WHAT IS A MATROID?

JAMES OXLEY

ABSTRACT. Matroids were introduced by Whitney in 1935 to try to capture abstractly the essence of dependence. Whitney's definition embraces a surprising diversity of combinatorial structures. Moreover, matroids arise naturally in combinatorial optimization since they are precisely the structures for which the greedy algorithm works. This survey paper introduces matroid theory, presents some of the main theorems in the subject, and identifies some of the major problems of current research interest.

1. INTRODUCTION

A paper with this title appeared in *Cubo* **5** (2003), 179–218. This paper is a revision of that paper. It contains some new material, including some exercises, along with updates of some results from the original paper. It was prepared for presentation at the Workshop on Combinatorics and its Applications run by the New Zealand Institute of Mathematics and its Applications in Auckland in July, 2004.

This survey of matroid theory will assume only that the reader is familiar with the basic concepts of linear algebra. Some knowledge of graph theory and field theory would also be helpful but is not essential since the concepts needed will be reviewed when they are introduced. The name "matroid" suggests a structure related to a matrix and, indeed, matroids were introduced by Whitney [61] in 1935 to provide a unifying abstract treatment of dependence in linear algebra and graph theory. Since then, it has been recognized that matroids arise naturally in combinatorial optimization and can be used as a framework for approaching a diverse variety of combinatorial problems. This survey is far from complete and reviews only certain aspects of the subject. Two other easily accessible surveys have been written by Welsh [58] and Wilson [64]. The reader seeking a further introduction to matroids is referred to these papers or to the author's book [34]. Frequent reference will be made to the latter throughout the paper as it contains most of the proofs that are omitted here.

This paper is structured as follows. In Section 2, Whitney's definition of a matroid is given, some basic classes of examples of matroids are introduced, and some important questions are identified. In Section 3, some alternative ways of defining matroids are given along with some basic constructions for

Date: October 31, 2014.
1991 *Mathematics Subject Classification.* 05B35.

matroids. Some of the tools are introduced for answering the questions raised in Section 2 and the first of these answers is given. Section 4 indicates why matroids play a fundamental role in combinatorial optimization by proving that they are precisely the structures for which the greedy algorithm works. In Section 5, the answers to most of the questions posed in Section 2 are given. Some areas of currently active research are discussed and some major unsolved problems are described. Section 6 presents what is probably the most important theorem ever proved in matroid theory, a decomposition theorem that not only describes the structure of a fundamental class of matroids but also implies a polynomial-time algorithm for a basic problem in combinatorial optimization. Section 7 provides a brief summary of some parts of matroid theory that were omitted from the earlier sections of this paper along with some guidance to the literature. It is hoped that the presence of exercises throughout the text will be helpful to the reader.

2. The Definition and Some Examples

In this section, matroids will be defined, some basic classes of examples will be given, and some fundamental questions will be identified.

2.1. **Example.** Consider the matrix

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Let E be the set $\{1,2,3,4,5,6,7\}$ of column labels of A and let $\mathcal{I}$ be the collection of subsets I of E for which the multiset of columns labelled by I is linearly independent over the real numbers $\mathbb{R}$. Then $\mathcal{I}$ consists of all subsets of $E-\{7\}$ with at most three elements except for $\{1,2,4\}$, $\{2,3,5\}$, $\{2,3,6\}$, and any subset containing $\{5,6\}$. The pair $(E,\mathcal{I})$ is a particular example of a matroid. The set E and the members of $\mathcal{I}$ are the *ground set* and *independent sets* of this matroid.

Now consider some of the properties of the set $\mathcal{I}$. Clearly

(I1) *$\mathcal{I}$ is non-empty.*

In addition, $\mathcal{I}$ is hereditary:

(I2) *Every subset of every member of $\mathcal{I}$ is also in $\mathcal{I}$.*

More significantly, $\mathcal{I}$ satisfies the following augmentation condition:

(I3) *If X and Y are in $\mathcal{I}$ and $|X| = |Y|+1$, then there is an element x in $X-Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$.*

Whitney's paper [61], "On the abstract properties of linear dependence", used conditions **(I1)–(I3)** to try to capture abstractly the essence of dependence. A *matroid* M is a pair $(E,\mathcal{I})$ consisting of a finite set E and a collection of subsets of E satisfying **(I1)–(I3)**.

2.2. **Exercise.** Show that if $\mathcal{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathcal{I})$ is a matroid if and only if, for all $X \subseteq E$, all maximal members of $\{I : I \in \mathcal{I} \text{ and } I \subseteq X\}$ have the same number of elements.

The name "matroid" has not always been universally admired. Indeed, Gian-Carlo Rota, whose many important contributions to matroid theory include coauthorship of the first book on the subject [9], mounted a campaign to try to change the name to "geometry", an abbreviation of "combinatorial geometry". At the height of this campaign in 1973, he wrote [22], "Several other terms have been used in place of geometry, by the successive discoverers of the notion; stylistically, these range from the pathetic to the grotesque. The only surviving one is "matroid", still used in pockets of the tradition-bound British Commonwealth." Today, almost thirty years since those words were written, both "geometry" and "matroid" are still in use although "matroid" certainly predominates.

What is the next number in the sequence $1, 2, 4, 8, \ldots$? The next example suggests one way to answer this and a second way will be given later.

2.3. **Example.** If $E = \emptyset$, then there is exactly one matroid on E, namely the one having $\mathcal{I} = \{\emptyset\}$. If $E = \{1\}$, then there are exactly two matroids on E, one having $\mathcal{I} = \{\emptyset\}$ and the other having $\mathcal{I} = \{\emptyset, \{1\}\}$. If $E = \{1, 2\}$, there are exactly five matroids on E, their collections of independent sets being $\{\emptyset\}$, $\{\emptyset, \{1\}\}$, $\{\emptyset, \{2\}\}$, $\{\emptyset, \{1\}, \{2\}\}$, and $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. But the second and third matroids, M_2 and M_3, have exactly the same structure. More formally, there is a bijection from the ground set of M_2 to the ground set of M_3 such that a set is independent in the first matroid if and only if its image is independent in the second matroid. Such matroids are called *isomorphic*, and we write $M_2 \cong M_3$. Since two of the five matroids on a 2-element set are isomorphic, we see that there are exactly four non-isomorphic matroids on such a set.

The answer to the second part of the next exercise will be given at the end of this section.

2.4. **Exercise.** Let $E = \{1, 2, 3\}$.

(i) Show that there are exactly eight non-isomorphic matroids on E.
(ii) How many non-isomorphic matroids are there on a 4-element set?

2.5. **Example.** Let E be an n-element set and, for an integer r with $0 \leq r \leq n$, let $\mathcal{I}$ be the collection of subsets of E with at most r elements. Then it is easy to verify that $(E, \mathcal{I})$ is a matroid. It is called the *uniform matroid* $U_{r,n}$. The three matroids on a set of size at most one are isomorphic to $U_{0,0}, U_{0,1}$, and $U_{1,1}$.

We have yet to verify that matrices do indeed give rise to matroids. We began with a matrix over $\mathbb{R}$. But we could have viewed A as a matrix over $\mathbb{C}$ and we would have obtained exactly the same matroid. Indeed, A yields the same matroid when viewed over any field. This is because, as is easily checked, all square submatrices of A have their determinants in $\{0, 1, -1\}$ so such a subdeterminant is zero over one field if and only if it is zero over every field. We shall say more about this property in Section 5, specifically in Exercise 5.17. In this paper, we shall be interested particularly in finite fields although we shall need very few of their properties. Recall that, for every prime number p and every positive integer k, there is a unique finite field $GF(p^k)$ having exactly p^k elements, and every finite field is of this form. When $k = 1$, these fields are relatively familiar: we can view $GF(p)$ as the set $\{0, 1, \dots, p-1\}$ with the operations of addition and multiplication modulo p. When $k > 1$, the structure of $GF(p^k)$ is more complex and is not the same as that of the set of integers modulo p^k. We shall specify the precise structure of $GF(4)$ in Exercise 2.9 and, in Section 5, the matroids arising from matrices over that field are characterized.

2.6. **Theorem.** *Let A be a matrix over a field $\mathbb{F}$. Let E be the set of column labels of A, and $\mathcal{I}$ be the collection of subsets I of E for which the multiset of columns labelled by I is linearly independent over $\mathbb{F}$. Then $(E, \mathcal{I})$ is a matroid.*

Proof. Certainly $\mathcal{I}$ satisfies **(I1)** and **(I2)**. To verify that **(I3)** holds, let X and Y be linearly independent subsets of E such that $|X| = |Y| + 1$. Let W be the vector space spanned by $X \cup Y$. Then $\dim W$, the dimension of W, is at least $|X|$. Suppose that $Y \cup \{x\}$ is linearly dependent for all x in $X - Y$. Then W is contained in the span of Y, so W has dimension at most $|Y|$. Thus $|X| \leq \dim W \leq |Y|$; a contradiction. We conclude that $X - Y$ contains an element x such that $Y \cup \{x\}$ is linearly independent, that is, **(I3)** holds. □

The matroid obtained from the matrix A as in the last theorem will be denoted by $M[A]$. This matroid is called the *vector matroid* of A. A matroid M that is isomorphic to $M[A]$ for some matrix A over a field $\mathbb{F}$ is called $\mathbb{F}$-*representable*, and A is called an $\mathbb{F}$-*representation* of M. It is natural to ask how well Whitney's axioms succeed in abstracting linear independence. More precisely:

2.7. **Question.** *Is every matroid representable over some field?*

Not every matroid is representable over *every* field as the next proposition will show. Matroids representable over the fields $GF(2)$ and $GF(3)$ are called *binary* and *ternary*, respectively.

2.8. **Proposition.** *The matroid $U_{2,4}$ is not binary but is ternary.*

Proof. Suppose that $U_{2,4}$ is represented over some field $\mathbb{F}$ by a matrix A. Then, since the largest independent set in $U_{2,4}$ has two elements, the *column space* of A, the vector space spanned by its columns, has dimension 2. A 2-dimensional vector space over $GF(2)$ has exactly four members, three of which are non-zero. Thus, if $\mathbb{F} = GF(2)$, then A does not have four distinct non-zero columns so A has a set of two columns that is linearly dependent and therefore A does not represent $U_{2,4}$ over $GF(2)$. Thus $U_{2,4}$ is not binary. The matrix $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ represents $U_{2,4}$ over $GF(3)$ since every two columns of this matrix are linearly independent. Hence $U_{2,4}$ is ternary. □

2.9. **Exercise.** Show that

(i) neither $U_{2,5}$ nor $U_{3,5}$ is ternary;
(ii) $U_{3,6}$ is representable over $GF(4)$, where the elements of this field are $0, 1, \omega, \omega + 1$ and, in this field, $\omega^2 = \omega + 1$ and $2 = 0$.

In light of the last proposition, we have the following:

2.10. **Question.** *Which matroids are regular, that is, representable over every field?*

Once we focus attention on specific fields, a number of questions arise. For example:

2.11. **Question.** *Which matroids are binary?*

2.12. **Question.** *Which matroids are ternary?*

All of Questions 2.7, 2.10, 2.11, and 2.12 will be answered later in the paper. As a hint of what is to come, we note that a consequence of these answers is that a matroid is representable over every field if and only if it is both binary and ternary.

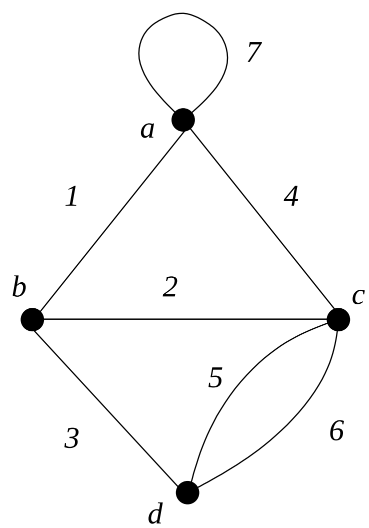

FIGURE 1. The graph G.

It was noted earlier that graph theory played an important role in motivating Whitney's founding paper in matroid theory and we show next how matroids arise from graphs. Consider the graph G with 4 vertices and 7 edges shown in Figure 1. Let E be the edge set of G, that is, $\{1, 2, 3, 4, 5, 6, 7\}$, and let $\mathcal{I}$ be the collection of subsets of E that do not contain all of the edges of any simple closed path or *cycle* of G. The cycles of G have edge sets $\{7\}, \{5,6\}, \{1,2,4\}, \{2,3,5\}, \{2,3,6\}, \{1,3,4,5\}$, and $\{1,3,4,6\}$.

2.13. **Exercise.** Show that the set $\mathcal{I}$ just defined coincides with the set of linearly independent sets of columns of the matrix A in Example 2.1.

A consequence of this exercise is that the pair $(E, \mathcal{I})$ is a matroid. As we shall show in the next theorem, we get a matroid on the edge set of every graph G by defining $\mathcal{I}$ as above. This matroid is called the *cycle matroid* of the graph G and is denoted by $M(G)$.

2.14. **Exercise.** Use a graph-theoretic argument to show that if G is a graph, then $M(G)$ is indeed a matroid.

A matroid that is isomorphic to the cycle matroid of some graph is called *graphic*. It is natural to ask:

2.15. **Question.** *Which matroids are graphic?*

We shall show next that every graphic matroid is binary. This proof will also show that every graphic matroid is actually a matroid. It will use the vertex-edge incidence matrix of a graph. For the graph G in Figure 1, this matrix A_G is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

We observe that the rows of A_G are indexed by the vertices a, b, c, and d of G; the columns are indexed by the edges of G; the column corresponding to the loop 7 is all zeros; and, for every other edge j, the entry in row i of column j is 1 if edge j meets vertex i, and is 0 otherwise.

2.16. **Theorem.** *Let G be a graph and A_G be its vertex-edge incidence matrix. When A_G is viewed over $GF(2)$, its vector matroid $M[A_G]$ has as its independent sets all subsets of $E(G)$ that do not contain the edges of a cycle. Thus $M[A_G] = M(G)$ and every graphic matroid is binary.*

Proof. It suffices to prove that a set X of columns of A_G is linearly dependent if and only if X contains the set of edges of a cycle of G. Assume that X contains the edge set of some cycle C. If C is a loop, then the corresponding column is the zero vector, so X is linearly dependent. When C is not a loop, each vertex that is met by C is met by exactly two edges of C. Thus the sum, modulo 2, of the columns of C is the zero vector. Hence X is linearly dependent. Conversely, suppose that X is a linearly dependent set of columns. Take a subset D of X that is minimal with the property of being linearly dependent, that is, D is linearly dependent but all of its proper subsets are linearly independent. If D contains a zero column, then D contains the edge set of a loop. Assume that D does not contain a zero column. Now $GF(2)$ has 1 as its only non-zero entry. As D is a minimal linearly dependent set, the sum, modulo 2, of the columns in D is the zero vector. This means that every vertex that meets an edge of D is met by at least two such edges. It follows that D contains the edges of a cycle. To see this, take an edge d_1 of D and let v_0 and v_1 be the vertices met by d_1. Clearly v_1 is met by another edge d_2 of D. Let v_2 be the other end-vertex of d_2. In this way, we define a sequence $d_1, d_2, \ldots$ of edges of D and a sequence $v_0, v_1, \ldots$ of vertices. Because the graph is finite, eventually one of the vertices v in the sequence must repeat. When this first occurs, a cycle in D has been found that starts and ends at v. □

2.17. **Exercise.** For a graph G, let A'_G be obtained from A_G by replacing the second 1 in each non-zero column by -1. Show that $M[A'_G]$ represents $M(G)$ over all fields.

We noted earlier that the number of non-isomorphic matroids on an n-element set behaves like the sequence 2^n for small values of n. As Table 1 shows, the sequence 2^n persists even longer when counting non-isomorphic binary matroids on an n-element set. Each of the matroids on a 3-element set is graphic.

