Super-minimally 3-connected matroids

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Abstract

A super-minimally k-connected matroid is a k-connected matroid having no proper k-connected restriction of size at least $2k-2$. This extends the corresponding concept for graphs. For $k=2$ and $k=3$, we determine the maximum size of a super-minimally k-connected rank-r matroid and characterize, in each case, those matroids attaining the extremal bound. These results parallel Murty's results for minimally 2-connected matroids and Oxley's results for minimally 3-connected matroids.

1 Introduction

The study of connectivity plays a central role in both graph theory and matroid theory, particularly in understanding the structure of objects that are minimally sufficient to maintain a given level of connectivity. Classical results of Dirac [3, (7)] and Halin [6, (7.6)] established sharp bounds on the size of minimally k-connected graphs for $k \in \{2,3\}$, and these ideas were later extended to matroids by Murty [10, Theorems 3.2 and 3.4] and Oxley [11, (4.7)], respectively.

Recently, Ge [4] introduced the class of *super-minimally k-connected graphs*, namely k-connected graphs that contain no proper k-connected subgraphs, and determined tight bounds on their density for $k \in \{2,3\}$. This raises the natural question as to whether analogous extremal phenomena occur for matroids under a similar strengthening of minimal connectivity.

Motivated by this question, we introduce the class of super-minimally k-connected matroids. Since a matroid with fewer than $2k-2$ elements cannot have a $(k-1)$-separation, we define a k-connected matroid M to be *super-minimally k-connected* if it has no proper k-connected restriction with at least $2k-2$ elements. This notion captures matroids whose connectivity is highly fragile in the sense that deleting every subset of the ground set destroys k-connectivity. When $k=2$, we make the following elementary observation.

Proposition 1.1. *A matroid M is super-minimally 2-connected if and only if M is isomorphic to $U_{1,1}$, or $U_{r,r+1}$ for some integer $r \geq 0$. In particular, $|E(M)| \leq r(M)+1$.*

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Our main result is the following analogue of Oxley's theorem [11, (4.7)] on the density of minimally 3-connected matroids. It extends Ge's bound [4, Theorem 1.10] for super-minimally 3-connected graphs to matroids.

Theorem 1.2. *Let M be a super-minimally 3-connected matroid with at least four elements. Then*

$$|E(M)| \leq 2r(M).$$

Moreover, equality holds if and only if $M \cong U_{2,4}$, or $M \cong M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 3$.

In the final section, we derive results on triads in super-minimally 3-connected matroids from Lemos's results on minimally 3-connected matroids [7, 8].

2 Preliminaries

2.1 Some connectivity results

We assume familiarity with the basic notions of matroid theory as found, for example, in [12]. In particular, Chapter 8 of [12] gives a comprehensive treatment of matroid connectivity. Let M be a matroid. We use λ_M to denote the connectivity function of M and will often abbreviate it as λ. We omit the elementary proof of the following.

Proposition 2.1. *Suppose $\{X, Y\}$ is a k-separation of a matroid M and e is an element of M in $\operatorname{cl}(X) \cup \operatorname{cl}^*(X)$. Then $\{X \cup e, Y - e\}$ is a k-separation of M if and only if $|Y - e| \geq k$.*

The following lemma, due to Tutte [14, (7.2)], has been used extensively in the study of 3-connected matroids and is often referred to as Tutte's Triangle Lemma (see, for example, [12, Lemma 8.7.7]).

Lemma 2.2. *Let M be a 3-connected matroid having at least four elements and suppose that $\{e, f, g\}$ is a triangle of M such that neither $M \setminus e$ nor $M \setminus f$ is 3-connected. Then M has a triad that contains e and exactly one of f and g.*

The next lemma is commonly referred to as Bixby's Lemma [1] (see also [12, Lemma 8.7.3]). Let M be a matroid. We use $\operatorname{si}(M)$ to denote the simple matroid associated with M. The *cosimplification*, $\operatorname{co}(M)$, of M is $(\operatorname{si}(M^*))^*$.

Lemma 2.3. *Let e be an element of a 3-connected matroid M. Then $\operatorname{si}(M/e)$ is 3-connected or $\operatorname{co}(M \setminus e)$ is 3-connected.*

One of the main tools in the study of 3-connected matroids is Tutte's Wheels-and-Whirls Theorem. When M is a 3-connected matroid, an element e is *essential* if neither $M \backslash e$ nor M/e is 3-connected. The next result follows immediately from an extension of Tutte's Wheels-and-Whirls Theorem that is due to Oxley and Wu [13, Theorem 1.6] (see also [12, Lemma 8.8.6]).

