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On the Structure of 3-connected Matroids and Graphs

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An element e of a 3-connected matroid M is essential if neither the deletion $M\backslash e$ nor the contraction M/e is 3-connected. Tutte's Wheels and Whirls Theorem proves that the only 3-connected matroids in which every element is essential are the wheels and whirls. In this paper, we consider those 3-connected matroids that have some non-essential elements, showing that every such matroid M must have at least two such elements. We prove that the essential elements of M can be partitioned into classes where two elements are in the same class if M has a fan, a maximal partial wheel, containing both. We also prove that if an essential element e of M is in more than one fan, then that fan has three or five elements; in the latter case, e is in exactly three fans. Moreover, we show that if M has a fan with $2k$ or $2k+1$ elements for some $k \geq 2$, then M can be obtained by sticking together a $(k+1)$-spoked wheel and a certain 3-connected minor of M. The results proved here will be used elsewhere to completely determine all 3-connected matroids with exactly two non-essential elements.

1. INTRODUCTION

One of the main tools in the study of 3-connected matroids is the following result, Tutte's Wheels and Whirls Theorem [32], a generalization of an earlier graph result also due to Tutte [31].

THEOREM 1.1. *The following statements are equivalent for a* 3*-connected matroid M.*

(i) *For every element e of M, neither the deletion nor the contraction of e from M is* 3*-connected.*
(ii) *M has rank at least three and is isomorphic to a whirl or the cycle matroid of a wheel.*

Tutte [32] calls an element e of a 3-connected matroid M *essential* if neither the deletion $M\backslash e$ nor the contraction M/e remains 3-connected. Evidently, Theorem 1.1 characterizes those 3-connected matroids in which no element is non-essential. This theorem has had a very strong influence on the development of a large body of theory for 3-connected matroids (see, for example, [1–6, 9–12, 15, 17–20, 29]). Moreover, a number of authors have generalized the theorem in various ways (see, for example, [13, 14, 16, 21, 26–30]).

In this paper, we begin a study of the 3-connected matroids and graphs in which the set of non-essential elements is small. We show, for example, that a 3-connected matroid cannot have exactly one non-essential element; and, in another paper [24], we determine all 3-connected matroids with exactly two non-essential elements and all 3-connected simple graphs with exactly three non-essential edges. However, we can say considerably more. In particular, we prove a result that describes the local structure about an essential element in a 3-connected matroid. Part of this structure was determined by Tutte [32]:

THEOREM 1.2. *An essential element in a* 3*-connected matroid is in either a triangle or a triad.*

Here, a *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. Our first structure theorem generalizes this result by showing that every essential element in a 3-connected matroid M is in a submatroid of M which can be viewed as a maximal partial wheel and which we call a fan. Our second structure theorem describes how to break off such a partial wheel from M to leave a smaller 3-connected matroid with at most one new non-essential element.

The rest of this section will be devoted to introducing some terminology and basic results needed to state the two main theorems. These statements, Theorems 1.6 and 1.8, appear towards the end of the section. Section 2 contains a number of properties of 3-connected matroids that are needed in the proofs of the main theorems. These proofs appear in Sections 3 and 4. The main results have several important applications some of which will be discussed elsewhere [24, 33]. In particular, these results can immediately be applied to give basic structural information for 3-connected simple graphs.

The matroid terminology used here will follow Oxley [21]. For a matroid M, the simple matroid and the cosimple matroid associated with M will be denoted by $\widetilde{M}$ and $\underset{\sim}{M}$, respectively. We call these matroids the *simplification* and the *cosimplification* of M. A basic property of matroids that we shall use repeatedly is that a circuit and a cocircuit cannot have exactly one common element. We shall refer to this property as *orthogonality*.

Suppose that $r \geq 2$. The *wheel* $\mathcal{W}_r$ *of rank* r is a graph having $r+1$ vertices, r of which lie on a cycle (the *rim*); the remaining vertex is joined by a single edge (a *spoke*) to each of the other vertices. The *rank-r whirl* $\mathcal{W}^r$ is a matroid on $E(\mathcal{W}_r)$ having as its circuits all cycles of $\mathcal{W}_r$ other than the rim as well as all sets of edges formed by adding a single spoke to the edges of the rim. The terms 'rim' and 'spoke' will be applied in the obvious way in both $M(\mathcal{W}_r)$ and $\mathcal{W}^r$. Moreover, we shall usually refer to the cycle matroid of a wheel as just a wheel. The smallest 3-connected whirl is $\mathcal{W}^2$, which is isomorphic to $U_{2,4}$; the smallest 3-connected wheel is $M(\mathcal{W}_3)$, which is isomorphic to $M(K_4)$. By contrast with wheels and whirls of larger rank, in $M(\mathcal{W}_3)$ and $\mathcal{W}^2$, we cannot distinguish rim elements from spokes by looking just at the matroid. In these two cases, we arbitrarily designate a 3-element circuit and a 2-element set, respectively, as the rim with the complementary set being the set of spokes.

A fundamental concept in the statement of our main results is that of a chain of triangles and triads [22]. Let $T_1, T_2, \ldots, T_k$ be a non-empty sequence of sets each of which is a triangle or a triad of a matroid M such that, for all i in $\{1, 2, \ldots, k-1\}$:

(a) in $\{T_i, T_{i+1}\}$, exactly one set is a triangle and exactly one set is a triad;
(b) $|T_i \cap T_{i+1}| = 2$; and
(c) $(T_{i+1} - T_i) \cap (T_1 \cup T_2 \cup \cdots \cup T_i)$ is empty.

Then we call $T_1, T_2, \ldots, T_k$ a *chain* of M of *length* k with *links* $T_1, T_2, \ldots, T_k$. Evidently, $T_1, T_2, \ldots, T_k$ is a chain of M if and only if it is a chain of M^*. The last assertion relies on statement (a) being self-dual. Statement (a) corrects the corresponding condition in [22], which was intended to be self-dual but which is not.

The next lemma can be proved by a straightforward induction argument using orthogonality.

LEMMA 1.3. *Let* $T_1, T_2, \ldots, T_k$ *be a chain in a matroid* M. *Then* M *has* $k+2$ *distinct elements* $a_1, a_2, \ldots, a_{k+2}$ *such that* $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ *for all* i *in* $\{1, 2, \ldots, k\}$.

In this paper, we shall be concentrating on 3-connected matroids. A useful, but elementary, fact about such matroids (see, for example, [21, Proposition 8.1.7]) is the following lemma.

LEMMA 1.4. *The only* 3*-connected matroid that has a triangle which is also a triad is* $U_{2,4}$.

Much of our interest here is in maximal chains in 3-connected matroids. The following extension of Tutte's Wheels and Whirls Theorem shows that such a chain has non-essential elements at both ends. The proof of this result, which extends Tutte's proof, will be delayed until Section 3.

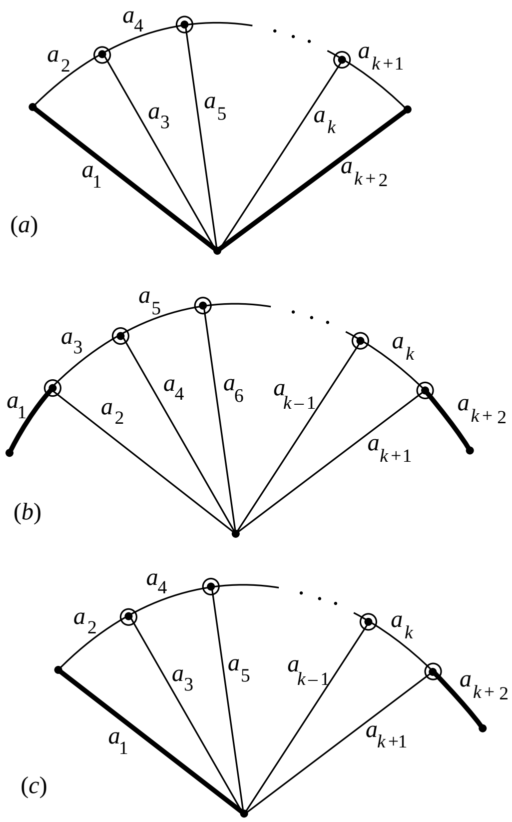

FIGURE 1. The three types of chain.

LEMMA 1.5. *Let M be a 3-connected matroid with at least four elements and suppose that M is not a wheel or a whirl. Let $T_1, T_2, \ldots, T_k$ be a maximal chain in M. Then the elements of $T_1 \cup T_2 \cup \cdots \cup T_k$ can be labelled such that neither a_1 nor a_{k+2} is essential where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i.*

Although chains can certainly occur in both non-graphic and graphic matroids, we follow Tutte [32] in keeping track of the triangles and triads in a chain by using graphs as in Figure 1. In each case, the chain is $T_1, T_2, \ldots, T_k$ where $T_i = \{a_i, a_{i+1}, a_{i+2}\}$. In (*a*), k is odd and T_1 is a triangle, hence T_k is also a triangle; in (*b*), k is odd and T_1 is a triad, hence T_k is also a triad; in (*c*), k is even, T_1 is a triangle and T_k is a triad. The remaining case, when k is even, T_1 is a triad, and T_k is a triangle, is, up to relabelling, the same as (*c*). In each of (*a*)–(*c*), every triangle in the graph is a triangle in the chain, while the triads in the chain correspond to circled vertices.

Now suppose that $T_1, T_2, \ldots, T_k$ is a maximal chain of a 3-connected matroid M where M is not a wheel or a whirl. We call this maximal chain a *fan* of M with *links* $T_1, T_2, \ldots, T_k$. Let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i. Then $\{a_1, a_2, \ldots, a_{k+2}\}$ is the *ground set* of the fan, and $a_1, a_2, \ldots, a_{k+2}$ are the *elements* of the fan. For $k \geq 2$, Lemma 1.5 implies that there are exactly two non-essential elements in $T_1 \cup T_2 \cup \cdots \cup T_k$, namely a_1 and a_{k+2}, for each of $a_2, a_3, \ldots, a_{k+1}$ is in both a triangle and a triad. We call a_1 and a_{k+2} the *ends* of the fan. When $k = 1$, the fan has T_1 as its ground set and contains either two or three non-essential elements of M. In the first case, we take the ends of the fan to be the non-essential elements in T_1; in the second case, we arbitrarily choose two of the elements of T_1 to be the ends of the fan. Figure 1(a), (b), and (c) show the three types of chains. Maximal chains of these three types will be called *type*-1, *type*-2, and *type*-3 *fans*, respectively. In the figure, the non-essential elements of these fans have been marked in bold. Two fans are *equal* if they have the same sets of links.