2.18. **Exercise.** Find 8 graphs each with 3 edges such that the associated cycle matroids are non-isomorphic.

We note here that non-isomorphic graphs can have isomorphic cycle matroids. For instance, the cycle matroid of any graph is unchanged by adding a collection of isolated vertices, that is, vertices that meet no edges. More significantly, the 3-vertex graph having a single loop meeting each vertex has the same cycle matroid as the single-vertex graph having three loops meeting the only vertex. In general, if a graph G has connected components $G_1, G_2, \ldots, G_k$ and v_i is a vertex of G_i for all i, then the graph that is obtained by identifying all of the vertices v_i has the same cycle matroid as G since the identifications specified do not alter the edge sets of any cycles. In a paper that preceded and doubtless motivated his paper introducing matroids, Whitney [60] determined precisely when two graphs have isomorphic cycle matroids (see also [34, Theorem 5.3.1]).

All 16 of the binary matroids on a 4-element set are graphic. The one non-binary matroid on a 4-element set is the one that we have already noted, $U_{2,4}$.

In spite of its early similarity to 2^n, the number $f(n)$ of non-isomorphic matroids on an n-element set behaves much more like 2^{2^n}. Indeed, by results of Piff [38] and Knuth [24], there are constants c_1 and c_2 and an integer N such that, for all $n \geq N$,

$$n - \tfrac{3}{2}\log_2 n + c_1 \log_2 \log_2 n \leq \log_2 \log_2 f(n) \leq n - \log_2 n + c_2 \log_2 \log_2 n.$$

n	0	1	2	3	4	5	6	7	8
matroids	1	2	4	8	17	38	98	306	1724
binary matroids	1	2	4	8	16	32	68	148	342

TABLE 1. Numbers of non-isomorphic matroids and non-isomorphic binary matroids on an n-element set.

Let $b(n)$ be the number of non-isomorphic binary matroids on an n-element set. One can obtain a crude upper bound on $b(n)$ by noting that every n-element binary matroid can be represented by an $n \times n$ matrix in which every entry is in $\{0,1\}$. Thus $b(n) \leq 2^{n^2}$. On combining this with the lower bound on $f(n)$, we deduce that most matroids are non-binary, that is, $\lim_{n\to\infty} \frac{b(n)}{f(n)} = 0$.

For functions g and h defined on the set of positive integers, g is *asymptotic to h*, written $g \sim h$, if $\lim_{n\to\infty} g/h = 1$. Let $\left[{n \atop k}\right]_2$ be the number of k-dimensional vector spaces of an n-dimensional vector space over $GF(2)$. Evidently $\left[{n \atop 0}\right]_2 = 1$ and it is not difficult to show by counting linearly independent sets (see, for example, [34, Proposition 6.1.4]) that, for all $k \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_2 = \frac{(2^n-1)(2^{n-1}-1)\dots(2^{n-k+1}-1)}{(2^k-1)(2^{k-1}-1)\dots(2-1)}.$$

In 1971, Welsh [57] raised the problem of finding the asymptotic behaviour of $b(n)$. Wild [62, 63] solved Welsh's problem by proving the next theorem. Wild's initial solution in [62] contained errors that were noted in the *Mathematical Reviews* entry for the paper, **MR2001i:94077**, and in a paper of Lax [28]. However, there errors were corrected in **MR2001i:94077** and [63]. Curiously, the asymptotic behaviour of $b(n)$ depends upon the parity of n.

2.19. **Theorem.** *The number $b(n)$ of non-isomorphic binary matroids on an n-element set satisfies*

$$b(n) \sim \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_2 .$$

Moreover, if $\beta(n) = 2^{n^2/4 - n\log_2 n + n \log_2 e - (1/2)log_2 n}$ for all positive integers n, then there are constants d_1 and d_2 such that

$$b(2n+1) \sim d_1\beta(2n+1) \text{ and } b(2n) \sim d_2\beta(2n).$$

Rounded to 6 decimal places, $d_1 = 2.940982$ and $d_2 = 2.940990$.

For the reader familiar with coding theory, it is worth noting that $b(n)$ equals the number of inequivalent binary linear codes of length n, where two such codes are *equivalent* if they differ only in the order of the symbols.

3. Circuits, Bases, Duals, and Minors

In this section, we consider alternative ways to define matroids together with some basic constructions for matroids. We also introduce some tools for answering the questions from the last section and give three answers to Question 2.11. A set in a matroid that is not independent is called *dependent*. The hereditary property, **(I2)**, means that a matroid is uniquely determined by its collection of maximal independent sets, which are called *bases*, or by its collection of minimal dependent sets, which are called *circuits*. Indeed, the cycle matroid $M(G)$ of a graph G is perhaps most naturally defined in terms of its circuits, which are precisely the edge sets of the cycles of G.

By using **(I1)**–**(I3)**, it is not difficult to show that the collection $\mathcal{C}$ of circuits of a matroid M has the following three properties:

- **(C1)** *The empty set is not in* $\mathcal{C}$.
- **(C2)** *No member of* $\mathcal{C}$ *is a proper subset of another member of* $\mathcal{C}$.
- **(C3)** *If* C_1 *and* C_2 *are distinct members of* $\mathcal{C}$ *and* $e \in C_1 \cap C_2$, *then* $(C_1 \cup C_2) - \{e\}$ *contains a member of* $\mathcal{C}$.

These three conditions characterize the collections of sets that can be the circuits of a matroid. More formally:

3.1. **Theorem.** *Let* M *be a matroid and* $\mathcal{C}$ *be its collection of circuits. Then* $\mathcal{C}$ *satisfies* **(C1)**–**(C3)**. *Conversely, suppose* $\mathcal{C}$ *is the collection of subsets of a finite set* E *satisfying* **(C1)**–**(C3)** *and let* $\mathcal{I}$ *be those subsets of* E *that contain no member of* $\mathcal{C}$. *Then* $(E, \mathcal{I})$ *is a matroid having* $\mathcal{C}$ *as its collection of circuits.*

We leave the proof of this result as an exercise noting that it may be found in [34, Theorem 1.1.4]. The next result characterizes matroids in terms of their collections of bases. Its proof may be found in [34, Theorem 1.2.3].

3.2. **Theorem.** *Let* $\mathcal{B}$ *be a set of subsets of a finite set* E. *Then* $\mathcal{B}$ *is the collection of bases of a matroid on* E *if and only if* $\mathcal{B}$ *satisfies the following conditions:*

- **(B1)** $\mathcal{B}$ *is non-empty.*
- **(B2)** *If* B_1 *and* B_2 *are members of* $\mathcal{B}$ *and* $x \in B_1 - B_2$, *then there is an element* y *of* $B_2 - B_1$ *such that* $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

It follows immediately from **(I3)** that, like the bases of a vector space, all bases of a matroid M have the same cardinality, $r(M)$, which is called the *rank* of M. Thus the rank of a vector matroid $M[A]$ is equal to the rank of the matrix A. If G is a connected graph, then the bases of $M(G)$ are the maximal sets of edges that do not contain a cycle. These sets are precisely the edge sets of spanning trees of G and, if G has m vertices, each spanning tree has exactly $m - 1$ edges, so $r(M(G)) = m - 1$.

Let us return to the graph G considered in Figure 1 to introduce a basic matroid operation. Evidently G is a plane graph, that is, it is embedded in the plane without edges crossing. To construct the *dual* G^* of G, we insert a single vertex of G^* in each *face* or region determined by G and, for each edge e of G, if e lies on the boundary of two faces, then we join the corresponding vertices of G^* by an edge labelled by e, while if e lies on the boundary of a single face, then we add a loop labelled by e at the corresponding vertex of G^*. This construction is illustrated in Figure 2. We observe from that figure that if we had begun with G^* instead of G and had constructed the dual of G^*, then we would have obtained G; that is, $(G^*)^* = G$. The last observation holds for all *connected* plane graphs G, that is, for all plane graphs in which every two vertices are joined by a path.

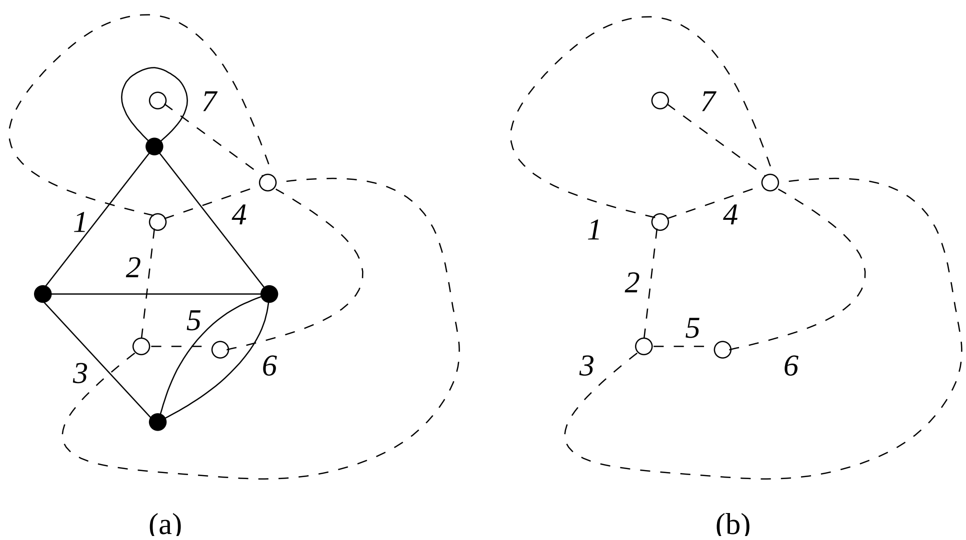

FIGURE 2. (a) Constructing the dual G^* of G. (b) G^*.

Now the edge sets of the graphs G^* and G are the same. The collection of circuits of the cycle matroid $M(G^*)$, which is the collection of edge sets of cycles of the graph G^*, equals

$$\{\{1,4\},\{1,2,3\},\{2,3,4\},\{3,5,6\},\{1,2,5,6\},\{2,4,5,6\}\}.$$

What do these sets correspond to in the original graph G? They are the minimal edge cuts of G, that is, the minimal sets of edges of G with the property that their removal increases the number of connected pieces or *components* of the graph. To see this, the key observation is that the set of edges of G corresponding to a cycle C of G^* consists of the edges that join a vertex of G that lies inside of C to a vertex of G that lies outside of C.

A minimal edge cut of a graph is also called a *bond* of the graph. We have seen how the bonds of a graph G are the circuits of a matroid on the edge set of G in the case that G is a plane graph. In fact, this holds for arbitrary graphs as can be proved using Theorem 3.1. Again we leave this as an exercise.

3.3. **Proposition.** *Let G be a graph with edge set $E(G)$. Then the set of bonds of G is the set of circuits of a matroid on $E(G)$.*

The matroid in the last proposition is called the *bond matroid* of G and is denoted by $M^*(G)$. This matroid is the *dual* of the cycle matroid $M(G)$. A matroid that is isomorphic to the bond matroid of some graph is called *cographic*. Every matroid M has a dual but it is easier to define this in terms of bases rather than circuits. In preparation for the next result, the reader is urged to check that the set of edge sets of spanning trees of the graph G in Figure 1 is

$$\{\{1,2,3\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\\ \{1,4,6\},\{2,3,4\},\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,4,6\}\}.$$

The dual G^* of G, which is shown in Figure 2, has as its spanning trees every set of the form $\{7\} \cup X$ where X is in the following set:

$$\{\{1,2,5\},\{1,2,6\},\{1,3,5\},\{1,3,6\},\{1,5,6\},\{2,3,5\},\{2,3,6\},\\ \{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{4,5,6\}\}.$$

Observe that the spanning trees of G^* are the complements of the spanning trees of G.

3.4. **Theorem.** *Let M be a matroid on a set E and $\mathcal{B}$ be the collection of bases of M. Let $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^*$ is the collection of bases of a matroid M^* on E.*

The proof of this theorem may be found in [34, Theorem 2.1.1]. The matroid M^* is called the *dual* of M. The bases and circuits of M^* are called *cobases* and *cocircuits*, respectively, of M. Evidently

3.5. $(M^*)^* = M$.

It can be shown that, for every graph G,

3.6. $(M(G))^* = M^*(G)$.

For the uniform matroid $U_{r,n}$, the set of bases is the set of r-element subsets of the ground set. Theorem 3.4 implies that the set of bases of the dual matroid is the set of $(n-r)$-element subsets of the ground set. Hence

3.7. $(U_{r,n})^* \cong U_{n-r,n}$.

3.8. **Exercise.** A matroid is *self-dual* if it is isomorphic to its dual.

(i) For all ranks r, give an example of a rank-r graphic matroid that is self-dual.
(ii) Give an example of a self-dual matroid that is not equal to its dual.

The set of cocircuits of $U_{r,n}$ consists of all $(n-r+1)$-element subsets of the ground set. Thus, in this case, the cocircuits are the minimal sets meeting every basis. We leave it as an exercise to show that this attractive property holds in general.

3.9. **Theorem.** *Let M be a matroid.*

(i) *A set C^* is a cocircuit of M if and only if C^* is a minimal set having non-empty intersection with every basis of M.*
(ii) *A set B is a basis of M if and only if B is a minimal set having non-empty intersection with every cocircuit of M.*

This blocking property suggests the following two-person game. Given a matroid M with ground set E, two players B and C alternately tag elements of E. The goal for B is to tag all the elements of some basis of M, while the goal for C is to prevent this. Equivalently, by the last result, C's goal is to tag all the elements of some cocircuit of M. We shall specify when B can win against all possible strategies of C. If B has a winning strategy playing second, then it will certainly have a winning strategy playing first. The next result is obtained by combining some attractive results of Edmonds [11] one of which extended a result of Lehman [30]. The game that we have described is a variant of Shannon's switching game (see [31]).

3.10. **Theorem.** *The following statements are equivalent for a matroid M with ground set E.*

(i) *Player C plays first and player B can win against all possible strategies of C.*
(ii) *The matroid M has 2 disjoint bases.*
(iii) *For all subsets X of E, $|X| \geq 2(r(M) - r(M\backslash X))$.*

Edmonds also specified when player C has a winning strategy but this is more complicated and we omit it. If the game is played on a connected graph G, then B's goal is to tag the edges of a spanning tree, while C's goal is to tag the edges of a bond. If we think of this game in terms of a communication network, then C's goal is to separate the network into pieces that are no longer connected to each other, while B is aiming to reinforce edges of the network to prevent their destruction. Each move for C consists of destroying one edge, while each move for B involves securing an edge against destruction. By applying the last theorem to the cycle matroid of G, we get the following result where the equivalence of (ii) and (iii) was first proved by Tutte [53] and Nash-Williams [32]. For a partition π of a set, we denote the number of classes in the partition by $|\pi|$.

3.11. **Corollary.** *The following statements are equivalent for a connected graph G.*

(i) *Player C plays first and player B can win against all possible strategies of C.*
(ii) *The graph G has 2 edge-disjoint spanning trees.*
(iii) *For all partitions π of the vertex set of G, the number of edges of G that join vertices in different classes of the partition is at least $2(|\pi| - 1)$.*

Proof. We shall show that (ii) implies (i) by describing a winning strategy for B when G has two edge-disjoint spanning trees T_1 and T_2. We may assume that C picks an edge in T_1 or T_2, otherwise B's task is simplified. Suppose C picks the edge c_1 of T_1. Consider $T_1 - \{c_1\}$ and T_2. Both these sets are independent in $M(G)$ and $|T_2| = |T_1 - \{c_1\}| + 1$. Thus, by **(I3)**, there is an edge b_1 of $T_2 - (T_1 - \{c_1\})$ so that $(T_1 - \{c_1\}) \cup \{b_1\}$ is independent in $M(G)$. Hence $(T_1 - \{c_1\}) \cup \{b_1\}$ is the edge set of a spanning tree in G. Now both $(T_1 - \{c_1\}) \cup \{b_1\}$ and T_2 are spanning trees of G containing b_1. Thus $T_1 - \{c_1\}$ and $T_2 - \{b_1\}$ are edge-disjoint spanning trees of the connected graph $G\backslash c_1/b_1$. Therefore, after one move each, B has preserved the property that the game is being played on a connected graph with two edge-disjoint spanning trees. Continuing in this way, it is clear that B will win. Hence (ii) implies (i). We omit the proofs of the remaining implications. □

In the last theorem and corollary, parts (ii) and (iii) remain equivalent if, in each part, we replace 2 by an arbitrary positive integer k.

