WEAK MAPS AND THE TUTTE POLYNOMIAL

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Abstract. Let M and N be matroids such that N is the image of M under a rank-preserving weak map. Generalizing results of Lucas, we prove that, for x and y positive, $T(M;x,y) \geq T(N;x,y)$ if and only if $x+y \geq xy$ or $M \cong N$. We give a number of consequences of this result.

1. Introduction

Terminology and notation used here will follow [7] unless otherwise stated. Given two rank-r matroids M and N, a bijective map from $E(M)$ to $E(N)$ is a *rank-preserving weak map* if every basis of N is the image of a basis of M. We write $M \xrightarrow{rp} N$ if N is a rank-preserving weak-map image of M.

The following theorem of Lucas [6] shows that the numbers of bases, independent sets, and spanning sets of M are greater than the corresponding numbers for N if $M \xrightarrow{rp} N$. Note that $T(M;1,1)$, $T(M;2,1)$, and $T(M;1,2)$ count the numbers of bases, independent sets, and spanning sets of M, respectively, where $T(M;x,y)$ is the Tutte polynomial of M.

Theorem 1. *If $M \ncong N$ and $M \xrightarrow{rp} N$, then*
(i) $T(M;1,1) > T(N;1,1)$;
(ii) $T(M;2,1) > T(N;2,1)$;
(iii) $T(M;1,2) > T(N;1,2)$;
(iv) $T(M;x,0) \geq T(N;x,0)$ *for all* $x > 0$ *unless* M *has a loop;*
(v) $T(M;0,y) \geq T(N;0,y)$ *for all* $y > 0$ *unless* M *has a coloop.*

The main result of the paper is the following generalization of the last theorem.

Theorem 2. *Let x and y be positive real numbers. Let M and N be matroids such that there is a rank-preserving weak map from M to N. Then $T(M;x,y) \geq T(N;x,y)$ if and only if $x+y \geq xy$ or $M \cong N$.*

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Moreover, if $M \ncong N$, then $T(M;x,y) > T(N;x,y)$ if and only if $x+y > xy$.

Jaeger, Vertigan, and Welsh [5] proved that the problem of evaluating the Tutte polynomial of a graphic matroid at a point (x, y) in the first quadrant of the real plane is #P-hard unless $x + y = xy$ or $(x, y) = (1, 1)$. Evidently $x + y = xy$ if and only if (x, y) is a point on the hyperbola H_1 defined by the equation $(x - 1)(y - 1) = 1$. It is straightforward to prove that $T(M; x, y) = (x - 1)^{r(M)} y^{|E|}$ for all $(x, y) \in H_1$. Therefore, for any two matroids M and N that have the same rank and the same ground set, $T(M; x, y) = T(N; x, y)$ for all $(x, y) \in H_1$, that is, for all (x, y) for which $x + y = xy$. Theorem 2 is summarized in the following figure.

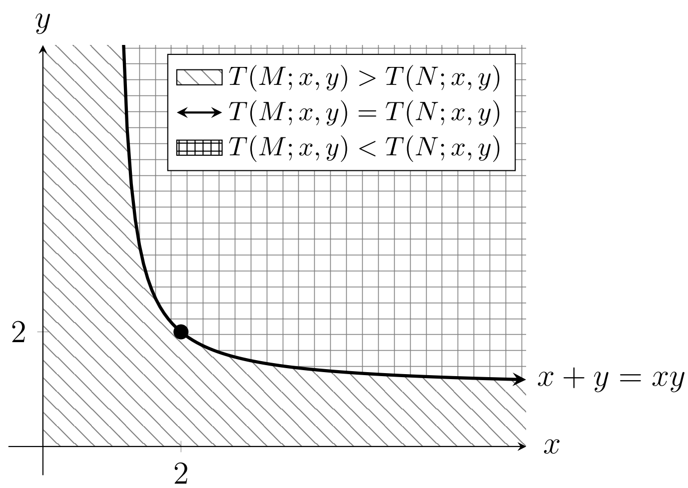

FIGURE 1. A summary of Theorem 2.

Let f and g be distinct elements in a matroid M. The element f is *freer* than the element g if g is contained in the closure of every circuit containing f. For example, in the rank-3 matroid M for which

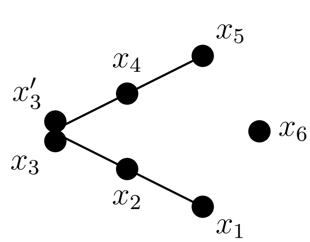

FIGURE 2. A rank-3 matroid M.

a geometric representation is shown in Figure 2, the element x_6 is *free*, that is, every circuit containing x_6 is spanning. Hence x_6 is freer than every element in $E(M) - \{x_6\}$. The element x_1 is freer than both x_3 and x_3'. Since $\{x_3, x_3'\}$ is a parallel class, it is straightforward to see that x_3 is freer than x_3', and x_3' is freer than x_3. Similarly, x_1 is freer than x_2, and x_2 is freer than x_1. By contrast, although x_1 and x_5 are symmetric in M, neither is freer than the other.