Theorem 2.4. *Suppose M is a 3-connected matroid with at least four elements that is not isomorphic to a whirl or to the cycle matroid of a wheel. Then M has at least two nonessential elements.*

The following lemmas will be useful in our arguments.

Lemma 2.5. *[11, (2.6)] Suppose that x and y are distinct elements of an n-connected matroid M where $n \geq 2$ and $|E(M)| \geq 2(n-1)$. Assume that $M \setminus x/y$ is n-connected but that $M \setminus x$ is not n-connected. Then M has a cocircuit of size n containing x and y.*

Lemma 2.6. *[11, (5.2)] Let M be a rank-r minimally 3-connected matroid with precisely $2r$ elements. If $3 \leq r \leq 5$, then M is isomorphic to either $M(\mathcal{W}_r)$ or $\mathcal{W}^r$.*

2.2 Small 3-connected matroids

Recall that a 3-connected matroid M is super-minimally 3-connected if M does not have a proper 3-connected restriction of size at least four. Therefore, if the size of M is at most four, then M is super-minimally 3-connected if and only if M is 3-connected. Table 1 lists all 3-connected matroids on at most four elements.

Number n of elements	**3-connected n-element matroids**
1	$U_{0,1}$, $U_{1,1}$
2	$U_{1,2}$
3	$U_{1,3}$, $U_{2,3}$
4	$U_{2,4}$

Table 1: All 3-connected matroids on at most four elements.

Although every super-minimally 3-connected graph is minimally 3-connected [4, Lemma 1.1], a super-minimally 3-connected matroid is not necessarily minimally 3-connected. For instance, the uniform matroid $U_{1,3}$ is super-minimally 3-connected but not minimally 3-connected, since each single-element deletion of it is isomorphic to $U_{1,2}$, which is 3-connected. The next proposition shows that this counterintuitive phenomenon occurs only when $|E(M)|$ is small. We omit its elementary proof.

Proposition 2.7. *Let $k \geq 2$ be an integer and let M be a matroid.*

(i) If $|E(M)| \leq 2k-2$, then M is super-minimally k-connected if and only if M is k-connected.

(ii) If M is super-minimally k-connected and $|E(M)| > 2k-2$, then M is minimally k-connected.

While Table 1 lists all super-minimally 3-connected matroids with at most four elements, in the remainder of the paper, we will focus primarily on those super-minimally 3-connected matroids with at least five elements. The following is an immediate consequence of Proposition 2.7 and a theorem of Oxley [11, (4.7)].

Lemma 2.8. *If M is a super-minimally 3-connected matroid with at least four elements and $r(M) \leq 6$, then*

$$|E(M)| \leq 2r(M).$$

3 Structural Lemmas

In this section, we prove some structural lemmas for super-minimally 3-connected matroids. The next lemma provides a characterization analogous to Bixby's Lemma for 3-connected matroids.

Lemma 3.1. *Suppose M is a super-minimally 3-connected matroid with at least five elements. If $e \in E(M)$, then either $\mathrm{si}(M/e)$ is 3-connected, or $\mathrm{co}(M\backslash e)$ is super-minimally 3-connected.*

Proof. Suppose that $\mathrm{si}(M/e)$ is not 3-connected. Then, for every 2-separation $\{A, B\}$ of M/e, we know $e \in \mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$. First we show the following.

3.1.1. *If $\{A, B\}$ is a 2-separation of M/e and $\{e, f, g\}$ is a triad T^* of M, then T^* meets each of A and B in exactly one element.*

Suppose $\{f, g\} \subseteq A$. Because $e \in \mathrm{cl}_M(B)$, there is a circuit C of M in $B \cup \{e\}$ containing e. However, $|C \cap T^*| = 1$, a contradiction. By symmetry, 3.1.1 follows.

Since $\mathrm{si}(M/e)$ is not 3-connected, M/e has a nonminimal 2-separation $\{X, Y\}$.

3.1.2. *If $\{e, f, g\}$ is a triad of M such that $f \in X$ and $g \in Y$, then $f \notin \mathrm{cl}_{M/e}(Y)$ and $g \notin \mathrm{cl}_{M/e}(X)$.*

To see this, observe that, as $\{X, Y\}$ is a nonminimal 2-separation, $\min\{|X|, |Y|\} \geq 3$. If $f \in \mathrm{cl}_{M/e}(Y)$, then, by Proposition 2.1, $\{X - \{f\}, Y \cup \{f\}\}$ is also a 2-separation of M/e. However, $\{f, g\} \subseteq Y \cup \{f\}$, contradicting 3.1.1. By symmetry, 3.1.2 follows.