3.12. **Exercise.** Let $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$ be matroids M_1 and M_2.

(i) Show that $(E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1,\ I_2 \in \mathcal{I}_2\}$. This matroid is the *union* $M_1 \vee M_2$ of M_1 and M_2.

(ii) Show that a matroid M has 2 disjoint bases if and only if $M \vee M$ has rank $2r(M)$.

(iii) Use the fact (and prove it if you can) that the rank of a set X in $M_1 \vee M_2$ is $\min\{r_1(Y) + r_2(Y) + |X - Y| : Y \subseteq X\}$ to show the equivalence of (ii) and (iii) in each of Theorem 3.10 and Corollary 3.11.

We deduce from (3.7) that the sum of the ranks of a uniform matroid and its dual equals the size of the ground set. This is true in general and follows immediately from Theorem 3.4.

3.13. *For a matroid M on an n-element set, $r(M) + r(M^*) = n$.*

Before considering how to construct the dual of a representable matroid, we look at how one can alter a matrix A without affecting the associated vector matroid $M[A]$. The next result follows without difficulty by using elementary linear algebra and is left as an exercise.

3.14. **Lemma.** *Suppose that the entries of a matrix A are taken from a field $\mathbb{F}$. Then $M[A]$ remains unaltered by performing any of the following operations on A.*

(i) *Interchange two rows.*
(ii) *Multiply a row by a non-zero member of $\mathbb{F}$.*
(iii) *Replace a row by the sum of that row and another.*
(iv) *Delete a zero row (unless it is the only row).*
(v) *Interchange two columns (moving the labels with the columns).*
(vi) *Multiply a column by a non-zero member of $\mathbb{F}$.*

If A is a zero matrix with n columns, then clearly $M[A]$ is isomorphic to $U_{0,n}$. Now suppose that A is non-zero having rank r. Then, by performing a sequence of operations (3.14)(i)–(v), we can transform A into a matrix in the form $[I_r|D]$, where I_r is the $r \times r$ identity matrix. The dual of $M[I_r|D]$ involves the transpose D^T of D.

3.15. **Theorem.** *Let M be an n-element matroid that is representable over a field $\mathbb{F}$. Then M^* is representable over $\mathbb{F}$. Indeed, if $M = M[I_r|D]$, then $M^* = [-D^T|I_{n-r}]$.*

3.16. **Exercise.** Using the facts that a matroid is determined by its set of bases and that one can use determinants to decide whether or not a certain set is a basis in a vector matroid, prove the last theorem. The details of this proof may be found in [34, Theorem 2.2.8].

The last result provides an attractive link between matroid duality and orthogonality in vector spaces. Recall that two vectors $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_n)$ are *orthogonal* if $\sum_{i=1}^{n} v_i w_i = 0$. Given a subspace W of a vector space V, the set $W^\perp$ of vectors of V that are orthogonal to every vector in W forms a subspace of V called the *orthogonal subspace* of W. It is not difficult to show that if W is the vector space spanned by the rows of the matrix $[I_r|D]$, then $W^\perp$ is the vector space spanned by the rows of $[-D^T|I_{n-r}]$. The reader familiar with coding theory will recognize that if W is a vector space over $GF(q)$, then W is just a linear code over that field. Moreover, the matrix $[I_r|D]$ is a generator matrix for this code, while $[-D^T|I_{n-r}]$ is a parity-check matrix for this code. The last matrix is also a generator matrix for the dual code, which coincides with $W^\perp$.

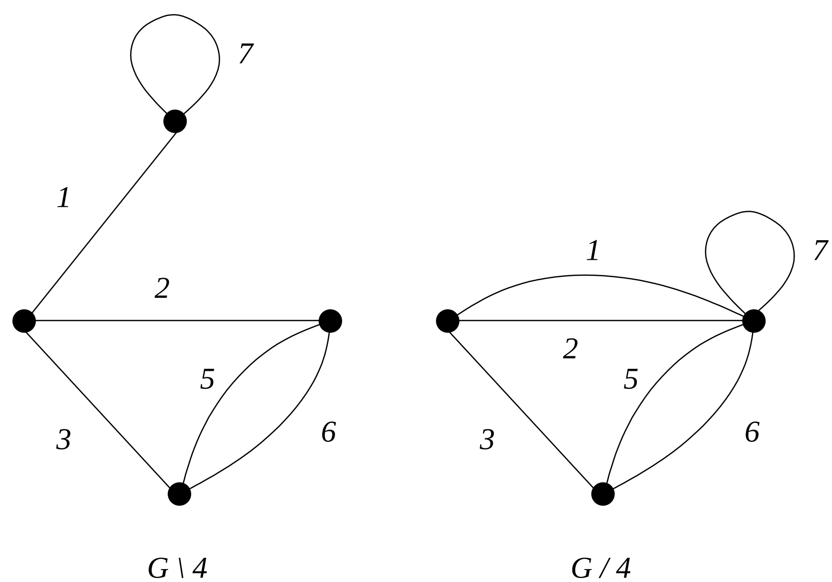

FIGURE 3. Deletion and contraction of an edge of a graph.

Taking duals is one of three fundamental matroid operations which generalize operations for graphs. The other two are deletion and contraction.

If e is an edge of a graph G, then the *deletion* $G\backslash e$ of e is the graph obtained from G by simply removing e. The *contraction* G/e of e is the graph that is obtained by identifying the endpoints of e and then deleting e. Figure 3 shows the graphs $G\backslash 4$ and $G/4$ where G is the graph from Figure 1. We note that the deletion and contraction of a loop are the same. These operations have a predictable effect on the independent sets of the cycle matroid $M(G)$: a set I is independent in $M(G\backslash e)$ if and only if $e \notin I$ and I is independent in $M(G)$; and, provided e is not a loop of G, a set I is independent in $M(G/e)$ if and only if $I \cup \{e\}$ is independent in $M(G)$. By generalizing this, we can define the operations of deletion and contraction for arbitrary matroids.

Let M be a matroid $(E, \mathcal{I})$ and e be an element of E. Let $\mathcal{I}' = \{I \subseteq E - \{e\} : I \in \mathcal{I}\}$. Then it is easy to check that $(E - \{e\}, \mathcal{I}')$ is a matroid. We denote this matroid by $M\backslash e$ and call it the *deletion* of e from M. If e is a *loop* of M, that is, $\{e\}$ is a circuit of M, then we define $M/e = M\backslash e$. If e is not a loop, then $M/e = (E - \{e\}, \mathcal{I}'')$ where $\mathcal{I}'' = \{I \subseteq E - \{e\} : I \cup \{e\} \in \mathcal{I}\}$. Again it is not difficult to show that M/e is a matroid. This matroid is the *contraction* of e from M. If e and f are distinct elements of a matroid M, then it is straightforward to check that

3.17. $M\backslash e\backslash f = M\backslash f\backslash e$; $M/e/f = M/f/e$; *and* $M\backslash e/f = M/f\backslash e$.

This means that, for disjoint subsets X and Y of E, the matroids $M\backslash X$, M/Y, and $M\backslash X/Y$ are well-defined. A *minor* of M is any matroid that can be obtained from M by a sequence of deletions or contractions, that is, any matroid of the form $M\backslash X/Y$ or, equivalently, of the form $M/Y\backslash X$. If $X \cup Y \neq \emptyset$, then $M\backslash X/Y$ is a *proper minor* of M.

The next result specifies the independent sets, circuits, and bases of $M\backslash T$ and M/T. The proof is left as an exercise.

3.18. **Proposition.** *Let M be a matroid on a set E and let T be a subset of E. Then $M\backslash T$ and M/T are matroids on $E - T$. Moreover, for a subset X of $E - T$,*

(i) *X is independent in $M\backslash T$ if and only if X is independent in M;*
(ii) *X is a circuit of $M\backslash T$ if and only if X is a circuit in M;*
(iii) *X is a basis of $M\backslash T$ if and only if X is a maximal subset of $E - T$ that is independent in M;*
(iv) *X is independent in M/T if and only if $X \cup B_T$ is independent in M for some maximal subset B_T of T that is independent in M;*
(v) *X is a circuit in M/T if and only if X is a minimal non-empty member of $\{C - T : C$ is a circuit of $M\}$;*
(vi) *X is a basis of M/T if and only if $X \cup B_T$ is a basis of M for some maximal subset B_T of T that is independent in M.*

Duality, deletion, and contraction are related through the following attractive result which can be proved, for example, by using (iii) and (vi) of the last proposition.

3.19. $M^*/T = (M\backslash T)^*$ *and* $M^*\backslash T = (M/T)^*$.

Certain important classes of matroids are *closed under minors*, that is, every minor of a member of the class is also in the class.

3.20. **Theorem.** *The classes of uniform, graphic, and cographic matroids are minor-closed. Moreover, for all fields $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids is minor-closed. In particular, the classes of binary and ternary matroids are minor-closed.*

Proof. If the uniform matroid $U_{r,n}$ has ground set E and $e \in E$, then

$$U_{r,n} \backslash e \cong \begin{cases} U_{r,n-1} \text{ if } r < n; \\ U_{r-1,n-1} \text{ if } r = n; \end{cases}$$

and

$$U_{r,n}/e \cong \begin{cases} U_{r-1,n-1} \text{ if } r > 0; \\ U_{r,n-1} \text{ if } r = 0. \end{cases}$$

Hence the class of uniform matroids is indeed minor-closed.

To see that the class of graphic matroids is minor-closed, it suffices to note that if e is an edge of a graph G, then

$$M(G)\backslash e = M(G\backslash e) \text{ and } M(G)/e = M(G/e).$$

On the other hand, the class of cographic matroids is minor-closed because, by (3.19), (3.6), and the last observation,

$$M^*(G)\backslash e = (M(G)/e)^* = (M(G/e))^* = M^*(G/e)$$

and

$$M^*(G)/e = (M(G)\backslash e)^* = (M(G\backslash e))^* = M^*(G\backslash e).$$

Finally, to see that the class of $\mathbb{F}$-representable matroids is minor-closed, we note that if $M = M[A]$ and e is an element of M, then $M\backslash e$ is represented over $\mathbb{F}$ by the matrix that is obtained by deleting column e from A. Thus the class of $\mathbb{F}$-representable matroids is closed under deletion. Since it is also closed under duality by Theorem 3.15, we deduce from (3.19) that it is closed under contraction. Hence it is minor-closed. □

From the last result, we know that, for all fields $\mathbb{F}$, every contraction M/e of an $\mathbb{F}$-representable matroid M is $\mathbb{F}$-representable. However, the construction of an $\mathbb{F}$-representation for M/e that can be derived from the last paragraph of the preceding proof is rather convoluted. There is a much more direct method, which we now describe. Let $M = M[A]$. If e is a loop of M, then e labels a zero column of A and M/e is represented by the matrix that is obtained by deleting this column. Now assume that e is not a loop of M. Then e labels a non-zero column of A. Suppose first that e labels a standard basis vector. For example, let e be the element 3 in the matrix A in Example 2.1. Then e determines a row of A, namely the one in which e has its unique non-zero entry. By deleting from A this row as well as the column labelled by e, it is not difficult to check using elementary linear algebra that we obtain a representation for M/e. In our example, the row in question is the third row of A and, by deleting from A both this row and the column labelled by 3, we obtain the matrix

$$\begin{array}{cccccc} 1 & 2 & 4 & 5 & 6 & 7 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

This matrix represents the contraction $M/3$.

What do we do if the non-zero column e is not a standard basis vector? By operations (3.14)(i)–(v), we can transform A into a matrix A' in which e does label a standard basis vector. Moreover, $M[A] = M[A']$ and we may now proceed as before to obtain an $\mathbb{F}$-representation for M/e.

Now that we know that certain basic classes of matroids are minor-closed, we can seek to describe such classes by a list of the minimal obstructions to membership of the class. Let $\mathcal{M}$ be a minor-closed class of matroids and let $\mathcal{EX}(\mathcal{M})$ be the collection of minor-minimal matroids not in $\mathcal{M}$, that is, $N \in \mathcal{EX}(\mathcal{M})$ if and only if $N \notin \mathcal{M}$ and every proper minor of N is in $\mathcal{M}$. The members of $\mathcal{EX}(\mathcal{M})$ are called *excluded minors* of $\mathcal{M}$. While the collection of excluded minors of a minor-closed class certainly exists, actually determining its members may be very difficult. Indeed, even determining whether it is finite or infinite may be hard. However, for the class $\mathcal{U}$ of uniform matroids, finding $\mathcal{EX}(\mathcal{U})$ is not difficult. To describe $\mathcal{EX}(\mathcal{U})$, it will be useful to introduce a way of sticking two matroids together.

3.21. **Proposition.** *Let M_1 and M_2 be the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ where E_1 and E_2 are disjoint. Let*

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}).$$

Then $M_1 \oplus M_2$ is a matroid.

We omit the proof of this proposition, which follows easily from **(I1)–(I3)**. The matroid $M_1 \oplus M_2$ is called the *direct sum* of M_1 and M_2. Evidently if G_1 and G_2 are disjoint graphs, then $M(G_1) \oplus M(G_2)$ is graphic since it is the cycle matroid of the graph obtained by taking the disjoint union of G_1 and G_2. Thus the class of graphic matroids is closed under direct sums. It is easy to check that, in general,

3.22. $(M_1 \oplus M_2)^* = M_1^* \oplus M_2^*$.

From this, it follows that the class of cographic matroids is also closed under direct sums. Moreover, the class of $\mathbb{F}$-representable matroids is closed under direct sums. To see this, note that if A_1 and A_2 are matrices over $\mathbb{F}$, then $M[A_1] \oplus M[A_2]$ is represented over $\mathbb{F}$ by the matrix $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

One consequence of the next result is that the class of uniform matroids is not closed under direct sums.

3.23. **Proposition.** *The unique excluded minor for the class $\mathcal{U}$ is $U_{0,1} \oplus U_{1,1}$.*

Proof. The matroid $U_{0,1} \oplus U_{1,1}$ is certainly not uniform since it has a 1-element independent set but not every 1-element set is independent. Moreover, every proper minor of $U_{0,1} \oplus U_{1,1}$ is easily seen to be uniform. Thus $U_{0,1} \oplus U_{1,1}$ is an excluded minor for $\mathcal{U}$.

Now suppose that N is an excluded minor for $\mathcal{U}$. We shall show that $N \cong U_{0,1} \oplus U_{1,1}$. Since N is not uniform, there is an integer k such that N has both a k-element independent set and a k-element dependent set. Pick the least such k and let C be a k-element dependent set. Then C is a circuit of M. Choose e in C and consider $C - \{e\}$. This is a $(k-1)$-element independent set of M. Since M has a k-element independent set, it follows by **(I3)** that M has an element f such that $(C - \{e\}) \cup \{f\}$ is independent. Now $M/(C - \{e\})$ has $\{e\}$ as a circuit and has $\{f\}$ as an independent set. Since M is an excluded minor for $\mathcal{U}$, we deduce that $M/(C - \{e\}) = M$ so $C - \{e\}$ is empty. If we now delete from M every element except e and f, we still have a matroid in which $\{e\}$ is a circuit and $\{f\}$ is an independent set. The fact that M is an excluded minor now implies that $E(M) = \{e, f\}$ and we conclude that $N \cong U_{0,1} \oplus U_{1,1}$. □

3.24. **Exercise.** Let M be a rank-r matroid.