The next result, one of the two main theorems of this paper, extends Theorem 1.2 by giving more detailed information concerning the structure around an essential element in a 3-connected matroid. The proof of this theorem will be given in Section 3.

THEOREM 1.6. *Let M be a* 3*-connected matroid that is not a wheel or a whirl. Suppose that e is an essential element of M. Then e is in a fan, both ends of which are non-essential. Moreover, this fan is unique unless:*

(a) *every fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$ or:*
(b) *every fan containing e consists of a single triad and any two such triads meet in $\{e\}$ or:*
(c) *e is in exactly three fans; these three fans are of the same type, each has five elements, together they contain a total of six elements; and, depending on whether these fans are of type-*1 *or type-*2*, the restriction or contraction, respectively, of M to this set of six elements is isomorphic to $M(K_4)$.*

This theorem implies that the fans in a 3-connected matroid other than a wheel or whirl induce a partition of the set of essential elements.

COROLLARY 1.7. *Let M be a* 3*-connected matroid that is not a wheel or a whirl. Then there is a partition of the set of essential elements of M such that two elements are in the same class if and only if there is a fan whose ground set contains both.*

Given a fan with at least five elements in a 3-connected matroid M, our second main result describes how M can be constructed by sticking together a wheel and a certain 3-connected minor of M. The operation used here to join these two matroids is relatively well known for graphs and even binary matroids but, for matroids in general, it may be less familiar. Let M_1 and M_2 be matroids such that $M_1|T = M_2|T$, where $T = E(M_1) \cap E(M_2)$. Let $N = M_1|T$ and suppose that $\widetilde{N}$ is a modular flat of the matroid $\widetilde{M}_1$. The *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. This construction was introduced by Brylawski [8] when M_1 and M_2 are simple matroids, but it extends easily to the more general case considered above (see, for example, [21, Section 12.4]). Brylawski identified numerous attractive properties of the construction. When $|T| = 1$, $P_N(M_1, M_2)$ is just the parallel connection $P(M_1, M_2)$ [7] of M_1 and M_2.

One special case of the generalized parallel connection will be of particular importance here. Let N be a triangle Δ in both M_1 and M_2 and suppose that Δ is a modular flat of $\widetilde{M}_1$. In this case, we shall write $P_\Delta(M_1, M_2)$ for $P_N(M_1, M_2)$. Since every triangle is a modular flat in a simple binary matroid [8], if M_1 is binary, then $P_\Delta(M_1, M_2)$ is certainly well-defined. Perhaps the best-known instance of this operation occurs when both M_1 and M_2 are binary. For example, let G_1 and G_2 be graphs whose sets of edge labels are disjoint except that each has a triangle Δ whose edges are labelled by e, f, and g. If G is the graph that is obtained by identifying these triangles such that edges with the same labels coincide, then the cycle matroid of G is precisely the matroid $P_\Delta(M(G_1), M(G_2))$. We remark here that the graph $G\backslash\{e, f, g\}$ is what Robertson and Seymour [25] call the 3-*sum* of G_1 and G_2.

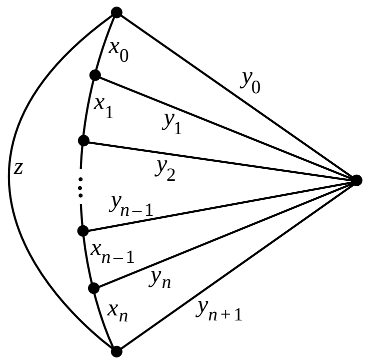

FIGURE 2. A labelled $\mathcal{W}_{n+2}$.

We are now ready to state the second main result of the paper, which will be proved in Section 4.

THEOREM 1.8. *Let M be a 3-connected matroid and suppose that, for some non-negative integer n, the sequence $\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \ldots, \{y_n, x_n, y_{n+1}\}$ is a chain in M in which $\{y_0, x_0, y_1\}$ is a triangle. Then*

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z$$

where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathcal{W}_{n+2}$ is labelled as in Figure 2; and M_1 is obtained from the matroid $M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots, y_n$ by relabelling x_n as z. Moreover, either:

(i) *M_1 is 3-connected; or*
(ii) *z is in a unique 2-circuit $\{z, h\}$ of M_1, and $M_1\backslash z$ is 3-connected.*

In the latter case,

$$M = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2)$$

where $\Delta_2 = \{y_0, y_{n+1}, h\}$; $\mathcal{W}_{n+2}$ is labelled as in Figure 2 with z relabelled as h; and M_2 is $M_1\backslash z$, which equals $M\backslash x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_n$.

An immediate consequence of this theorem is that the restriction of M to $\{x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_{n+1}\}$, the ground set of the chain, is equal to the cycle matroid of the graph shown in Figure 2 with the edge z deleted.

In Section 4, we shall describe how the essential elements behave when a wheel is broken off as above. In particular, we shall show that, for i in $\{1, 2\}$, if M_i is 3-connected, then an element of M_i that is essential in M remains essential in M_i.

2. PRELIMINARIES

The purpose of this section is to present a number of results for 3-connected matroids that will be used in the proofs of the main results to be given in Sections 3 and 4.

We begin with three known results due to Bixby [4], Tutte [32], and Seymour [26], respectively. The first two of these results will be used frequently throughout the paper.

LEMMA 2.1. *Let e be an element of a 3-connected matroid M. Then either $\underset{\smile}{M\backslash e}$ or $\widetilde{M/e}$ is 3-connected.*

The following lemma is often called Tutte's Triangle Lemma.

LEMMA 2.2. *Let $\{e, f, g\}$ be a triangle of a 3-connected matroid M. If both e and f are essential, then M has a triad containing e and exactly one of f and g.*

The next lemma has been stated explicitly by Seymour [26], though it is also implicit in Tutte's proof of the Wheels and Whirls Theorem [32].

LEMMA 2.3. *For some $n \geq 2$, let $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ be disjoint subsets of the ground set of a connected matroid M. Suppose that, for all i in $\{1, 2, \dots, n\}$, $\{e_i, f_i, e_{i+1}\}$ is a triangle and $\{f_i, e_{i+1}, f_{i+1}\}$ is a triad, where all subscripts are read modulo n. Then $E(M) = \{e_1, f_1, e_2, f_2, \dots, e_n, f_n\}$ and M is isomorphic to $M(\mathcal{W}_n)$ or $\mathcal{W}^n$.*

In the last lemma, the hypothesis that $\{e_n, f_n, e_1\}$ is a triangle can be eliminated if we know, for example, that M is 3-connected.

LEMMA 2.4. *For some $n \geq 2$, let $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ be disjoint subsets X and Y of the ground set of a 3-connected matroid M. Suppose that, for all i in $\{1, 2, \dots, n-1\}$ and all j in $\{1, 2, \dots, n\}$, $\{e_i, f_i, e_{i+1}\}$ is a triangle and $\{f_j, e_{j+1}, f_{j+1}\}$ is a triad, where all subscripts are read modulo n. Then M is isomorphic to $M(\mathcal{W}_n)$ or $\mathcal{W}^n$.*

PROOF. Clearly $r(X \cup Y) \leq r(X) + 1 = n + 1$, and $r^*(X \cup Y) \leq |Y| = n$. Hence

$$r(X \cup Y) + r^*(X \cup Y) - |X \cup Y| \leq 1,$$

that is,

$$r(X \cup Y) + r(E(M) - (X \cup Y)) - r(M) \leq 1.$$

Since M is 3-connected, it follows that $|E(M) - (X \cup Y)| \leq 1$. We now distinguish two cases:

(i) $|E(M) - (X \cup Y)| = 1$; and
(ii) $E(M) = X \cup Y$.

In case (i), let $E(M) - (X \cup Y) = \{d\}$. Since Y spans $E(M) - d$ in M^* and M^* is connected, Y spans M^*. In addition, since all of $\{e_1, f_1, e_2\}, \{e_2, f_2, e_3\}, \dots, \{e_{n-1}, f_{n-1}, e_n\}$ are cocircuits of M^*, and Y meets each of these cocircuits in a single element, Y is independent in M^*. Thus Y is a basis of M^*. Let C^* be the fundamental circuit of d with respect to this basis of M^*. By orthogonality, $C^* \subseteq \{d, f_n\}$. This is a contradiction since M is 3-connected having at least four elements.

In case (ii), X spans $E(M) - f_n$ in M. Hence X spans M. Moreover, X is independent in M since it meets each of the n triads of the hypothesis in a single element. Thus X is a basis of M. Let C be the fundamental circuit of f_n with respect to this basis. Then $C \subseteq X \cup f_n$. By orthogonality with the n known triads, we deduce that none of $e_2, e_3, \dots, e_{n-1}$ is in C. Hence $C \subseteq \{e_n, f_n, e_1\}$. Since $|C| \geq 3$, we conclude that $C = \{e_n, f_n, e_1\}$. It now follows by Lemma 2.3 that M is isomorphic to $M(\mathcal{W}_n)$ or $\mathcal{W}^n$. □

Now we show that a $U_{2,4}$-restriction in a 3-connected matroid avoids all essential elements of the matroid.

LEMMA 2.5. *Let M be a 3-connected matroid. If $X \subseteq E(M)$ and $M|X \cong U_{2,4}$, then no element of X is essential in M.*

PROOF. Suppose that some element x of X is essential in M. Then $M\backslash x$ has a 2-separation $\{S, T\}$. Without loss of generality, we may assume that $|S \cap (X - x)| \geq 2$. Then $\{S \cup x, T\}$ is a 2-separation of M; a contradiction. □

The last three results in this section concern connectivity. The first [23] will be used only in the case $n = 3$.

LEMMA 2.6. *Let n be an integer exceeding one and M be a matroid having no circuits with fewer than n elements. If $M|X$ and $M|Y$ are n-connected and $|\mathrm{cl}(X) \cap \mathrm{cl}(Y)| \geq n - 1$, then $M|(X \cup Y)$ is n-connected.*

COROLLARY 2.7. *Let M_1 and M_2 be matroids whose ground sets meet in a set Δ that is a triangle in both matroids and a modular flat of M_1. Then $P_\Delta(M_1, M_2)$ is 3-connected if and only if both M_1 and M_2 are 3-connected.*

PROOF. By the last lemma, if M_1 and M_2 are both 3-connected, then so is $P_\Delta(M_1, M_2)$. Now suppose that $P_\Delta(M_1, M_2)$ is 3-connected. Assume that, for some i in $\{1, 2\}$ and some $k \leq 2$, the matroid M_i has a k-separation, $\{S, T\}$. Then we may assume, without loss of generality, that $|S \cap \Delta| \geq 2$. It follows easily that $\{S \cup (E(M_j) - \Delta), T\}$ is a k-separation of $P_\Delta(M_1, M_2)$ where $\{i, j\} = \{1, 2\}$; a contradiction. □

The next result is an extension of this corollary.