3.1.3. *Suppose T_1^* and T_2^* are two distinct triads of M that both contain e. Then $T_1^* \cap T_2^* = \{e\}$.*

Suppose $T_1^* = \{f_1, e, g\}$ and $T_2^* = \{f_2, e, g\}$. Without loss of generality, assume $g \in Y$. Then, by 3.1.1, we have $\{f_1, f_2\} \subseteq X$. By circuit elimination, $\{f_1, f_2, e\}$ is a triad of M. But $|\{f_1, f_2, e\} \cap X| = 2$, contradicting 3.1.1. This establishes 3.1.3.

Since $\mathrm{si}(M/e)$ is not 3-connected, by Bixby's Lemma, it follows that $\mathrm{co}(M\backslash e)$ is 3-connected. Because M is super-minimally 3-connected, the matroid $M\backslash e$

contains at least one series pair. Therefore, M has at least one triad containing e. Suppose M has exactly k triads $T_1^*, T_2^*, \ldots, T_k^*$ containing e. By 3.1.1 and 3.1.3, we know, for each $i \in \{1, 2, \ldots, k\}$, the triad T_i^* has the form $\{e, f_i, g_i\}$ such that $f_i \in X$ and $g_i \in Y$. Moreover, $f_i = f_j$ or $g_i = g_j$ if and only if $i = j$ (see Figure 1 for an illustrative diagram). Thus, we may assume that $\text{co}(M\backslash e) = M\backslash e/\{f_1, f_2, \ldots, f_k\}$.

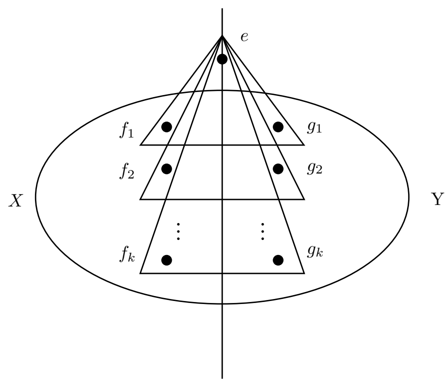

Figure 1: Every triad of M containing e has the form $\{e, f_i, g_i\}$.

Suppose $\text{co}(M\backslash e)$ is not super-minimally 3-connected. Then it has a proper 3-connected restriction N such that $|E(N)| \geq 4$. Next we show that

3.1.4. *$E(N)$ contains g_i for some $i \in \{1, 2, \ldots, k\}$.*

Suppose that $E(N) \cap \{g_1, g_2, \ldots, g_k\} = \emptyset$. We know that

$$N = \text{co}(M\backslash e)|E(N) = M\backslash e/\{f_1, f_2, \ldots, f_k\}\backslash\{g_1, g_2, \ldots, g_k\}|E(N).$$

Since $\{e, f_i, g_i\}$ is a triad for all $i \in \{1, 2, \ldots, k\}$, it follows that

$$N = M\backslash\{e, f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k\}|E(N),$$

which is a restriction of M, contradicting M being super-minimally 3-connected. Therefore 3.1.4 holds.

Let $I = \{i \in \{1, 2, \ldots, k\} : g_i \in E(N)\}$, and let $F_I = \{f_i : i \in I\}$ and $G_I = \{g_i : i \in I\}$. By 3.1.4, without loss of generality, we may assume that $g_1 \in G_I$. Let L be $M|(E(N) \cup F_I \cup \{e\})$ and recall that $\{f_1, f_2, \ldots, f_k\} \subseteq X$ and $\{g_1, g_2, \ldots, g_k\} \subseteq Y$. We show next that

3.1.5. *e is not a coloop of L.*

It is clear that $N = L\backslash e/F_I$. Suppose e is a coloop of L. Then

$$N = L/e/F_I = (M/e/F_I)|E(N),$$

which is a 3-connected minor of M/e with at least four elements. Recall that $\{X, Y\}$ is a 2-separation of M/e. Then $\min\{|E(N) \cap X|, |E(N) \cap Y|\} \leq 1$.

By 3.1.4, $|E(N) \cap Y| \geq |G_I| \geq 1$. Suppose $|E(N) \cap Y| = 1$. Then $E(N) \cap Y = \{g_1\}$ and $g_1 \in \mathrm{cl}_N(E(N) \cap X)$. Because $F_I \subseteq X$, it follows that $g_1 \in \mathrm{cl}_{M/e}(X)$, contradicting 3.1.2.