As a consequence of Theorem 2, we deduce that if f is freer than g in a matroid M, then the numbers of bases, circuits, and hyperplanes of M containing f are at least as large as the corresponding numbers of sets containing g. The next section presents some preliminaries. The main result is proved in Section 3. The last section contains consequences of the main theorem.

2. Preliminaries

The *nullity* of a matroid M is equal to $|E(M)| - r(M)$. For matroids M and N, a bijection $\varphi : E(M) \to E(N)$ is a *weak map* if $\varphi^{-1}(I) \in \mathcal{I}(M)$ whenever $I \in \mathcal{I}(N)$. If $r(M) = r(N)$, then φ is a *rank-preserving weak map* from M to N. Although it is not required that a weak map be bijective, we will only consider bijective weak maps. Such maps have the following attractive property (see, for example, [7, Corollary 7.3.13]).

Lemma 3. *If $\varphi : M \to N$ is a rank-preserving weak map from M to N, then φ is a rank-preserving weak map from M^* to N^*.*

For a matroid M with ground set E, the *Tutte polynomial* $T(M; x, y)$ of M is defined by

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

It is well known that $T(M^*; x, y) = T(M; y, x)$ for a matroid M and its dual M^*. In-depth accounts of the Tutte polynomial and its applications can be found in [1] and [3]. In [1, Exercise 6.10(b)], it is noted that $T(M; x, y) = T(N; x, y) - xy + x + y$ if M is obtained from N by relaxing a circuit-hyperplane. Since relaxation is an example of a rank-preserving weak map, this adds to the plausibility of the main result.

Before proving the main result in general, we prove it in the specific case when N is comprised solely of loops and coloops. Observe that if there is a rank-preserving weak map from a matroid M to a matroid N, then every coloop of M is a coloop of N, while every loop of M is a loop of N. The proof of the following lemma uses the sign function $\text{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ where

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Lemma 4. *Let $x > 0$ and $y > 0$. Let M be a matroid with rank k, nullity m, and $|E| \geq 2$. Then $T(M; x, y) \geq x^k y^m$ if and only if $x + y \geq xy$ or $M \cong U_{k,k} \oplus U_{0,m}$. Moreover, if $M \not\cong U_{k,k} \oplus U_{0,m}$, then $T(M; x, y) > x^k y^m$ if and only if $x + y > xy$.*

Proof. Suppose $M \not\cong U_{k,k} \oplus U_{0,m}$. We argue by induction on $|E|$ that $\text{sgn}(T(M) - x^k y^m) = \text{sgn}(x + y - xy)$, where we have abbreviated $T(M; x, y)$ as $T(M)$.

If $|E| = 2$, then $M \cong U_{1,2}$ and $T(M; x, y) = x + y$. Therefore $\text{sgn}(T(M) - xy) = \text{sgn}(x + y - xy)$, and the result holds. Assume that the result holds for $|E| < n$ and let $|E| = n \geq 3$. Since $M \not\cong U_{k,k} \oplus U_{0,m}$, there is an element e of M that is neither a loop nor a coloop. Thus $T(M) = T(M \backslash e) + T(M/e)$.

Suppose $M \backslash e \cong U_{k,k} \oplus U_{0,m-1}$. Then $M \cong U_{s,s+1} \oplus U_{k-s,k-s} \oplus U_{0,m-1}$ for some s with $1 \leq s \leq k$. If $s = 1$, then $M/e \cong U_{k-1,k-1} \oplus U_{0,m}$. Thus $T(M \backslash e) = x^k y^{m-1}$ and $T(M/e) = x^{k-1} y^m$. Since $x > 0$ and $y > 0$,

$$\begin{aligned} \text{sgn}(T(M) - x^k y^m) &= \text{sgn}(x^k y^{m-1} + x^{k-1} y^m - x^k y^m) \\ &= \text{sgn}(x^{k-1} y^{m-1}(x + y - xy)) = \text{sgn}(x + y - xy), \end{aligned}$$

as desired.

Suppose $s \geq 2$. Then $M/e \not\cong U_{k-1,k-1} \oplus U_{0,m}$ and, by the induction assumption,

$$\text{sgn}(T(M/e) - x^{k-1} y^m) = \text{sgn}(x + y - xy).$$

Now, by the deletion-contraction formula for $T(M)$,

$$\begin{aligned} \text{sgn}(T(M) - x^k y^m) &= \text{sgn}(T(M/e) + x^k y^{m-1} - x^k y^m) \\ &= \text{sgn}(T(M/e) - x^{k-1} y^{m-1}(xy - x)). \end{aligned}$$

If $\text{sgn}(x + y - xy) = 1$, then $y > xy - x$ and $T(M/e) > x^{k-1} y^{m-1}(y)$. Hence $\text{sgn}(T(M/e) - x^{k-1} y^{m-1}(xy - x)) = 1 = \text{sgn}(T(M) - x^k y^m)$. Applying analogous arguments to the remaining two cases, it follows that $\text{sgn}(T(M) - x^k y^m) = \text{sgn}(x + y - xy)$, as desired.

