrm{ccl}(A)$ is forked, then B is forked. □

The next two lemmas are taken from [13, Lemmas 2.4 and 6.1].

Lemma 8.8. *Let x be an element of a matroid M.*

(i) *Let X be a k–separating set of $M\backslash x$. If $x \in \mathrm{cl}(X)$, then $X \cup \{x\}$ is a k–separating set of M.*
(ii) *Let X be a k–separating set of M/x. If $x \in \mathrm{cl}^*(X)$, then $X \cup \{x\}$ is a k–separating set of M.*

Lemma 8.9. *Let $\{A, B\}$ be a 3–separation of a 3–connected matroid M, and suppose that A is fully closed. Then there are elements a_1, a_2 of A such that, for each i in $\{1, 2\}$, either $M\backslash a_i$ or M/a_i is 3–connected.*

A matroid M is *k–connected up to separators of size l* if, whenever A is a $(k-1)$–separating set in M, either $|A| \leq l$ or $|E(M) - A| \leq l$.

Lemma 8.10. *Let M be an excluded minor for the class of forked matroids. Then M is 4–connected up to separators of size six.*

Proof. Let $\{A, B\}$ be a 3–separating partition of M. If both A and B are forked, then, by Lemma 8.4, M is forked; a contradiction. Thus, without loss of generality, we may assume that B is not forked. We prove the lemma by showing that $\min\{|A|, |B|\} \leq 6$. Assume the contrary. Since B is not forked, it follows by Lemma 8.7 that $B - \mathrm{ccl}(A)$ is not forked. Hence we may also assume that A is fully closed.

Let $A = \{a_1, a_2, \ldots, a_m\}$. Then $m \geq 7$. We consider two cases:

(I) A is the ground set of a fan in M; and
(II) A is not the ground set of a fan in M.

Consider the first case, letting $\mathcal{F}$ be a fan with ground set A. Since $m \geq 5$, it follows by Lemma 7.3 that there is, up to reversal, a unique ordering $a_1, a_2, \ldots, a_m$ of the elements of A such that every consecutive triple is either a triangle or a triad of M. Furthermore, as $m \geq 7$, there is an integer i such that $3 \leq i \leq m-3$ and a_i is a spoke of $\mathcal{F}$. Note that a_{i+1} is a rim element of $\mathcal{F}$. By Lemma 7.4, $M\backslash a_i/a_{i+1}$ is 3–connected. Furthermore, $A - \{a_i, a_{i+1}\}$ is the ground set of a fan of $M\backslash a_i/a_{i+1}$, and so $\{A - \{a_i, a_{i+1}\}, B\}$ is a 3–separating partition of $M\backslash a_i/a_{i+1}$.

Let $B = \{b_1, b_2, \ldots, b_k\}$, and suppose that $\{A - \{a_i, a_{i+1}\}, B\}$ can be displayed in some fork-decomposition of $M\backslash a_i/a_{i+1}$. Then the partitioned matroid induced by $M\backslash a_i/a_{i+1}$ on $\{A - \{a_i, a_{i+1}\}, \{b_1\}, \{b_2\}, \ldots, \{b_k\}\}$ has a fork-decomposition. By relabelling the leaf $A - \{a_i, a_{i+1}\}$ of this fork-decomposition with A, and observing that

$$r_M(A \cup B') = r_{M\backslash a_i/a_{i+1}}((A - \{a_i, a_{i+1}\}) \cup B') + 1$$

for all subsets B' of B, we can easily check that the resulting tree is a fork-decomposition of the partitioned matroid induced by M on $\{A, \{b_1\}, \ldots, \{b_k\}\}$. But this implies that B is forked in M; a contradiction. Hence $\{A - \{a_i, a_{i+1}\}, B\}$ cannot be displayed in any fork-decomposition of $M\backslash a_i/a_{i+1}$. Thus, by Lemma 8.2, B is not the ground set of a fan of $M\backslash a_i/a_{i+1}$. Thus, by Theorem 6.2, $M\backslash a_i/a_{i+1}$ has a fork-decomposition T as shown in Figure 20, where B is the disjoint union of non-empty sets B_1 and B_2, and, for all j in $\{1, 2, \ldots, i-1, i+2, \ldots, m\}$,

(i) v_j is a guts vertex if a_j is a spoke of the fan of $M\backslash a_i/a_{i+1}$ with ground set $A - \{a_i, a_{i+1}\}$; and
(ii) v_j is a coguts vertex if a_j is a rim element of the fan of $M\backslash a_i/a_{i+1}$ with ground set $A - \{a_i, a_{i+1}\}$.

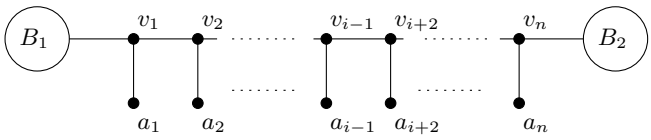


FIGURE 20

Let $\hat{T}$ be the tree obtained from T by subdividing the edge $\{v_{i-1}, v_{i+2}\}$; inserting two new vertices v_i and v_{i+1} with v_i adjacent to v_{i-1}; adding a new leaf adjacent to each of v_i and v_{i+1}; and labelling the new leaves a_i and a_{i+1}, respectively. We shall show that $\hat{T}$ is a fork-decomposition of M, where v_i is a guts vertex and v_{i+1} is a coguts vertex.

To show that $\hat{T}$ is a width–3 branch-decomposition of M, let f be an interior edge of $\hat{T}$. Let $\{C, D\}$ be the partition of $E(M)$ that is displayed by f. First assume that $C \subseteq B_1 \cup \{a_1, a_2, \ldots, a_{i-1}\}$. Since $D - \{a_i, a_{i+1}\}$ is 3–separating in $M\backslash a_i/a_{i+1}$ and $a_{i+1} \in \mathrm{cl}^*(\{a_{i+2}, a_{i+3}\})$, it follows by Lemma 8.8(ii) that $D-\{a_i\}$ is 3–separating in $M\backslash a_i$. This in turn implies, by Lemma 8.8(i) that D is 3–separating in M as $a_i \in \mathrm{cl}(\{a_{i+1}, a_{i+2}\})$. Thus, in this case, f has width at most three. By a similar argument, if $C \subseteq B_2 \cup \{a_m, a_{m-1}, \ldots, a_{i+2}\}$, then f also has width at most three. The case when $C = B_1 \cup \{a_1, a_2, \ldots, a_i\}$ is treated by noting that $B_1 \cup \{a_1, a_2, \ldots, a_{i-1}\}$ is 3–separating and $a_i \in \mathrm{cl}(B_1 \cup \{a_1, a_2, \ldots, a_{i-1}\})$, and then applying [13, Lemma 2.3]. Hence $\hat{T}$ is a width–3 branch-decomposition of M.

We show next that every interior vertex v of $\hat{T}$ satisfies either the strong guts or the strong coguts condition. If $v \in \{v_1, v_2, \ldots, v_m\}$, then at least one of the sets displayed by v is not 3–separating. Thus, by Lemma 4.3(i), v satisfies either the strong guts or the strong coguts condition. We may now assume that $v \notin \{v_1, v_2, \ldots, v_m\}$. Then, noting that

$$r_M(A \cup B') = r_{M\backslash a_i/a_{i+1}}((A - \{a_i, a_{i+1}\}) \cup B') + 1$$

for all subsets B' of B, we can easily check that v satisfies either the strong guts or the strong coguts condition. Hence $\hat{T}$ is a fork-decomposition of M; a contradiction.