We now know that $|E(N) \cap X| \leq 1$ and $|E(N) \cap Y| \geq 3$. Because e is a coloop of L and N is 3-connected, f_i and g_i are in series in $L\backslash e$ for all $i \in I$. Therefore,

$$L/e/((F_I \cup \{g_1\}) - \{f_1\})$$

is a 3-connected minor N' of M/e that is isomorphic to N. Because $\{X, Y\}$ is a 2-separation of M/e, we know $\min\{|E(N') \cap X|, |E(N') \cap Y|\} \leq 1$. Clearly, $|E(N') \cap X| \geq 1$ and $|E(N') \cap Y| \geq 2$. Hence $E(N') \cap X = \{f_1\}$. If $F_I = \{f_1\}$, then $f_1 \in \mathrm{cl}_{N'}(E(N') \cap Y)$. Because $(F_I \cup \{g_1\}) - \{f_1\} = \{g_1\}$ and $g_1 \in Y$, it follows that $f_1 \in \mathrm{cl}_{M/e}(Y)$, contradicting 3.1.2. Thus $F_I \neq \{f_1\}$. Without loss of generality, we may assume that $f_2 \in F_I$. Let N'' be the matroid

$$L/e/((F_I \cup \{g_1, g_2\}) - \{f_1, f_2\}).$$

Clearly, $N'' \cong N$. It is straightforward to check that $|E(N'') \cap X| = 2$ and $|E(N'') \cap Y| \geq 2$, a contradiction as N is 3-connected. We conclude that 3.1.5 holds.

In the original matroid M, the set $\{e, f_i, g_i\}$ is a triad for all $i \in \{1, 2, \ldots, k\}$. The restriction L of M contains e, but neither f_i nor g_i if $i \in \{1, 2, \ldots, k\} - I$. If $\{1, 2, \ldots, k\} - I \neq \emptyset$, then the element e will be a coloop in L, contradicting 3.1.5. Therefore, $I = \{1, 2, \ldots, k\}$. We show next that

3.1.6. *L is 3-connected.*

Because $L\backslash e$ has $\{f_i, g_i\}$ as a series pair for all $i \in I$, and $L\backslash e/F_I = N$, which is a 3-connected matroid with at least four elements, it is clear that $L\backslash e$ is 2-connected. Moreover, since e is neither a loop nor a coloop, L is also 2-connected. Suppose $\{C, D\}$ is a 2-separation of L. Because the series pairs in $L\backslash e$ are precisely $\{f_i, g_i\}$ for all $i \in I$ and each $\{e, f_i, g_i\}$ is a triad in L, we know L has no series pairs. Clearly, L has no parallel pairs either, so $\min\{|C|, |D|\} \geq 3$. Without loss of generality, we may assume that $e \in C$. By Proposition 2.1, we may also assume that, for each $i \in I$, the set $\{f_i, g_i\}$ is contained in either C or D. Let $s = |C \cap F_I|$ and $t = |D \cap F_I|$. Clearly $s + t = |I| = k$. Because $N = L\backslash e/F_I$, we see that

$$\begin{aligned} r_N(C - \{e\} - F_I) + r_N(D - F_I) - r(N) &\leq (r_L(C) - s) + (r_L(D) - t) - (r(L) - k) \\ &= r_L(C) + r_L(D) - r(L) \\ &\leq 1. \end{aligned}$$

Moreover, $|C - \{e\} - F_I| \geq \max\{2-s, s\}$ and $|D - F_I| \geq \max\{3-t, t\}$. Since N is 3-connected and $\{C - \{e\} - F_I, D - F_I\}$ is not a 2-separation, the only possibility is that $C = \{e, f_j, g_j\}$ for some $j \in I$. However, $\{e, f_j, g_j\}$ is a triad and L is not isomorphic to $U_{2,4}$, so $r_L(C) + r^*_L(C) - |C| = 2$. Thus $\{C, D\}$ is not a 2-separation, a contradiction. Therefore 3.1.6 holds.

Because $|E(L)| \geq |E(N)| \geq 4$, we see that L is a 3-connected proper restriction of M with at least four elements. As M is super-minimally 3-connected, this contradiction finishes the proof. □

Lemma 3.2. *Suppose M is a super-minimally 3-connected matroid with at least seven elements and $\{e, f, g\}$ is a triangle of M. Then M has an element x of $\{e, f, g\}$ such that $\mathrm{co}(M \backslash x)$ is super-minimally 3-connected.*

Proof. Since $|E(M)| \geq 7$, Proposition 2.7 implies that M is minimally 3-connected. Consequently, none of $M \backslash e$, $M \backslash f$, or $M \backslash g$ is 3-connected. After possibly relabeling e, f, and g, we deduce by Tutte's Triangle Lemma that there is an element h such that $\{e, f, h\}$ is a triad of M. Now we show that