COROLLARY 2.8. *Let the ground sets of the matroids M_1 and M_2 meet in a set Δ that is a triangle of both matroids and a modular flat of M_1. Suppose that both $|E(M_1)|$ and $|E(M_2)|$ exceed three and let z be an element of Δ. Then $P_\Delta(M_1, M_2)\backslash z$ is 3-connected if and only if: (i) M_1 is 3-connected; and (ii)(a) M_2 is 3-connected, or (b) M_2 has a unique 2-circuit, which contains z, and $M_2\backslash z$ is 3-connected.*

PROOF. Suppose first that $P_\Delta(M_1, M_2)\backslash z$ is 3-connected. Then either $P_\Delta(M_1, M_2)$ is 3-connected, or z is parallel to some element y of $P_\Delta(M_1, M_2)$. In the first case, by the previous corollary, (i) and (ii)(a) hold. In the second case, as Δ is a modular flat of M_1 containing z, it follows that $y \in E(M_2)$, so $\{z, y\}$ is a circuit of M_2. Moreover, $P_\Delta(M_1, M_2)\backslash z \cong P_\Delta(M_1, M_2\backslash y)$. Thus, by Corollary 2.7 again, M_1 and $M_2\backslash y$ are 3-connected, and it follows easily that (i) and (ii)(b) hold.

A similar argument shows that if (i) and (ii)(b) hold, then $P_\Delta(M_1, M_2)\backslash z$ is 3-connected. Now assume that (i) and (ii)(a) hold. By Corollary 2.7, $P_\Delta(M_1, M_2)$ is 3-connected. Suppose that $P_\Delta(M_1, M_2)\backslash z$ is not 3-connected. Then this matroid has a 2-separation $\{S, T\}$. If $r(M_2) = 2$, then we may assume, without loss of generality, that $|S \cap E(M_2\backslash z)| \geq 2$. Hence $\{S \cup z, T\}$ is a 2-separation of $P_\Delta(M_1, M_2)$; a contradiction. We conclude that $r(M_2) > 2$. Clearly $r(M_1) > 2$. Thus the simplification of $P_\Delta(M_1, M_2)/z$ is a 2-sum of two matroids of rank at least two and so is not 3-connected. Hence, by Lemma 2.1, the cosimplification of $P_\Delta(M_1, M_2)\backslash z$ is 3-connected. Since $P_\Delta(M_1, M_2)\backslash z$ is not 3-connected itself, z must be in a triad T^* of $P_\Delta(M_1, M_2)$. By orthogonality, $|T^* \cap \Delta| \geq 2$. However, each of M_1 and M_2 is a restriction of $P_\Delta(M_1, M_2)$. Thus, for $i = 1, 2$, the set $T^* \cap E(M_i)$ contains a cocircuit of M_i and so, as M_i is 3-connected, $|T^* \cap E(M_i)| \geq 3$. Therefore $T^* \cap E(M_1) = T^* \cap E(M_2) = \Delta$, so $T^* = \Delta$. It follows that Δ is both a triangle and a triad of M_1. Hence, by Lemma 1.4, $r(M_1) = 2$; a contradiction. □

3. Proof of the Fan Theorem

In this section, we shall prove Theorem 1.6. Evidently, the properties of chains in 3-connected matroids will be important in this proof. We begin this section with four lemmas describing such properties. The first of these is essentially a converse of Lemma 1.5. Its straightforward proof is omitted.

LEMMA 3.1. *Let $T_1, T_2, \ldots, T_k$ be a chain in a 3-connected matroid M. Suppose that $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i. If a_1 is non-essential and $k \geq 2$, then there is no 3-element subset T_0 of $E(M)$ such that $T_0, T_1, T_2, \ldots, T_k$ is a chain in M.*

LEMMA 3.2. *Let T_1, T_2, T_3 be a chain in a 3-connected matroid M. Then there is at most one set T_4 such that T_1, T_2, T_3, T_4 is a chain.*

PROOF. Let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for $i = 1, 2, 3$. Assume that T_1, T_2, T_3, T_4 is a chain both when $T_4 = \{a_4, a_5, b\}$ and when $T_4 = \{a_4, a_5, c\}$ where $b \neq c$. Then the restriction of M or M^* to $\{a_4, a_5, b, c\}$ is isomorphic to $U_{2,4}$. However, a_4 is in two links of a chain and hence is essential, contradicting Lemma 2.5. □

LEMMA 3.3. *Let $e_1, e_2, e_3, e_4, e_5, e_6$ be distinct elements of a 3-connected matroid M. Suppose that T_1, T_2, T_3, T_4 is a chain in M such that $T_i = \{e_i, e_{i+1}, e_{i+2}\}$ for all i, and e_1 is non-essential. Then the only triangles or triads of M containing e_2 are T_1 and T_2.*

PROOF. By switching to the dual if necessary, we may assume that T_1 is a triangle. Suppose that M has a triangle T containing e_2 but different from $\{e_1, e_2, e_3\}$. Then, by orthogonality with the triad $\{e_2, e_3, e_4\}$, we deduce that T contains e_3 or e_4. If $e_3 \in T$, then $M|(T \cup \{e_1, e_2, e_3\}) \cong U_{2,4}$. Since e_2 is essential, this contradicts Lemma 2.5. Thus we may assume that $e_3 \notin T$. Hence $e_4 \in T$. By orthogonality with the triad $\{e_4, e_5, e_6\}$, we deduce that T contains e_5 or e_6. If $e_5 \in T$, then $M|\{e_2, e_3, e_4, e_5\} \cong U_{2,4}$, a contradiction to Lemma 2.5. Hence $e_5 \notin T$ and so $T = \{e_2, e_4, e_6\}$. Thus $M|\{e_1, e_2, \ldots, e_6\}$ is spanned by $\{e_1, e_3, e_5\}$. Since $\{e_2, e_3, e_4\}$ is a triad of M, it contains a cocircuit of $M|\{e_1, e_2, \ldots, e_6\}$. Thus $\{e_1, e_5, e_6\}$, which avoids $\{e_2, e_3, e_4\}$, is dependent in $M|\{e_1, e_2, \ldots, e_6\}$ and hence is a circuit of M. However now, by Lemma 2.4, $M \cong M(\mathcal{W}_3)$ or $\mathcal{W}^3$ so every element, including e_1, is essential. This contradiction implies that $\{e_1, e_2, e_3\}$ is the only triangle of M containing e_2.

Now suppose that T^* is a triad of M that contains e_2 but is different from $\{e_2, e_3, e_4\}$. By orthogonality with the circuit $\{e_1, e_2, e_3\}$, we must have that e_1 or e_3 is in T^*. The first possibility is out as e_1 is non-essential. Hence $e_3 \in T^*$. However, then $M^*|(T^* \cup \{e_2, e_3, e_4\}) \cong U_{2,4}$; a contradiction to Lemma 2.5 since e_2 is essential. □

LEMMA 3.4. *Let e_1, e_2, e_3, e_4, e_5 be distinct elements of a 3-connected matroid M that is not isomorphic to $M(\mathcal{W}_3)$. Suppose that $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are triangles and $\{e_2, e_3, e_4\}$ is a triad of M. Then these two triangles and this one triad are the only triangles and triads of M containing e_3.*

PROOF. Suppose first that T^* is a triad of M containing e_3 but that $T^* \neq \{e_2, e_3, e_4\}$. Then, by orthogonality and symmetry, we may assume that T^* is $\{e_1, e_3, e_4\}$ or $\{e_1, e_3, e_5\}$. In the first case, $M^*|\{e_1, e_2, e_3, e_4\} \cong U_{2,4}$. Since e_2 is essential, this contradicts Lemma 2.5. Hence $T^* = \{e_1, e_3, e_5\}$. Let $A = \{e_1, e_2, e_3, e_4, e_5\}$. Then $\{e_1, e_3, e_5\}$ spans A in M, and $\{e_1, e_2, e_3\}$ spans A in M^*. Thus

$$r(A) + r^*(A) - |A| = 1.$$

Since M is 3-connected, it follows that $|E(M) - A| = 1$, so $|E(M)| = 6$. Using the known triangles and triads of M, it is now routine to show that $M \cong M(\mathcal{W}_3)$; a contradiction. We conclude that $\{e_2, e_3, e_4\}$ is the only triad of M containing e_3.

Suppose next that T is a triangle of M containing e_3 but different from $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$. Then, by orthogonality with the triad $\{e_2, e_3, e_4\}$ and by symmetry, we may assume that $e_2 \in T$. Thus $M|(\{e_1, e_2, e_3\} \cup T) \cong U_{2,4}$. Since e_2 is essential, we have a contradiction to Lemma 2.5. □

Next we insert the proof of Lemma 1.5 which was delayed from Section 1.

PROOF OF LEMMA 1.5. We may assume that $k \geq 2$ for if $k = 1$, the lemma follows easily by using the maximality of the chain along with Tutte's Triangle Lemma.

As noted in Lemma 1.3, the elements of $T_1 \cup T_2 \cup \cdots \cup T_k$ can certainly be labelled by $a_1, a_2, \ldots, a_{k+2}$ such that $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i. It remains to show that this labelling can be adjusted such that neither a_1 nor a_{k+2} is essential. Suppose that this is not the case. Then, by reversing the ordering on $a_1, a_2, \ldots, a_{k+2}$ if necessary, we may assume that a_{k+2} is essential. Moreover, by duality, we may also assume that T_k is a triangle. Then, as neither $M \backslash a_{k+1}$ nor $M \backslash a_{k+2}$ is 3-connected, Lemma 2.2 implies that M has a triad T^* containing a_{k+2} and exactly one of a_k and a_{k+1}.

Suppose that $a_{k+1} \in T^*$. Then the maximality of the chain $T_1, T_2, \ldots, T_k$ implies that T^* must also contain one of $a_1, a_2, \ldots, a_{k-1}$. Thus if $k = 2$, then $a_1 \in T^*$; if $k > 2$, then each of $a_2, a_3, \ldots, a_{k-1}$ is in a triangle of the chain $T_1, T_2, \ldots, T_k$ that avoids $\{a_{k+1}, a_{k+2}\}$, so again $a_1 \in T^*$. It follows by orthogonality and Lemma 2.4 that M is a wheel or a whirl; a contradiction.