Now consider case (II). Let A' be the set of elements e of A for which $M\backslash e$ or M/e is 3–connected. Since A is fully closed, Lemma 8.9 implies that A' is nonempty. Let x be an arbitrary element of A'. By duality, we may assume that $M\backslash x$ is 3–connected. Thus $\{A - x, B\}$ is a 3–separation of $M\backslash x$. Therefore $r(A - x) = r(A)$ and so, if B is forked in $M\backslash x$, then it is forked in M; a contradiction. We deduce that B is not forked in $M\backslash x$. Thus, by Lemma 8.2, B is not the ground set of a fan of $M\backslash x$. If $A - x$ is not the ground set of a fan of $M\backslash x$, then, by Theorem 6.2, there is a fork-decomposition of $M\backslash x$ that displays B. Thus B is forked in $M\backslash x$; a contradiction. We conclude that $A - x$ is the ground set of a fan of $M\backslash x$. Since x

was arbitrarily chosen in A', it follows that A is a minimal non-fan. Therefore, by Lemma 7.5, $|A| \leq 6$; a contradiction. □

The proof of Theorem 8.12 will combine the last lemma with the following lemma which was proved in [13].

Lemma 8.11. *Let M be a matroid that is k–connected up to separators of size l. Then, for all x in $E(M)$, either $M\backslash x$ or M/x is k–connected up to separators of size $2l$.*

Theorem 8.12. *Let M be an excluded minor for the class of forked matroids. Then M has at most 37 elements.*

Proof. From Lemma 8.10, M is 4–connected up to separators of size 6. Let $x \in E(M)$. Then, by Lemma 8.11, either $M\backslash x$ or M/x is 4–connected up to separators of size 12. By duality, we may assume the former. Since $M\backslash x$ is forked, there is a reduced fork-decomposition T of $M\backslash x$. Furthermore, by [13, Lemma 3.1], there is an edge e of T such that each of the sets B_1 and B_2 displayed by e has at least $\frac{1}{3}|E(M\backslash x)|$ elements. But B_1 and B_2 are 3–separating sets of $M\backslash x$, so either $|B_1| \leq 12$ or $|B_2| \leq 12$. Since $|B_1|, |B_2| \geq \frac{1}{3}|E(M\backslash x)|$, it follows that $|E(M\backslash x)| \leq 36$. Therefore $|E(M)| \leq 37$. □

9. A Characterization of Forked Matroids

In this section, we give a characterization of forked matroids in terms of an operation introduced by Oxley, Semple, and Vertigan [20]. This operation, segment-cosegment exchange, is a generalization of the familiar graph and matroid operation of $\Delta - Y$ exchange.

Let M be a matroid. A segment or cosegment of M is *strict* if it is exactly 3–separating. Suppose that $A = \{a_1, a_2, \ldots, a_k\}$ is a strict segment of M. We denote by $\Delta_A(M)$ the matroid on $E(M)$ in which a subset B of $E(M)$ is a basis of $\Delta_A(M)$ precisely if B is a member of one of the following sets:

(i) $\{A \cup B' : B' \text{ is a basis of } M/A\}$;
(ii) $\{(A - a_i) \cup B'' : 1 \leq i \leq k \text{ and } B'' \text{ is a basis of } M/a_i\backslash(A - a_i)\}$; or
(iii) $\{(A - \{a_i, a_j\}) \cup B''' : 1 \leq i < j \leq k \text{ and } B''' \text{ is a basis of } M\backslash A\}$.

The fact that $\Delta_A(M)$ is actually a matroid follows from [20, Lemma 2.9]. We say that $\Delta_A(M)$ has been obtained from M by a *Δ_A–exchange* or a *segment-cosegment exchange on A*. Observe that, in $\Delta_A(M)$, the set A is a cosegment. Moreover, if $|A| = 2$, then $\Delta_A(M) \cong M$.

Next we describe an alternative definition of $\Delta_A(M)$, whose equivalence with the definition above is established in [20]. This equivalent definition uses the operation of generalized parallel connection [2] (see, for example, [17]). First we define a matroid Θ_k for $k \geq 3$ as follows. In $PG(k-1, \mathbb{R})$, let $\{b_1, b_2, \ldots, b_k\}$ be a basis B and let L be a line that is freely placed relative to B. For each i in $\{1, 2, \ldots, k\}$,

the hyperplane of $PG(k-1,\mathbb{R})$ that is spanned by $B-b_i$ meets L in a single point a_i. Let $A=\{a_1,a_2,\ldots,a_k\}$ and Θ_k be the restriction of $PG(k-1,\mathbb{R})$ to $A\cup B$. In Θ_k, the set A is a modular line. Thus, if M is a matroid and $\{a_1,a_2,\ldots,a_k\}$ is a strict segment A of M, then the generalized parallel connection $P_A(\Theta_k,M)$ is well-defined. To obtain $\Delta_A(M)$ from this matroid, we delete A and relabel each b_i in $E(\Theta_k)-A$ by a_i. Thus $\Delta_A(M)\cong P_A(\Theta_k,M)\backslash A$.

To illustrate a segment-cosegment exchange, note that $U_{4,6}$ can be obtained from $U_{2,6}$ by a segment-cosegment exchange on any 4–element subset of its ground set. Furthermore, if $|A|=3$, then the matroid $\Delta_A(M)$ is precisely the matroid obtained by performing a $\Delta-Y$ exchange on M at A.

The dual of a segment-cosegment exchange is a cosegment-segment exchange and is define as follows. For a strict cosegment A of a matroid M, let $\nabla_A(M)$ be the matroid $(\Delta_A(M^*))^*$. We say that $\nabla_A(M)$ has been obtained from M by a ∇_A*–exchange* or a *cosegment-segment exchange on* A. In terms of the generalized parallel connection, $\nabla_A(M)\cong(P_A(\Theta_k,M^*)\backslash A)^*$.

For the purposes of this paper, we need to extend the definition of segment-cosegment exchange to partitioned matroids. Let P be a partitioned matroid. A segment or cosegment of P is *strict* if it is exactly 3–separating. Observe that if A is such a segment or cosegment of P, then A is a strict segment or strict cosegment, respectively, of the underlying matroid M of P. Suppose that A is a strict segment of P. We denote by $\Delta_A(P)$ the partitioned matroid with ground set $E(P)$ and underlying matroid $\Delta_A(M)$, and say that $\Delta_A(P)$ has been obtained from P by a *segment-cosegment exchange on* A. Dually, if A is a strict cosegment of P, let $\nabla_A(P)$ be the partitioned matroid $(\Delta_A(P^*))^*$. We say that $\nabla_A(P)$ has been obtained from P by a ∇_A*–exchange* or a *cosegment-segment exchange on* A.

The next sequence of lemmas is needed for the proof of our characterization of forked matroids. The first of these lemmas is a straightforward consequence of the definition of a segment-cosegment exchange.

Lemma 9.1. *Let P be a partitioned matroid, and let X be a subset of $E(P)$.*

(i) *If A is a strict segment of P, then*
  (a) $r_{\Delta_A(P)}(X)=r_P(X)+|A|-2$ *if X contains A, and*
  (b) $r_{\Delta_A(P)}(X)=r_P(X)$ *if X is disjoint from A.*

(ii) *If A is a strict cosegment of P, then*
  (a) $r_{\nabla_A(P)}(X)=r_P(X)-|A|+2$ *if X contains A, and*
  (b) $r_{\nabla_A(P)}(X)=r_P(X)$ *if X is disjoint from A.*

A partitioned matroid P is 3*–connected up to parallel pairs* if, whenever $\{W,B\}$ is a 2–separation of P, either W or B is a parallel pair of matroid elements. Dually, P is 3*–connected up to series pairs* if, whenever $\{W,B\}$ is a 2–separation of P, either W or B is a series pair of matroid elements.

Lemma 9.2. *Let P be a 3–connected partitioned matroid, and let A be a subset of $E(P)$.*

(i) *If A is a strict segment of P, then $\Delta_A(P)$ is 3–connected up to series pairs.*
(ii) *If A is a strict cosegment of P, then $\nabla_A(P)$ is 3–connected up to parallel pairs.*

This lemma is an immediate consequence of the following result.