3.2.1. $\mathrm{si}(M/g)$ *is not 3-connected.*

Suppose $\mathrm{si}(M/g)$ is 3-connected. Let $H = E(M) - \{e, f, h\}$, which is a hyperplane of M (see Figure 2). Since $|E(M)| \geq 7$, we have $|H| \geq 4$. Because M is simple and has no $U_{2,4}$-restriction, $r(H) \geq 3$, and hence $r(M) \geq 4$. Therefore $r(M/g) = r(\mathrm{si}(M/g)) \geq 3$. By Table 1, we know $\mathrm{si}(M/g)$ has at least five elements. Note that e and f are parallel in M/g. In the simplification, deleting one of them (say f) leaves e and h in series in $\mathrm{si}(M/g)$. Since $\mathrm{si}(M/g)$ has at least five elements, $\{e, h\}$ is a 2-separating set, a contradiction. Therefore 3.2.1 holds.

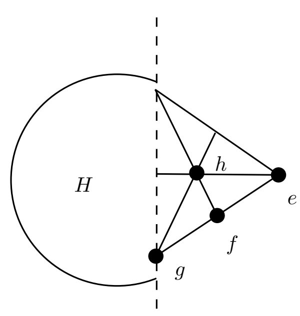

Figure 2: An illustrative geometric representation of M

By Lemma 3.1, it follows that $\mathrm{co}(M \backslash g)$ is super-minimally 3-connected, which completes the proof. □

The next result extends a result of Oxley [11, (5.1)].

Lemma 3.3. *Suppose M is a minimally 3-connected matroid such that $M\backslash x/y$ is isomorphic to $M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 4$. Then M is isomorphic to $M(\mathcal{W}_{k+1})$ or $\mathcal{W}^{k+1}$.*

Proof. Because both M and $M\backslash x/y$ are 3-connected, and $M\backslash x$ is not 3-connected, by Lemma 2.5, x and y are in a triad $\{x, y, z\}$ of M. We note that $M\backslash x$ is a series-extension of $\mathcal{W}_k$ or $\mathcal{W}^k$. Let

$$E(M\backslash x/y) = \{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\},$$

for some integer $k \geq 4$, such that the triangles of $M\backslash x/y$ are exactly

$$\{b_i, a_{i+1}, b_{i+1}\} \quad (i \in \mathbb{Z}_k),$$

and the triads of $M\backslash x/y$ are exactly

$$\{a_i, b_i, a_{i+1}\} \quad (i \in \mathbb{Z}_k).$$

We know, in M, for each $i \in \mathbb{Z}_k$, either

$$\{b_i, a_{i+1}, b_{i+1}\} \quad \text{or} \quad \{b_i, a_{i+1}, b_{i+1}, y\}$$

is a circuit, and either

$$\{a_i, b_i, a_{i+1}\} \quad \text{or} \quad \{a_i, b_i, a_{i+1}, x\}$$

is a cocircuit. Therefore, it is elementary to see the following.

3.3.1. *If T^* is a triad of M, then*

(1) $T^* = \{a_i, b_i, a_{i+1}\}$ for some $i \in \mathbb{Z}_k$, or

(2) $T^* = \{x, y, z\}$, or

(3) T^* contains y but not x and, in particular, T^* is also a triad of $M\backslash x$.

Because of the symmetries of $M(\mathcal{W}_k)$ and $\mathcal{W}^k$, it suffices to consider the following two cases:

(i) $z = b_1$, or

(ii) $z = a_1$.

Suppose that (i) holds, that is, $\{x, y, b_1\}$ is a triad in M. For each $i \in \{2, 3, \ldots, k-1\}$, because $|\{b_i, a_{i+1}, b_{i+1}, y\} \cap \{x, y, b_1\}| = 1$, we know $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle in M. Next we show that

3.3.2. *$\{a_i, b_i, a_{i+1}\}$ is a triad of M, for each $i \in \{2, 3, \ldots, k\}$.*

For each $i \in \{2, 3, \ldots, k-1\}$, we know b_i is in the triangle $\{b_i, a_{i+1}, b_{i+1}\}$. Since M is minimally 3-connected, by Tutte's Triangle Lemma, M has a triad T^* containing b_i and exactly one of $\{a_{i+1}, b_{i+1}\}$. If $y \in T^*$, then $(T^* \cup \{x, y, z\}) - \{y\}$ contains a cocircuit containing x. Thus $M\backslash x/y$ has $(T^* - \{y\}) \cup \{z\}$ as a triad, which contradicts 3.3.1. Hence, by 3.3.1, $\{a_i, b_i, a_{i+1}\}$ is a triad in M for all $i \in \{2, 3, \ldots, k-1\}$. Moreover, it follows by symmetry that $\{a_k, b_k, a_1\}$ is a triad. Hence 3.3.2 holds.

