# Super-minimally 3-connected matroids

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**Abstract**

A super-minimally $k$-connected matroid is a $k$-connected matroid having no proper $k$-connected restriction of size at least $2k-2$. This extends the corresponding concept for graphs. For $k=2$ and $k=3$, we determine the maximum size of a super-minimally $k$-connected rank-$r$ matroid and characterize, in each case, those matroids attaining the extremal bound. These results parallel Murty's results for minimally 2-connected matroids and Oxley's results for minimally 3-connected matroids.

## 1 Introduction

The study of connectivity plays a central role in both graph theory and matroid theory, particularly in understanding the structure of objects that are minimally sufficient to maintain a given level of connectivity. Classical results of Dirac [3, (7)] and Halin [6, (7.6)] established sharp bounds on the size of minimally $k$-connected graphs for $k \in \{2,3\}$, and these ideas were later extended to matroids by Murty [10, Theorems 3.2 and 3.4] and Oxley [11, (4.7)], respectively.

Recently, Ge [4] introduced the class of *super-minimally $k$-connected graphs*, namely $k$-connected graphs that contain no proper $k$-connected subgraphs, and determined tight bounds on their density for $k \in \{2,3\}$. This raises the natural question as to whether analogous extremal phenomena occur for matroids under a similar strengthening of minimal connectivity.

Motivated by this question, we introduce the class of super-minimally $k$-connected matroids. Since a matroid with fewer than $2k-2$ elements cannot have a $(k-1)$-separation, we define a $k$-connected matroid $M$ to be *super-minimally $k$-connected* if it has no proper $k$-connected restriction with at least $2k-2$ elements. This notion captures matroids whose $k$-connectivity is highly fragile, in the sense that every proper restriction fails to be $k$-connected unless it has too few elements to admit a $(k-1)$-separation. When $k=2$, we make the following elementary observation.

**Proposition 1.1.** *A matroid $M$ is super-minimally 2-connected if and only if $M$ is isomorphic to $U_{1,1}$, or $U_{r,r+1}$ for some integer $r \geq 0$. In particular, $|E(M)| \leq r(M)+1$.*

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Our main result is the following analogue of Oxley's theorem [11, (4.7)] on the density of minimally 3-connected matroids. It extends Ge's bound [4, Theorem 1.10] for super-minimally 3-connected graphs to matroids. For $k \geq 3$, let $\mathcal{W}_k$ denote the *k-spoked wheel* obtained from a $k$-cycle $C_k$ by adding a vertex that is adjacent to every vertex of $C_k$. In $M(\mathcal{W}_k)$, the edge set of $C_k$ is both a circuit and a hyperplane. The rank-$k$ *whirl* $\mathcal{W}^k$ is obtained from $M(\mathcal{W}_k)$ by relaxing the circuit-hyperplane $E(C_k)$ [12, p. 317]. Thus $\mathcal{W}^k$ has the same ground set as $M(\mathcal{W}_k)$ while the set of bases of $\mathcal{W}^k$ consists of all basis of $M(\mathcal{W}_k)$ together with the set $E(C_k)$.

**Theorem 1.2.** *Let $M$ be a super-minimally 3-connected matroid with at least four elements. Then*

$$|E(M)| \leq 2r(M).$$

*Moreover, equality holds if and only if $M \cong U_{2,4}$, or $M \cong M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 3$.*

In the final section, we derive results on triads in super-minimally 3-connected matroids from Lemos's results on minimally 3-connected matroids [7, 8].

# 2 Preliminaries

## 2.1 Some connectivity results

We assume familiarity with the basic notions of matroid theory as found, for example, in [12]. In particular, Chapter 8 of [12] gives a comprehensive treatment of matroid connectivity.

Let $M$ be a matroid. The *connectivity function* $\lambda_M$ of $M$ is defined on the set of subsets $X$ of $E(M)$ by $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M)$. We often abbreviate $\lambda_M$ as $\lambda$. For a positive integer $k$, a partition $\{X, Y\}$ of $E(M)$ is a *k-separation* if $|X|, |Y| \geq k$ and $\lambda_M(X) \leq k-1$. The matroid $M$ is *k-connected* if for all $j$ in $\{1, 2, \ldots, k-1\}$, it has no $j$-separations. A *triangle* is a three-element circuit, and a *triad* is a three-element cocircuit. We omit the elementary proof of the following.

**Proposition 2.1.** *Suppose $\{X, Y\}$ is a $k$-separation of a matroid $M$ and $e$ is an element of $M$ in $\mathrm{cl}(X) \cup \mathrm{cl}^*(X)$. Then $\{X \cup e, Y - e\}$ is a $k$-separation of $M$ if and only if $|Y - e| \geq k$.*

The following lemma, due to Tutte [14, (7.2)], has been used extensively in the study of 3-connected matroids and is often referred to as Tutte's Triangle Lemma (see, for example, [12, Lemma 8.7.7]).

**Lemma 2.2.** *Let $M$ be a 3-connected matroid having at least four elements and suppose that $\{e, f, g\}$ is a triangle of $M$ such that neither $M \setminus e$ nor $M \setminus f$ is 3-connected. Then $M$ has a triad that contains $e$ and exactly one of $f$ and $g$.*

The next lemma is commonly referred to as Bixby's Lemma [1] (see also [12, Lemma 8.7.3]). Let $M$ be a matroid. We use $\mathrm{si}(M)$ to denote the simple matroid associated with $M$. The *cosimplification*, $\mathrm{co}(M)$, of $M$ is $(\mathrm{si}(M^*))^*$.