Lemma 9.3. *Let M be a 3–connected matroid, and let A be a subset of $E(M)$ having at least three elements.*

(i) *If A is a strict segment of M, then, for all subsets A' of A, the matroid $P_A(\Theta_k, M)\backslash A'$ is 3–connected up to series pairs.*
(ii) *If A is a strict cosegment of M, then, for all subsets A' of A, the matroid $(P_A(\Theta_k, M^*)\backslash A')^*$ is 3–connected up to parallel pairs.*

Proof. By duality, it suffices to prove (i). It is not difficult to check that Θ_k is 3–connected [20]. Since M is also 3–connected, it follows that $P_A(\Theta_k, M)$ is also 3–connected (see, for example, [17, Ex. 12.4.10]). Now the set B is a cosegment in $P_A(\Theta_k, M)$ and remains a cosegment in $P_A(\Theta_k, M)\backslash A$, which is isomorphic to $\Delta_A(M)$. Thus B is a cosegment of $P_A(\Theta_k, M)\backslash A'$. Therefore no series class of $P_A(\Theta_k, M)\backslash A'$ contains more than one element of B. If $P_A(\Theta_k, M)\backslash A'$ has a cocircuit C^* with at most two elements such that $C^* \subseteq E(M) - A'$, then the 3–connected matroid $P_A(\Theta_k, M)$ has a cocircuit that is properly contained in $C^* \cup A'$. Thus Θ_k, which is the restriction of $P_A(\Theta_k, M)$ to $A \cup B$, has a cocircuit that is contained in A'; a contradiction since A' avoids the basis B of Θ_k. We deduce that $P_A(\Theta_k, M)\backslash A'$ has no coloops and has no series classes with more than two elements.

Now suppose that the cosimplification of $P_A(\Theta_k, M)\backslash A'$ is not 3–connected. Then, by [17, p. 283], there is a partition $\{X, Y\}$ of $E(P_A(\Theta_k, M)\backslash A')$ such that

$$r(X) + r(Y) - r(P_A(\Theta_k, M)\backslash A') \leq 1,$$

where both X and Y contain circuits of $P_A(\Theta_k, M)\backslash A'$. Choose such a partition $\{X, Y\}$ so that $\min\{|X \cap B|, |Y \cap B|\}$ is minimal. Suppose that this minimum occurs for $X \cap B$ and is at least 1, and let $x \in X \cap B$. Then, since B is a cosegment of $P_A(\Theta_k, M)\backslash A'$, the element x is a coloop of X. It follows that $\{X - x, Y \cup x\}$ contradicts the choice of $\{X, Y\}$. We deduce that $X \cap B$ is empty. Thus $Y \supseteq B$ and it follows that $\{X, Y \cup A'\}$ is a 2–separation of the 3–connected matroid $P_A(\Theta_k, M)$. This contradiction completes the proof that the cosimplification of $P_A(\Theta_k, M)\backslash A'$ is indeed 3–connected and thereby finishes the proof of the lemma. □

The next lemma is an immediate consequence of Lemmas 2.1 and 3.1.

Lemma 9.4. *Let T be a fork-decomposition of a partitioned matroid P, and let T^* denote the tree obtained from T by interchanging the labels g and c on the internal vertices of T. Then T^* is a fork-decomposition of P^*.*

Lemma 9.5. *Let T be a fork-decomposition of a 3–connected partitioned matroid P, and let e be an edge of T of width 3. Let T' be a branch of e displaying a set D of matroid elements of P.*

(i) *If all internal vertices of T' are guts vertices, then D is a strict segment of P.*
(ii) *If all internal vertices of T' are coguts vertices, then D is a strict cosegment of P.*

Proof. By Lemma 9.4, it suffices to prove (i). Thus suppose that all internal vertices of T' are guts vertices. We argue by induction on the size of D. Since e has width 3, it follows that $|D| \geq 2$ and that if D is a segment, then it is a strict segment. Hence it suffices to show that D is a segment. This is certainly true if $|D| = 2$. Now assume that $|D| \geq 3$ and that (i) holds for all sets with fewer elements than D. Let v be the end-vertex of e that is contained in T'. Then two of the sets displayed by v induce a partition $\{D_1, D_2\}$ of D where $|D_1| \geq |D_2|$. By induction, either

(I) $r(D_1) = r(D_2) = 2$, or
(II) $r(D_1) = 2$ and $|D_2| = 1$.

In (I), each of the edges incident with e has width 3. Therefore, as v is a guts vertex, Lemma 4.6(i) implies that $r(D_1 \cup D_2) = r(D_1) + r(D_2) - 2 = 2$. Thus, in this case, D is a segment of P. Now assume that (II) holds. Then two of the edges incident with v have width 3 while the third has width 2. Therefore, as v is a guts vertex, it follows, by Lemma 4.4(ii)(a), that $r(D_1 \cup D_2) = r(D_1) = 2$, and so D is again a segment of P. □

For a converse of Lemma 9.5, we have the following lemma.

Lemma 9.6. *Let P be a forked partitioned matroid, and let A be a subset of $E(P)$ that can be displayed in a reduced fork-decomposition T of P.*

(i) *If A is a segment of P, then the tree obtained from T by relabelling every internal vertex of the branch of T that displays A with g is a fork-decomposition of P.*
(ii) *If A is a cosegment of P, then the tree obtained from T by relabelling every internal vertex of the branch of T that displays A with c is a fork-decomposition of P.*

Proof. By Lemma 9.4, it suffices to prove (i). Suppose that A is a segment of P. We shall show that every internal vertex v of the branch of T that displays A satisfies the strong guts condition. Let $\{A_1, A_2, B\}$ be the partition of $E(P)$ displayed by v, where $A_1, A_2 \subseteq A$. It follows, since A is a segment of P, that $r(A_1 \cup A_2) = 2$. Therefore

$$r(A_1 \cup A_2) + r(A_1 \cup B) + r(A_2 \cup B) \leq 2 + 2r(P).$$

Thus the strong guts condition holds at v, as required. □

Lemma 9.7. *Let P be a forked partitioned matroid, and let A be a subset of $E(P)$ that can be displayed in a reduced fork-decomposition T of P.*

(i) *If A is a strict segment of P, then $\Delta_A(P)$ is forked. Moreover, a fork-decomposition of $\Delta_A(P)$ is obtained from T by relabelling with a c each internal vertex of the branch of T that displays A.*

(ii) *If A is a strict cosegment of P, then $\nabla_A(P)$ is forked. Moreover, a fork-decomposition of $\nabla_A(P)$ is obtained from T by relabelling with a g each internal vertex of the branch of T that displays A.*

Proof. By Lemma 9.4, it suffices to show that (i) holds. Assume that A is a strict segment of P, and let T' be the tree obtained from T by relabelling with a c each internal vertex of the branch of T that displays A. To prove (i), we shall show that T' is a fork-decomposition of $\Delta_A(P)$.

We begin by showing that T' is a branch-decomposition of $\Delta_A(P)$. Observe that this is equivalent to showing that T is a branch-decomposition of $\Delta_A(P)$ since, in obtaining T' from T, only vertex labels were changed. Let e be an edge of T', and let $\{Y, Z\}$ be the partition of $E(P)$ that is displayed by e. Since A is displayed by T, one of the blocks of this partition, Y say, has the property that either $Y \subseteq A$ or $Y \supseteq A$. If $Y \subseteq A$, then either $|Y| = 1$, or $|Y| \geq 2$ and Y is a cosegment of $\Delta_A(P)$. In both cases, Y is a 3–separating set of $\Delta_A(P)$. Now assume that $Y \supseteq A$. Then, by Lemma 9.1, we have $r_{\Delta_A(P)}(Y) = r_P(Y) + |A| - 2$, $r(\Delta_A(P)) = r(P) + |A| - 2$, and $r_{\Delta_A(P)}(Z) = r_P(Z)$. A routine check using these three equations and the fact Y that is a 3–separating set of P shows that Y is a 3–separating set of $\Delta_A(P)$. Thus T' is indeed a branch-decomposition of $\Delta_A(P)$.