Let $\{b_1, b_2, \ldots, b_k, y\}$ be a basis B_y of M and let C_x be the fundamental circuit of x with respect to B_y. Because $|E(M)| \geq 10$, we know $C_x \neq \{x, y, b_1\}$, otherwise

$$\lambda_M(\{x, y, b_1\}) = r_M(\{x, y, b_1\}) + r^*_M(\{x, y, b_1\}) - |\{x, y, b_1\}| = 1,$$

which implies that $\{x, y, b_1\}$ is a 2-separating set, a contradiction. Therefore, C_x contains b_j for some $j \neq 1$. Since $|C_x \cap \{a_j, b_j, a_{j+1}\}| = 1$, we deduce that $\{a_j, b_j, a_{j+1}\}$ is not a cocircuit, a contradiction to 3.3.2.

We may now suppose that (i) fails and (ii) holds, that is, $\{x, y, a_1\}$ is a triad. For each $i \in \{1, 2, \ldots, k-1\}$, since $|\{b_i, a_{i+1}, b_{i+1}, y\} \cap \{x, y, a_1\}| = 1$, we deduce that $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of M. Next we show that

3.3.3. *$\{a_i, b_i, a_{i+1}\}$ is a triad of M, for each $i \in \{2, 3, \ldots, k-1\}$.*

Since $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of M and M is minimally 3-connected, by Tutte's Triangle Lemma, b_i is in a triad containing exactly one of a_{i+1} and b_{i+1}. Because $\{b_j, a_{j+1}, b_{j+1}\}$ is a triangle of M, for each $j \in \{1, 2, \ldots, k-1\}$, it is straightforward to check that neither $\{b_i, a_{i+1}, y\}$ nor $\{b_i, b_{i+1}, y\}$ is a triad of M. Therefore, by 3.3.1, $\{a_i, b_i, a_{i+1}\}$ is a triad. Hence 3.3.3 holds.

3.3.4. *M has $\{b_1, a_2, a_3, b_3\}$ as a circuit.*

As $\{b_1, a_2, b_2\}$ and $\{b_2, a_3, b_3\}$ are circuits, M has a circuit C containing b_1 that is contained in $\{b_1, a_2, a_3, b_3\}$. As $\{a_2, b_2, a_3\}$ and $\{a_3, b_3, a_4\}$ are cocircuits, it follows, by orthogonality, that $C = \{b_1, a_2, a_3, b_3\}$. Thus 3.3.4 holds.

3.3.5. *M has $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$ as a triad.*

Since $\{b_1, a_2, b_2\}$ is a triangle, by Tutte's Triangle Lemma, M has a triad T^* containing b_1 and exactly one element of $\{a_2, b_2\}$. By 3.3.1, T^* is one of $\{a_1, b_1, a_2\}$, $\{y, b_1, b_2\}$, or $\{y, b_1, a_2\}$. As $\{b_1, a_2, a_3, b_3\}$ is a circuit of M, it follows, by orthogonality, that T^* is $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$.

Let $\{b_1, b_2, \ldots, b_k, x\}$ be a basis B_x of M. Let C_{a_1} be the fundamental circuit of a_1 with respect to B_x, and let C_y be the fundamental circuit of y with respect to B_x. By 3.3.3, we deduce that C_{a_1} is a subset of $\{a_1, x, b_1, b_k\}$ and C_y is a subset of $\{y, x, b_1, b_k\}$. Because M is simple and has $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$ as a triad, at least one of C_{a_1} and C_y is a triangle. Moreover, since $\{x, y, a_1\}$ is a triad of M, it is clear that x is contained in both C_{a_1} and C_y. Recall that, for each of $i \in \{1, 2, \ldots, k-1\}$, the set $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of M. Thus, there is at most one element of M that is not contained in a triangle. Since M is minimally 3-connected, there is at most one element of M that is nonessential. By Theorem 2.4, M is a whirl or the cycle matroid of a wheel. □

Combining the last lemma with Lemma 2.6, we immediately obtain the following result.