**Lemma 2.3.** *Let $e$ be an element of a 3-connected matroid $M$. Then $\mathrm{si}(M/e)$ is 3-connected or $\mathrm{co}(M \setminus e)$ is 3-connected.*

One of the main tools in the study of 3-connected matroids is Tutte's Wheels-and-Whirls Theorem. When $M$ is a 3-connected matroid, an element $e$ is *essential* if neither $M\backslash e$ nor $M/e$ is 3-connected. The next result follows immediately from an extension of Tutte's Wheels-and-Whirls Theorem that is due to Oxley and Wu [13, Theorem 1.6] (see also [12, Lemma 8.8.6]).

**Theorem 2.4.** *Suppose $M$ is a 3-connected matroid with at least four elements that is not isomorphic to a whirl or to the cycle matroid of a wheel. Then $M$ has at least two nonessential elements.*

The following lemmas will be useful in our arguments.

**Lemma 2.5.** *[11, (2.6)] Suppose that $x$ and $y$ are distinct elements of an $n$-connected matroid $M$ where $n \geq 2$ and $|E(M)| \geq 2(n-1)$. Assume that $M \setminus x/y$ is $n$-connected but that $M \setminus x$ is not $n$-connected. Then $M$ has a cocircuit of size $n$ containing $x$ and $y$.*

**Lemma 2.6.** *[11, (5.2)] Let $M$ be a rank-$r$ minimally 3-connected matroid with precisely $2r$ elements. If $3 \leq r \leq 5$, then $M$ is isomorphic to either $M(\mathcal{W}_r)$ or $\mathcal{W}^r$.*

## 2.2 Small 3-connected matroids

Recall that a 3-connected matroid $M$ is super-minimally 3-connected if $M$ does not have a proper 3-connected restriction of size at least four. Therefore, if the size of $M$ is at most four, then $M$ is super-minimally 3-connected if and only if $M$ is 3-connected. Table 1 lists all 3-connected matroids on at most four elements.

| **Number $n$ of elements** | **3-connected $n$-element matroids** |
|---|---|
| 1 | $U_{0,1}$, $U_{1,1}$ |
| 2 | $U_{1,2}$ |
| 3 | $U_{1,3}$, $U_{2,3}$ |
| 4 | $U_{2,4}$ |

Table 1: All 3-connected matroids on at most four elements.

Although every super-minimally 3-connected graph is minimally 3-connected [4, Lemma 1.1], a super-minimally 3-connected matroid is not necessarily minimally 3-connected. For instance, the uniform matroid $U_{1,3}$ is super-minimally 3-connected

but not minimally 3-connected, since each single-element deletion of it is isomorphic to $U_{1,2}$, which is 3-connected. The next proposition shows that this counterintuitive phenomenon occurs only when $|E(M)|$ is small. We omit its elementary proof.

**Proposition 2.7.** *Let $k \geq 2$ be an integer and let $M$ be a matroid.*

*(i) If $|E(M)| \leq 2k-2$, then $M$ is super-minimally $k$-connected if and only if $M$ is $k$-connected.*

*(ii) If $M$ is super-minimally $k$-connected and $|E(M)| > 2k-2$, then $M$ is minimally $k$-connected.*

While Table 1 lists all super-minimally 3-connected matroids with at most four elements, in the remainder of the paper, we will focus primarily on those super-minimally 3-connected matroids with at least five elements. The following is an immediate consequence of Proposition 2.7 and a theorem of Oxley [11, (4.7)].

**Lemma 2.8.** *If $M$ is a super-minimally 3-connected matroid with at least four elements and $r(M) \leq 6$, then*

$$|E(M)| \leq 2r(M).$$

# 3 Structural Lemmas

The results in this section provide the reduction and recognition steps that we use in the proof of Theorem 1.2. Let $M$ be a super-minimally 3-connected matroid, in the presence of a triangle, we seek an element $e$ of $M$ for which $\mathrm{co}(M\backslash e)$ is super-minimally 3-connected. Lemma 3.1 is analogous to Bixby's Lemma and may be of independent interest. It strengthens the deletion alternative in Bixby's Lemma when the original matroid is not only 3-connected but also super-minimally 3-connected. Lemma 3.2 then gives the required reduction step for matroids with at least seven elements. Lemma 3.3 and Corollary 3.4 enable us to determine when equality occurs in the bound in Theorem 1.2. The triangle-free case is treated separately in Section 4.

**Lemma 3.1.** *Suppose $M$ is a super-minimally 3-connected matroid with at least five elements. If $e \in E(M)$, then either $\mathrm{si}(M/e)$ is 3-connected, or $\mathrm{co}(M\backslash e)$ is super-minimally 3-connected.*

*Proof.* Suppose that $\mathrm{si}(M/e)$ is not 3-connected. Then, for every 2-separation $\{A, B\}$ of $M/e$, we know $e \in \mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$. First we show the following.

**3.1.1.** *If $\{A, B\}$ is a 2-separation of $M/e$ and $\{e, f, g\}$ is a triad $T^*$ of $M$, then $T^*$ meets each of $A$ and $B$ in exactly one element.*

Suppose $\{f, g\} \subseteq A$. Because $e \in \mathrm{cl}_M(B)$, there is a circuit $C$ of $M$ in $B \cup \{e\}$ containing $e$. However, $|C \cap T^*| = 1$, a contradiction. By symmetry, 3.1.1 follows.