We may now assume that $a_{k+1} \notin T^*$. Then $a_k \in T^*$. If $k \geq 3$, then, taking $e_3 = a_k$ in Lemma 3.4, we obtain the contradiction that $T^* = \{a_{k-1}, a_k, a_{k+1}\}$. Thus we may assume that $k = 2$. Let $T^* = \{a_2, a_4, z\}$. If $z \neq a_1$, then T_1, T_2, T^* is a chain contradicting the maximality of the chain T_1, T_2. Thus $z = a_1$. However, then $M^*|\{a_1, a_2, a_3, a_4\} \cong U_{2,4}$ and a_2 is essential; a contradiction to Lemma 2.5. □

The following extension of Theorem 1.1 is an immediate consequence of Lemma 1.5.

COROLLARY 3.5. *Let M be a 3-connected matroid with at least four elements. Then either M is a wheel or a whirl, or M has at least two non-essential elements.*

Among the results in [24] is a specification of all the 3-connected matroids in which the set of non-essential elements has rank two.

We are now ready to prove the fan theorem, and the remainder of the section will be devoted to presenting this proof.

PROOF OF THEOREM 1.6. Because M has an essential element and is not a whirl, $|E(M)| \geq 5$. By Theorem 1.2, since e is essential, it is in a triangle or a triad of M. Thus M has a chain containing e. Let $T_1, T_2, \ldots, T_k$ be a maximal chain of M such that $e \in T_1 \cup T_2 \cup \cdots \cup T_k$. Then $T_1, T_2, \ldots, T_k$ is a fan $\mathcal{F}$ of M containing e.

Now let $T'_1, T'_2, \ldots, T'_n$ be another fan $\mathcal{F}'$ of M containing e; let $T_1 \cup T_2 \cup \cdots \cup T_k = E(\mathcal{F})$; let $T'_1 \cup T'_2 \cup \cdots \cup T'_n = E(\mathcal{F}')$; and let $T_i = \{a_i, a_{i+1}, a_{i+2}\}$ for all i. The proof of the various assertions concerning the fans containing e will be broken into the following four cases:

(i) $|E(\mathcal{F})| = 3$;
(ii) $|E(\mathcal{F})| = 4$;
(iii) $|E(\mathcal{F})| = 5$; and
(iv) $|E(\mathcal{F})| \geq 6$.

Consider case (i) assuming, without loss of generality, that T_1 is a triangle of M. Then e is not in a triad of M otherwise the chain whose single link is T_1 is not maximal. Since, in a maximal chain of length at least two, every essential element is in a triad, we deduce that the fan $\mathcal{F}'$ also has just one link, a triangle. Moreover, the fans $\mathcal{F}$ and $\mathcal{F}'$ meet in $\{e\}$ otherwise $M|(E(\mathcal{F}) \cup E(\mathcal{F}')) \cong U_{2,4}$, contrary to Lemma 2.5. We conclude that if $|E(\mathcal{F})| = 3$, then (a) or (b) holds.

Now assume that (ii) holds and suppose, without loss of generality, that $e = a_3$ and that T_1 is a triangle. Clearly a_1 and a_4 are non-essential. If $|E(\mathcal{F}')| = 3$, then we may apply case (i) interchanging $\mathcal{F}$ and $\mathcal{F}'$ to obtain a contradiction. Thus we may assume that $|E(\mathcal{F}')| \geq 4$. Hence the links of $\mathcal{F}'$ include a triangle T and a triad T^* both containing a_3. As a_4 is non-essential and is in the triad T_2, it follows that $a_4 \notin T$. By orthogonality with the triad $\{a_2, a_3, a_4\}$, we deduce that T contains a_2 and a_3. Since these two elements are essential, Lemma 2.5 implies that $T = \{a_1, a_2, a_3\}$. A dual argument establishes that $T^* = \{a_2, a_3, a_4\}$. Hence, in this case, $\mathcal{F} = \mathcal{F}'$; that is, e is in a unique fan.

We shall assume next that (iv) occurs. From cases (i) and (ii), we may assume that $|E(\mathcal{F}')| \geq 5$. Next we shall distinguish the following two subcases of (iv):

(I) e is in just two of $T_1, T_2, \dots, T_k$; and
(II) e is in at least three of $T_1, T_2, \dots, T_k$.

Suppose that (I) occurs. Then we may assume, without loss of generality, that $e = a_2$. By Lemma 3.3, the only triangles or triads of M containing e are T_1 and T_2. Since $|E(\mathcal{F}')| \geq 5$ and e is essential, e is in at least two links of $\mathcal{F}'$. Hence T_1 and T_2 must both be links of $\mathcal{F}'$. Moreover, since a_1 is non-essential, it follows by Lemma 3.1 that a_1 is an end of $\mathcal{F}'$. Thus we may assume that $T'_1 = T_1$ and $T'_2 = T_2$. Taking a_3 equal to e_3 in Lemma 3.4 and using the fact that $T'_1, T'_2, \dots, T'_n$ is a maximal chain, we deduce that $T'_3 = T_3$. Again using the fact that $T'_1, T'_2, \dots, T'_n$ is a maximal chain, this time with Lemma 3.2, we find that $T'_j = T_j$ for all j in $\{4, 5, \dots, k\}$. Now, since $T_1, T_2, \dots, T_k$ is a maximal chain, it follows that $k = n$ and hence that $\mathcal{F} = \mathcal{F}'$.

To complete the proof in case (iv), we need to treat subcase (II). Thus assume that $e = a_j$ for some j in $\{3, 4, \dots, k\}$. By Lemma 3.4 or its dual, the only triangles or triads of M containing e are T_{j-2}, T_{j-1}, and T_j. If all three of these sets are links of $\mathcal{F}'$, then, since $\mathcal{F}'$ is a maximal chain, repeated applications of Lemma 3.2 yield that $\mathcal{F}'$ has exactly the same set of links as $\mathcal{F}$. Thus we may assume, without loss of generality, that T_{j-2} and T_{j-1} are links of $\mathcal{F}'$ but T_j is not. Hence e is in exactly two links of $\mathcal{F}'$, so e is in a link of $\mathcal{F}'$ with some non-essential element. Therefore a_{j-2} or a_{j+1} is non-essential. However, a_{j+1} is in both T_{j-1} and T_j so it is essential. Hence a_{j-2} is non-essential, so $a_{j-2} = a_1$. Applying Lemma 3.3 to the chain T_1, T_2, T_3, T_4, we find that T_1 and T_2 are the only triangles or triads of M containing a_2. However, now T_1 is a link of $\mathcal{F}'$ containing two elements, a_2 and a_3, each of which is in just two links of $\mathcal{F}'$. Hence $\mathcal{F}'$ has exactly two links, a contradiction to the fact that $|E(\mathcal{F}')| \geq 5$.

It now remains to treat case (iii). First we note that $|E(\mathcal{F}')| = 5$ otherwise we can obtain the result by applying one of cases (i), (ii), and (iv) with $\mathcal{F}$ and $\mathcal{F}'$ interchanged. Now either:

(I) e is in all three of T_1, T_2, and T_3; or
(II) e is in exactly two of T_1, T_2, and T_3.

We may assume, by switching to the dual if necessary, that T_1 and T_3 are triangles.

Consider (I). Evidently $e = a_3$. By Lemma 3.4, T_1, T_2, and T_3 are the only triangles or triads of M containing a_3. Since $|E(\mathcal{F}')| = 5$ and a_3 is essential, the links of $\mathcal{F}'$ include both a triangle and a triad containing a_3. Without loss of generality, we may assume that T_1 and T_2 are links of $\mathcal{F}'$. Since a_1 is non-essential, the remaining link of $\mathcal{F}'$ is a triangle T containing a_4 and exactly one of a_2 and a_3. If $a_3 \in T$, then it follows by Lemma 2.5 that $T = T_3$; that is, $\mathcal{F} = \mathcal{F}'$. If $a_3 \notin T$, then $T = \{a_2, a_4, z\}$ for some $z \neq a_3$. The dual of Lemma 2.5 can now be used to show that $z \neq a_1$ and $z \neq a_5$. Consider $M|(E(\mathcal{F}) \cup E(\mathcal{F}'))$. This matroid has ground set $\{a_1, a_2, a_3, a_4, a_5, z\}$. It has $\{a_2, a_3, a_4\}$ as a basis and $\{a_2, a_4, z\}$ as a circuit. Thus, by Lemma 2.5, this matroid has $\{a_1, a_3, a_5\}$ as a cocircuit. Since $M|(E(\mathcal{F}) \cup E(\mathcal{F}'))$ is clearly 3-connected, it follows by Lemma 3.4 that $M|(E(\mathcal{F}) \cup E(\mathcal{F}')) \cong M(K_4)$. Moreover, one easily sees that, apart from $\mathcal{F}$ and $\mathcal{F}'$, the only fan containing e is $\{z, a_2, a_4\}, \{a_2, a_4, a_3\}, \{a_4, a_3, a_5\}$.

Finally consider (iii)(II). Without loss of generality, we may assume that $e = a_2$. If $a_3 \in E(\mathcal{F}')$, then a_3 is an essential element that is in all three links of $\mathcal{F}$ and is also an element of $\mathcal{F}'$. Hence we may apply (iii)(I) to obtain the desired result. We may now assume that a_3 is not in $E(\mathcal{F}')$. Certainly a_2 is in a triad T^* that is a link of $\mathcal{F}'$. By orthogonality with the triangle $\{a_1, a_2, a_3\}$, we deduce that $a_1 \in T^*$. Thus a_1 is essential; a contradiction. □

4. Breaking off Wheels

In this section, we shall prove Theorem 1.8. In addition, we shall describe how the essential elements behave when a wheel is broken off as in that theorem. We begin with a straightforward result showing that if one wants to perform two successive generalized parallel connections across triangles, then the order in which these operations are performed does not matter.

Lemma 4.1. *Let M_1, M_2, and M_3 be matroids, the first two of which are binary. Suppose that $E(M_1)$ and $E(M_2)$ meet in a set Δ which is a triangle of both M_1 and M_2, that $E(M_2)$ and $E(M_3)$ meet in a set Δ' which is a triangle of both M_2 and M_3, and that $E(M_1)$ and $E(M_3)$ meet in $\Delta \cap \Delta'$. Then*

$$P_\Delta(M_1, P_{\Delta'}(M_2, M_3)) = P_{\Delta'}(P_\Delta(M_1, M_2), M_3).$$

Proof. First we remark that both sides of the asserted equation are well-defined. To see this, note that, as M_1 and M_2 are binary, Δ and Δ' are modular flats of $\widetilde{M}_1$ and $\widetilde{M}_2$, respectively. Moreover, $P_\Delta(M_1, M_2)$ is also binary and hence has Δ' as a modular flat of its simplification.