We now show that T' is a fork-decomposition of $\Delta_A(P)$. Let v be an internal vertex of T', and let $\{X, Y, Z\}$ denote the partition of $E(P)$ displayed by v. Since A is displayed in T, there are two cases to consider:

(I) v is an internal vertex of the branch of T' that displays A; and
(II) v is not an internal vertex of the branch of T' that displays A.

In case (I), we may assume that $Y \cup Z \subseteq A$, so $Y \cup Z$ is independent in $\Delta_A(P)$. Also, $Y \cup Z$ is 3–separating in $\Delta_A(P)$ since T' is a branch-decomposition of $\Delta_A(P)$. Thus

$$\begin{aligned} 2 &\geq r_{\Delta_A(P)}(X) + r_{\Delta_A(P)}(Y \cup Z) - r(\Delta_A(P)) \\ &= r_{\Delta_A(P)}(X) + r_{\Delta_A(P)}(Y) + r_{\Delta_A(P)}(Z) - r(\Delta_A(P)). \end{aligned}$$

Thus $\{X, Y, Z\}$ satisfies the strong coguts condition.

Now consider (II). In this case, we may assume that Y contains A, and so both $X \cap A$ and $Z \cap A$ are empty. Then, by Lemma 9.1, the ranks of X, Z, and $X \cup Z$ are the same in $\Delta_A(P)$ as in P. Furthermore, by the same lemma, the ranks of each of Y, $X \cup Y$, $Y \cup Z$, and $E(P)$ increase by $|A| - 2$ in moving from P to $\Delta_A(P)$. It is now easily checked that if $\{X, Y, Z\}$ satisfies the strong guts or strong coguts condition in P, then $\{X, Y, Z\}$ satisfies the same condition in $\Delta_A(P)$. Hence T' is indeed a fork-decomposition of $\Delta_A(P)$, as required. □

The next theorem gives us one direction of our characterization of forked matroids. Let M and N be matroids. A Δ_A*–reduction* or *segment-cosegment reduction* on a strict segment A of M is obtained by first performing a Δ_A-exchange, and then cosimplifying the resulting matroid. Dually, a ∇_A*–reduction* or *cosegment-segment*

reduction on a strict cosegment A of M is obtained by first performing a ∇_A-exchange, and then simplifying the resulting matroid. Observe that, by Lemma 9.2, if M is 3–connected, then any matroid obtained from M by a Δ–reduction or a ∇–reduction is also 3–connected. The matroid M is $\Delta-\nabla$*–reducible* to N if there is a sequence $M_0, M_1, \ldots, M_k$ of matroids such that, for each i in $\{1, 2, \ldots, k\}$, the matroid M_i is obtained from M_{i-1} by either a Δ–reduction or a ∇–reduction, $M_0 = M$, and $M_k \cong N$.

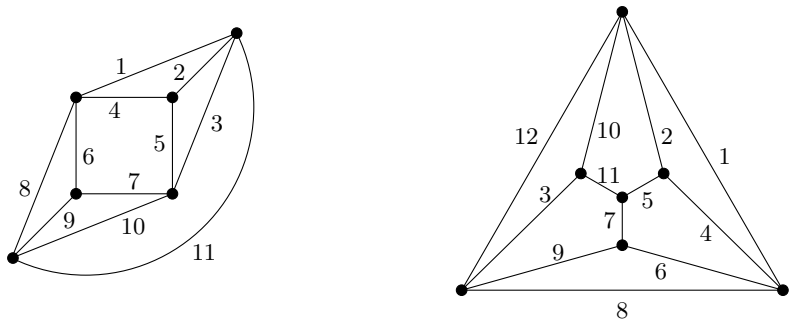
Theorem 9.8. *Let M be a 3–connected matroid, and suppose that $|E(M)| \geq 3$. If M is forked, then M is $\Delta-\nabla$–reducible to either $U_{2,n}$ or $U_{n-2,n}$, for some $n \geq 3$. More particularly, there is a sequence $M_0, M_1, \ldots, M_k$ of 3–connected forked matroids such that $M_0 = M$ and M_k is isomorphic to $U_{2,n}$ or $U_{n-2,n}$, for some $n \geq 3$; for each i in $\{1, 2, \ldots, k\}$, the matroid M_i is obtained from M_{i-1} by either a Δ_A–reduction or a ∇_A–reduction, where A is displayed in some fork-decomposition of M_{i-1}.*

Proof. Let T be a reduced fork-decomposition of M. For the purpose of the proof, we call an edge of T alternating if one end-vertex is labelled c and the other end-vertex is labelled g. The proof is by induction on the number of alternating edges of T. If T has no such edges, then it follows by Lemma 9.5(i) that M is either a segment or a cosegment of M depending on whether the internal vertices of T are all labelled g or c, respectively. Therefore, M is isomorphic to either $U_{2,n}$ or $U_{n-2,n}$, for some $n \geq 3$. Now assume that T has at least one alternating edge and that the result holds for all 3–connected forked matroids with a fork-decomposition having fewer alternating edges than T. Let e be an alternating edge of T such that one of the branches displayed by e, say T_1, has no alternating edges. Clearly, such an edge exists. Furthermore, as M is 3–connected, the width of e must be 3. Let A denote the set displayed by T_1. By duality, we may assume that all of the internal vertices of T in T_1 are labelled g. Then, by Lemma 9.5(i), A is a strict segment of M. Let T' be the tree obtained from T by relabelling with a c all of the internal vertices of T in T_1. By Lemma 9.7(i), T' is a fork-decomposition of $\Delta_A(M)$. Moreover, T' has fewer alternating edges than T. By Lemma 9.2(i), $\Delta_A(M)$ is 3–connected up to series pairs. By deleting the leaf of T' corresponding to exactly one element of every series pair of $\Delta_A(M)$, and then reducing the resulting tree, we get a fork-decomposition of $\mathrm{co}(\Delta_A(M))$ that has fewer alternating edges than T. Since $\mathrm{co}(\Delta_A(M))$ is 3–connected, it follows by our induction assumption that $\mathrm{co}(\Delta_A(M))$, and hence M, is $\Delta-\nabla$–reducible to either $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 3$. Moreover, the corresponding sequence of Δ– and ∇–reductions has the properties specified in the theorem. □

With the aim of obtaining a converse to Theorem 9.8, consider the situation for graphs. In this case, the only non-trivial segment-cosegment and cosegment-segment reductions are the familiar $\Delta-Y$ and $Y-\Delta$ reductions. It is known (see, for example, [28]) that all planar graphs are $\Delta-Y$–reducible to a triangle. Moreover, as planar graphs can have arbitrarily high branch-width, their cycle matroids need not be forked. Thus, the converse of Theorem 9.8 fails. The source of this failure is that one can move from a graph whose cycle matroid is forked to one whose cycle matroid is not forked by performing a $\Delta-Y$ exchange on a triangle that cannot be displayed in a fork-decomposition. Indeed, if we restrict attention to exchanges

on sets that can be displayed in a fork-decomposition, the next theorem shows that we do obtain a converse to Theorem 9.8.

As a concrete illustration of the failure of the full converse of Theorem 9.8, consider the graph G in Figure 21(a). Order the edges of G as follows: $1, 2, 4, 5, 3, 11, 10, 8, 6, 7, 9$ and let $M = M(G)$. Then $\{1, 2\}$ is forked. Moreover, in the specified



(a) G (b) H

FIGURE 21

ordering on the edges of G, every element after $1, 2$ is in the closure or the coclosure of its set of predecessors in the sequence. Thus, by Lemma 8.7, $E(M)$, which is the full closure of $\{1, 2\}$, is forked. Hence M is forked. To show that the triangle $\{3, 10, 11\}$ cannot be displayed in a fork-decomposition of M, we use the following result.