Since $\mathrm{si}(M/e)$ is not 3-connected, $M/e$ has a nonminimal 2-separation $\{X, Y\}$.

**3.1.2.** *If $\{e, f, g\}$ is a triad of $M$ such that $f \in X$ and $g \in Y$, then $f \notin \mathrm{cl}_{M/e}(Y)$ and $g \notin \mathrm{cl}_{M/e}(X)$.*

To see this, observe that, as $\{X, Y\}$ is a nonminimal 2-separation, $\min\{|X|, |Y|\} \geq 3$. If $f \in \mathrm{cl}_{M/e}(Y)$, then, by Proposition 2.1, $\{X - \{f\}, Y \cup \{f\}\}$ is also a 2-separation of $M/e$. However, $\{f, g\} \subseteq Y \cup \{f\}$, contradicting 3.1.1. By symmetry, 3.1.2 follows.

**3.1.3.** *Suppose $T_1^*$ and $T_2^*$ are two distinct triads of $M$ that both contain $e$. Then $T_1^* \cap T_2^* = \{e\}$.*

Suppose $T_1^* = \{f_1, e, g\}$ and $T_2^* = \{f_2, e, g\}$. Without loss of generality, assume $g \in Y$. Then, by 3.1.1, we have $\{f_1, f_2\} \subseteq X$. By circuit elimination, $\{f_1, f_2, e\}$ is a triad of $M$. But $|\{f_1, f_2, e\} \cap X| = 2$, contradicting 3.1.1. This establishes 3.1.3.

Since $\mathrm{si}(M/e)$ is not 3-connected, by Bixby's Lemma, it follows that $\mathrm{co}(M\backslash e)$ is 3-connected. Because $M$ is super-minimally 3-connected, the matroid $M\backslash e$ contains at least one series pair. Therefore, $M$ has at least one triad containing $e$. Suppose $M$ has exactly $k$ triads $T_1^*, T_2^*, \ldots, T_k^*$ containing $e$. By 3.1.1 and 3.1.3, we know, for each $i \in \{1, 2, \ldots, k\}$, the triad $T_i^*$ has the form $\{e, f_i, g_i\}$ such that $f_i \in X$ and $g_i \in Y$. Moreover, $f_i = f_j$ or $g_i = g_j$ if and only if $i = j$ (see Figure 1 for an illustrative diagram). Thus, we may assume that $\mathrm{co}(M\backslash e) = M\backslash e/\{f_1, f_2, \ldots, f_k\}$.

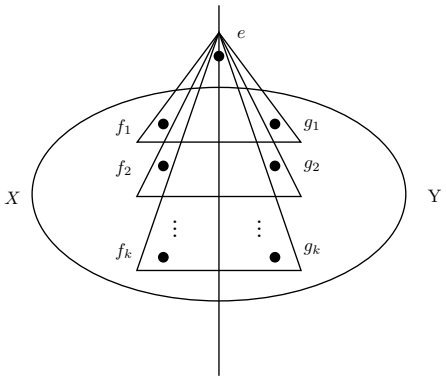



Figure 1: Every triad of $M$ containing $e$ has the form $\{e, f_i, g_i\}$.

Suppose $\mathrm{co}(M\backslash e)$ is not super-minimally 3-connected. Then it has a proper 3-connected restriction $N$ such that $|E(N)| \geq 4$. Next we show that

**3.1.4.** *$E(N)$ contains $g_i$ for some $i \in \{1, 2, \ldots, k\}$.*

Suppose that $E(N) \cap \{g_1, g_2, \ldots, g_k\} = \emptyset$. We know that

$$N = \mathrm{co}(M\backslash e)|E(N) = M\backslash e/\{f_1, f_2, \ldots, f_k\}\backslash\{g_1, g_2, \ldots, g_k\}|E(N).$$

Since $\{e, f_i, g_i\}$ is a triad for all $i \in \{1, 2, \ldots, k\}$, it follows that

$$N = M\backslash\{e, f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k\}|E(N),$$

which is a restriction of $M$, contradicting $M$ being super-minimally 3-connected. Therefore 3.1.4 holds.

Let $I = \{i \in \{1, 2, \ldots, k\} : g_i \in E(N)\}$, and let $F_I = \{f_i : i \in I\}$ and $G_I = \{g_i : i \in I\}$. By 3.1.4, without loss of generality, we may assume that $g_1 \in G_I$. Let $L$ be $M|(E(N) \cup F_I \cup \{e\})$ and recall that $\{f_1, f_2, \ldots, f_k\} \subseteq X$ and $\{g_1, g_2, \ldots, g_k\} \subseteq Y$. We show next that

**3.1.5.** *$e$ is not a coloop of $L$.*

It is clear that $N = L\backslash e/F_I$. Suppose $e$ is a coloop of $L$. Then

$$N = L/e/F_I = (M/e/F_I)|E(N),$$

which is a 3-connected minor of $M/e$ with at least four elements. Recall that $\{X, Y\}$ is a 2-separation of $M/e$. Then $\min\{|E(N) \cap X|, |E(N) \cap Y|\} \leq 1$.