Next we observe that $P_\Delta(M_1, P_{\Delta'}(M_2, M_3))$ and $P_{\Delta'}(P_\Delta(M_1, M_2), M_3)$ have the same ground set, namely $E(M_1) \cup E(M_2) \cup E(M_3)$. To complete the proof that these two matroids are equal, we show that they have the same sets of flats. By definition, F is a flat of $P_\Delta(M_1, P_{\Delta'}(M_2, M_3))$ if and only if

$$F \cap E(M_1) \text{ is a flat of } M_1 \tag{1}$$

and

$$F \cap E(P_{\Delta'}(M_2, M_3)) \text{ is a flat of } P_{\Delta'}(M_2, M_3). \tag{2}$$

However, (2) holds if and only if

$$F \cap E(M_2) \text{ is a flat of } M_2 \tag{3}$$

and

$$F \cap E(M_3) \text{ is a flat of } M_3. \tag{4}$$

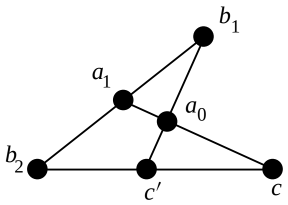

FIGURE 3. A labelled $M(K_4)$.

Thus F is a flat of $P_\Delta(M_1, P_{\Delta'}(M_2, M_3))$ if and only if (1), (3), and (4) hold. A similar argument shows that F is a flat of $P_{\Delta'}(P_\Delta(M_1, M_2), M_3)$ if and only if (1), (3), and (4) hold. The lemma follows immediately. □

The next lemma, an extension of a result of Akkari and Oxley [3], will be used to prove Theorem 1.8 in the case $n = 1$. Following this lemma, we prove the theorem.

LEMMA 4.2. *Let a_1 and b_1 be distinct elements of a* 3*-connected matroid M. Suppose that $\{a_0, b_1, a_1\}$ is a triad, $\{b_1, a_1, b_2\}$ is a triangle, and M is not isomorphic to $U_{2,4}$. Then $M = P_\Delta(M(K_4), M')\backslash\{c, c'\}$ where $\Delta = \{c, c', b_2\}$; $M(K_4)$ is labelled as in Figure* 3*; and M' is obtained from M/a_0 by relabelling a_1 and b_1 as c and c', respectively. Moreover, one of the following holds:*

(i) *M' is* 3*-connected;*
(ii) *a_0 is in a unique triangle of M, this triangle contains a_1, and $M'\backslash c$ is* 3*-connected;*
(iii) *a_0 is in a unique triangle of M, this triangle contains b_1, and $M'\backslash c'$ is* 3*-connected;*
(iv) *a_0 is in exactly two triangles of M, one of which also contains a_1 and the other of which also contains b_1, and $M'\backslash c, c'$ is* 3*-connected.*

PROOF. The first part of the lemma is proved in [3]. It only remains to show that one of (i)–(iv) holds. Suppose that (i) fails. Then M/a_0 is not 3-connected. Since M/a_1 has a 2-circuit and has rank exceeding one, it is not 3-connected. Applying the dual of Tutte's Triangle Lemma to the triad $\{a_0, a_1, b_1\}$ of M, we deduce that M has a triangle containing a_0 and exactly one of a_1 and b_1. Moreover, as both a_1 and b_1 are essential, Lemma 2.5 implies that each of $\{a_0, a_1\}$ and $\{a_0, b_1\}$ is in at most one triangle of M.

Now, by Lemma 2.1, $M\backslash a_0$ or M/a_0 is 3-connected. If the latter occurs, it follows without difficulty from the preceding paragraph that one of (ii)–(iv) holds. Hence we may assume that $M\backslash a_0$ is 3-connected. However, $M\backslash a_0$ has $\{a_1, b_1\}$ as a cocircuit and $M\backslash a_0/a_1$ has $\{b_1, b_2\}$ as a circuit. Therefore $M\backslash a_0$ is not simple. As $M\backslash a_0$ is 3-connected, this matroid is isomorphic to $U_{1,2}$ or $U_{1,3}$. Thus M is a 3-connected matroid having corank equal to two or three. Since M has both a triangle and a triad and $M \not\cong U_{2,4}$, it follows that $r^*(M) \neq 2$. Hence $r^*(M) = 3$. Thus M^* has rank three and has $\{a_1, b_1, b_2\}$ as a triad. The remaining elements of M^* lie on a line, and it is straightforward to check that one of (i)–(iv) holds. □

PROOF OF THEOREM 1.8. We argue by induction on n. When $n = 0$, $M_1 \cong M$ and the theorem holds. Now suppose that $n = 1$. Then, by Lemma 4.2, $M = P_\Delta(M(\mathcal{W}_3), M')\backslash\{z, z'\}$ where $\Delta = \{z, z', y_2\}$; $M(\mathcal{W}_3)$ is labelled as in Figure 4(a); and M' is obtained from M/x_0 by relabelling x_1 and y_1 as z and z', respectively. However, y_0 and y_1 are in parallel in M/x_0. Thus

$$M = P_{\Delta_1}(M(\mathcal{W}_3), M')\backslash\{z, z'\}$$

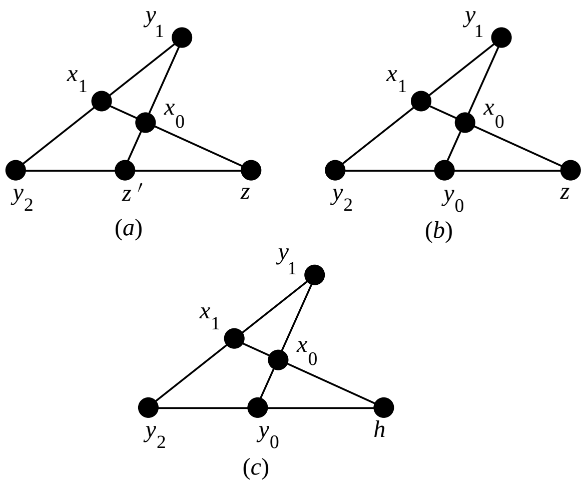

FIGURE 4. Three labellings of $M(\mathcal{W}_3)$.

where $\Delta_1 = \{z, y_0, y_2\}$; $M(\mathcal{W}_3)$ is labelled as in Figure 4(*b*). Therefore

$$M = P_{\Delta_1}(M(\mathcal{W}_3), M'\backslash z')\backslash z. \tag{5}$$

Now, clearly $M'\backslash z'$ is $M/x_0\backslash y_1$ with x_1 relabelled as z; that is, $M'\backslash z'$ is M_1. Since x_0 is in a triangle of M, the matroid M' is not 3-connected. Thus, by Lemma 4.2 again, either: (i) $M'\backslash z'$ is 3-connected; or (ii) $M'\backslash z'$ has a unique 2-circuit $\{z, h\}$ containing z, and $M'\backslash z', z$ is 3-connected. However, $M'\backslash z' = M_1$, so, in the first case, M_1 is 3-connected, and, in the second case, $M_1\backslash z$ is 3-connected. Moreover, in the second case, since z and h are parallel in $M'\backslash z'$, it follows by (5) that

$$M = P_{\Delta_2}(M(\mathcal{W}_3), M'\backslash z'\backslash z)$$

where $\Delta_2 = \{y_0, y_2, h\}$ and $M(\mathcal{W}_3)$ is labelled as in Figure 4(*c*). Now, $M'\backslash z'\backslash z = M/x_0\backslash y_1\backslash x_1$, and x_0 is a coloop of $M\backslash x_1, y_1$, so $M'\backslash z'\backslash z = M\backslash x_0, x_1, y_1$. This completes the proof of the theorem in the case $n = 1$.

Now assume that the theorem holds for $n < k$ and let $n = k \geq 2$. By the induction assumption,

$$M = P_{\Delta}(M(\mathcal{W}_{k+1}), M')\backslash z_1 \tag{6}$$

where $\Delta = \{y_0, y_k, z_1\}$; $\mathcal{W}_{k+1}$ is labelled as in Figure 5; and M' is $M/x_0, x_1, \ldots, x_{k-2}\backslash y_1, y_2, \ldots, y_{k-1}$ with x_{k-1} relabelled as z_1. Moreover, either:

(i) M' is 3-connected; or
(ii) $M'\backslash z_1$ is 3-connected.

However, $\{x_{k-1}, y_k, x_k\}$ is a triad of M, and hence $\{y_k, x_k\}$ contains a cocircuit of $M'\backslash z_1$. Since the last matroid has at least four elements, we conclude that (ii) does not hold.

We may now assume that M' is 3-connected. Then, by repeated application of circuit elimination and orthogonality, we deduce that $\{y_0, x_0, x_1, \ldots, x_{k-1}, y_k\}$ is a circuit of M. It follows that $\{y_0, z_1, y_k\}, \{z_1, y_k, x_k\}, \{y_k, x_k, y_{k+1}\}$ is a chain in M' in which $\{y_0, z_1, y_k\}$ is a triangle. Thus, by the induction assumption,

$$M' = P_{\Delta_1}(M(\mathcal{W}_3), M_1)\backslash z \tag{7}$$

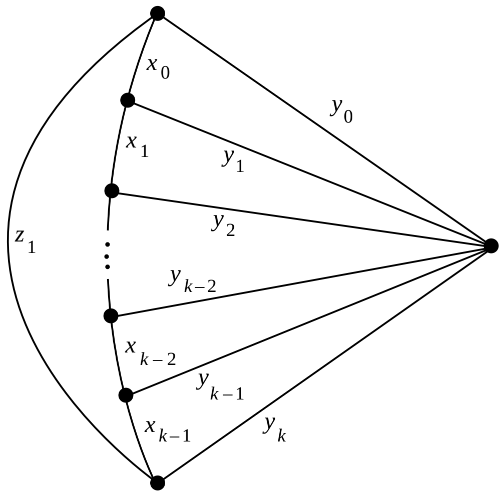

FIGURE 5. A labelled $\mathcal{W}_{k+1}$.