(i) Show that the following statements are equivalent:
   (a) M is uniform;
   (b) every circuit of M has at least $r+1$ elements; and
   (c) every circuit of M meets every cocircuit of M.

(ii) The matroid M is *paving* if and only if every circuit has at least r elements. Show that M is paving if and only if it has no minor isomorphic to $U_{0,1} \oplus U_{2,2}$.

Finding the collections of excluded minors for the various other classes of matroids that we have considered is not as straightforward. It is worth noting that once we know the excluded minors for the class of graphic matroids, we simply take the duals of these excluded minors to get the excluded minors for the class of cographic matroids. Another useful general observation is that if $\mathcal{M}$ is a class of matroids that is closed under both minors and duals, then the dual of every excluded minor for $\mathcal{M}$ is also an excluded minor for $\mathcal{M}$. In Section 5, we shall answer the following question:

3.25. **Question.** *What is the collection of excluded minors for the class of graphic matroids?*

We showed in Proposition 2.8 that $U_{2,4}$ is not binary. In fact, $U_{2,4}$ is an excluded minor for the class of binary matroids because if e is an element of $U_{2,4}$, then $U_{2,4}\backslash e \cong U_{2,3}$ and $U_{2,4}/e \cong U_{1,3}$. Both $U_{2,3}$ and $U_{1,3}$ are binary being represented by the matrices $\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]$ and $[111]$, respectively. Tutte [50] established a number of interesting properties of binary matroids and thereby showed that $U_{2,4}$ is the unique excluded minor for the class:

3.26. **Theorem.** *The following statements are equivalent for a matroid M.*

(i) *M is binary.*
(ii) *For every circuit C and cocircuit C^* of M, $|C \cap C^*|$ is even.*
(iii) *If C_1 and C_2 are distinct circuits of M, then $(C_1 \cup C_2) - (C_1 \cap C_2)$ is a disjoint union of circuits.*
(iv) *M has no minor isomorphic to $U_{2,4}$.*

3.27. **Exercise.** In the last theorem, show that (i) implies (ii).

The last theorem gives several different answers to Question 2.11. A proof of the equivalence of (i) and (iv) will be given later in Theorem 5.15. In view of this equivalence, it is natural to ask:

3.28. **Question.** *What is the collection of excluded minors for the class of ternary matroids?*

Many of the attractive properties of binary matroids are not shared by ternary matroids. Nevertheless, the collection of excluded minors for the latter class has been found. As we shall see in Theorem 5.14, it contains exactly four members. Motivated in part by the knowledge of the excluded minors for the classes of binary and ternary matroids, Rota [40] made the following conjecture in 1970 and this conjecture has been a focal point for matroid theory research ever since, particularly in the last five years.

3.29. **Conjecture.** *For every finite field $GF(q)$, the collection of excluded minors for the class of matroids representable over $GF(q)$ is finite.*

As we shall see in Section 5, until recently, progress on this conjecture has been relatively slow and it has only been settled for one further case. By contrast, it is known that, for all infinite fields, $\mathbb{F}$, there are infinitely many excluded minors for $\mathbb{F}$-representability. Theorem 5.9 establishes this for an important collection of fields including $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

4. Matroids and Combinatorial Optimization

Matroids play an important role in combinatorial optimization. In this section, we briefly indicate the reason for this by showing first how matroids occur naturally in scheduling problems and then how the definition of a matroid arises inevitably from the greedy algorithm. A far more comprehensive treatment of the part played by matroids in optimization can be found in the survey of Bixby and Cunningham [4] or the book by Cook, Cunningham, Pulleyblank, and Schrijver [8, Chapter 8]. We begin with another example of a class of matroids. Suppose that a supervisor has m one-worker one-day jobs $J_1, J_2, \dots, J_m$ that need to be done. The supervisor controls n workers $1, 2, \dots, n$, each of whom is qualified to perform some subset of the jobs. The supervisor wants to know the maximum number of jobs the workers can do in one day. As we shall see, this number is the rank of a certain matroid.

Let $\mathcal{A}$ be a collection $(A_1, A_2, \dots, A_m)$ of subsets of a finite set E. For example, let $\mathcal{A} = (\{1, 2, 4\}, \{2, 3, 5, 6\}, \{5, 6\}, \{7\})$. A subset $\{x_1, x_2, \dots, x_k\}$

of E is a *partial transversal* of $\mathcal{A}$ if there is a one-to-one mapping ϕ from $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, m\}$ such that $x_i \in A_{\phi(i)}$ for all i. A partial transversal with $k = m$ is called a *transversal*. In our specific example, $\{2, 3, 6, 7\}$ is a transversal because $2, 3, 6$, and 7 are in A_1, A_2, A_3, and A_4, respectively.

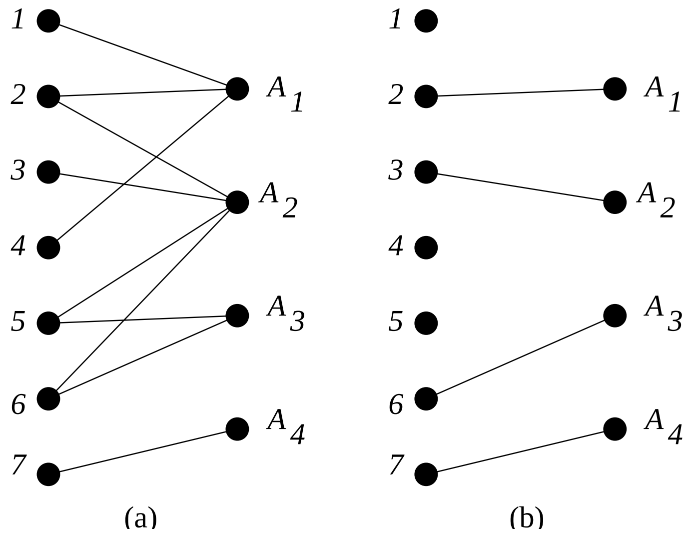

FIGURE 4. (a) $\Delta(\mathcal{A})$. (b) A matching in $\Delta(\mathcal{A})$.

4.1. **Theorem.** *Let $\mathcal{A}$ be a collection of subsets of a finite set E. Let $\mathcal{I}$ be the collection of all partial transversals of $\mathcal{A}$. Then $(E, \mathcal{I})$ is a matroid.*

Proof. Clearly every subset of a partial transversal is a partial transversal, and the empty set is a partial transversal of the empty family of subsets of $\mathcal{A}$. Thus **(I2)** and **(I1)** hold. To prove that $\mathcal{I}$ satisfies **(I3)**, we associate a bipartite graph $\Delta(\mathcal{A})$ with $\mathcal{A}$ as follows. Label one vertex class of the bipartite graph by the elements of E and the other vertex class by the sets $A_1, A_2, \ldots, A_m$ in $\mathcal{A}$. Put an edge from an element e of E to a set A_j if and only if $e \in A_j$. As an example, the bipartite graph associated with the specific family listed above is shown in Figure 4(a). A partial transversal of $\mathcal{A}$ corresponds to a *matching* in $\Delta(\mathcal{A})$, that is, a set of edges no two of which meet at a common vertex. The matching associated with the partial transversal $\{2, 3, 6, 7\}$ noted above is shown in Figure 4(b).

Let X and Y be partial transversals of $\mathcal{A}$ where $|X| = |Y| + 1$. Consider the matchings in $\Delta(\mathcal{A})$ corresponding to X and Y and colour the edges of these matchings blue and red, respectively, where an edge that is in both matchings is coloured purple. Thus there are $|X - Y|$ blue edges and $|Y - X|$ red edges, and $|X - Y| = |Y - X| + 1$. Focussing on the red edges and blue edges only, we see that each vertex of the subgraph H induced by these edges either meets a single edge or meets both a red edge and a blue edge.

It is a straightforward exercise in graph theory to show that each component of H is a path or a cycle where in each case, the edges alternate in colour. Because $\Delta(\mathcal{A})$ is a bipartite graph, every cycle in H is even and so has the same number of red and blue edges. Since there are more blue edges than red in H, there must be a component H' of H that is path that begins and ends with blue edges. In H', interchange the colours red and blue. Then the edges of $\Delta(\mathcal{A})$ that are now coloured red or purple form a matching, and it is not difficult to check that the subset of E that is met by an edge of this matching is $Y \cup \{x\}$ for some x in $X - Y$. We conclude that $\mathcal{I}$ satisfies **(I3)** and so $(E, \mathcal{I})$ is a matroid. □

We denote the matroid obtained in the last theorem by $M[\mathcal{A}]$ and call a matroid that is isomorphic to such a matroid *transversal*. We leave it to the reader to check that when $\mathcal{A}$ is the family $(\{1,2,4\},\{2,3,5,6\},\{5,6\},\{7\})$ considered above, the transversal matroid $M[\mathcal{A}]$ is isomorphic to the cycle matroid of the graph G^* in Figure 2. This can be achieved by showing, for example, that the list of edge sets of spanning trees of G^*, which was compiled just before Theorem 3.4, coincides with the list of transversals of $\mathcal{A}$.

Returning to the problem with which we began the section, if we let A_i be the set of workers that are qualified to do job J_i, then the maximum number of jobs that can be done in a day is the rank of $M[\mathcal{A}]$. This is given by the following result, a consequence of a theorem of Ore [33].

4.2. **Theorem.** *Let $\mathcal{A}$ be a family $(A_1, A_2, \ldots, A_m)$ of subsets of a finite set E. Then the rank of $M[\mathcal{A}]$ is*

$$\min\{|\cup_{j\in J} A_j| - |J| + m : J \subseteq \{1,2,\ldots,m\}\}.$$

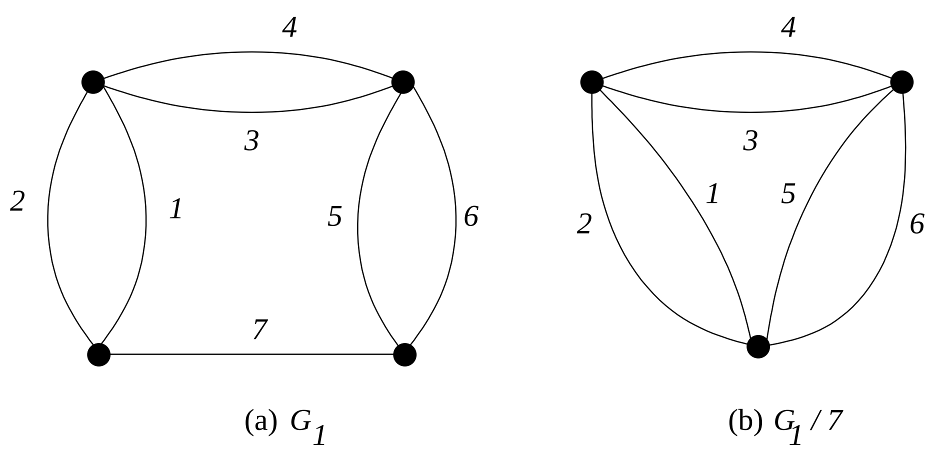

FIGURE 5. (a) $M(G_1)$ is transversal. (b) $M(G_1/7)$ is not transversal.

The class of transversal matroids differs from the other classes that we have considered in that it is not closed under minors.

4.3. **Example.** Consider the graphs G_1 and $G_1/7$ shown in Figure 5. The cycle matroid $M(G_1)$ is transversal since, as is easily checked, $M(G_1) = M[\mathcal{A}]$ where $\mathcal{A} = (\{1,2,7\},\{3,4,7\},\{5,6,7\})$. On the other hand, $M(G_1)/7$, which equals $M(G_1/7)$, is not transversal. To see this, we note that if $M(G_1/7)$ is transversal, then there is a family $\mathcal{A}'$ of sets such that $M(G_1/7) = M[\mathcal{A}']$. Each single-element subset of $\{1,2,\dots,6\}$ is independent but $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$ are dependent. This means that each of $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$ is a subset of exactly one member of $\mathcal{A}'$. Let these sets be A'_1, A'_2, and A'_3, respectively. Then, since $\{1,3\}$, $\{1,5\}$, and $\{3,5\}$ are all independent, A'_1, A'_2, and A'_3 are distinct. Thus $\{1,3,5\}$ is a partial transversal of $\mathcal{A}'$ and so is independent in $M(G_1/7)$; a contradiction. We conclude that the class of transversal matroids is not closed under contraction, although it is clearly closed under deletion.

4.4. **Exercise.** Show that

(i) every uniform matroid is transversal;
(ii) $M(K_4)$ is not transversal.

We have seen a number of examples of structures that give rise to matroids. We now look at one that does not. If G is a graph and $\mathcal{I}$ is the collection of edge sets of matchings in G, then $(E(G),\mathcal{I})$ need not be a matroid. To see this, let G be a 3-edge path whose edges are labelled, in order, $1,2,3$. Let $X = \{1,3\}$ and $Y = \{2\}$. Then X and Y satisfy the hypotheses but not the conclusion of **(I3)**. In fact, if one wants to use the matchings in a graph G to define a matroid, then this matroid should be defined on the vertex set $V(G)$ of G. Indeed, Edmonds and Fulkerson [12] proved the following result.

4.5. **Theorem.** *Let G be a graph and $\mathcal{I}$ be the set of subsets X of $V(G)$ such that G has a matching whose set of endpoints contains X. Then $(V(G),\mathcal{I})$ is a matroid.*

Interestingly, Edmonds and Fulkerson [12] also proved that the class of matroids arising as in the last theorem coincides with the class of transversal matroids.

While matroids arise in a number of places in combinatorial optimization, their most striking appearance relates to the greedy algorithm. Let G be a connected graph and suppose that each edge e of G has an assigned positive real weight $w(e)$. Let $\mathcal{I}$ be the collection of independent sets of $M(G)$. Kruskal's Algorithm [25], which is described next, finds a maximum-weight spanning tree of G, that is, a spanning tree such that the sum of the weights of the edges is a maximum. It is attractive because, by pursuing a locally greedy strategy, it finds a global maximum.

4.6. **The Greedy Algorithm.**

(i) *Set $B_G = \emptyset$.*

(ii) *While there exists* $e \notin B_G$ *for which* $B_G \cup \{e\} \in \mathcal{I}$, *choose such an* e *with* $w(e)$ *maximum, and replace* B_G *by* $B_G \cup \{e\}$.

Now let M be a matroid $(E, \mathcal{I})$ and assume that each element e of E has an associated positive real weight $w(e)$. Then the Greedy Algorithm also works for M.

4.7. **Lemma.** *When the Greedy Algorithm is applied to* M, *the set* B_G *it produces is a maximum-weight independent set and hence a maximum-weight basis of* M.