By 3.1.4, $|E(N) \cap Y| \geq |G_I| \geq 1$. Suppose $|E(N) \cap Y| = 1$. Then $E(N) \cap Y = \{g_1\}$ and $g_1 \in \mathrm{cl}_N(E(N) \cap X)$. Because $F_I \subseteq X$, it follows that $g_1 \in \mathrm{cl}_{M/e}(X)$, contradicting 3.1.2.

We now know that $|E(N) \cap X| \leq 1$ and $|E(N) \cap Y| \geq 3$. Because $e$ is a coloop of $L$ and $N$ is 3-connected, $f_i$ and $g_i$ are in series in $L\backslash e$ for all $i \in I$. Therefore,

$$L/e/((F_I \cup \{g_1\}) - \{f_1\})$$

is a 3-connected minor $N'$ of $M/e$ that is isomorphic to $N$. Because $\{X, Y\}$ is a 2-separation of $M/e$, we know $\min\{|E(N') \cap X|, |E(N') \cap Y|\} \leq 1$. Clearly, $|E(N') \cap X| \geq 1$ and $|E(N') \cap Y| \geq 2$. Hence $E(N') \cap X = \{f_1\}$.

If $F_I = \{f_1\}$, then $f_1 \in \mathrm{cl}_{N'}(E(N') \cap Y)$. Because $(F_I \cup \{g_1\}) - \{f_1\} = \{g_1\}$ and $g_1 \in Y$, it follows that $f_1 \in \mathrm{cl}_{M/e}(Y)$, contradicting 3.1.2. Thus $F_I \neq \{f_1\}$. Without loss of generality, we may assume that $f_2 \in F_I$. Let $N''$ be the matroid

$$L/e/((F_I \cup \{g_1, g_2\}) - \{f_1, f_2\}).$$

Clearly, $N'' \cong N$. Because $E(N') \cap X = \{f_1\}$, we conclude that $E(N'') \cap X = \{f_1, f_2\}$. It follows that $|E(N'') \cap X| = 2$ and $|E(N'') \cap Y| \geq 2$, a contradiction as $N$ is 3-connected. Therefore 3.1.5 holds.

In the original matroid $M$, the set $\{e, f_i, g_i\}$ is a triad for all $i \in \{1, 2, \ldots, k\}$. The restriction $L$ of $M$ contains $e$, but neither $f_i$ nor $g_i$ if $i \in \{1, 2, \ldots, k\} - I$. If $\{1, 2, \ldots, k\} - I \neq \emptyset$, then the element $e$ will be a coloop in $L$, contradicting 3.1.5. Therefore, $I = \{1, 2, \ldots, k\}$. We show next that

**3.1.6.** *$L$ is 3-connected.*

Because $L\backslash e$ has $\{f_i, g_i\}$ as a series pair for all $i \in I$, and $L\backslash e/F_I = N$, which is a 3-connected matroid with at least four elements, it is clear that $L\backslash e$ is 2-connected. Moreover, since $e$ is neither a loop nor a coloop, $L$ is also 2-connected. Suppose $\{C, D\}$ is a 2-separation of $L$. Because the series pairs in $L\backslash e$ are precisely $\{f_i, g_i\}$ for all $i \in I$ and each $\{e, f_i, g_i\}$ is a triad in $L$, we know $L$ has no series pairs. Clearly, $L$ has no parallel pairs either, so $\min\{|C|, |D|\} \geq 3$. Without loss of generality, we may assume that $e \in C$. By Proposition 2.1, we may also assume that, for each $i \in I$, the set $\{f_i, g_i\}$ is contained in either $C$ or $D$. Let $s = |C \cap F_I|$ and $t = |D \cap F_I|$. Clearly $s + t = |I| = k$. Because $N = L\backslash e/F_I$, we see that

$$\begin{aligned} r_N(C - \{e\} - F_I) + r_N(D - F_I) - r(N) &\leq (r_L(C) - s) + (r_L(D) - t) - (r(L) - k) \\ &= r_L(C) + r_L(D) - r(L) \\ &\leq 1. \end{aligned}$$

Moreover, $|C - \{e\} - F_I| \geq \max\{2 - s, s\}$ and $|D - F_I| \geq \max\{3 - t, t\}$. Since $N$ is 3-connected and $\{C - \{e\} - F_I, D - F_I\}$ is not a 2-separation, the only possibility is that $C = \{e, f_j, g_j\}$ for some $j \in I$. However, $\{e, f_j, g_j\}$ is a triad and $L$ is not isomorphic to $U_{2,4}$, so $r_L(C) + r_L^*(C) - |C| = 2$. Thus $\{C, D\}$ is not a 2-separation, a contradiction. Therefore 3.1.6 holds.

Because $|E(L)| \geq |E(N)| \geq 4$, we see that $L$ is a 3-connected proper restriction of $M$ with at least four elements. As $M$ is super-minimally 3-connected, this contradiction finishes the proof. □

**Lemma 3.2.** *Suppose $M$ is a super-minimally 3-connected matroid with at least seven elements and $\{e, f, g\}$ is a triangle of $M$. Then $M$ has an element $x$ of $\{e, f, g\}$ such that $\mathrm{co}(M\backslash x)$ is super-minimally 3-connected.*

*Proof.* Since $|E(M)| \geq 7$, Proposition 2.7 implies that $M$ is minimally 3-connected. Consequently, none of $M\backslash e$, $M\backslash f$, or $M\backslash g$ is 3-connected. After possibly relabeling $e$, $f$, and $g$, we deduce by Tutte's Triangle Lemma that there is an element $h$ such that $\{e, f, h\}$ is a triad of $M$. Now we show that