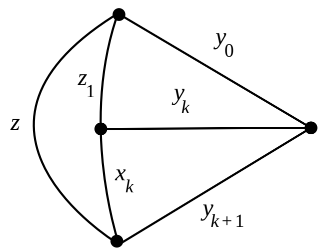

FIGURE 6. A labelled $\mathcal{W}_{k+1}$.

where $\Delta_1 = \{y_0, y_{k+1}, z\}$; $\mathcal{W}_3$ is labelled as in Figure 6; and M_1 is $M'/z_1\backslash y_k$ with x_k relabelled as z. Since M' is $M/x_0, x_1, \ldots, x_{k-2}\backslash y_1, y_2, \ldots, y_{k-1}$ with x_{k-1} relabelled as z_1, the matroid M_1 is

$$M/x_0, x_1, \ldots, x_{k-2}, x_{k-1}\backslash y_1, y_2, \ldots, y_{k-1}, y_k$$

with x_k relabelled as z. By (6) and (7),

$$\begin{aligned} M &= P_\Delta(M(\mathcal{W}_{k+1}), P_{\Delta_1}(M(\mathcal{W}_3), M_1)\backslash z)\backslash z_1 \\ &= P_\Delta(M(\mathcal{W}_{k+1}), P_{\Delta_1}(M(\mathcal{W}_3), M_1))\backslash\{z, z_1\}, \end{aligned} \tag{8}$$

where $\mathcal{W}_{k+1}$ and $\mathcal{W}_3$ are labelled as in Figures 5 and 6, respectively. Thus, by Lemma 4.1,

$$\begin{aligned} M &= P_{\Delta_1}(P_\Delta(M(\mathcal{W}_{k+1}), M(\mathcal{W}_3)), M_1)\backslash\{z, z_1\} \\ &= P_{\Delta_1}(P_\Delta(M(\mathcal{W}_{k+1}), M(\mathcal{W}_3))\backslash z_1, M_1)\backslash z. \end{aligned}$$

In addition, $P_\Delta(M(\mathcal{W}_{k+1}), M(\mathcal{W}_3))\backslash z_1$ is $M(\mathcal{W}_{k+2})$ where $\mathcal{W}_{k+2}$ is labelled as in Figure 2 with $n = k$. Hence $M = P_{\Delta_1}(M(\mathcal{W}_{k+2}), M_1)\backslash z$ where $\Delta_1 = \{y_0, y_{k+1}, z\}$ and M_1 is $M/x_0, x_1, \ldots, x_{k-1}\backslash y_1, y_2, \ldots, y_k$ with x_k relabelled as z. We conclude that, when $n = k$, M is as asserted in the theorem.

We now need to check that either (i) or (ii) holds. Hence we may assume that M_1 is not 3-connected. We noted above that M' is 3-connected, that $M' = P_{\Delta_1}(M(\mathcal{W}_3), M_1)\backslash z$, and that M_1 is $M'/z_1\backslash y_k$ with x_k relabelled as z. By the induction assumption applied to the chain $\{y_0, z_1, y_k\}, \{z_1, y_k, x_k\}, \{y_k, x_k, y_{k+1}\}$ of M', we deduce that z is in a unique 2-circuit $\{z, h\}$ of M_1, and $M_1\backslash z$ is 3-connected. Moreover,

$$M' = P_{\Delta_2}(M(\mathcal{W}_3), M_2)$$

where $\Delta_2 = \{y_0, y_{k+1}, h\}$; $\mathcal{W}_3$ is labelled as in Figure 6 with z relabelled as h; and M_2 is $M'\backslash z_1, x_k, y_k$. Thus, by (6),

$$M = P_\Delta(M(\mathcal{W}_{k+1}), P_{\Delta_2}(M(\mathcal{W}_3), M_2))\backslash z_1 \tag{9}$$

where $\Delta = \{y_0, y_k, z_1\}$; $\Delta_2 = \{y_0, y_{k+1}, h\}$; $\mathcal{W}_{k+1}$ is labelled as in Figure 5; $\mathcal{W}_3$ is labelled as in Figure 6 with z relabelled as h; and M_2 is $M'\backslash z_1, x_k, y_k$. Since M' is $M/x_0, x_1, \ldots, x_{k-2}\backslash y_1, y_2, \ldots, y_{k-1}$ with x_{k-1} relabelled as z_1, we deduce that

$$M_2 = (M/x_0, x_1, \ldots, x_{k-2}\backslash x_{k-1}, x_k)\backslash y_1, y_2, \ldots, y_k.$$

However, in $M\backslash y_1, y_2, \ldots, y_k$, each of $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}$ is a union of cocircuits. Hence $M\backslash y_1, y_2, \ldots, y_k\backslash x_{k-1}, x_k$ has x_{k-2} as a coloop, and so has x_{k-3} as a coloop. Continuing in this way, we deduce that all of $x_{k-2}, x_{k-3}, \ldots, x_0$ are coloops. Thus

$$M_2 = M\backslash x_0, x_1, \ldots, x_k, y_1, y_2, \ldots, y_k.$$

By Lemma 4.1 and (9), we have that

$$\begin{aligned} M &= P_{\Delta_2}(P_\Delta(M(\mathcal{W}_{k+1}), M(\mathcal{W}_3)), M_2)\backslash z_1 \\ &= P_{\Delta_2}(P_\Delta(M(\mathcal{W}_{k+1}), M(\mathcal{W}_3))\backslash z_1, M_2) \\ &= P_{\Delta_2}(M(\mathcal{W}_{k+2}), M_2) \end{aligned}$$

where $\mathcal{W}_{k+2}$ is labelled as in Figure 2 with $n = k$ and with z relabelled as h; and $M_2 = M\backslash x_0, x_1, \ldots, x_k, y_1, y_2, \ldots, y_k$. We conclude, by induction, that the theorem holds for all positive integers n. □

COROLLARY 4.3. *Let M be a* 3*-connected matroid that is not a wheel or a whirl, and let e be an essential element that is in more than one type-*1 *fan with five or more elements. Then e is in a unique triad T^* of M. Moreover, M has a triangle Δ such that $M|(T^* \cup \Delta) \cong M(K_4)$ and*

$$M = P_\Delta(M|(T^* \cup \Delta), M\backslash T^*).$$

In addition, $M\backslash T^*$ is 3*-connected.*

PROOF. By Theorem 1.6, e is in exactly three fans each of which is of type-1 having five elements. Moreover, these three fans contain a total of six elements, and the restriction of M to these six elements is isomorphic to $M(K_4)$. Let the elements of this $M(K_4)$ be labelled as shown in Figure 7 where $\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}$ is one of the type-1 fans containing e. It follows without difficulty from Theorem 1.8 that

$$M = P_\Delta(M(K_4), M\backslash x_0, y_1, x_1)$$

where $M(K_4)$ is labelled as indicated, $\Delta = \{y_0, x_2, y_2\}$, and $M\backslash x_0, y_1, x_1$ is 3-connected. Moreover, by applying Lemma 3.4 to one of the three type-1 fans containing e, it follows that $\{x_0, y_1, x_1\}$ is the unique triad of M containing e. □

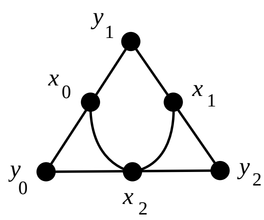

FIGURE 7. A labelled $M(K_4)$.

We shall now describe how essential elements behave when a wheel is broken off as in Theorem 1.8. In that theorem, the resulting 3-connected matroid is M_1 or M_2. We shall first consider the latter.

PROPOSITION 4.4. *Let $M = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2)$ where n is a positive integer and Δ_2 is a triangle. Suppose that M_2 is 3-connected having at least four elements, and let e be an element of M_2. Then:*

(a) *M_2/e is 3-connected if and only if either M/e is 3-connected or $M_2 \cong U_{2,4}$;*
(b) *$M\backslash e$ is 3-connected if and only if either $M_2\backslash e$ is 3-connected or $e \in \Delta_2$.*

Hence if e *is essential in M, then* e *is essential in M_2; and if e is non-essential in M, then* e *is non-essential in M_2 or* e $\in \Delta_2$.

PROOF. By Corollary 2.7, M is 3-connected. We shall first prove (a) by breaking the argument into the two cases: (i) $e \notin \text{cl}_M(\Delta_2)$; and (ii) $e \in \text{cl}_M(\Delta_2)$. In case (i), $M/e = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2/e)$ and (a) follows easily by Corollary 2.7. In case (ii), M/e is non-simple having at least four elements so M/e is not 3-connected. If M_2/e is 3-connected, then, since this matroid is non-simple, but M_2 has a triangle, it follows that $M_2 \cong U_{2,4}$. Conversely, if $M_2 \cong U_{2,4}$, then M_2/e is 3-connected. Hence (a) holds in case (ii), so (a) is proved.

We break the proof of (b) into the two cases: (i) $e \notin \Delta_2$; and (ii) $e \in \Delta_2$. In case (i), $M\backslash e = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2\backslash e)$ and (b) follows easily by Corollary 2.7. In case (ii), $M\backslash e = P_{\Delta_2}(M(\mathcal{W}_{n+2}), M_2)\backslash e$, so, by Corollary 2.8, $M\backslash e$ is 3-connected. Hence (b) holds in case (ii), so (b) is proved.

On combining (a) and (b), we deduce that if e is essential in M, then either e is essential in M_2, or $M_2 \cong U_{2,4}$. However, the latter cannot occur otherwise $M_2\backslash e$ is 3-connected and so, by (b), $M\backslash e$ is 3-connected; a contradiction.

If e is non-essential in M, then, by (a) and (b) again, e is non-essential in M_2 or $e \in \Delta_2$. □

The next result shows that, when the matroid M_1 in Theorem 1.8 is 3-connected, every essential element of M that is in M_1 is also essential in M_1. However, the behaviour of the non-essential elements of M is less straightforward.

PROPOSITION 4.5. *Let M be a 3-connected matroid that is not a whirl. Suppose that there is a positive integer n such that*

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z$$

where $\Delta_1 = \{y_0, y_{n+1}, z\}$ and $\mathcal{W}_{n+2}$ is labelled as in Figure 2. Let M_1 be 3-connected and e be an element of $E(M_1) - z$. Then:

(a) *M/e is 3-connected if and only if either M_1/e is 3-connected; or $e \notin \Delta_1$, there is a unique triangle of M_1 containing $\{e, z\}$, and $M_1/e\backslash z$ is 3-connected;*
(b) *$M\backslash e$ is 3-connected if and only if either $M_1\backslash e$ is 3-connected; or $e \in \Delta_1$ and e is not in a triad of M.*

Hence if e is essential in M, then e is essential in M_1. However, if e is non-essential in M, then either e is non-essential in M_1; or $e \in \Delta_1$ and e is not in a triad of M; or $e \notin \Delta_1$, there is a unique triangle of M_1 containing $\{e, z\}$, and $M_1/e\backslash z$ is 3-connected.

The proof of this proposition will use the following lemma, the straightforward proof of which is omitted.