Proof. Since all weights are positive, a maximum-weight independent set B of M must be a basis of M. Moreover, the set B_G is also a basis of M. Let $B_G = \{e_1, e_2, \ldots, e_r\}$ where the elements are chosen in the order listed. Then $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_r)$. Let $B = \{f_1, f_2, \ldots, f_r\}$ where $w(f_1) \geq w(f_2) \geq \ldots \geq w(f_r)$. We shall show that $w(e_j) \geq w(f_j)$ for all j in $\{1, 2, \ldots, r\}$. Assume the contrary and let $k+1$ be the least integer for which $w(e_{k+1}) < w(f_{k+1})$. Let $Y = \{e_1, e_2, \ldots, e_k\}$ and $X = \{f_1, f_2, \ldots, f_{k+1}\}$. Since $|X| = |Y|+1$, **(I3)** implies that $Y \cup \{f_i\} \in \mathcal{I}$ for some i in $\{1, 2, \ldots, k+1\}$. But $w(f_i) \geq w(f_{k+1}) > w(e_{k+1})$. Hence the Greedy Algorithm would have chosen f_i in preference to e_{k+1}; a contradiction. We conclude that we do indeed have $w(e_j) \geq w(f_j)$ for all j. Thus $\sum_{j=1}^{r} w(e_j) \geq \sum_{j=1}^{r} w(f_j)$; that is, B_G has weight at least that of B. Since B has maximum weight, so does B_G. □

While it is interesting that the Greedy Algorithm extends from graphs to matroids, the particularly striking result here is that matroids are the only non-empty hereditary structures for which the Greedy Algorithm works.

4.8. **Theorem.** *Let* $\mathcal{I}$ *be a collection of subsets of a finite set* E. *Then* $(E, \mathcal{I})$ *is a matroid if and only if* $\mathcal{I}$ *satisfies* **(I1)**, **(I2)**, *and*

(G) *for all positive real weight functions* w *on* E, *the Greedy Algorithm produces a maximum-weight member of* $\mathcal{I}$.

Proof. If $(E, \mathcal{I})$ is a matroid, then it follows from the definition and the last lemma that **(I1)**, **(I2)**, and **(G)** hold. For the converse, assume that $\mathcal{I}$ satisfies **(I1)**, **(I2)**, and **(G)**. We need to show that $\mathcal{I}$ satisfies **(I3)**. Assume it does not and let X and Y be members of $\mathcal{I}$ such that $|X| = |Y| + 1$ but that $Y \cup \{e\} \notin \mathcal{I}$ for all e in $X - Y$. Now $|X - Y| = |Y - X| + 1$ and $Y - X$ is non-empty, so we can choose a real number ε such that $0 < \varepsilon < 1$ and

$$0 < (1 + 2\varepsilon)|Y - X| < |X - Y|.$$

Define a weight function w on E by

$$w(e) = \begin{cases} 2, & \text{if } e \in X \cap Y; \\ \frac{1}{|Y-X|}, & \text{if } e \in Y - X; \\ \frac{1+2\varepsilon}{|X-Y|}, & \text{if } e \in X - Y; \\ \frac{\varepsilon}{|X-Y||E-(X\cup Y)|}, & \text{if } e \in E - (X \cup Y) \neq \emptyset. \end{cases}$$

The Greedy Algorithm will first pick all the elements of $X \cap Y$ and then all the elements of $Y - X$. By assumption, it cannot then pick any element of $X - Y$. Thus the remaining elements of B_G will be in $E - (X \cup Y)$. Hence $w(B_G)$, the sum of the weights of the elements of B_G, satisfies

$$\begin{aligned} w(B_G) &\leq 2|X \cap Y| + \frac{|Y - X|}{|Y - X|} + \frac{|E - (X \cup Y)|\varepsilon}{|X - Y||E - (X \cup Y)|} \\ &\leq 2|X \cap Y| + 1 + \varepsilon. \end{aligned}$$

But, by **(I2)**, X is contained in a maximal member X' of $\mathcal{I}$, and

$$\begin{aligned} w(X') \geq w(X) &= 2|X \cap Y| + |X - Y| \frac{1 + 2\varepsilon}{|X - Y|} \\ &= 2|X \cap Y| + 1 + 2\varepsilon. \end{aligned}$$

Thus $w(X') > w(B_G)$, that is, the Greedy Algorithm fails for this weight function. This contradiction completes the proof of the theorem. □

A number of proofs of the last result have been published. Curiously, what seems to be the first of these was obtained by Borůvka [5] in 1926 nearly a decade before Whitney introduced matroids.

5. Excluded-minor Theorems

In this section, we answer many of the questions that were raised earlier by giving excluded-minor characterizations of each of the classes of ternary, regular, graphic, and cographic matroids. In addition, some problems that are the focus of current research attention are identified. Most of the results in this section concern matroid minors. For a more detailed survey of this topic, see Seymour [47]. Very few proofs are included here but many may be found in [34]. We begin this section by describing another way to represent certain matroids.

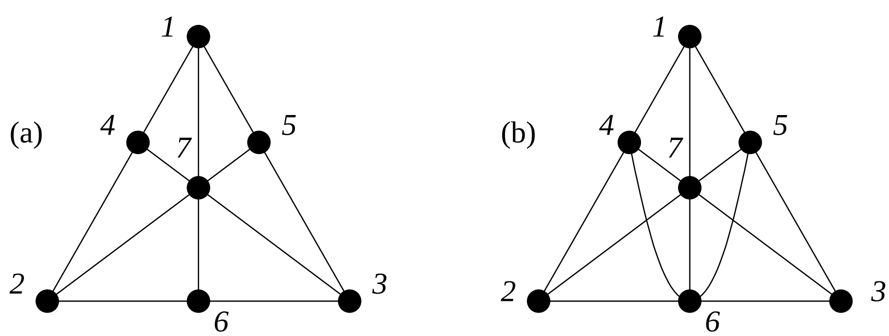

Figure 6. (a) The non-Fano matroid. (b) The Fano matroid.

Consider the diagram in Figure 6(a). Let E be the set $\{1, 2, \ldots, 7\}$ of points and let $\mathcal{I}$ be the collection of subsets X of E such that $|X| \leq 3$ and X does not contain 3 collinear points. Then it is not difficult to check that

$(E, \mathcal{I})$ is a matroid. Indeed, this matroid is represented over $GF(3)$ by the matrix

$$A_7 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Now suppose that we view A_7 as a matrix over $GF(2)$. Then $M[A_7]$ has $\{4,5,6\}$ as a circuit. We can represent this new matroid as in Figure 6(b) where 4,5, and 6 lie on a curved line as shown. This configuration of 7 points and 7 lines is known as the Fano projective plane, $PG(2,2)$. The corresponding matroid is called the *Fano matroid* and is denoted by F_7. The matroid in Figure 6(a), which does not have the curved line, is denoted by F_7^- and is called the *non-Fano matroid*. The Fano matroid has more symmetry than Figure 6(b) may suggest. For example, if we add row 1 to row 2 in A_7, then, modulo 2, we recover A_7 with its columns reordered. Thus F_7 has a symmetry that interchanges 1 with 4 and 5 with 7. It follows that, up to symmetry, all the points of F_7 look the same, as do all the lines.

In general, suppose we have a finite set E of points in the plane and a distinguished collection of subsets of the points, called lines, such that any two distinct lines have at most one common point.

5.1. **Exercise.** Show that we get a matroid on E having as its independent sets all subsets of E of size at most 3 that do not contain 3 points from a common line.

Another example of a matroid that is obtained as in the last exercise is the 13-point matroid shown in Figure 7, which has thirteen lines including $\{1,2,4,5\}, \{5,6,8,11\}, \{5,7,9,10\}, \{1,8,10,13\}$, and $\{2,6,10,12\}$. The reader may recognize this diagram as the 13-point projective plane, $PG(2,3)$. We leave it as an exercise to check that this matroid is the vector matroid of the following matrix over $GF(3)$:

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \end{array}.$$

The diagrams of the matroids that appear in Figures 6 and 7 are called *geometric representations* of the matroids. If we delete the point 6 in Figure 6(b), we obtain the diagram in Figure 8(a). It is not difficult to check that this is a geometric representation for $M(K_4)$ where K_4 is the graph labelled as in Figure 8(b). The symmetry of the Fano matroid implies that all of its single-element deletions are isomorphic to $M(K_4)$ and hence are graphic.

If B is a basis of a matroid M with ground set E and $e \in E - B$, then $B \cup \{e\}$ contains a circuit $C(e,B)$. Moreover, by **(C3)**, this circuit is unique. We call $C(e,B)$ the *fundamental circuit of e with respect to B*. Now suppose

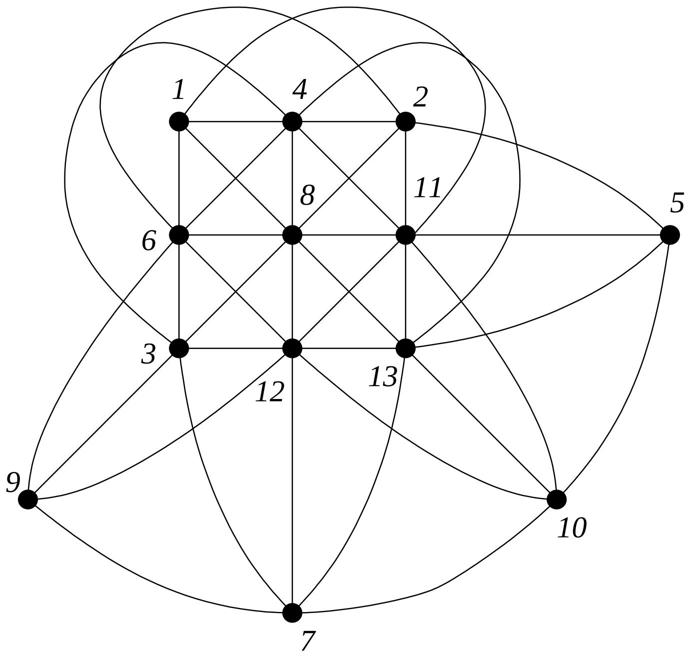

FIGURE 7. A 13-point matroid with 13 lines.

that M is represented over a field $\mathbb{F}$ by the matrix $[I_r|D]$ where the first r columns $b_1, b_2, \ldots, b_r$ of this matrix correspond to the basis B. Suppose e labels a column of D and let $C(e,B) = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}, e\}$. Then some linear combination of the columns $b_{i_1}, b_{i_2}, \ldots, b_{i_k}, e$ must be the zero vector. Moreover, since $C(e,B)$ is a minimal dependent set, no coefficient in this linear combination is zero. It follows that column e is non-zero in row j if and only if $j \in \{i_1, i_2, \ldots, i_k\}$. Hence the fundamental circuits of B completely determine the pattern of zero and non-zero entries in D. In particular, if $\mathbb{F}$ is $GF(2)$, then, because $GF(2)$ has a single non-zero element, the fundamental circuits uniquely determine D. Formally, we have the following:

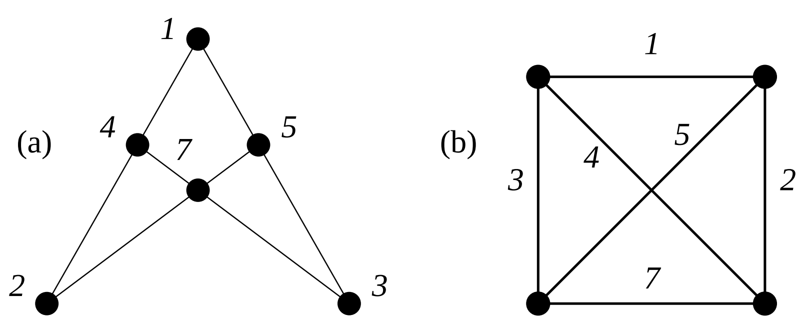

FIGURE 8. (a) A geometric representation for $M(K_4)$. (b) K_4.

5.2. **Lemma.** *If M is a binary matroid with ground set E and a basis B, then M is uniquely determined by B and the set of circuits $C(e, B)$ such that $e \in E - B$.*

We shall use the Fano and non-Fano matroids to show that there is a matroid that is not representable over any field. The next result uses the notion of the *characteristic* of a field $\mathbb{F}$. This is the least positive integer m such that $m \cdot 1 = 0$ in $\mathbb{F}$; if no such integer m exists, then $\mathbb{F}$ has characteristic 0. Thus, for example, for all primes p, the field $GF(p^k)$ has characteristic p, while the fields $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ all have characteristic 0. Moreover, by considering the elements that can be produced by sums, differences, products, and quotients starting with 1, it is not difficult to see that every field of prime characteristic p has $GF(p)$ as a subfield, while every field of characteristic 0 has $\mathbb{Q}$ as a subfield.

5.3. **Proposition.** *Let $\mathbb{F}$ be a field.*

(i) *F_7 is $\mathbb{F}$-representable if and only if the characteristic of $\mathbb{F}$ is two; and*

(ii) *F_7^- is $\mathbb{F}$-representable if and only if the characteristic of $\mathbb{F}$ is not two.*

Proof. Suppose that $M \in \{F_7, F_7^-\}$ and that M is $\mathbb{F}$-representable for some field $\mathbb{F}$. Because we know that M is represented over some field by the matrix A_7, it follows, from considering fundamental circuits, that an $\mathbb{F}$-representation of M has the same pattern of zeros and non-zeros as A_7. Thus we may assume that M has an $\mathbb{F}$-representation A' of the form

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} * & 0 & 0 & * & * & 0 & * \\ 0 & * & 0 & * & 0 & * & * \\ 0 & 0 & * & 0 & * & * & * \end{bmatrix}, \end{array}$$

where each $*$ represents some non-zero member of $\mathbb{F}$ and two different $*$-entries need not be equal. Now, by multiplying columns of A' by non-zero members of $\mathbb{F}$, we may assume that the first $*$-entry in each column is 1. Then, by multiplying rows 2 and 3 and then columns 2, 3, and 6 by non-zero elements of $\mathbb{F}$, we can make all entries in column 7 equal to 1 while maintaining the fact that the first $*$-entry in each column is 1. Hence we may assume that

$$A' = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & a & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & b & c & 1 \end{bmatrix}, \end{array}$$

where a, b, and c are non-zero elements of $\mathbb{F}$. Because M has each of $\{3, 4, 7\}, \{2, 5, 7\}$, and $\{1, 6, 7\}$ as a circuit, it follows that each of a, b, and c is 1. Thus $A' = A_7$. We conclude that if M is $\mathbb{F}$-representable, then M is represented over $\mathbb{F}$ by the matrix A_7. Now the 3×3 matrix labelled by columns $4, 5$, and 6 has determinant -2. In F_7, the set $\{4, 5, 6\}$ is a circuit, while, in F_7^-, it is a basis. Thus if F_7 is $\mathbb{F}$-representable, then $-2 = 0$ in $\mathbb{F}$, so $\mathbb{F}$ has characteristic 2. Similarly, if F_7^- is $\mathbb{F}$-representable, then $-2 \neq 0$ so $\mathbb{F}$ has characteristic not 2. By its definition, F_7 is $GF(2)$-representable so it is representable over all fields of characteristic 2, and we deduce that (i) holds. To complete the proof of (ii), we just need to show that $M[A_7]$ and F_7^- have the same set of circuits when A_7 is viewed over any field of characteristic other than 2. But since we already know that $M[A_7]$ and F_7^- share as circuits all sets consisting of 3 collinear points in Figure 8(a), this leaves little to check and (ii) follows without difficulty. □

We are now able to answer Question 2.7.

5.4. **Corollary.** *The matroid $F_7 \oplus F_7^-$ is not representable.*

Proof. Both F_7 and F_7^- are minors of $F_7 \oplus F_7^-$. The corollary now follows immediately from the last proposition. □

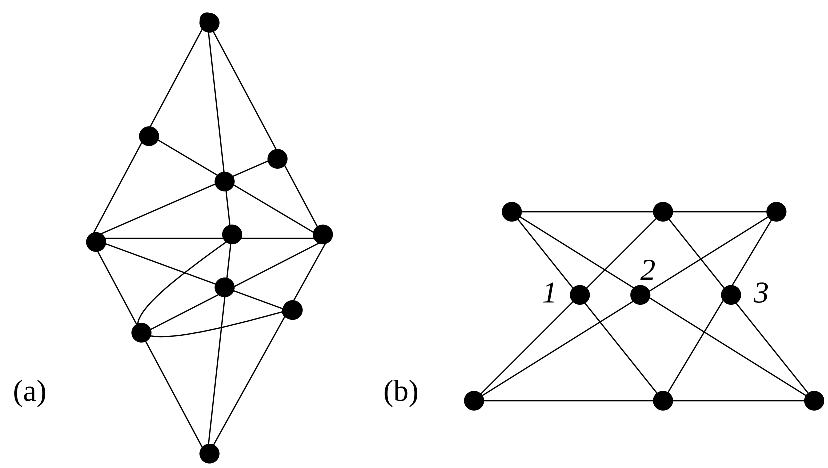

FIGURE 9. Non-representable matroids with 11 and 9 elements.