**3.2.1.** $\mathrm{si}(M/g)$ *is not 3-connected.*

Suppose $\mathrm{si}(M/g)$ is 3-connected. Let $H = E(M) - \{e, f, h\}$, which is a hyperplane of $M$ (see Figure 2). Since $|E(M)| \geq 7$, we have $|H| \geq 4$. Because $M$ is simple and has no $U_{2,4}$-restriction, $r(H) \geq 3$, and hence $r(M) \geq 4$. Therefore $r(M/g) = r(\mathrm{si}(M/g)) \geq 3$. By Table 1, we know $\mathrm{si}(M/g)$ has at least five elements. Note that $e$ and $f$ are parallel in $M/g$. In the simplification, deleting one of them (say $f$) leaves $e$ and $h$ in series in $\mathrm{si}(M/g)$. Since $\mathrm{si}(M/g)$ has at least five elements, $\{e, h\}$ is a 2-separating set, a contradiction. Therefore 3.2.1 holds.

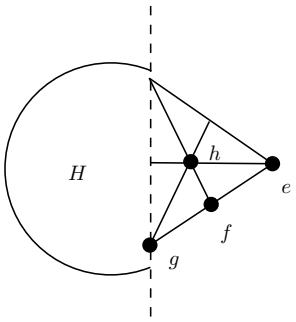



Figure 2: An illustrative geometric representation of $M$

By Lemma 3.1, it follows that $\mathrm{co}(M\backslash g)$ is super-minimally 3-connected, which completes the proof. □

We now consider the recognition steps needed to establish when equality holds in the bound in Theorem 1.2. When the reduction consists of one deletion and one contraction, we need to determine whether obtaining a whirl or the cycle matroid of a wheel forces the original matroid to have the same form. The next lemma, which extends [11, (5.1)], establishes this under the weaker assumption of minimal 3-connectivity. Its proof uses the triangles and triads of $M\backslash x/y$ to constrain how $x$ and $y$ can be restored.

**Lemma 3.3.** *Suppose $M$ is a minimally 3-connected matroid such that $M\backslash x/y$ is isomorphic to $M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 4$. Then $M$ is isomorphic to $M(\mathcal{W}_{k+1})$ or $\mathcal{W}^{k+1}$.*

*Proof.* Because both $M$ and $M\backslash x/y$ are 3-connected, and $M\backslash x$ is not 3-connected, by Lemma 2.5, $x$ and $y$ are in a triad $\{x, y, z\}$ of $M$. We note that $M\backslash x$ is a series extension of $M(\mathcal{W}_k)$ or $\mathcal{W}^k$. Let

$$E(M\backslash x/y) = \{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\},$$

for some integer $k \geq 4$, such that the triangles of $M\backslash x/y$ are exactly

$$\{b_i, a_{i+1}, b_{i+1}\} \quad (i \in \mathbb{Z}_k),$$

and the triads of $M\backslash x/y$ are exactly

$$\{a_i, b_i, a_{i+1}\} \quad (i \in \mathbb{Z}_k).$$

We know, in $M$, for each $i \in \mathbb{Z}_k$, either

$$\{b_i, a_{i+1}, b_{i+1}\} \quad \text{or} \quad \{b_i, a_{i+1}, b_{i+1}, y\}$$

is a circuit, and either

$$\{a_i, b_i, a_{i+1}\} \quad \text{or} \quad \{a_i, b_i, a_{i+1}, x\}$$

is a cocircuit. Therefore, it is elementary to see the following.

**3.3.1.** *If $T^*$ is a triad of $M$, then*

*(1) $T^* = \{a_i, b_i, a_{i+1}\}$ for some $i \in \mathbb{Z}_k$, or*

*(2) $T^* = \{x, y, z\}$, or*

*(3) $T^*$ contains $y$ but not $x$ and, in particular, $T^*$ is also a triad of $M\backslash x$.*

Because of the symmetries of $M(\mathcal{W}_k)$ and $\mathcal{W}^k$, it suffices to consider the following two cases:

(i) $z = b_1$, or

(ii) $z = a_1$.

Suppose that (i) holds, that is, $\{x, y, b_1\}$ is a triad in $M$. For each $i \in \{2, 3, \ldots, k-1\}$, because $|\{b_i, a_{i+1}, b_{i+1}, y\} \cap \{x, y, b_1\}| = 1$, we know $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle in $M$. Next we show that

**3.3.2.** *$\{a_i, b_i, a_{i+1}\}$ is a triad of $M$, for each $i \in \{2, 3, \ldots, k\}$.*

For each $i \in \{2, 3, \ldots, k-1\}$, we know $b_i$ is in the triangle $\{b_i, a_{i+1}, b_{i+1}\}$. Since $M$ is minimally 3-connected, by Tutte's Triangle Lemma, $M$ has a triad $T^*$ containing $b_i$ and exactly one of $\{a_{i+1}, b_{i+1}\}$. If $y \in T^*$, then $(T^* \cup \{x, y, z\}) - \{y\}$ contains a cocircuit containing $x$. Thus $M\backslash x/y$ has $(T^* - \{y\}) \cup \{z\}$ as a triad. However, $(T^* - \{y\}) \cup \{z\}$ contains both $b_1$ and $b_i$, whereas every triad of $M\backslash x/y$ has the form $\{a_j, b_j, a_{j+1}\}$, a contradiction. Hence, by 3.3.1, $\{a_i, b_i, a_{i+1}\}$ is a triad in $M$ for all $i \in \{2, 3, \ldots, k-1\}$. Moreover, it follows by symmetry that $\{a_k, b_k, a_1\}$ is a triad. Hence 3.3.2 holds.