Lemma 9.9. *Let M be a 3–connected forked matroid having at least five elements. Suppose that $\{X, Y, Z\}$ is displayed by some vertex v in a fork-decomposition T of M.*

(i) *If X is a triangle, then $M\backslash X$ or M/X is disconnected.*
(ii) *If $|X| = |Y| = 2$, then $X \cup Y$ is a segment or a cosegment.*

Proof. Let X be a triangle of M. Suppose that v is a coguts vertex of T. Then $r(X) + r(Y) + r(Z) - r(M) \leq 2$. Since X is not a triad, $r(M\backslash X) = r(M)$, so $r(Y) + r(Z) - r(M\backslash X) \leq 0$, that is, $M\backslash X$ is disconnected. Now suppose that v is a guts vertex of T. Then $r(X \cup Y) + r(X \cup Z) + r(Y \cup Z) - 2r(M) \leq 2$. Thus $r_{M/X}(Y) + r_{M/X}(Z) - r(M/X) \leq 0$, so M/X is disconnected. Thus (i) holds. Part (ii) follows by a similar argument: if the coguts condition holds at v, then $X \cup Y$ is a cosegment; if the guts condition holds, then $X \cup Y$ is a segment. □

For $M = M(G)$ where G is the graph in Figure 21(a) and $X = \{3, 10, 11\}$, neither $M\backslash X$ nor M/X is disconnected. Hence, by Lemma 9.9(i), X cannot be displayed in a fork-decomposition of M. Now add 12 in parallel to 11 in G and perform a $\Delta - Y$ exchange on X to produce the cycle matroid of a graph H, which can be drawn as in Figure 21(b). Suppose that $M(H)$ is forked having T as a fork-decomposition. By Lemma 9.9(i) and its dual and using symmetry, it is not

difficult to show that none of the triangles or triads of $M(H)$ can be displayed in T. Therefore no 3–element subset of $E(H)$ is displayed in T. It follows that T must have a vertex such that two of the sets that it displays have exactly two elements. Since $M(H)$ is binary, we obtain a contradiction to Lemma 9.9(ii). We conclude that $M(H)$ is not forked. Incidentally, if we perform a $\Delta - Y$ exchange on $\{1, 8, 12\}$ in $M(H)$, we obtain the cycle matroid of the cube, which Dharmatilake [4] has shown is an excluded minor for the class of matroids of branch-width at most 3.

Let M be a 3–connected forked matroid, and let A be a subset of $E(M)$. If A is a strict segment of M, a *segment-cosegment move on* A is achieved by the following sequence of operations.

(i) Choose a (possibly empty) subset of A and, for each element a of this subset, add a single element in parallel with a.
(ii) Perform a segment-cosegment exchange on A on the matroid obtained in (i), and then cosimplify the resulting matroid.

Dually, if A is a strict cosegment of M, a *cosegment-segment move on* A is achieved by the following sequence of operations.

(i) Choose a (possibly empty) subset of A and, for each element a of this subset, add a single element in series with a.
(ii) Perform a cosegment-segment exchange on A on the matroid obtained in (i), and then simplify the resulting matroid.

Lastly, a segment-cosegment or cosegment-segment move on A is *allowable* if there is a fork-decomposition of M that displays A. The following theorem is our characterization of forked matroids.

Theorem 9.10. *Let M be a 3–connected matroid with at least three elements. Then M is forked if and only if M can be obtained from $U_{2,n}$ or $U_{n-2,n}$, for some $n \geq 3$, by a sequence of allowable segment-cosegment and cosegment-segment moves.*

Proof. Suppose that M is forked. Then, by Theorem 9.8, for some $n \geq 3$, either $U_{2,n}$ or $U_{n-2,n}$ can be obtained from M by a sequence of allowable segment-cosegment and cosegment-segment moves. But each of these moves can be reversed so that M can be obtained from $U_{2,n}$ or $U_{n-2,n}$ by a sequence of allowable cosegment-segment and segment-cosegment moves.

Now, for all $n \geq 3$, both $U_{2,n}$ and $U_{n-2,n}$ are 3–connected and forked. Therefore, to prove the converse of the theorem, it suffices to show, by duality, that performing an allowable segment-cosegment move on a 3–connected forked matroid N preserves the property of being 3–connected and forked. Since no allowable moves can be performed on $U_{2,3}$ or $U_{1,3}$, we may assume that $|E(N)| \geq 4$. Let A be a strict segment of N, and suppose that A is displayed in a fork-decomposition T of N. Let N' be a matroid that is obtained from N by choosing a subset of A and adding an element a' in parallel to each element a of the subset. Let A' be the set of added elements. Note that A is a strict segment of N'. It is easily seen that a

fork-decomposition T' of N' can be obtained as follows. For each element a' of A', subdivide the edge of T incident with the leaf labelled by the element a that is parallel to a' and insert a new vertex v labelled g; add a new leaf labelled by a' adjacent to v. The tree T' is a fork-decomposition of N' that displays $A \cup A'$, but does not necessarily display A. Choose an element a' of A'. Then, as A is a strict segment of N, we have that $a \in \mathrm{cl}_N(E(N) - A)$, and so $a' \in \mathrm{cl}_{N'}(E(N) - (A \cup A'))$. It now follows by Lemma 5.1 that T' can be modified to obtain a fork-decomposition of N' that displays $A \cup (A' - a')$. By repeating this process, we eventually obtain a fork-decomposition T'' of N' that displays A. Therefore, by Lemma 9.7(i), $\Delta_A(N')$ is forked. Since the class of forked matroids is closed under minors, the cosimplification of $\Delta_A(N')$ is also forked. Finally, by Lemma 9.3, the cosimplification of $\Delta_A(N')$ is 3–connected, and the theorem follows. □

For all $k \geq 4$, the class Λ_k of matroids that can be obtained from $U_{2,k}$ by a sequence of segment-cosegment and cosegment-segment exchanges is studied in [20]. These matroids have numerous attractive properties. For example, they can be used to show that the number of excluded minors for representability over a fixed finite field is at least exponential in the size of the field. Moreover, in [10], it is shown that the matroids in this class are precisely the totally free matroids with no $U_{3,6}$–minor. A consequence of the latter result is that, for any finite field $GF(q)$, the matroids representable over $GF(q)$ with no $U_{3,6}$–minor have a bounded number of inequivalent $GF(q)$–representations. Since every segment of $U_{2,k}$ can be displayed in some fork-decomposition of this matroid, it is not difficult to see that the exchanges used to build the matroids in Λ_k are examples of allowable segment-cosegment and cosegment-segment moves. The difference between Λ_k and the class of forked matroids is that, in constructing the members of the former class, one never performs parallel extensions, simplifications, series extensions, or cosimplifications. The class Λ_k is a fundamental subclass of the class of forked matroids.

10. Fork–Connected Matroids

A matroid M is *fork-connected* if it is 3–connected and, whenever $\{A, B\}$ is a 3–separation of M, either A or B is forked. An immediate consequence of Lemma 8.3 is that every 3–connected forked matroid is fork-connected. The converse of this fails. For instance, $U_{4,8}$ is a fork-connected matroid that is not forked. The purpose of this section is to prove the next theorem, which can be seen as a generalization of Theorem 9.8. Recall that, in this paper, a matroid is *vertically* 4*–connected* if it is 3–connected and, whenever A is an exactly 3–separating set of M, either A or $E(M) - A$ is a segment.

Theorem 10.1. *Let M be a fork-connected matroid. Then M is $\Delta - \nabla$–reducible to a vertically 4–connected matroid.*

Before proving Theorem 10.1, we establish some preliminary lemmas. Let M be a matroid, and let $X = \{x_1, x_2, \ldots, x_n\}$ be a forked 3–separating set of M. We refer to any fork-decomposition of the partitioned matroid induced by M on $\{E(M) - X, x_1, x_2, \ldots, x_n\}$ as a *fork-decomposition of X*.