One may ask whether $F_7 \oplus F_7^-$ is a smallest non-representable matroid and that question is easily resolved. If we stick F_7 and F_7^- together in the plane along a line as in Figure 9(a), then we obtain an 11-element matroid having both F_7 and F_7^- as minors. This matroid is also non-representable. As an aside for the reader familiar with projective geometry, we note that, by Pappus's Theorem, if the configuration shown in Figure 9(b) exists in a projective geometry over a field, then the points 1, 2, and 3 must be collinear. It follows from this that the 9-element rank-3 matroid for which Figure 9(b) is a geometric representation is non-representable. But there are even smaller non-representable matroids and we now describe one of these. This construction will use the following result, which can be proved using Theorem 3.2.

5.5. **Proposition.** *Let M be a matroid with ground set E and collection of bases $\mathcal{B}$. If C is a circuit of M such that $E - C$ is a cocircuit of M, then $\mathcal{B} \cup \{C\}$ is the set of bases of a matroid M'_C on E.*

The matroid M'_C in the last proposition is said to be obtained from M by *relaxing* C. Thus, for example, the non-Fano matroid is obtained from the Fano matroid by relaxing the line $\{4, 5, 6\}$. The proof of the next result is not difficult.

5.6. **Lemma.** *Let M be a matroid with ground set E and let C be a circuit of M such that $E - C$ is a cocircuit of M.*

(i) *If $e \in C$, then $M'_C \backslash e = M \backslash e$, and if $|C| \geq 2$, then $C - \{e\}$ is a circuit of M/e whose complement in M/e is a cocircuit of M/e, and M'_C/e is obtained from M/e by relaxing $C - \{e\}$.*

(ii) *If $f \in E - C$, then $M'_C/f = M/f$, and if $|E - C| \geq 2$, then C is a circuit of $M \backslash f$ whose complement in $M \backslash f$ is a cocircuit of $M \backslash f$, and $M'_C \backslash f$ is obtained from $M \backslash f$ by relaxing C.*

Consider the matroid $AG(3, 2)$ that is represented over $GF(2)$ by the matrix

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus the columns of $AG(3, 2)$ consist of all the vectors $(x_1, x_2, x_3, x_4)^T$ in the 4-dimensional vector space over $GF(2)$ such that $x_1 + x_2 + x_3 + x_4 \neq 0$. Evidently $AG(3, 2)/4 \cong F_7$. Moreover, $\{1, 4, 5, 8\}$ is a circuit C of $AG(3, 2)$ whose complement is a cocircuit. Thus we can relax C to obtain $AG(3, 2)'_C$ and, by the last lemma, $AG(3, 2)'_C/2 = AG(3, 2)/2 \cong F_7$. Moreover, $AG(3, 2)'_C/1$ is the matroid that is obtained from $AG(3, 2)/1$ by relaxing $\{4, 5, 8\}$. But $AG(3, 2)/1 \cong F_7$ and it follows, by the symmetry of F_7, that $AG(3, 2)'_C/1 \cong F_7^-$. We conclude that $AG(3, 2)'_C$ has both F_7 and F_7^- as minors so it is non-representable. It is a smallest non-representable matroid, for Fournier [13] proved the following:

5.7. **Theorem.** *Every matroid on a set of at most 7 elements is representable. Moreover, every non-representable matroid on an 8-element set has rank 4.*

We have seen how to define matroids using points and lines in the plane. One can also define matroids using points, lines, and planes in 3-dimensional space. In particular, if E is a finite set of points in $\mathbb{R}^3$ and $\mathcal{I}$ consists of all subsets of E that contain at most four points but do not contain three collinear points or four coplanar points, then $(E, \mathcal{I})$ is a matroid. This construction can be generalized so that, as in Exercise 5.1, one relaxes the definition of a line and a plane. The rules governing these structures are geometrically intuitive and can be found, for example, in [34, p. 42]. Indeed, geometrical reasoning is a fundamental tool in matroid theory. The next exercise, the first part of which is somewhat vague, gives some of the flavour of the role of geometry in the subject.

5.8. **Exercise.** Show that

(i) by sticking F_7 and F_7^- together along a 3-point line to give a rank-4 matroid and then deleting the three points of this line, one can construct an 8-element non-representable matroid that is different from $AG(3,2)'$.
(ii) $M(K_5)$ can be represented geometrically as described above by ten points in 3-space. The reader familiar with projective geometry will recognize this configuration of 10 points and 10 lines as the 3-dimensional *Desargues configuration*.

We noted at the end of Section 3 that, for all infinite fields, the set of excluded minors for representability over that field is infinite. The next result shows this for all fields of characteristic 0 and hence, in particular, for $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Let J'_k be the $k \times k$ matrix that has zeros on the main diagonal and ones elsewhere. Lazarson [29] proved the following result.

5.9. **Theorem.** *Let $\mathbb{F}$ be a field of characteristic 0. For all prime numbers p, let L_p be the vector matroid of the matrix $[I_{p+1}|J'_{p+1}]$ viewed over $GF(p)$. Then L_p is an excluded minor for $\mathbb{F}$-representability.*

The model for all theorems that characterize classes of matroids by excluded minors is Wagner's modification [56] of Kuratowski's famous characterization of planar graphs [27]. The graphs K_5 and $K_{3,3}$ are shown in Figure 10. A *minor* of a graph G is a graph H that can be obtained from G by deleting or contracting edges, or deleting isolated vertices.

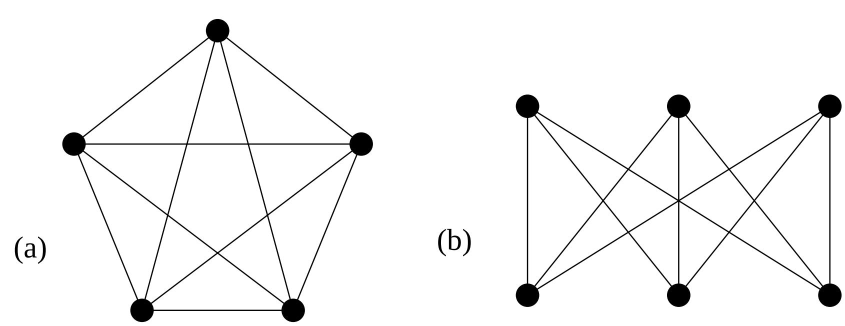

FIGURE 10. (a) K_5. (b) $K_{3,3}$.

5.10. **Theorem.** *A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

Tutte [51] generalized this theorem to give an excluded-minor characterization of graphic matroids. The fact that neither K_5 nor $K_{3,3}$ is planar means that the bond matroids of these two graphs are not graphic although this is not immediate (see, for example, [34, Theorem 5.2.2]). These two bond matroids are among the five excluded minors for the class of graphic matroids. Of the other three, one, namely $U_{2,4}$, has been shown in Proposition 2.8 to be non-binary and so, by Theorem 2.16, is non-graphic. The other two excluded minors are F_7 and F_7^*. To see that F_7 is non-graphic, recall from Figure 8 that every single-element deletion of F_7 is isomorphic to the cycle matroid of the complete graph K_4. As F_7 is simple having the same rank as $M(K_4)$, we deduce that F_7 is non-graphic. If F_7^* is graphic, then it is isomorphic to $M(G)$ for some connected 5-vertex graph G. Clearly G has 7 edges and so has average degree less than 3. Thus G has a vertex of degree at most 2, so F_7^* has a cocircuit of size at most 2. This implies that F_7 has a circuit of size at most 2, and this contradiction establishes that F_7^* is non-graphic.

5.11. **Theorem.** *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}, F_7, F_7^*, M^*(K_5)$, or $M^*(K_{3,3})$.*

5.12. **Corollary.** *A matroid is cographic if and only if it has no minor isomorphic to $U_{2,4}, F_7, F_7^*, M(K_5)$, or $M(K_{3,3})$.*

Finding a list of candidates for the excluded minors for the class of ternary matroids is not difficult. We know that F_7 is non-ternary. Moreover, all its proper minors are ternary, so F_7 is an excluded minor. Since the dual of every ternary matroid is ternary, F_7^* is also an excluded minor. From Exercise 2.9, two other excluded minors are $U_{2,5}$ and its dual $U_{3,5}$. Indeed, as the reader will easily show:

5.13. **Lemma.** *The matroid $U_{2,n}$ is representable over a field $\mathbb{F}$ if and only if $\mathbb{F}$ has at least $n-1$ elements.*

Although this lemma completely settles the question of when a rank-2 uniform matroid is representable over a field, it is an open problem to determine all the fields over which an arbitrary $U_{r,n}$ is representable. This problem has received considerable attention in projective geometry where uniform matroids are called *n-arcs*. The history of the problem and progress towards its solution are described in [18, 19]

The last lemma means that we have now identified four excluded minors for the class of ternary matroids and these four were conjectured to be the only such matroids. In 1971, at a National Science Foundation Advanced Science Seminar held at Bowdoin College, Maine, Ralph Reid gave a lecture in which he announced a proof of this conjecture that was based on techniques introduced by Tutte [50]. However, Reid never published his proof. In 1975, Bob Bixby and Paul Seymour, working independently, obtained two different proofs of the conjecture. Indeed, Bixby called his paper "On Reid's characterization of the ternary matroids". Both proofs appeared in the same issue of the *Journal of Combinatorial Theory Series B* in 1979, and several more proofs of this result have appeared since. None is elementary enough for inclusion here. However, following the theorem statement, we prove the excluded-minor characterization of binary matroids, using a proof that gives some flavour of the ternary-matroid result.

5.14. **Theorem.** *A matroid is ternary if and only if it has no minor isomorphic to $U_{2,5}, U_{3,5}, F_7$, or F_7^*.*

5.15. **Theorem.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

Proof. We showed following Question 3.25 that $U_{2,4}$ is an excluded minor for the class of binary matroids. Now let M be an arbitrary excluded minor for the class of binary matroids. Then M has no 1- or 2-element circuits and, since M^* is also an excluded minor, M has no 1- or 2-element cocircuits. Thus if x and y are distinct elements of M, then $M\backslash\{x,y\}$ has the same rank as M. Let $[I_r|D]$ be a matrix representing $M\backslash\{x,y\}$ over $GF(2)$. As $M\backslash x$ and $M\backslash y$ are binary, there are column vectors v_x and v_y such that $[I_r|D|v_x]$ and $[I_r|D|v_y]$ represent $M\backslash y$ and $M\backslash x$, respectively, over $GF(2)$. Now let M' be the matroid that is represented over $GF(2)$ by $[I_r|D|v_x|v_y]$. Then $M\backslash x = M'\backslash x$ and $M\backslash y = M'\backslash y$. Since $M \neq M'$, there is a set that is independent in one of M and M' and dependent in the other. Take Z to be a minimal such set. Then Z is an independent set in one of M and M', say M_I, and a circuit in the other, M_C. As $M_I\backslash x = M_C\backslash x$ and $M_I\backslash y = M_C\backslash y$, we have that $\{x,y\} \subseteq Z$.

We shall show that

5.15.1. $Z = \{x,y\}$*; and*

5.15.2. $r(M_I) = 2$.

Both of these assertions are consequences of the following:

5.15.3. *Let J be an independent set of M_I containing Z. Then $J = \{x,y\}$*

To show this, suppose that $J-\{x,y\}$ is non-empty. This set is independent in $M_I\backslash\{x,y\}$ and so in $M_C\backslash\{x,y\}$. Hence if we contract $J-\{x,y\}$ from M_I and M_C, we get matroids N_I and N_C of the same rank. Since one of M_I and M_C is binary and the other is an excluded minor for the class of binary matroids, both N_I and N_C are binary. Clearly,

$$N_I \neq N_C$$

since $\{x,y\}$ is independent in N_I and dependent in N_C. But $N_I\backslash x = N_C\backslash x$ and $N_I\backslash y = N_C\backslash y$. Consider $N_I\backslash\{x,y\}$, which equals $N_C\backslash\{x,y\}$. Take a basis B of this matroid. Because N_I and N_C have the same rank, either B is a basis of both these matroids, or it is a basis of neither. In the latter case, by Theorem 3.9, $\{x,y\}$ contains a cocircuit of both N_I and N_C. Thus, by (3.19) and Proposition 3.18(ii), $\{x,y\}$ contains a cocircuit of both M_I and M_C. This contradicts the fact that M has no cocircuit of size at most 2.

We now know that B is a basis of both N_I and N_C. Since $N_I \backslash x = N_C \backslash x$ and $N_I \backslash y = N_C \backslash y$, it follows that if $e \in E(N_I) - B$, then $C_{N_I}(e, B) = C_{N_C}(e, B)$. Hence, by Lemma 5.2, $N_I = N_C$. This contradiction implies that (5.15.3) holds.

Clearly (5.15.1) follows immediately from (5.15.3). To get (5.15.2), note that, as Z is independent in M_I, it is contained in a basis J of M_I. By (5.15.3), $J = \{x, y\}$ so $r(M_I) = 2$.

We now know that all of $M_I, M_C, M_I \backslash \{x, y\}$, and $M_C \backslash \{x, y\}$ have rank 2. Thus M, which is M_I or M_C, has rank 2, at least four elements, and no 1- or 2-element circuits. Hence $U_{2,4}$ is a deletion of M and we conclude that $M \cong U_{2,4}$. □

The matroid in Example 2.1 is representable over every field. We recall that such matroids are called regular. Several attractive characterizations of regular matroids were proved by Tutte [50].

5.16. **Theorem.** *The following statements are equivalent for a matroid M.*

(i) *M is regular.*
(ii) *M is both binary and ternary.*
(iii) *M is representable over $GF(2)$ and some field of characteristic other than 2.*
(iv) *M is representable over $\mathbb{R}$ by a matrix all of whose square submatrices have determinants in $\{0, 1, -1\}$.*
(v) *M has no minor isomorphic to $U_{2,4}, F_7$, or F_7^*.*

A matrix A that obeys the condition in (iv) is called *totally unimodular.* Such a matrix simultaneously represents M over all fields where, of course, $-1 = 1$ over fields of characteristic 2.

5.17. **Exercise.** Let A be a totally unimodular matrix.

(i) Let a_{ij} be a non-zero entry of A. Show that if we convert the jth column of A to the ith standard basis vector by potentially multiplying row i by -1 and then adding or subtracting row i from the other rows, another totally unimodular matrix is obtained.
(ii) Show that a totally unimodular matrix represents the same matroid over all fields thereby verifying that (iv) implies (i) in the last theorem.

It follows from Exercise 2.17 that every graphic matroid is regular. Although we shall not prove in general that (i) implies (iv) in the last theorem, it does follow for graphic matroids using the same construction as in Exercise 2.17 together with the following result of Poincaré [39], whose proof is left as an exercise.

5.18. **Lemma.** *Let A be a real matrix with every entry in $\{0, 1, -1\}$ such that every column has at most one 1 and one -1. Then A is totally unimodular.*

In the next section, we shall note a deep structural theorem of Seymour [46] for the class of regular matroids which, because of the link between regular matroids and totally unimodular matrices, has important implications for combinatorial optimization.