Let $\{b_1, b_2, \ldots, b_k, y\}$ be a basis $B_y$ of $M$ and let $C_x$ be the fundamental circuit of $x$ with respect to $B_y$. Because $|E(M)| \geq 10$, we know $C_x \neq \{x, y, b_1\}$, otherwise

$$\lambda_M(\{x, y, b_1\}) = r_M(\{x, y, b_1\}) + r^*_M(\{x, y, b_1\}) - |\{x, y, b_1\}| = 1,$$

which implies that $\{x, y, b_1\}$ is a 2-separating set, a contradiction. Therefore, $C_x$ contains $b_j$ for some $j \neq 1$. Since $|C_x \cap \{a_j, b_j, a_{j+1}\}| = 1$, we deduce that $\{a_j, b_j, a_{j+1}\}$ is not a cocircuit, a contradiction to 3.3.2.

We may now suppose that (i) fails and (ii) holds, that is, $\{x, y, a_1\}$ is a triad. For each $i \in \{1, 2, \ldots, k-1\}$, since $|\{b_i, a_{i+1}, b_{i+1}, y\} \cap \{x, y, a_1\}| = 1$, we deduce that $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of $M$. Next we show that

**3.3.3.** *$\{a_i, b_i, a_{i+1}\}$ is a triad of $M$, for each $i \in \{2, 3, \ldots, k-1\}$.*

Since $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of $M$ and $M$ is minimally 3-connected, by Tutte's Triangle Lemma, $b_i$ is in a triad containing exactly one of $a_{i+1}$ and $b_{i+1}$. Because $\{b_j, a_{j+1}, b_{j+1}\}$ is a triangle of $M$, for each $j \in \{1, 2, \ldots, k-1\}$, it is straightforward to check that neither $\{b_i, a_{i+1}, y\}$ nor $\{b_i, b_{i+1}, y\}$ is a triad of $M$. Therefore, by 3.3.1, $\{a_i, b_i, a_{i+1}\}$ is a triad. Hence 3.3.3 holds.

**3.3.4.** *$M$ has $\{b_1, a_2, a_3, b_3\}$ as a circuit.*

As $\{b_1, a_2, b_2\}$ and $\{b_2, a_3, b_3\}$ are circuits, $M$ has a circuit $C$ containing $b_1$ that is contained in $\{b_1, a_2, a_3, b_3\}$. As $\{a_2, b_2, a_3\}$ and $\{a_3, b_3, a_4\}$ are cocircuits, it follows, by orthogonality, that $C = \{b_1, a_2, a_3, b_3\}$. Thus 3.3.4 holds.

**3.3.5.** *$M$ has $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$ as a triad.*

Since $\{b_1, a_2, b_2\}$ is a triangle, by Tutte's Triangle Lemma, $M$ has a triad $T^*$ containing $b_1$ and exactly one element of $\{a_2, b_2\}$. By 3.3.1, $T^*$ is one of $\{a_1, b_1, a_2\}$, $\{y, b_1, b_2\}$, or $\{y, b_1, a_2\}$. As $\{b_1, a_2, a_3, b_3\}$ is a circuit of $M$, it follows, by orthogonality, that $T^*$ is $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$.

Set $B_x = \{b_1, b_2, \ldots, b_k, x\}$. Remember that $z = a_1$ and that $\{x, y, z\}$ is a triad of $M$. As $B_x \cap \{x, y, z\} = \{x\}$, orthogonality implies that $B_x - x$ does not span $x$ in $M$ and so $B_x$ is a basis of $M$. Let $C_{a_1}$ be the fundamental circuit of $a_1$ with respect to $B_x$, and let $C_y$ be the fundamental circuit of $y$ with respect to $B_x$. By 3.3.3, we deduce that $C_{a_1}$ is a subset of $\{a_1, x, b_1, b_k\}$ and $C_y$ is a subset of $\{y, x, b_1, b_k\}$. Because $M$ is simple and has $\{a_1, b_1, a_2\}$ or $\{y, b_1, a_2\}$ as a triad, at least one of $C_{a_1}$ and $C_y$ is a triangle. Moreover, since $\{x, y, a_1\}$ is a triad of $M$, it is clear that $x$ is contained in both $C_{a_1}$ and $C_y$. Recall that, for each of $i \in \{1, 2, \ldots, k-1\}$, the set $\{b_i, a_{i+1}, b_{i+1}\}$ is a triangle of $M$. Thus, there is at most one element of $M$ that is not contained in a triangle. Since $M$ is minimally 3-connected, there is at most one element of $M$ that is nonessential. By Theorem 2.4, $M$ is a whirl or the cycle matroid of a wheel. □

Combining the last lemma with Lemma 2.6, we immediately obtain the following result.