Corollary 3.4. *Suppose M is a minimally 3-connected matroid such that $M\backslash x/y$ is $M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 2$. Then M is either $M(\mathcal{W}_{k+1})$ or $\mathcal{W}^{k+1}$.*

4 Brittle matroids

A simple graph G is *fragile* if G has no 3-connected subgraphs. Mader [9] showed that every fragile graph is 4-degenerate and thus 5-colorable. Recently, Bonnet et al. [2] proved that every fragile graph is 4-colorable, and they also showed the following [2, 3.1].

Proposition 4.1. *If G is a fragile graph, then*

$$|E(G)| \leq 2.5|V(G)| - 5.$$

In this section, we extend the concept of fragile graphs to matroids and establish several lemmas that will be used in subsequent proofs. Since the term "fragile" has been used with a different meaning in matroid theory, we will instead use the term "brittle" throughout this paper. A simple matroid M is *brittle* if it has no 3-connected restriction with at least four elements.

Lemma 4.2. *Suppose M is a brittle matroid. Then one of the following holds.*

(i) $M = M_1 \oplus M_2$ for some brittle matroids M_1 and M_2.

(ii) $M = M_3 \oplus_2 M_4$ at basepoint p, for some 2-connected matroids M_3 and M_4, such that $|E(M_i)| \geq 3$ and $M_i\backslash p$ is brittle for each $i \in \{3, 4\}$.

Proof. If M is not 2-connected, then $M = M_1 \oplus M_2$ for some matroids M_1 and M_2. Since both M_1 and M_2 are restrictions of M, they are also brittle. Hence (i) holds.

Now assume that M is 2-connected. As M is brittle and hence not 3-connected, there are 2-connected matroids M_3 and M_4 such that $M = M_3 \oplus_2 M_4$ such that $|E(M_3)| \geq 3$ and $|E(M_4)| \geq 3$ (see, for example, [12, Theorem 8.3.1]). Let p be the basepoint of this 2-sum. Since both $M_3\backslash p$ and $M_4\backslash p$ are restrictions of M, they are also brittle. Therefore, (ii) holds. □

Observe that M_1, M_2, $M_3\backslash p$, and $M_4\backslash p$ in the previous lemma are all restrictions of M. Therefore, the next result follows immediately.

Corollary 4.3. *Suppose M is a triangle-free brittle matroid. Then one of the following holds.*

(i) $M = M_1 \oplus M_2$ *for some triangle-free brittle matroids* M_1 *and* M_2.

(ii) $M = M_3 \oplus_2 M_4$ *at basepoint* p, *for some 2-connected matroids* M_3 *and* M_4, *such that* $|E(M_i)| \geq 3$ *and* $M_i \backslash p$ *is triangle-free and brittle for each* $i \in \{3, 4\}$.

The following is a matroid analogue of a result of Bonnet et al. [2, 3.2].

Lemma 4.4. *If* M *is a triangle-free brittle matroid, then*

$$|E(M)| \leq 2r(M) - 2,$$

unless $M \cong U_{1,1}$.

Proof. We proceed by induction on $|E(M)|$. When $|E(M)| = 1$, the only brittle matroid with one element is $U_{1,1}$, and

$$|E(U_{1,1})| = 2r(U_{1,1}) - 1.$$

When $|E(M)| = 2$, the only brittle matroid with two elements is $U_{2,2}$, and clearly

$$|E(U_{2,2})| = 2r(U_{2,2}) - 2.$$

Hence, the result holds for $|E(M)| \in \{1, 2\}$.

Now assume that $|E(M)| \geq 3$, and that the statement holds for all triangle-free brittle matroids with fewer than $|E(M)|$ elements. In particular, for every such matroid N, we have

4.4.1. $|E(N)| \leq 2r(N) - 1$.

Suppose first that M is not 2-connected. Then $M = M_1 \oplus M_2$ for some triangle-free brittle matroids M_1 and M_2. Since $|E(M_i)| < |E(M)|$ for each $i \in \{1, 2\}$, the inductive hypothesis gives

$$\begin{aligned} |E(M)| &= |E(M_1)| + |E(M_2)| \\ &\leq (2r(M_1) - 1) + (2r(M_2) - 1) && \text{(by 4.4.1)} \\ &= 2r(M) - 2. \end{aligned}$$

Next, suppose that M is 2-connected. By Corollary 4.3, we may assume $M = M_3 \oplus_2 M_4$ at basepoint p, where M_3 and M_4 are 2-connected matroids such that $|E(M_i)| \geq 3$ and $M_i \backslash p$ is triangle-free and brittle for each $i \in \{3, 4\}$. Clearly, $|E(M_i)| < |E(M)|$, and since $|E(M_i)| \geq 3$, neither $M_3 \backslash p$ nor $M_4 \backslash p$ is isomorphic to $U_{1,1}$. Thus, by the inductive hypothesis,

4.4.2. $|E(M_i \backslash p)| \leq 2r(M_i \backslash p) - 2$ *for each* $i \in \{3, 4\}$.