The proof given above of the excluded-minor characterization of binary matroids relied crucially on Lemma 5.2. All the known proofs of Theorem 5.14 rely on a similar result, namely that a ternary matroid arises from an essentially unique matrix.

5.19. **Theorem.** *Let A_1 and A_2 be matrices over $GF(3)$ such that the columns of these matrices are labelled by the same set E. If $M[A_1] = M[A_2]$ and A_1 has no more rows than A_2, then A_1 can be obtained from A_2 by a sequence of operations (3.14)(i)–(vi).*

This theorem fails, for example, if we replace $GF(3)$ by $GF(4)$. We noted earlier that the latter does not have the same structure as the ring of integers modulo 4. Recall that we are taking the elements of $GF(4)$ to be $0, 1, \omega, \omega + 1$ where, in this field, $\omega^2 = \omega + 1$ and $2 = 0$. This field has an automorphism that maps each element to its square. If we replace every entry in a $GF(4)$-representation of a matroid M by its image under this automorphism, we obtain another $GF(4)$-representation for M. Two $\mathbb{F}$-representations A_1 and A_2 of a matroid are *equivalent* if one can be obtained from the other by a sequence of operations each consisting of one of (3.14)(i)–(vi) or the following:

(vii) Replace each entry of the matrix by its image under an automorphism of $\mathbb{F}$.

The reader unfamiliar with field automorphisms should note that, when p is prime, $GF(p)$ has the identity map as its only automorphism. In general, for all positive integers k, the field $GF(p^k)$ has exactly k automorphisms, namely the maps that take each element x to x^{p^i} for all i in $\{0, 1, \dots, k-1\}$. The following two matrices A_1 and A_2 are both $GF(4)$-representations of the same matroid M but they are not equivalent:

$$A_1 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \omega & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \omega & 1 \end{bmatrix} \end{array} \text{ and}$$

$$A_2 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \omega & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \omega+1 & 1 \end{bmatrix} \end{array}.$$

The matroid M can be broken apart in a simple way. In fact, M can be represented geometrically as in Figure 11. Thus M can be obtained by sticking together two 4-point lines at a common point and then deleting that point. To formalize this idea, let M_1 and M_2 be matroids on sets E_1 and E_2, each having at least three elements, and let $E_1 \cap E_2 = \{p\}$. Assume that, for each i, the set $\{p\}$ is neither a circuit nor a cocircuit of M_i. Then the 2-*sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 is the matroid whose ground set is $(E_1 \cup E_2) - \{p\}$ and whose set of circuits consists of all circuits of $M_1 \backslash p$ together with all circuits of $M_2 \backslash p$ and all sets of the form $(C_1 \cup C_2) - \{p\}$ where each C_i is a circuit of M_i containing p.

5.20. **Exercise.** Show that

(i) $M_1 \oplus_2 M_2$ is actually a matroid;
(ii) each of M_1 and M_2 is isomorphic to a minor of $M_1 \oplus_2 M_2$;
(iii) $(M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^*$.

Note that the 6-element matroid M above is isomorphic to $U_{2,4} \oplus_2 U_{2,4}$ where the two copies of $U_{2,4}$ have ground sets $\{1, 2, 3, p\}$ and $\{p, 4, 5, 6\}$.

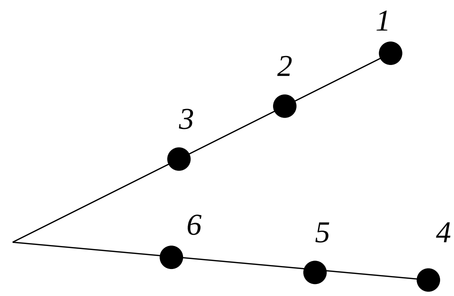

FIGURE 11. A geometric representation for $U_{2,4} \oplus_2 U_{2,4}$.

A matroid M is *connected* if it cannot be written as the direct sum of two non-empty matroids. If M is connected and cannot be written as the 2-sum of two matroids, then M is 3-*connected*. If G is a connected graph with at least 4 vertices, then $M(G)$ is a 3-connected matroid if and only if the graph G is 3-connected and G is simple, that is, G cannot be disconnected by removing 2 vertices, and G has no cycles with fewer than 3 edges. Extending Theorem 5.19, Kahn [21] proved the following:

5.21. **Theorem.** *If M is a 3-connected $GF(4)$-representable matroid, then all $GF(4)$-representations of M are equivalent.*

This theorem was a crucial tool in the proof of the excluded-minor characterization of *quaternary*, that is, $GF(4)$-representable, matroids, which was obtained very recently by Geelen, Gerards, and Kapoor [15]. Although we have already met some of the excluded minors, there are three others that have yet to be introduced here. The first of these, P_8, is the matroid that is represented over $GF(3)$ by the matrix

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

In P_8, the complementary sets $\{1,4,5,8\}$ and $\{2,3,6,7\}$ are both circuits and are both cocircuits. If we relax both of these circuits, then we get the matroid P_8''. The matroid P_6 is represented geometrically as in Figure 12.

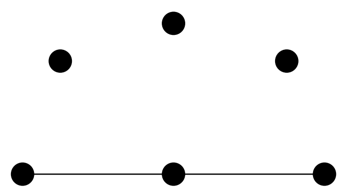

FIGURE 12. A geometric representation for P_6.

5.22. **Theorem.** *A matroid is quaternary if and only if it has no minor isomorphic to* $U_{2,6}, U_{4,6}, F_7^-, (F_7^-)^*, P_6, P_8$, *or* P_8''.

The last theorem means that Rota's conjecture (3.29) has now been proved for $q = 2, 3$, and 4. Comparing Theorems 3.26, 5.14, and 5.22, we see that, for $q \leq 4$, the number of excluded minors for the class of $GF(q)$-representable matroids increases with q. Oxley, Semple, and Vertigan [36] showed that, in general, this number is at least exponential in q.

5.23. **Theorem.** *For all prime powers q, there are at least 2^{q-4} excluded minors for the class of $GF(q)$-representable matroids.*

Rota's conjecture remains open for values of q larger than 4. Consider the case when $q = 5$. The matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & a \\ 0 & 0 & 1 & 1 & b \end{bmatrix}$$

represents $U_{3,5}$ over $GF(5)$ for all a and b in $GF(5) - \{0,1\}$ such that $a \neq b$. There are 6 such matrices and they are all inequivalent. Since $U_{3,5}$ is 3-connected, the analogue of Theorem 5.21 does not hold for $GF(5)$-representable matroids. Kahn [21] conjectured that, for all q, there is a fixed number $n(q)$ such that every 3-connected $GF(q)$-representable matroid has at most $n(q)$ inequivalent representations. Oxley, Vertigan, and Whittle [37] proved this conjecture when $q = 5$ but showed that it fails for all larger values of q.

5.24. **Theorem.** *Every 3-connected $GF(5)$-representable matroid has at most 6 inequivalent $GF(5)$-representations. For all integers N and all prime powers $q > 5$, there is a 3-connected $GF(q)$-representable matroid that has at least N inequivalent $GF(q)$-representations.*

This theorem means that if further progress is to be made on Rota's conjecture, then techniques will need to be developed that go beyond 3-connected matroids. One direction in which Geelen, Gerards, and Whittle [17] have had some recent success is in proving a weakened form of

Kahn's conjecture in which the connectivity condition is strengthened. This is an important result for it enables one to regain control of the number of inequivalent representations. Another direction that has been explored involves using the parameter *branch-width*, which was introduced for graphs by Robertson and Seymour [42] as a relative of their better-known *tree-width*. Loosely speaking, for each of these parameters, the smaller the value of the parameter the more tree-like is the structure. Geelen and Whittle [14] have proved that, for all finite fields $GF(q)$ and all positive integers k, there are only finitely many excluded minors for $GF(q)$-representability that have branch width at most k. This work is part of an effort that is being made to extend Robertson and Seymour's graph minors project (see, for example, [41]) to matroids. Among the many important contributions of this project is the following very deep result, which appears in the twentieth paper [43] of the series!

5.25. **Theorem.** *In every infinite set of finite graphs, there is always one that is isomorphic to a minor of another.*

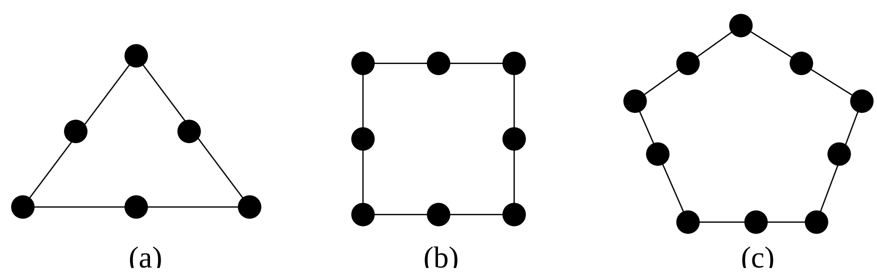

FIGURE 13. Geometric representations for (a) M_3, (b) M_4, and (c) M_5.

This theorem fails if we replace "graphs" by "matroids". For example, the matroids L_p in Theorem 5.9 are all excluded minors for $\mathbb{R}$-representability and so none is a minor of another. As another example, let $M_3, M_4, M_5, \ldots$ be the sequence of rank-3 matroids for which geometric representations are shown in Figure 13. None of these matroids is isomorphic to a minor of another. To see this, observe that, since these matroids all have the same rank and contraction drops rank, if M_i is a minor of M_j, then M_i must be a deletion of M_j. But once we delete an element from M_j, we destroy the ring of 3-point lines common to all the M_k's and this cannot be recovered by further deletions. Thus Theorem 5.25 does not extend to the class of all matroids.

5.26. **Exercise.** For $r \geq 2$, let N_r be the binary matroid that is represented by the $2r \times 4r$ matrix $[I_{2r}|J'_{2r}]$ where J'_{2r} is the matrix with zeros on the main diagonal and ones elsewhere. Let the columns of this matrix be labelled $1, 2, \ldots, 4r$ in order.

(i) Show that each of $\{2, 3, \ldots, 2r+1\}$ and $\{1, 2r+2, 2r+3, \ldots, 4r\}$ is a circuit of N_r whose complement is a cocircuit.

(ii) Let N_r'' be obtained from N_r by relaxing both of these circuits. Show that $N_2'', N_3'', N_4'' \ldots$ is a sequence of matroids none of which is isomorphic to a minor of another.

Among the biggest unsolved problems in matroid theory and one that has been the focus of much recent research attention is the following:

5.27. **Question.** *Is there an infinite set of binary matroids none of which is isomorphic to a minor of another?*

For all prime powers q, the corresponding question for the class of $GF(q)$-representable matroids is also open. Indeed, it is generally believed that the answer to this question will be the same irrespective of which finite field is considered. Geelen, Gerards, and Whittle [16] have answered this question negatively for all prime powers q provided that the branch-width of all the matroids in the set is bounded above.

6. Decomposition of regular matroids

Probably the most important theorem ever proved in matroid theory is a deep and important structure theorem for regular matroids due to Seymour [46]. Not only does this theorem provide a beautiful decomposition of regular matroids, but it also has profound implications for combinatorial optimization in that it leads to a polynomial-time algorithm to determine whether a real matrix is totally unimodular. We begin the discussion of this theorem with some background from linear programming.

6.1. **Example.** Suppose that we seek to maximize $cx + dy$ subject to the constraints:

$$\begin{aligned} x + y &\leq 4 \\ -x + y &\leq 3 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

These four inequalities determine a *feasible region* P of the plane where all four are satisfied. This region is shaded in Figure 14. Irrespective of the choice of c and d, the maximum value of $cx + dy$ will always be attained at some vertex of P. So, for instance, the maximum value of $-x + 2y$ is 6.5 and it occurs at the point $(0.5, 3.5)$.

We can rewrite our problem as: maximize $(c, d)^T \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ subject to the constraints $\left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right] \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \geq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. This is an example of a linear programming problem where, in general, such a problem seeks to find $\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}\}$ where $\mathbf{c}, \mathbf{x}, \mathbf{b}$, and $\mathbf{0}$ are real $1 \times n$ column vectors and A is a real $m \times n$ matrix. For a constant δ, the equation $\mathbf{c}^T\mathbf{x} = \delta$ defines a hyperplane in $\mathbb{R}^n$. Possibly the feasible region P is unbounded, but if it is

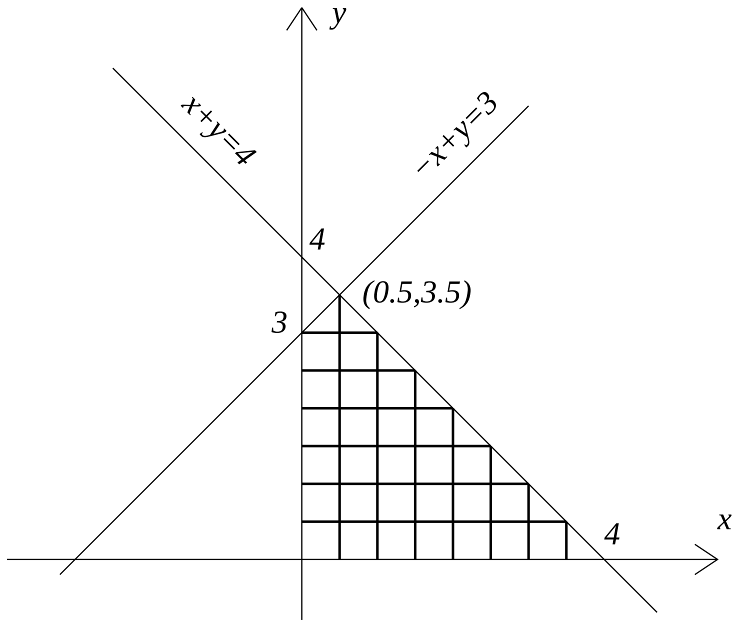

FIGURE 14. The feasible region P.

bounded, then by moving this hyperplane in the direction orthogonal to the vector $\mathbf{c}$, we will find a point of P at which $\mathbf{c}^T\mathbf{x}$ is maximized. Intuitively, the maximum will always occur at a vertex of P.

In integer linear programming problems, one imposes the additional requirement that the optimal solution $\mathbf{x}$ must be *integral*, that is, have all its coordinates integers. Whereas ordinary linear programming problems can be solved in poynomial time [23], results of Cook [7] (see Schrijver [44, Theorem 18.1]) imply that the integer linear programming problem is of the same level of difficulty computationally as, say, determining whether a graph is Hamiltonian. For an introduction to computational complexity, we refer the reader to the book of Cook, Cunningham, Pulleyblank, and Schrijver [8].

6.2. **Theorem.** *The problem:*
Given an integral matrix A and an integral vector $\mathbf{b}$, does the polyhedron $\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}\}$ contain an integral vector $\mathbf{x}$?
is $\mathcal{N}P$-complete.

By contrast, for an *integral polyedron*, that is, one all of whose vertices have only integer coordinates, we have (see Schrijver [44, Theorem 16.2]):

6.3. **Theorem.** *There is a polynomial-time algorithm which, given a rational system $A\mathbf{x} \leq \mathbf{b}$ defining a bounded integral polyhedron and given a rational vector $\mathbf{c}$, finds an optimum solution for the integer linear program* $\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}; \mathbf{x}$ *is integral*$\}$

Obviously, then, there is considerable interest in knowing when integral polyhedra arise. Hoffman and Kruskal [20] proved the following characterization.

6.4. **Theorem.** *Let A be an integral matrix. The polyhedron $\{\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}\}$ is integral for every integral vector $\mathbf{b}$ if and only if the matrix A is totally unimodular.*

In Example 6.1, the polyhedron is not integral and the matrix A, which is $\left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$, has determinant 2. The last theorem provides the link between integer programming and regular matroids for we recall, from Theorem 5.16, that a matroid is regular if and only if it can be represented by a totally unimodular matrix.