**Corollary 3.4.** *Suppose $M$ is a minimally 3-connected matroid such that $M\backslash x/y$ is $M(\mathcal{W}_k)$ or $\mathcal{W}^k$ for some $k \geq 2$. Then $M$ is either $M(\mathcal{W}_{k+1})$ or $\mathcal{W}^{k+1}$.*

# 4 Brittle matroids

A simple graph $G$ is *fragile* if $G$ has no 3-connected subgraphs. Mader [9] showed that every fragile graph is 4-degenerate and thus 5-colorable. Recently, Bonnet et al. [2] proved that every fragile graph is 4-colorable, and they also showed the following [2, 3.1].

**Proposition 4.1.** *If $G$ is a fragile graph, then*

$$|E(G)| \leq 2.5|V(G)| - 5.$$

In this section, we extend the concept of fragile graphs to matroids and establish several lemmas that will be used in subsequent proofs. Since the term "fragile" has been used with a different meaning in matroid theory, we will instead use the term "brittle" throughout this paper. A simple matroid $M$ is *brittle* if it has no

3-connected restriction with at least four elements. Let $M_1$ and $M_2$ be matroids on at least three elements. Suppose that $E(M_1) \cap E(M_2) = \{p\}$, and that neither $M_1$ nor $M_2$ has $p$ as a loop or a coloop. The 2-*sum*, $M_1 \oplus_2 M_2$ of $M_1$ and $M_2$, is the matroid having ground set $(E(M_1) \cup E(M_2)) - \{p\}$ whose circuits are the circuits of $M_1$ or of $M_2$ that avoid $p$, together with the sets $(C_1 \cup C_2) - \{p\}$, where each $C_i$ is a circuit of $M_i$ containing $p$. The element $p$ is the *basepoint* of the 2-sum.

**Lemma 4.2.** *Suppose $M$ is a brittle matroid. Then one of the following holds.*

*(i)* $M = M_1 \oplus M_2$ *for some brittle matroids* $M_1$ *and* $M_2$.

*(ii)* $M = M_3 \oplus_2 M_4$ *at basepoint* $p$, *for some 2-connected matroids* $M_3$ *and* $M_4$, *such that* $|E(M_i)| \geq 3$ *and* $M_i \backslash p$ *is brittle for each* $i \in \{3, 4\}$.

*Proof.* If $M$ is not 2-connected, then $M = M_1 \oplus M_2$ for some matroids $M_1$ and $M_2$. Since both $M_1$ and $M_2$ are restrictions of $M$, they are also brittle. Hence (i) holds.

Now assume that $M$ is 2-connected. As $M$ is brittle and hence not 3-connected, there are 2-connected matroids $M_3$ and $M_4$, each with at least three elements, such that $M = M_3 \oplus_2 M_4$ (see, for example, [12, Theorem 8.3.1]). Let $p$ be the basepoint of this 2-sum. Since both $M_3 \backslash p$ and $M_4 \backslash p$ are restrictions of $M$, they are also brittle. Therefore, (ii) holds. □

Observe that $M_1$, $M_2$, $M_3 \backslash p$, and $M_4 \backslash p$ in the previous lemma are all restrictions of $M$. Therefore, the next result follows immediately.

**Corollary 4.3.** *Suppose $M$ is a triangle-free brittle matroid. Then one of the following holds.*

*(i)* $M = M_1 \oplus M_2$ *for some triangle-free brittle matroids* $M_1$ *and* $M_2$.

*(ii)* $M = M_3 \oplus_2 M_4$ *at basepoint* $p$, *for some 2-connected matroids* $M_3$ *and* $M_4$, *such that* $|E(M_i)| \geq 3$ *and* $M_i \backslash p$ *is triangle-free and brittle for each* $i \in \{3, 4\}$.

The following is a matroid analogue of a result of Bonnet et al. [2, 3.2].

**Lemma 4.4.** *If $M$ is a triangle-free brittle matroid, then*

$$|E(M)| \leq 2r(M) - 2,$$

*unless* $M \cong U_{1,1}$.

*Proof.* We proceed by induction on $|E(M)|$. When $|E(M)| = 1$, the only brittle matroid with one element is $U_{1,1}$, and

$$|E(U_{1,1})| = 2r(U_{1,1}) - 1.$$

When $|E(M)| = 2$, the only brittle matroid with two elements is $U_{2,2}$, and clearly

$$|E(U_{2,2})| = 2r(U_{2,2}) - 2.$$

Hence, the result holds for $|E(M)| \in \{1, 2\}$.

Now assume that $|E(M)| \geq 3$, and that the statement holds for all triangle-free brittle matroids with fewer than $|E(M)|$ elements. In particular, for every such matroid $N$, we have

**4.4.1.** $|E(N)| \leq 2r(N) - 1$.

Suppose first that $M$ is not 2-connected. Then $M = M_1 \oplus M_2$ for some triangle-free brittle matroids $M_1$ and $M_2$. Since $|E(M_i)| < |E(M)|$ for each $i \in \{1, 2\}$, the inductive hypothesis gives

$$\begin{aligned} |E(M)| &= |E(M_1)| + |E(M_2)| \\ &\leq (2r(M_1) - 1) + (2r(M_2) - 1) && \text{(by 4.4.1)} \\ &= 2r(M) - 2. \end{aligned}$$

Next, suppose that $M$ is 2-connected. By Corollary 4.3, we may assume $M = M_3 \oplus_2 M_4$ at basepoint $p$, where $M_3$ and $M_4$ are 2-connected matroids such that $|E(M_i)| \geq 3$ and $M_i \backslash p$ is triangle-free and brittle for each $i \in \{3, 4\}$. Clearly, $|E(M_i)| < |E(M)|$, and since $|E(M_i)| \geq 3$, neither $M_3 \backslash p$ nor $M_4 \backslash p$ is isomorphic to $U_{1,1}$. Thus, by the inductive hypothesis,