LEMMA 4.6. *Let k be an integer exceeding two and suppose that $E(M(\mathcal{W}_k))$ and $E(U_{2,4})$ meet in a set Δ that is a triangle of both matroids. Let z be an element of Δ that is a rim element of $\mathcal{W}_k$. Then*

$$P_\Delta(M(\mathcal{W}_k), U_{2,4})\backslash z \cong \mathcal{W}^k.$$

PROOF OF PROPOSITION 4.5. Certainly $|E(M_1)| \geq 3$. If $|E(M_1)| = 3$, then M is a single-element deletion of $M(\mathcal{W}_{n+2})$ and so is not 3-connected. Thus $|E(M_1)| \geq 4$. If $|E(M_1)| = 4$, then, as M_1 is 3-connected, $M_1 \cong U_{2,4}$ and so, by Lemma 4.6, M is a whirl; a contradiction. Hence we may assume that $|E(M_1)| \geq 5$. Therefore, as M_1 has a triangle, $r^*(M_1) \geq 3$ and so $r^*(M) \geq 4$.

We shall break the proof of (a) into the two cases: (i) $e \in \mathrm{cl}_M(\Delta_1)$; and (ii) $e \notin \mathrm{cl}_M(\Delta_1)$. In case (i), neither M/e nor M_1/e is 3-connected since each has a 2-circuit and at least four elements. Moreover, if $e \notin \Delta_1$, then either $r(M_1) = 2$, in which case $\{e, z\}$ is not in a unique triangle of M_1, or $r(M_1) > 2$, in which case $M_1/e\backslash z$ has rank at least two and has a 2-circuit, and so is not 3-connected. We conclude that (a) holds in case (i). In case (ii), we certainly have that $e \notin \Delta_1$. By Corollary 2.8, M/e is 3-connected if and only if either M_1/e is 3-connected, or M_1/e has a unique 2-circuit, which contains z, and $M_1/e\backslash z$ is 3-connected. Since M_1 is 3-connected, (a) follows easily in case (ii).

The proof of (b) will be broken into the two cases: (i) $e \notin \Delta_1$; and (ii) $e \in \Delta_1$. In case (i), since M_1 is 3-connected having at least five elements, Corollary 2.8 implies that $M\backslash e$ is 3-connected if and only if $M_1\backslash e$ is 3-connected. Thus (b) holds in case (i). Now assume that (ii) holds. By symmetry, we may suppose that $e = y_0$. If $M\backslash e$ is 3-connected, then e is not in a triad of M. Thus the forward implication of (b) holds in case (ii). To prove the reverse implication, suppose that $M\backslash e$ is not 3-connected, letting $\{X, Y\}$ be a 2-separation of it. Then, since $|E(M_1) - \{z, e\}| \geq 3$, we may assume that $|X \cap (E(M_1) - \{z, e\})| \geq 2$. Therefore, if $r(M_1) = 2$, then $\{X \cup e, Y\}$ is a 2-separation of M; a contradiction. Hence we may assume that $r(M_1) > 2$.Thus the simplification of $P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z/e$ is a parallel connection of two matroids of rank at least two and so is not 3-connected. Thus, by Lemma 2.1, the cosimplification of $P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z\backslash e$, which equals $\widetilde{M\backslash e}$, is 3-connected. Since $M\backslash e$ is not 3-connected, it follows that M has a triad T^* containing e. It remains to show that $M_1\backslash e$ is not 3-connected. As $e = y_0$ and $\{x_0, y_0, y_1\}$ is a triangle of M, orthogonality implies that T^* contains x_0 or y_1. Evidently T^* or $T^* \cup z$ is a cocircuit of $P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)$. Since M_1 is a restriction of the last matroid, it follows that $(T^* \cup z) \cap E(M_1)$ contains a cocircuit of M_1 containing e. However, $|(T^* \cup z) \cap E(M_1)| \leq 3$ yet M_1 is 3-connected having at least five elements, so M_1 has a triad containing e. Therefore $M_1\backslash e$ is not 3-connected. This completes the proof of (b) in case (ii).

The conclusions concerning essential and non-essential elements follow immediately on combining (a) and (b). □

Propositions 4.4 and 4.5 will be used in [24] to investigate those 3-connected matroids in which the set of non-essential elements is small. If we denote by $\nu(Q)$ the set of non-essential elements of a 3-connected matroid Q, then, by Proposition 4.4,

$$\nu(M_2) \subseteq E(M_2) \cap \nu(M) \subseteq \nu(M_2) \cup \Delta_2.$$

In Proposition 4.5, the situation is less straightforward. The set of non-essential elements of M_1 may include z, which is not in $E(M)$, but otherwise this set is a subset of $\nu(M)$. Moreover, if we suppose that the chain $\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \dots, \{y_n, x_n, y_{n+1}\}$ in Theorem 1.8 is a fan, then y_0 and y_{n+1} are non-essential elements of M. The element z is on the line of M_1 through y_0 and y_{n+1}. Thus, although $\nu(M_1)$ need not be a subset of $E(M_1) \cap \nu(M)$, we do have that

$$\nu(M_1) \subseteq \mathrm{cl}_{M_1}(E(M_1) \cap \nu(M)).$$

This fact will be very useful in [24].

Theorem 1.6 indicates how one can break off a wheel from a 3-connected matroid having a chain of odd length exceeding two. In fact, that theorem explicitly describes this break off when the chain has a triangle as its first link and hence has a triangle as its last link. If the chain has triads as its first and last links, then one can reduce to the case described in Theorem 1.6 by taking duals. For chains of even length, the situation is slightly different. The result in this case is stated in the next theorem, a generalization of Lemma 4.2. The reader will observe that, in this case, it is slightly more difficult to recover a 3-connected matroid in what is left after the break off.

THEOREM 4.7. *Let M be a 3-connected matroid which is not a wheel or a whirl. Suppose that, for some non-negative integer n, the sequence*

$$\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \dots, \{y_n, x_n, y_{n+1}\}, \{x_n, y_{n+1}, x_{n+1}\}$$

is a chain in M in which $\{y_0, x_0, y_1\}$ is a triangle. Then

$$M = P_\Delta(M(\mathcal{W}_{n+3}), M_3)\backslash\{z', y'_{n+1}\}$$

where $\Delta = \{y_0, z', y'_{n+1}\}$; $\mathcal{W}_{n+3}$ is labelled as in Figure 8; and M_3 is obtained from the matroid $M/x_0, x_1, \dots, x_{n-1}\backslash y_1, y_2, \dots, y_n/x_{n+1}$ by relabelling x_n and y_{n+1} as z' and y'_{n+1}. Moreover:

(i) *M_3 is 3-connected; or*
(ii) *z' is in a unique 2-circuit of M_3, and $M_3\backslash z'$ is 3-connected; or*
(iii) *y'_{n+1} is in a unique 2-circuit of M_3, and $M_3\backslash y'_{n+1}$ is 3-connected; or*
(iv) *each of z' and y'_{n+1} is in a unique 2-circuit of M_3, and $M_3\backslash z', y'_{n+1}$ is 3-connected.*

The proof of this will use another variant of Lemma 2.3.

LEMMA 4.8. *For some non-negative integer n, suppose that $\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \dots, \{y_n, x_n, y_{n+1}\}, \{x_n, y_{n+1}, x_{n+1}\}$ is a chain in a simple matroid M. If $E(M) = \{x_0, y_0, x_1, y_1, \dots, x_{n+1}, y_{n+1}\}$, then M is isomorphic to $M(\mathcal{W}_{n+2})$ or $\mathcal{W}^{n+2}$.*

PROOF. Since the chain has an even number of links, we may assume that the first link is a triangle. Thus $\{x_i, y_{i+1}, x_{i+1}\}$ is a triad of M for all i in $\{0, 1, \dots, n\}$. Since each of $y_1, y_2, \dots, y_{n+1}$ is in exactly one of these triads and M is simple, it follows that $\{y_0, y_1, \dots, y_{n+1}\}$ is independent in M. As this set clearly spans $E(M) - \{x_{n+1}\}$ and x_{n+1} is not a coloop

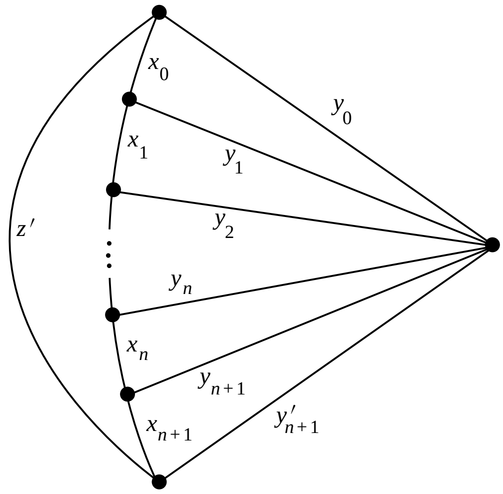

FIGURE 8. A labelled $\mathcal{W}_{n+3}$.

of M, we deduce that $\{y_0, y_1, \ldots, y_{n+1}\}$ is a basis of M. Consider the fundamental circuit of x_{n+1} with respect to this basis. By orthogonality and the simplicity of M, it follows that this circuit is $\{y_{n+1}, x_{n+1}, y_0\}$.

Now $\{y_1, y_2, \ldots, y_{n+1}\}$ spans a hyperplane of M that also contains all of $x_1, x_2, \ldots, x_n$ but avoids y_0 and hence avoids x_0 and x_{n+1}. Thus $\{y_0, x_0, x_{n+1}\}$ is a triad of M and the lemma follows by Lemma 2.3. □

PROOF OF THEOREM 4.7. If $n = 0$, the theorem is just a restatement of Lemma 4.2. Now suppose that $n > 0$. By the last lemma, $|E(M)| \geq 2n + 5$. Moreover, Theorem 1.8 implies that

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z$$

where $\Delta_1 = \{y_0, y_{n+1}, z\}$; $\mathcal{W}_{n+2}$ is labelled as in Figure 2; and M_1 is obtained from the matroid $M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots, y_n$ by relabelling x_n as z. Moreover, either: (i) M_1 is 3-connected, or (ii) $M_1\backslash z$ is 3-connected. In the latter case, $M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots,$ y_n, x_n is 3-connected. However, M has $\{x_n, y_{n+1}, x_{n+1}\}$ as a cocircuit, so the last matroid has a cocircuit contained in $\{y_{n+1}, x_{n+1}\}$. Since this matroid has at least four elements, it cannot be 3-connected. Therefore (ii) cannot occur. We conclude that M_1 is 3-connected.