To describe Seymour's regular-matroids decomposition theorem, we need to describe the building blocks together with the operations used for joining them. We begin with the former. We have already noted that the class of graphic matroids is contained in the class of regular matroids. Since the latter class is closed under duality, it also contains the class of cographic matroids. One sporadic regular matroid, which was found by Bixby [1], is the vector matroid R_{10} of the following totally unimodular matrix:

$$\left[\begin{array}{c|ccccc} & -1 & 1 & 0 & 0 & 1 \\ & 1 & -1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & -1 & 1 & 0 \\ & 0 & 0 & 1 & -1 & 1 \\ & 1 & 0 & 0 & 1 & -1 \end{array}\right].$$

Among the special properties of this matroid are that it is isomorphic to its dual, every single-element deletion is isomorphic to $M(K_{3,3})$, every single-element contraction is isomorphic to $M^*(K_{3,3})$, and R_{10} can also be represented over $GF(2)$ by the ten 5-tuples that have exactly three ones.

Two of the three operations used to join the building blocks for regular matroids are direct sum and 2-sum. The third operation corresponds to sticking two disjoint graphs together across a 3-edge cycle and then deleting the edges of the cycle. Let M_1 and M_2 be binary matroids with ground sets E_1 and E_2, respectively, each having at least seven elements. Suppose that $E_1 \cap E_2 = T$, where T is a 3-element circuit in both M_1 and M_2, and T does not contain a cocircuit in either matroid. The *3-sum* $M_1 \oplus_3 M_2$ of M_1 and M_2 is the matroid on $(E_1 \cup E_2) - T$ whose set of circuits consists of all circuits of $M_1 \backslash T$, all circuits of $M_2 \backslash T$, and all minimal non-empty sets of the form $(C_1 \cup C_2) - T$ where C_i is a circuit of M_i such that $C_1 \cap T = C_2 \cap T \neq \emptyset$. We omit the proof that this operation does actually produce a matroid.

6.5. **Exercise.** Let M_1, M_2, and T be as above.

(i) Show that if $T = \{t_1, t_2, t_3\}$, then there are matrices A_1 and A_2 that represent M_1 and M_2 over $GF(2)$ such that A_1 and A_2 are, respectively,

$$
\begin{array}{c}
\begin{array}{cc|ccc} & & t_1 & t_2 & t_3 \end{array}\\
\left[\begin{array}{c|ccc}
D_1 & & 0 & \\
v_1 & 1 & 0 & 1\\
w_1 & 0 & 1 & 1
\end{array}\right]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{ccc} t_1 & t_2 & t_3 \end{array}\\
\left[\begin{array}{ccc|c}
1 & 0 & 0 & v_2\\
0 & 1 & 0 & w_2\\
 & 0 & & D_2
\end{array}\right].
\end{array}
$$

(ii) Show that $M_1 \oplus_3 M_2$ is represented over $GF(2)$ by the matrix

$$
\left[\begin{array}{c|c}
D_1 & 0\\
v_1 & v_2\\
w_1 & w_2\\
0 & D_2
\end{array}\right].
$$

(iii) Show that the 3-sum of two cographic matroids need not be cographic.

Seymour [46] showed that every regular matroid can be built by piecing together graphic matroids, cographic matroids, and copies of R_{10}. Combining his theorem with earlier work of Brylawski [6] gives the following result.

6.6. **Theorem.** *The class of regular matroids coincides with the class of matroids that can be constructed using direct sums, 2-sums, and 3-sums beginning with graphic matroids, cographic matroids, and copies of R_{10}.*

6.7. **Exercise.** Show that R_{10} has no 3-element circuits and so is not involved in any 3-sums in the last theorem.

As an important consequence of this theorem, we have the following result which, as we have noted above, has significant implications for combinatorial optimization.

6.8. **Corollary.** *There is a polynomial-time algorithm to determine whether a given real matrix is totally unimodular.*

Proof. We shall only sketch the main ideas of the proof here. The reader interested in a detailed proof can find one in Schrijver [44, Theorem 20.3] or Truemper [49, Section 11.4]. A *k-separation* of a matroid M is a partition (X, Y) of $E(M)$ into sets with at least k elements such that $r(X) + r(Y) < r(M) + k$. When M is the direct sum, 2-sum, or 3-sum of M_1 and M_2, then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is, respectively, a 1-, 2-, or 3-separation of M. The main parts of the algorithm testing total unimodularity are polynomial subroutines that will

(i) test whether a vector matroid M is (a) graphic; (b) cographic; and (c) isomorphic to R_{10}; and
(ii) find 1-, 2- and 3-separations in a vector matroid.

Evidently one can easily check if $M \cong R_{10}$. Several authors including Tutte [52], and Bixby and Cunningham [3] have given algorithms to test whether a given vector matroid M is graphic. Applying such an algorithm to M^* will determine whether M is cographic. Finally, Cunningham and Edmonds [10] have given a polynomial-time algorithm for finding k-separations for fixed k. The reader interested in optimizing this test for total unimodularity will find a very efficient version of the test in the work of Truemper [48]. □

7. Conclusion

In terms of the research results highlighted, this paper has focussed mainly on representable matroids. Another important and active research direction in matroid theory involves the numerous links between matroids and graphs. So deep are these links that Tutte [55] wrote: "If a theorem about graphs can be expressed in terms of edges and circuits only it probably exemplifies a more general theorem about matroids." A recent survey of this area, which concentrates particularly on connectivity results, appears in [35]. Yet another very active and rich part of matroid theory centres on the Tutte polynomial, its properties, and its numerous interesting evaluations throughout combinatorics. A recent survey of work in this area appears in Welsh [59]. For the history of matroid theory and a reprinting of some of the most influential papers in the subject, the reader is referred to Kung [26]. Many mathematicians in the 1930s and before were led to formulate abstract axiom systems for dependence. As Kung [26, p. 15] notes, "it was an early testimony to the naturalness and inevitability of the concept of a matroid that all these axiomatizations, discovered independently by very different mathematicians, are all equivalent." The fact that the concept of a matroid has endured is a present-day testimony to its versatility and utility.

Acknowledgements. The author thanks Bogdan Oporowski and Charles Semple for helpful discussions during the preparation of this paper. The author's work was partially supported by a grant from the National Security Agency.

References

[1] Bixby, R. E., Kuratowski's and Wagner's theorems for matroids, *J. Combin. Theory Ser. B* **22** (1977), 31–53.
[2] Bixby, R. E., On Reid's characterization of the ternary matroids, *J. Combin. Theory Ser. B* **26** (1979), 174–204.
[3] Bixby, R. E. and Cunningham, W. H., Converting linear programs to network problems, *Math. Oper. Res.* **5** (1980), 321–357.

[4] Bixby, R. E. and Cunningham, W. H., Matroid optimization and algorithms, in *Handbook of Combinatorics* (eds. R. Graham, M. Grötschel, and L. Lovász), Elsevier, Amsterdam; MIT Press, Cambridge, 1995, pp. 551–609.
[5] Borůvka, O., O jistém problému minimálním, *Práce Mor. Přírodověd Spol. v. Brně (Acta Societ. Scient. Natur. Moravicae)* **3** (1926), 37–58.
[6] Brylawski, T. H., Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.
[7] Cook, S. A., The complexity of theorem-proving procedures, *Proc. Third Annual ACM Symposium on Theory of Computing* (Shaker Heights, Ohio, 1971), ACM, New York, 1971, pp. 151–158.
[8] Cook, W. J., Cunningham, W. H., Pulleyblank, W. R., and Schrijver, A., *Combinatorial Optimization*, Wiley, New York, 1998.
[9] Crapo, H. H. and Rota, G.-C., *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, Preliminary edition, MIT Press, Cambridge, 1970.
[10] Cunningham, W. H. and Edmonds, J., A combinatorial decomposition theory, *Canad. J. Math.* **32** (1980), 734–765.
[11] Edmonds, J., Lehman's switching game and a theorem of Tutte and Nash-Williams, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 67–72.
[12] Edmonds, J. and Fulkerson, D. R., Transversals and matroid partition, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965), 147–153.
[13] Fournier, J.-C., Représentation sur un corps des matroïdes d'ordre ≤ 8, in *Théorie des Matroïdes* (ed. C. P. Bruter), Lecture Notes in Math. **211**, Springer-Verlag, Berlin, 1974, pp. 50–61.
[14] Geelen, J. and Whittle, G., Branch-width and Rota's conjecture, *J. Combin. Theory Ser. B* **86** (2002), 315–330.
[15] Geelen, J. F., Gerards, A. M. H., and Kapoor, A., The excluded minors for $GF(4)$–representable matroids, *J. Combin. Theory Ser. B* **79** (2000), 247–299.
[16] Geelen, J., Gerards, A. M. H., and Whittle, G., Branch-width and well-quasi-ordering in matroids and graphs, *J. Combin. Theory Ser. B* **84** (2002), 270–290.
[17] Geelen, J., Gerards, A. M. H., and Whittle, G., Kahn's Conjecture for 4-connected matroids, in preparation.
[18] Hirschfeld, J. W. P., Maximum sets in finite projective spaces, *Surveys in Combinatorics (Southampton 1983)*, London Math. Soc. Lecture Notes **82**, Cambridge University Press, Cambridge, 1983, pp. 55–76.
[19] Hirschfeld, J. W. P., Complete arcs, *Discrete Math.* **174** (1997), 177–184.
[20] Hoffman, A. J. and Kruskal, J. B., Integral boundary points of convex polyhedra, in *Linear Inequalities and related Systems* (eds. H. W. Kuhn and A. W. Tucker), Princeton Univ. Press, Princeton, 1956, pp. 223–246.
[21] Kahn, J., On the uniqueness of matroid representation over $GF(4)$, *Bull. London Math. Soc.* **20** (1988), 5–10.
[22] Kelly, D. and Rota, G.-C., Some problems in combinatorial geometry, in *A Survey of Combinatorial Theory* (eds. J. N. Srivastava et al), North-Holland, Amsterdam, 1973, pp. 309–312.
[23] Khachiyan, L. G., A polynomial algorithm in linear programming (in Russian), *Dokl. Akad. Nauk SSR* **244** (1979), 1093–1096 [English translation: *Soviet Mathematics Doklady* **20** (1979), 191–194.]
[24] Knuth, D. E., The asymptotic number of geometries, *J. Combin. Theory Ser. A* **17** (1974), 398–401.
[25] Kruskal, J. B., On the shortest spanning tree of a graph and the traveling salesman problem, *Proc. Amer. Math. Soc.* **7** (1956), 48–50.
[26] Kung, J. P. S., *A Source Book in Matroid Theory*, Birkhäuser, Boston, 1986.
[27] Kuratowski, K., Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.

[28] Lax, R. F., On the character of S_n acting on subspaces of $\mathbb{F}_q^n$, submitted.
[29] Lazarson, T., The representation problem for independence functions, *J. London Math. Soc.* **33** (1958), 21–25.
[30] Lehman, A., A solution to the Shannon switching game, *J. Soc. Indust. Appl. Math.* **12** (1964), 687–725.
[31] Minsky, M., Steps toward artificial intelligence, *Proc. IRE* **49** (1961), 8–30.
[32] Nash-Williams, C. St.J. A, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.
[33] Ore, O., Graphs and matching theorems, *Duke Math. J.* **22** (1955), 625–639.
[34] Oxley, J. G., *Matroid Theory*, Oxford University Press, New York, 1992.
[35] Oxley, J. G., On the interplay between graphs and matroids, *Surveys in Combinatorics, 2001* (ed. J. W. P. Hirschfeld), London Math. Soc. Lecture Notes **288**, Cambridge University Press, Cambridge, 2001, pp. 199–239.
[36] Oxley, J., Semple, C., and Vertigan, D., Generalized $\Delta - Y$ exchange and k-regular matroids, *J. Combin. Theory Ser. B* **79** (2000), 1–65.
[37] Oxley, J., Vertigan, D., and Whittle, G., On inequivalent representations of matroids, *J. Combin. Theory Ser. B* **67** (1996), 325–343.
[38] Piff, M. J., An upper bound for the number of matroids, *J. Combin. Theory Ser. B* **14** (1973), 241–245.
[39] Poincaré, H., Second complément à l'analysis situs, *Proc. London Math. Soc.* **32** (1900), 277–308.
[40] Rota, G.-C., Combinatorial theory, old and new, in *Proc. Internat. Cong. Math.* (Nice, Sept. 1970), Gauthier-Villars, Paris, 1971, pp. 229–233.
[41] Robertson, N. and Seymour, P. D., Graph minors – a survey, *Surveys in Combinatorics 1985* (ed. I. Anderson), London Math. Soc. Lecture Notes **103**, Cambridge University Press, Cambridge, 1985, pp. 153–171.
[42] Robertson, N. and Seymour, P. D., Graph minors. X. Obstructions to tree-decomposition, *J. Combin. Theory Ser. B* **52** (1991), 153–190.
[43] Robertson, N. and Seymour, P. D., Graph minors. XX. Wagner's conjecture, *J. Combin. Theory Ser. B*, to appear.
[44] Schrijver, A., *Theory of Integer and Linear Programming*, Wiley, Chichester, 1986.
[45] Seymour, P. D., Matroid representation over $GF(3)$, *J. Combin. Theory Ser. B* **26** (1979), 159–173.
[46] Seymour, P. D., Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.
[47] Seymour, P. D., Matroid minors, in *Handbook of Combinatorics* (eds. R. Graham, M. Grötschel, and L. Lovász), Elsevier, Amsterdam; MIT Press, Cambridge, 1995, pp. 527–550.
[48] Truemper, K., A decomposition theory for matroids. V. Testing of matrix total unimodularity, *J. Combin. Theory Ser. B* **49** (1990), 241–281.
[49] Truemper, K., *Matroid Decomposition*, Academic Press, Boston, 1992.
[50] Tutte, W. T., A homotopy theorem for matroids I, II, *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
[51] Tutte, W. T., Matroids and graphs, *Trans. Amer. Math. Soc.* **90** (1959), 527–552.
[52] Tutte, W. T., An algorithm for determining whether a given binary matroid is graphic, *Proc. Amer. Math. Soc.* **11** (1960), 905–917.
[53] Tutte, W. T., On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* **36** (1961), 221–230.
[54] Tutte, W. T., Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.
[55] Tutte, W. T., *Selected Papers of W. T. Tutte, Volume II* (eds. D. McCarthy and R. G. Stanton), Charles Babbage Research Centre, Winnipeg, 1979.
[56] Wagner, K., Über eine Erweiterung eines Satzes von Kuratowski, *Deut. Math.* **2** (1937), 280–285.

[57] Welsh, D. J. A., Combinatorial problems in matroid theory, in *Combinatorial Mathematics and its Applications* (ed. D. J. A. Welsh), Academic Press, London, 1971, pp. 291–306.
[58] Welsh, D. J. A., Matroids: Fundamental concepts, in *Handbook of Combinatorics* (eds. R. Graham, M. Grötschel, and L. Lovász), Elsevier, Amsterdam; MIT Press, Cambridge, 1995, pp. 481–526.
[59] Welsh, D. J. A., The Tutte polynomial, *Random Structures Algorithms* **15** (1999), 210–228.
[60] Whitney, H., 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.
[61] Whitney, H., On the abstract properties of linear dependence, *Amer. J. Math.* **57** (1935), 509–533.
[62] Wild, M., The asymptotic number of inequivalent binary codes and nonisomorphic binary matroids, *Finite Fields Appl.* **6** (2000), 192–202.
[63] Wild, M., The asymptotic number of inequivalent binary codes revisited, submitted.
[64] Wilson, R. J., An introduction to matroid theory, *Amer. Math. Monthly* **80** (1973), 500–525.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803–4918, USA
E-mail address: oxley@math.lsu.edu