Lemma 10.2. *Let M be a 3–connected matroid, and let B be a forked exactly 3–separating set of M. If a subset A of B is 3–separating, then A is forked.*

Proof. We may assume that $A \neq B$, that $|A| \geq 3$, and that A is not the ground set of a fan, otherwise A is certainly forked. Consider a fork-decomposition of B, and let P denote the 3–connected partitioned matroid corresponding to this fork-decomposition. Since B is exactly 3–separating, $E(M) - B$ is not a matroid element of P. Therefore $E(P) - A$ does not consist entirely of matroid elements. It now follows by Theorem 6.2 that the 3–separating partition $\{A, E(P) - A\}$ can be displayed in a fork-decomposition T of P. Replacing the branch of T displaying $E(P) - A$ by a single vertex labelled by $E(P) - A$ gives a fork-decomposition of A. Hence A is forked in M. □

The next lemma, which is elementary, is an immediate consequence of [13, Lemma 2.6].

Lemma 10.3. *Let M be a matroid, and let X and $X \cup a$ be exactly 3–separating sets of M. Then either $a \in \mathrm{cl}(X)$ or $a \in \mathrm{cl}^*(X)$.*

Note that in the figures that follow, a large circle labelled Z in a tree indicates the branch of any fork-decomposition of Z for which the set of leaf labels is Z.

Lemma 10.4. *Let M be a matroid, and let $a \in E(M)$. Suppose that X is a forked 3–separating set of M.*

(i) *If $X \cup a$ is 3–separating, then $X \cup a$ is forked.*
(ii) *If either $a \in \mathrm{cl}(X)$ or $a \in \mathrm{cl}^*(X)$, then $X \cup a$ is forked.*

Proof. To prove (i), suppose that $X \cup a$ is 3–separating. Then, as X is forked, it follows by Lemma 4.3 that the tree shown in Figure 22 is a fork-decomposition of $X \cup a$. Thus, in (i), $X \cup a$ is forked. We obtain (ii) immediately from (i) by using

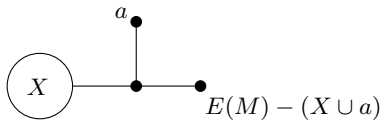


FIGURE 22

Lemma 8.6. □

Lemma 10.5. *Let M be a 3–connected matroid. Let A be an exactly 3–separating set of M. If X and Y are both forked 3–separating sets of M that contain A, and A can be displayed in a fork-decomposition of X, then $X \cup Y$ is a forked 3–separating set of M.*

Proof. Since A is exactly 3–separating, $|A| \geq 2$, and so $|X \cap Y| \notin \{0, 1\}$. Therefore, as M is 3–connected, it follows by Lemma 4.1 that $X \cup Y$ is 3–separating unless

$|X \cap Y| \in \{|E(M)|, |E(M)| - 1\}$. But, in each of the two exceptional cases, $X \cup Y$ is certainly 3–separating. Thus it remains to show that $X \cup Y$ is forked. For convenience, set $Z = X \cap Y$, $X' = X - Z$, $Y' = Y - Z$, and $D = E(M) - (X \cup Y)$.

If either X' or Y' is empty, then $X \cup Y$ is certainly forked. Assume that $|X'| = 1$. Then, as $X \cup Y$ is 3–separating, and Y is a forked 3–separating set, it follows by Lemma 10.4(i) that $X \cup Y$ is forked. Similarly, if $|Y'| = 1$, then $X \cup Y$ is forked. Thus we may assume that $|X'|, |Y'| \geq 2$, in which case, X and Y are both exactly 3–separating.

We show next that $X \cup Y$ is forked if $|D| \leq 1$. Assume that D is empty. Then $E(M) - X = Y'$, and so Y' is a 3–separating set of M. Therefore, by Lemma 10.2, Y' is forked. By Lemma 8.4, this implies that M is forked, and therefore, as $X \cup Y = E(M)$, we see that $X \cup Y$ is forked. Now assume that $|D| = 1$, and let $D = \{d\}$. By Lemma 10.2, X' is forked. Since $Y = E(M) - (X' \cup d)$, the set $X' \cup d$ is 3–separating set in M and so, by Lemma 10.4, $X' \cup d$ is forked. Thus, as $E(M)$ is the disjoint union of Y and $X' \cup d$, both of which are forked, Lemma 8.4 implies that M is forked. Hence $X \cup Y$, which equals $E(M) - d$, is forked. Thus we may assume that $|D| \geq 2$.

Since $A \subseteq Z$ and A is exactly 3–separating, $|Z| \geq 2$. As both X and Y are 3–separating, it is now straightforward to deduce using Lemma 4.1 that all of X', Y', Z, and D are exactly 3–separating sets of M. Let $X' = \{x_1, x_2, \ldots, x_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, and let P denote the partitioned matroid induced by M on $\{E(M) - X, x_1, \ldots, x_m, z_1, \ldots, z_n\}$. Since M is 3–connected, P is also 3–connected. By the hypothesis of the lemma, there is a reduced fork-decomposition T of P that displays A. Since A is an exactly 3–separating set of M, and therefore of P, the edge of T that displays A has width 3. Furthermore, the edge of T whose end-vertex is labelled by $E(M) - X$ also has width 3. Thus, as Z is a 3–separating set of M, and therefore of P, it follows by Lemma 5.2 that there is a fork-decomposition of P that displays Z. Moreover, this also implies that Z is forked in M. Now let P' be the partitioned matroid induced by M on $\{E(M) - X, x_1, x_2, \ldots, x_m, Z\}$. The partitioned matroid P' is 3–connected as M is 3–connected. Let T' be the tree obtained from T by replacing the branch of T that displays Z with a single vertex labelled by Z. As T is a fork-decomposition of P, it follows that T' is a fork-decomposition of P', and so P' is forked. Consider the 3–separating partition $\{\{x_1, x_2, \ldots, x_m\}, \{E(M) - X, Z\}\}$ of P'. We complete the proof of the lemma by considering the following two cases:

(I) X' can be displayed in a fork-decomposition of P'; and
(II) X' cannot be displayed in a fork-decomposition of P'.

In case (I), X' must be forked in M and there is a fork-decomposition of P' with a vertex that displays the 3–separating partition $\{\{x_1, x_2, \ldots, x_m\}, E(M) - X, Z\}$ of P'. Therefore this partition satisfies either the strong guts or the strong coguts condition in P'. Since $Y' \cup D = E(M) - X$, this in turn implies that the exactly 3–separating partition $\{X', Y' \cup D, Z\}$ of M satisfies either the strong guts or the strong coguts condition in M. Since $\{X' \cup D, Y', Z\}$ is an exactly 3–separating partition of M, it follows by Lemma 4.7 that $\{X' \cup D, Y', Z\}$ satisfies either the

strong guts or the strong coguts condition in M. By Lemma 4.7 again, this implies that $\{D, Y', X' \cup Z\}$, an exactly 3–separating partition of M, also satisfies either the strong guts or the strong coguts condition in M. Now let T_1 be the 6–vertex tree with exactly two degree–3 vertices v_1 and v_2. Label the leaves adjacent v_1 by X' and Z, and the leaves adjacent to v_2 by Y' and D. It follows that T_1 is a fork-decomposition of the partitioned matroid induced by M on $\{X', Z, Y', D\}$. As Y is forked in M, it follows by Lemma 10.2 that Y' is forked in M. Moreover, as both X' and Z are forked in M, it is easily seen that we can combine T_1 with fork-decompositions of Y', X', and Z to obtain a fork-decomposition of $X \cup Y$. Hence, in (I), $X \cup Y$ is indeed forked.