Now $\{y_0, y_{n+1}, z\}$ is a circuit of M_1 and $\{y_{n+1}, z, x_{n+1}\}$ is a cocircuit of M_1. Thus, by Lemma 4.2 $M_1 = P_\Delta(M(K_4), M_2)\backslash\{z', y'_{n+1}\}$ where $\Delta = \{y_0, z', y'_{n+1}\}$; $M(K_4)$ is labelled as in Figure 9; and M_2 is obtained from M_1/x_{n+1} by relabelling z and y_{n+1} as z' and y'_{n+1}, respectively.

By Lemma 4.1

$$P_{\Delta_1}(M(\mathcal{W}_{n+2}), P_\Delta(M(K_4), M_2)) = P_\Delta(P_{\Delta_1}(M(\mathcal{W}_{n+2}), M(K_4)), M_2), \tag{10}$$

where $M(\mathcal{W}_{n+2})$, $M(K_4)$, and M_2 are labelled as above. Since

$$M = P_{\Delta_1}(M(\mathcal{W}_{n+2}), M_1)\backslash z \qquad \text{and} \qquad M_1 = P_\Delta(M(K_4), M_2)\backslash\{z', y'_{n+1}\},$$

it follows that

$$\begin{aligned} M &= P_{\Delta_1}(M(\mathcal{W}_{n+2}), P_\Delta(M(K_4), M_2)\backslash\{z', y'_{n+1}\})\backslash z \\ &= P_{\Delta_1}(M(\mathcal{W}_{n+2}), P_\Delta(M(K_4), M_2))\backslash\{z', y'_{n+1}, z\}. \end{aligned}$$

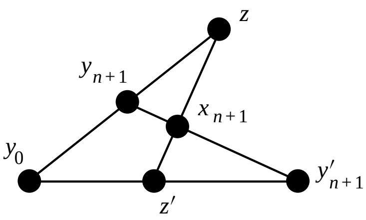

FIGURE 9. A labelled $M(K_4)$.

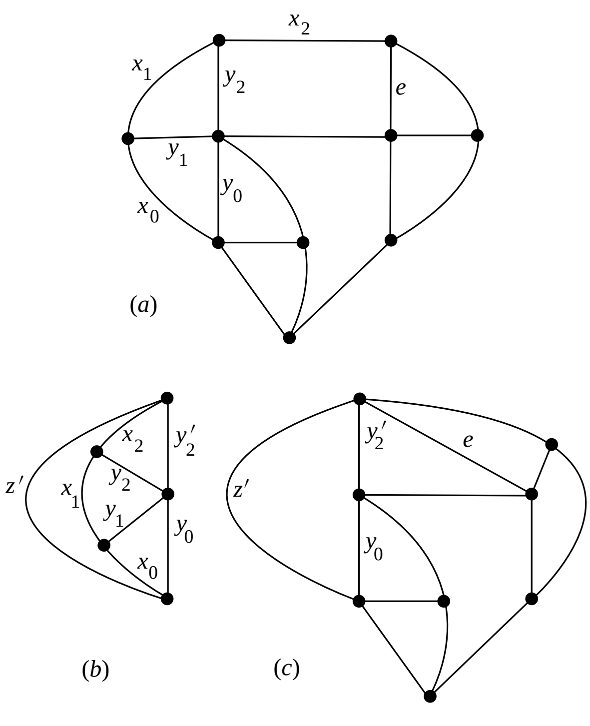

FIGURE 10. An essential element becomes non-essential.

Thus, by (10),

$$M = P_\Delta(P_{\Delta_1}(M(\mathcal{W}_{n+2}), M(K_4)), M_2)\backslash\{z', y'_{n+1}, z\}$$
$$= P_\Delta(P_{\Delta_1}(M(\mathcal{W}_{n+2}), M(K_4))\backslash z, M_2)\backslash\{z', y'_{n+1}\}.$$

It is not difficult to see that

$$P_{\Delta_1}(M(\mathcal{W}_{n+2}), M(K_4))\backslash z = M(\mathcal{W}_{n+3}),$$

where $\mathcal{W}_{n+3}$ is labelled as in Figure 8. Therefore

$$M = P_\Delta(M(\mathcal{W}_{n+3}), M_2)\backslash\{z', y'_{n+1}\}$$

where M_2 is obtained from M_1/x_{n+1} by relabelling z and y_{n+1} as z' and y'_{n+1}, respectively. However, M_1 is obtained from $M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots, y_n$ by relabelling x_n as z. Thus M_2 is obtained from

$$M/x_0, x_1, \ldots, x_{n-1}\backslash y_1, y_2, \ldots, y_n/x_{n+1}$$

by relabelling x_n and y_{n+1} as z' and y'_{n+1}, respectively. Hence $M_2 = M_3$ and the first part of the theorem is proved.

The fact that one of (i)–(iv) holds follows immediately by applying the second part of Lemma 4.2 to the 3-connected matroid M_1. □

Propositions 4.4 and 4.5 tell us that, in breaking off a wheel as in Theorem 1.8, an element that is essential in M remains essential in the resulting 3-connected matroid, M_1 or M_2. However, the corresponding result need not hold when one breaks off a wheel as in Theorem 4.7. For example, let M be the cycle matroid of the graph shown in Figure 10(*a*). Then $\{y_0, x_0, y_1\}, \{x_0, y_1, x_1\}, \{y_1, x_1, y_2\}, \{x_1, y_2, x_2\}$ is a chain in this matroid. By Theorem 4.7, $M = P_\Delta(M(\mathcal{W}_4), M_3)\backslash\{z', y'_2\}$ where $\Delta = \{y_0, z', y'_2\}$; $\mathcal{W}_4$ is labelled as in Figure 10(*b*); and M_3 is $M/x_0\backslash y_1/x_2$ with x_1 and y_2 relabelled as z' and y'_2 (see Figure 10(*c*)). The element e, which is essential in M, is non-essential in M_3. Moreover, e is not even in the flat of M_3 that is spanned by those non-essential elements of M that are in M_3. In [24] where we shall be examining the 3-connected matroids with a small number of non-essential elements, Theorem 1.8 will be of more use than Theorem 4.7 because the former enables us to better keep track of essential elements.

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References

1. S. Akkari, A minimal 3-connectedness result for matroids, *Discrete Math.*, **103** (1992), 221–232.
2. S. Akkari and J. Oxley, Some extremal connectivity results for matroids, *J. Comb. Theory, Ser. B*, **52** (1991), 301–320.
3. S. Akkari and J. Oxley, Some local extremal connectivity results for matroids, *Comb. Probab. Comput.*, **2** (1993), 367–384.
4. R. E. Bixby, A simple theorem on 3-connectivity, *Linear Algebra Appl.*, **45** (1982), 123–126.

5. R. E. Bixby and C. R. Coullard, On chains of 3-connected matroids, *Discrete Appl. Math.*, **15** (1986), 155–166.
6. R. E. Bixby and C. R. Coullard, Finding a smallest 3-connected minor maintaining a fixed minor and a fixed element, *Combinatorica*, **7** (1987), 231–242.
7. T. H. Brylawski, A combinatorial model for series-parallel networks, *Trans. Am. Math. Soc.*, **154** (1971), 1–22.
8. T. H. Brylawski, Modular constructions for combinatorial geometries, *Trans. Am. Math. Soc.*, **203** (1975), 1–44.
9. C. R. Coullard, L. L. Gardner and D. K. Wagner, Decomposition of 3-connected graphs, *Combinatorica*, **13** (1993), 7–30.
10. C. R. Coullard and J. G. Oxley, Extensions of Tutte's wheels-and-whirls theorem, *J. Comb. Theory, Ser. B*, **56** (1992), 130–140.
11. W. H. Cunningham, On matroid connectivity, *J. Comb. Theory, Ser. B*, **30** (1981), 94–99.
12. B. S. Gubser, Planar graphs with no 6-wheel minor, *Discrete Math.*, **120** (1993), 59–73.
13. M. Lemos, On 3-connected matroids, *Discrete Math.*, **73** (1989), 273–283.
14. S. Negami, A characterization of 3-connected graphs containing a given graph, *J. Comb. Theory, Ser. B*, **32** (1982), 69–74.
15. J. G. Oxley, On matroid connectivity, *Q. J. Math. Oxford Ser. (2)*, **32** (1981), 193–208.
16. J. G. Oxley, On 3-connected matroids, *Can. J. Math.*, **33** (1981), 20–27.
17. J. G. Oxley, On nonbinary 3-connected matroids, *Trans. Am. Math. Soc.*, **300** (1987), 663–679.
18. J. G. Oxley, The binary matroids with no 4-wheel minor, *Trans. Am. Math. Soc.*, **301** (1987), 63–75.
19. J. G. Oxley, A characterization of the ternary matroids with no $M(K_4)$-minor, *J. Comb. Theory, Ser. B*, **42** (1987), 212–249.
20. J. G. Oxley, The regular matroids with no 5-wheel minor, *J. Comb. Theory, Ser. B*, **46** (1989), 292–305.
21. J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.
22. J. G. Oxley and T. J. Reid, The smallest rounded sets of binary matroids, *Europ. J. Combinatorics*, **11** (1990), 47–56.
23. J. G. Oxley and H. Wu, A note on matroid connectivity, *Discrete Math.*, **146** (1995), 321–324.
24. J. G. Oxley and H. Wu, Matroids and graphs with few non-essential elements, *Graphs Combin.*, to appear.
25. N. Robertson and P. D. Seymour, Generalizing Kuratowski's theorem, *Congressus Numerantium*, **45** (1984), 129–138.
26. P. D. Seymour, Decomposition of regular matroids, *J. Comb. Theory, Ser. B*, **28** (1980), 305–359.
27. P. D. Seymour, Matroid minors, in: *Handbook of Combinatorics*, R. Graham, M. Grötschel and L. Lovász (eds), Elsevier, Amsterdam, 1995, pp. 527–550.
28. J. J.-M. Tan, Matroid 3-connectivity, Ph. D. Thesis, Carleton University, 1981.
29. K. Truemper, Partial matroid representations, *Europ. J. Combinatorics*, **5** (1984), 377–394.
30. K. Truemper, *Matroid Decomposition*, Academic Press, New York, 1992.
31. W. T. Tutte, A theory of 3-connected graphs, *Nederl. Akad. Wetensch. Proc. Ser. A*, **64** (1961), 441–455.
32. W. T. Tutte, Connectivity in matroids, *Can. J. Math.*, **18** (1966), 1301–1324.
33. H. Wu, On contractible and vertically contractible elements in 3-connected matroids and graphs, *Discrete Math.*, **179** (1998), 185–203.

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