Therefore,

$$\begin{aligned} |E(M)| &= |E(M_3)| + |E(M_4)| - 2 \\ &= |E(M_3 \backslash p)| + |E(M_4 \backslash p)| \\ &\leq 2r(M_3 \backslash p) + 2r(M_4 \backslash p) - 4 && \text{(by 4.4.2)} \\ &= 2r(M_3) + 2r(M_4) - 4 && \text{(as } M_3 \text{ and } M_4 \text{ are 2-connected)} \\ &= 2r(M) - 2. \end{aligned}$$

This completes the proof. □

With Proposition 2.7 and Table 1, one may check that if M is a super-minimally 3-connected matroid with at least four elements, then $M\backslash e$ is brittle for all $e \in E(M)$. The next result follows immediately from the previous lemma.

Corollary 4.5. *If M is a triangle-free super-minimally 3-connected matroid with at least four elements, then*

$$|E(M)| \leq 2r(M) - 1.$$

5 Density of super-minimally 3-connected matroids

In this section, we prove a tight upper bound on the number of elements in a rank-r super-minimally 3-connected matroid.

Proof of Theorem 1.2. Suppose there is a super-minimally 3-connected matroid M with at least four elements such that

$$|E(M)| \geq 2r(M),$$

and that M is not isomorphic to a whirl of rank at least two or the cycle matroid of a wheel of rank at least three. Choose such a matroid of minimum rank. By Proposition 2.7, M is a minimally 3-connected matroid. Thus, by Lemma 2.8 and Lemma 2.6, $r(M) \geq 6$ and hence $|E(M)| \geq 12$. Suppose M is triangle-free. By Corollary 4.5, we know $|E(M)| \leq 2r(M) - 1$, a contradiction. Therefore, we may assume that M has a triangle $\{e, f, g\}$. By Lemma 3.2, M has an element x in $\{e, f, g\}$ such that $\text{co}(M\backslash x)$ is super-minimally 3-connected. Note that, since M is super-minimally 3-connected, $\text{co}(M\backslash x) = M\backslash x/\{y_1, y_2, \ldots, y_k\}$ for some $k \geq 1$. If $k \geq 2$, then $|E(\text{co}(M\backslash x))| > 2r(\text{co}(M\backslash x))$ and $r(\text{co}(M\backslash x)) < r(M)$, contradicting the minimality of M. Thus, we may assume that $\text{co}(M\backslash x) = M\backslash x/y$. Then,

$$|E(M\backslash x/y)| = |E(M)| - 2 \geq 2r(M) - 2 = 2r(M\backslash x/y).$$

Since $M\backslash x/y$ is super-minimally 3-connected, the choice of M means that $M\backslash x/y$ is either a whirl or the cycle matroid of a wheel. By Corollary 3.4, M is either a whirl or the cycle matroid of a wheel, a contradiction. □

6 Triads in super-minimally 3-connected matroids

In addition to determining the extremal density of minimally 3-connected graphs, Halin [5, Satz 6] proved that every such graph G has at least $\frac{2|V(G)|+6}{5}$ vertices of degree three, and he showed that this bound is sharp. For super-minimally 3-connected graphs, Ge [4, Theorem 1.7] established the analogous sharp bound by showing that every such graph G has at least $\frac{|V(G)|+3}{2}$ vertices of degree three.

For matroids, triads play an analogous role to vertices of degree three in graphs. Lemos [7, 8] proved the following results concerning triads in minimally 3-connected matroids.

Theorem 6.1. *Let M be a minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least*

$$\frac{5|E(M)|+30}{9}.$$

Theorem 6.2. *Let M be a minimally 3-connected matroid with at least four elements. Then M has at least*

$$\frac{r(M)+6}{4}$$

triads.

Both bounds are sharp with extremal examples given in [7, p. 172] and [8, p. 940], respectively. It is straightforward to verify that these extremal examples are not only minimally 3-connected but also super-minimally 3-connected. Thus, using Proposition 2.7, we obtain the following analogous sharp bounds for super-minimally 3-connected matroids.

Corollary 6.3. *Let M be a super-minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least*

$$\frac{5|E(M)|+30}{9}.$$

Corollary 6.4. *Let M be a super-minimally 3-connected matroid with at least four elements. Then M has at least*

$$\frac{r(M)+6}{4}$$

triads.

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