Now consider (II). If $|X'| \geq 4$, then, by Theorem 6.2, $\{x_1, x_2, \ldots, x_m\}$ is the ground set of a fan of P', and P' has a fork-decomposition of the form shown in Figure 23, where each of the edges on the path from the vertex labelled Z to the vertex labelled $E(M) - X$ has width 3. For each $i \in \{1, 2, \ldots, m\}$, it follows by Lemma 10.3 that $x_i \in \mathrm{cl}^{(*)}(Z \cup \{x_1, x_2, \ldots, x_{i-1}\})$. Therefore, as Y is forked in

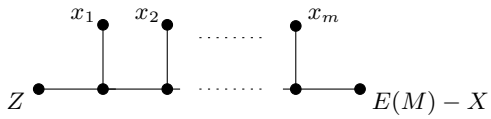


FIGURE 23

M, we deduce, by Lemma 10.2 and repeated applications of Lemma 10.4(ii), that $Y \cup X'$ is forked in M, that is, $X \cup Y$ is forked in M.

Now assume that $|X'| \in \{2, 3\}$. If there is a fork-decomposition of P' of the form shown in Figure 23, then we can argue as in the case $|X'| \geq 4$ to deduce that $X \cup Y$ is forked in M. Thus we may assume there is no such fork-decomposition. Then, as we are in (II), $|X'| = 3$ and every fork-decomposition of P' is of the form shown in Figure 24 where $\{U, V\} = \{Z, E(M) - X\}$. Since X' is an exactly 3–separating set of P', it is either a triangle or a triad of P'. Since $\{x_2, x_3\} \subseteq X'$

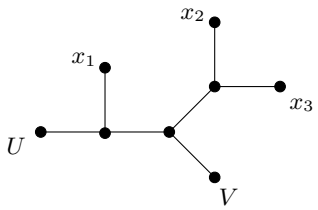


FIGURE 24

and $E(M) - X \subseteq E(P') - X'$, it follows from Lemma 5.1 that there is a fork-decomposition of P' that displays X'. This contradiction completes the proof of the lemma. □

A 3–separation $\{A, B\}$ of a matroid is *forked* if either A or B is forked.

Lemma 10.6. *Let M be a fork-connected matroid, and let A be a strict cosegment of M that can be displayed in a fork-decomposition of a maximal forked 3–separating set containing A. Then, for every 3–separation $\{X, Y\}$ of $\mathrm{si}(\nabla_A(M))$, either X or Y is forked.*

Proof. To ease notation, let N denote the matroid $\mathrm{si}(\nabla_A(M))$. By Lemma 9.2, N is 3–connected and we may assume that $A \subseteq E(N)$. Furthermore, A is an exactly 3–separating set of M. Now suppose that $\{X, Y\}$ is a 3–separation of N. Since N is 3–connected, $\{X, Y\}$ is an exact 3–separation of N. Furthermore, if $|A| = 2$, then $\nabla_A(M) \cong M$, and so $\nabla_A(M)$ is fork-connected. Hence we may assume that $|A| \geq 3$. If X or Y is contained in A, then X or Y is a segment of N and is therefore forked. Thus we may also assume that neither X nor Y is contained in A.

Next we establish the lemma when X or Y contains A.

10.6.1. *If $\{W, B\}$ is an exact 3–separation of N and A is contained in W or B, then W or B is forked.*

Proof. Without loss of generality, we may assume that $A \subseteq W$. By the definition of a cosegment-segment exchange, all parallel classes of $\nabla_A(M)$ contain an element of A. Thus $\{E(M) - B, B\}$ is an exact 3–separation of $\nabla_A(M)$. Therefore, by Lemma 9.1, $\{E(M) - B, B\}$ is an exact 3–separation of M. Thus, as M is fork-connected, either $E(M) - B$ or B is forked in M. In the second case, as B is disjoint from A, it is straightforward to deduce, using Lemma 9.1(ii), that a fork-decomposition of B in M is also a fork-decomposition of B in N. Hence, in this case, B is forked in N.

Now assume that $E(M) - B$ is forked in M. By the hypothesis of the lemma, there exists a maximal forked 3–separating set Z of M such that some fork-decomposition of Z displays A. Since Z is maximal, it follows by Lemma 10.5 that $E(M) - B \subseteq Z$. Now either (i) Z is not an exactly 3–separating set of M, or (ii) Z is an exactly 3–separating set of M. Suppose that (i) holds. Then the maximality of Z implies that $Z = E(M)$, and so M is forked. Thus A is displayed in a fork-decomposition of M, and therefore, by Lemma 9.7 and the fact that the class of forked matroids is minor-closed, it follows that N is forked. Hence, by Lemma 8.3, either W or B is forked in N. Now assume that (ii) holds and let $Z' = Z \cap E(N)$. By Lemma 9.1(ii), since $Z \supseteq A$, it follows that Z' is exactly 3–separating in N. Therefore, by Lemma 9.7, Z' is forked in N, and thus, by Lemma 10.2, W is forked in N. □

It remains to show that the lemma holds when both X and Y have a non-empty intersection with A. Suppose that $|X \cap A| = 1$, and let $X \cap A = \{x\}$. Then, as $|A| \geq 3$ and A is a segment of N, it follows that $x \in \mathrm{cl}_N(Y)$. Since $|X| \geq 3$, we deduce that $\{X - x, Y \cup x\}$ is an exact 3–separation of N. But then, as $A \subseteq Y \cup x$, we deduce by (10.6.1) that either $X - x$ or $Y \cup x$ is forked in N. First assume that $X - x$ is forked in N. Then, as $X - x$ and Y are both 3–separating sets of N, it follows by Lemma 4.3 that the tree shown in Figure 25(a) is a fork-decomposition of X in N, and so X is forked in N. Now assume that $Y \cup x$ is forked in N. Since $X - x$ and X are both exactly 3–separating sets of N, it follows by Lemma 10.3

FIGURE 25

that either $x \in \mathrm{cl}_N(X - x)$ or $x \in \mathrm{cl}_N^*(X - x)$. By Lemma 8.5, the latter case implies that $x \notin \mathrm{cl}_N(Y)$. It now follows by Lemma 5.1 that, in both cases, the tree shown in Figure 25(b) is a fork-decomposition of $Y \cup x$ in N, and so Y is forked in N. Similarly, if $|Y \cap A| = 1$, either X or Y is forked in N.

Now suppose that $|X \cap A|, |Y \cap A| \geq 2$, and let $X' = X - A$ and $Y' = Y - A$. Assume that $|X'| = 1$, and let $X' = \{x'\}$ and $X \cap A = \{a_1, a_2, \ldots, a_n\}$. Then, as $X \cap A$ is a segment of N, and both $X \cap A$ and X are exactly 3–separating sets of N, we deduce that the tree shown in Figure 26 is a fork-decomposition of X. Thus X is forked in N. Similarly, if $|Y'| = 1$, then Y is forked in N. Hence we may

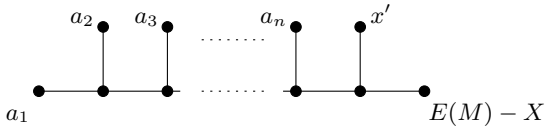


FIGURE 26

assume that $|X'|, |Y'| \geq 2$. Now Y and A are both 3–separating sets of N, and $\lambda_N(Y \cap A) \geq 3$ since N is 3–connected and $\min\{|X \cap A|, |Y \cap A)| \geq 2$. Therefore, by Lemma 4.1, $Y \cup A$ is 3–separating, and so, as N is 3–connected, $\{Y \cup A, X'\}$ is an exact 3–separation of N. Since $A \subseteq Y \cup A$, it again follows by (10.6.1) that either (i) $Y \cup A$ is forked in N, or (ii) X' is forked in N. If (i) holds, then, by Lemma 10.2, Y is forked in N. Now assume that (ii) holds. Let T be the tree shown in Figure 27, where $X \cap A = \{a_1, a_2, \ldots, a_n\}$. We assert that T is a fork-decomposition of X in N. To see this, observe that, as A is a segment of N and $|Y \cap A| \geq 2$, for all

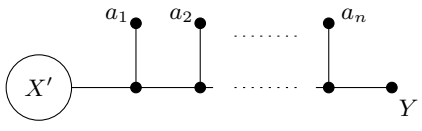


FIGURE 27

$i \in \{1, 2, \ldots, n\}$,

$$r_N(Y \cup \{a_n, a_{n-1}, \ldots, a_{i+1}\}) + r_N(X' \cup \{a_1, a_2, \ldots, a_i\}) - r(N) + 1 \leq r_N(Y) + r_N(X) - r(N) + 1.$$

Therefore, as $\{X, Y\}$ is an exact 3–separation of N, it follows that $X' \cup \{a_1, \ldots, a_i\}$ is a 3–separating set of N for all i. Thus, by repeated applications of Lemma 10.4(i), we obtain that X is forked in N, and the lemma follows. □

At last, we are in a position to prove the main result of this section.