**4.4.2.** $|E(M_i \backslash p)| \leq 2r(M_i \backslash p) - 2$ *for each* $i \in \{3, 4\}$.

Therefore,

$$\begin{aligned} |E(M)| &= |E(M_3)| + |E(M_4)| - 2 \\ &= |E(M_3 \backslash p)| + |E(M_4 \backslash p)| \\ &\leq 2r(M_3 \backslash p) + 2r(M_4 \backslash p) - 4 && \text{(by 4.4.2)} \\ &= 2r(M_3) + 2r(M_4) - 4 && \text{(as } M_3 \text{ and } M_4 \text{ are 2-connected)} \\ &= 2r(M) - 2. \end{aligned}$$

This completes the proof. □

With Proposition 2.7 and Table 1, one may check that if $M$ is a super-minimally 3-connected matroid with at least four elements, then $M \backslash e$ is brittle for all $e \in E(M)$. The next result follows immediately from the previous lemma.

**Corollary 4.5.** *If $M$ is a triangle-free super-minimally 3-connected matroid with at least four elements, then*

$$|E(M)| \leq 2r(M) - 1.$$

# 5 Density of super-minimally 3-connected matroids

In this section, we prove a tight upper bound on the number of elements in a rank-$r$ super-minimally 3-connected matroid.

*Proof of Theorem 1.2.* Suppose there is a super-minimally 3-connected matroid $M$ with at least four elements such that

$$|E(M)| \geq 2r(M),$$

and that $M$ is not isomorphic to a whirl of rank at least two or the cycle matroid of a wheel of rank at least three. Choose such a matroid of minimum rank. By Proposition 2.7, $M$ is a minimally 3-connected matroid. Thus, by Lemma 2.8 and Lemma 2.6, $r(M) \geq 6$ and hence $|E(M)| \geq 12$. Suppose $M$ is triangle-free. By Corollary 4.5, we know $|E(M)| \leq 2r(M) - 1$, a contradiction. Therefore, we may assume that $M$ has a triangle $\{e, f, g\}$. By Lemma 3.2, $M$ has an element $x$ in $\{e, f, g\}$ such that $\mathrm{co}(M\backslash x)$ is super-minimally 3-connected. Note that, since $M$ is super-minimally 3-connected, $\mathrm{co}(M\backslash x) = M\backslash x/\{y_1, y_2, \ldots, y_k\}$ for some $k \geq 1$. If $k \geq 2$, then $|E(\mathrm{co}(M\backslash x))| > 2r(\mathrm{co}(M\backslash x))$ and $r(\mathrm{co}(M\backslash x)) < r(M)$, contradicting the minimality of $M$. Thus, we may assume that $\mathrm{co}(M\backslash x) = M\backslash x/y$. Then,

$$|E(M\backslash x/y)| = |E(M)| - 2 \geq 2r(M) - 2 = 2r(M\backslash x/y).$$

Since $M\backslash x/y$ is super-minimally 3-connected, the choice of $M$ means that $M\backslash x/y$ is either a whirl or the cycle matroid of a wheel. By Corollary 3.4, $M$ is either a whirl or the cycle matroid of a wheel, a contradiction. □

Using this theorem, we recover the following graph-theoretic result of Ge [4, Theorem 1.10], originally proved using purely graph-theoretic methods.

**Corollary 5.1.** *If $G$ is a super-minimally 3-connected graph, then*

$$|E(G)| \leq 2|V(G)| - 2.$$

*Moreover, equality is attained if and only if $G$ is a wheel with at least three spokes.*

# 6 Triads in super-minimally 3-connected matroids

In addition to determining the extremal density of minimally 3-connected graphs, Halin [5, Satz 6] proved that every such graph $G$ has at least $\frac{2|V(G)|+6}{5}$ vertices of degree three, and he showed that this bound is sharp. For super-minimally 3-connected graphs, Ge [4, Theorem 1.7] established the analogous sharp bound by showing that every such graph $G$ has at least $\frac{|V(G)|+3}{2}$ vertices of degree three.

For matroids, triads play an analogous role to vertices of degree three in graphs. Lemos [7, 8] proved the following results concerning triads in minimally 3-connected matroids.

**Theorem 6.1.** *Let $M$ be a minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least*

$$\frac{5|E(M)| + 30}{9}.$$

**Theorem 6.2.** *Let $M$ be a minimally 3-connected matroid with at least four elements. Then $M$ has at least*

$$\frac{r(M)+6}{4}$$

*triads.*

Both bounds are sharp with extremal examples given in [7, p. 172] and [8, p. 940], respectively. It is straightforward to verify that these extremal examples are not only minimally 3-connected but also super-minimally 3-connected. Thus, using Proposition 2.7, we obtain the following analogous sharp bounds for super-minimally 3-connected matroids.

**Corollary 6.3.** *Let $M$ be a super-minimally 3-connected matroid with at least eight elements. Then the number of elements contained in at least one triad is at least*

$$\frac{5|E(M)|+30}{9}.$$

**Corollary 6.4.** *Let $M$ be a super-minimally 3-connected matroid with at least four elements. Then $M$ has at least*

$$\frac{r(M)+6}{4}$$

*triads.*

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