Proof of Theorem 10.1. Assume that the theorem fails and let M be a counterexample for which $(|E(M)|, r(M))$ is lexicographically minimal. Certainly M is not vertically 4–connected.

Suppose that M is forked. Then, by Theorem 9.8, M is $\Delta - \nabla$–reducible to either $U_{2,n}$, or $U_{n-2,n}$ for some $n \geq 3$. Evidently $U_{2,n}$ is vertically 4–connected for all $n \geq 3$. Also $U_{n-2,n}$ is vertically 4–connected for $n \leq 5$. But, if $n \geq 6$, then $U_{n-2,n}$ is not vertically 4–connected. Consider this case, letting $\{A, B\}$ be a partition of $U_{n-2,n}$ into sets of size at least 3. Clearly both A and B are strict cosegments. Moreover, it is easily checked that $U_{2,n} = \nabla_A(\nabla_B(U_{n-2,n}))$, so that $U_{n-2,n}$ is $\Delta - \nabla$–reducible to a vertically 4–connected matroid. Hence the result holds if M is forked.

We may now assume that M is not forked. Since M is not vertically 4–connected, it has a 3–separation $\{X, Y\}$ with $r(X), r(Y) \geq 3$. As M is fork-connected, either X or Y is forked. Thus M has a maximal forked 3–separating set Z of rank at least 3. Since M is not forked, $|E(M) - Z| \geq 2$. Let P be the partitioned matroid induced by M on $\{E(M) - Z, z_1, z_2, \ldots, z_n\}$, where $Z = \{z_1, z_2, \ldots, z_n\}$. Since Z is forked, there is a reduced fork-decomposition of P. Choose such a fork-decomposition T in which the number of guts vertices is maximized. If every internal vertex of T is a guts vertex, then, by Lemma 9.5, Z is a segment, so $r(Z) = 2$; a contradiction. Thus we may assume that not every internal vertex of T is a guts vertex.

Let v_0 be the internal vertex of T that is adjacent to the leaf labelled by $E(P) - Z$. Let v_1 be a coguts vertex of T such that all internal vertices of T in the branches B_1 and B_2 of $T - v_1$ not containing v_0 are guts vertices. Now B_1 or B_2 has more than one vertex otherwise, by Lemma 4.3 and the choice of T, the vertex v_1 is a guts vertex. Let Z_1 and Z_2 be the sets of elements of M that label leaves in B_1 and B_2, respectively. We may assume that $|Z_1| \geq |Z_2|$, so $|Z_1| \geq 2$ and Z_1 is exactly 3–separating. Now, by Lemma 9.5, Z_1 is a strict segment of P. Moreover, by the dual of Lemma 10.6, every 3–separation of $\mathrm{co}(\Delta_{Z_1}(M))$ is forked. By Lemma 9.2, $\mathrm{co}(\Delta_{Z_1}(M))$ is 3–connected. Thus the last matroid is fork-connected. If it has fewer elements than M, then it is $\Delta - \nabla$–reducible to a vertically 4–connected matroid. Therefore so is M; a contradiction. Thus $\mathrm{co}(\Delta_{Z_1}(M)) = \Delta_{Z_1}(M)$. Lemma 9.7 implies that, by relabelling each internal vertex of B_1 by c, we obtain from T a fork-decomposition T' of $\Delta_{Z_1}(M)$.

If $|Z_2| \geq 2$, then we may argue as above using $\Delta_{Z_1}(M)$ and Z_2 in place of M and Z_1 to deduce that $\Delta_{Z_2}(\Delta_{Z_1}(M))$ is fork-connected and that by relabelling by c each internal vertex of B_2, we obtain a fork-decomposition T'' of $\Delta_{Z_2}(\Delta_{Z_1}(M))$. Moreover, by applying Lemma 9.5 to T'', we obtain that $Z_1 \cup Z_2$ is a strict cosegment of $\Delta_{Z_2}(\Delta_{Z_1}(M))$.

If $|Z_2| = 1$, we let $T'' = T'$. Then T'' is a fork-decomposition of $\Delta_{Z_1}(M)$ and $Z_1 \cup Z_2$ is a strict cosegment of this matroid. We conclude that both when $|Z_2| \geq 2$ and when $|Z_2| = 1$, there is a fork-connected matroid N having $Z_1 \cup Z_2$ as a strict cosegment that is displayed in a fork-decomposition T'' of N. Moreover,

$$r(N) = r(M) + (|Z_1| - 1) + (|Z_2| - 1).$$

Construct $\nabla_{Z_1 \cup Z_2}(N)$. It has rank

$$r(N) - (|Z_1 \cup Z_2| - 1),$$

which is less than $r(M)$. Moreover, by Lemmas 9.2 and 10.6, $\mathrm{si}(\nabla_{Z_1 \cup Z_2}(N))$ is fork-connected. Since the last matroid has either fewer elements or lower rank than M, the choice of M implies that $\mathrm{si}(\nabla_{Z_1 \cup Z_2}(N))$ is $\Delta - \nabla$–reducible to a vertically 4–connected matroid and therefore so is M. This contradiction completes the proof of the theorem. □

To see that the converse of Theorem 10.1 fails, it suffices to modify the example given to show the failure of the converse of Theorem 9.8. Let H be the graph in Figure 21(b). Let H' be obtained from H by relabelling by i' all the edges i in $E(H) - \{1, 8, 12\}$. Let M be the cycle matroid of the graph that is obtained by taking the 3–sum of H and H' across the triangle $\{1, 8, 12\}$. Since the 3–sum of H and H' is planar, and thus $\Delta - Y$–reducible to a triangle, M is $\Delta - \nabla$–reducible to $U_{2,3}$, which is vertically 4–connected. But $E(H) - \{1, 8, 12\}$ is a 3–separating set of M. If it or its complement in $E(M)$ is forked, then it follows easily that $M(H)$ is forked. But we showed following Lemma 9.9 that $M(H)$ is not forked. We conclude that M is not fork-connected and so the converse of Theorem 10.1 fails.

Theorem 10.1 enable us to achieve our goal of showing that, for applications in matroid representation theory, fork-connectivity is essentially no weaker than vertical 4–connectivity. Let $\mathbb{F}$ be a field. It is shown in [10] that if M' is obtained from M via a single segment-cosegment or cosegment-segment exchange, then M is $\mathbb{F}$–representable if and only if M' is. It is also shown there that the $\mathbb{F}$–representations of M are in one-to-one correspondence with those of M'. It is easily seen that if M' is obtained from M by either simplification, cosimplification, a parallel extension or a series extension, then the $\mathbb{F}$–representations of M are also in one-to-one correspondence with those of M'. The following corollary is obtained by combining these remarks with Theorem 10.1.

Corollary 10.7. *For all prime powers q, all members of the class of vertically 4–connected $GF(q)$–representable matroids have at most ν_q inequivalent $GF(q)$–representations if and only if all members of the class of fork-connected $GF(q)$–representable matroids have at most ν_q inequivalent $GF(q)$–representations.*

Our final result is obtained by combining the remarks in the paragraph preceding Corollary 10.7 with Theorem 9.10.

Corollary 10.8. *If M is a forked matroid, then there is an integer $n(M)$ such that M is representable over all fields with at least $n(M)$ elements.*

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