We may now assume that $M \backslash e \not\cong U_{k,k} \oplus U_{0,m-1}$. By duality, we may also assume that $M/e \not\cong U_{k-1,k-1} \oplus U_{0,m}$. By the induction assumption,

$$\text{sgn}(T(M \backslash e) - x^k y^{m-1}) = \text{sgn}(T(M/e) - x^{k-1} y^m) = \text{sgn}(x + y - xy),$$

so $\operatorname{sgn}(T(M\backslash e) - x^k y^{m-1} + T(M/e) - x^{k-1}y^m) = \operatorname{sgn}(x + y - xy)$. It follows that

$$\begin{aligned}\operatorname{sgn}(x + y - xy) &= \operatorname{sgn}(T(M\backslash e) + T(M/e) - x^k y^{m-1} - x^{k-1}y^m)\\ &= \operatorname{sgn}(T(M) - x^{k-1}y^{m-1}(x + y))\\ &= \operatorname{sgn}(T(M) - x^{k-1}y^{m-1}(xy))\\ &= \operatorname{sgn}(T(M) - x^k y^m),\end{aligned}$$

where the third equality follows by checking each of the three possibilities for $\operatorname{sgn}(x + y - xy)$. We conclude that the lemma holds. □

3. Proof of the Main Theorem

The following argument follows the same general structure as the proof of Lemma 4. We again use the abbreviation $T(M)$ in place of $T(M; x, y)$.

Proof of Theorem 2. It suffices to prove the result when M and N have a common ground set E and the rank-preserving weak map from M to N is the identity map. We will argue by induction on $|E|$ that $\operatorname{sgn}(T(M) - T(N)) = \operatorname{sgn}(x + y - xy)$ whenever $M \neq N$.

Let $|E| = 2$. Since $U_{1,2}$ and $U_{1,1} \oplus U_{0,1}$ are the only 2-element matroids of equal rank, we must have that $M \cong U_{1,2}$ and $N \cong U_{1,1} \oplus U_{0,1}$. As $T(U_{1,2}; x, y) = x + y$ and $T(U_{1,1} \oplus U_{0,1}; x, y) = xy$, we see that the result holds for $|E| = 2$.

Assume the result holds for $|E| < n$ and let $|E| = n \geq 3$. Take $e \in E$. If e is a coloop of M, then e is a coloop of N, so $T(M) = xT(M\backslash e)$ and $T(N) = xT(N\backslash e)$. Therefore, as $x > 0$,

$$\operatorname{sgn}(T(M) - T(N)) = \operatorname{sgn}(T(M\backslash e) - T(N\backslash e)) = \operatorname{sgn}(x + y - xy).$$

Applying a similar argument to the dual, we see that the assertion holds if M has a loop.

Suppose e is not a loop or a coloop of N. Then

$$\begin{aligned}\operatorname{sgn}(T(M) - T(N)) &= \operatorname{sgn}(T(M\backslash e) + T(M/e) - T(N\backslash e) - T(N/e))\\ &= \operatorname{sgn}(T(M\backslash e) - T(N\backslash e)\\ &\qquad + T(M/e) - T(N/e)). \qquad (3.1)\end{aligned}$$

Since $M \neq N$, we have that $M\backslash e \neq N\backslash e$ or $M/e \neq N/e$.

Suppose that $M\backslash e \neq N\backslash e$ and $M/e \neq N/e$. Then, by the induction assumption,

$$\operatorname{sgn}(T(M\backslash e) - T(N\backslash e)) = \operatorname{sgn}(x + y - xy) = \operatorname{sgn}(T(M/e) - T(N/e)).$$

Thus $\operatorname{sgn}(T(M\backslash e) - T(N\backslash e) + T(M/e) - T(N/e)) = \operatorname{sgn}(x + y - xy)$, that is, $\operatorname{sgn}(T(M) - T(N)) = \operatorname{sgn}(x + y - xy)$.

Now suppose that $M\backslash e = N\backslash e$ or $M/e = N/e$. Then, by (3.1),

$$\begin{aligned}\operatorname{sgn}(T(M) - T(N)) &= \begin{cases}\operatorname{sgn}(T(M\backslash e) - T(N\backslash e)) & \text{if } M/e = N/e,\\ \operatorname{sgn}(T(M/e) - T(N/e)) & \text{if } M\backslash e = N\backslash e,\end{cases}\\ &= \operatorname{sgn}(x + y - xy)\end{aligned}$$

where the last step follows by the induction assumption.

Finally, if every element of N is a loop or a coloop, then $N \cong U_{k,k} \oplus U_{0,m}$, so $T(N; x, y) = x^k y^m$. Since $M \xrightarrow{rp} N$, the matroid M has rank k and nullity m. The result follows immediately from Lemma 4. $\square$

4. Consequences

A flat F of a matroid M is *cyclic* if F is a union of circuits. Given distinct elements f and g of a matroid M, it is well known that f is freer than g if g is contained in every cyclic flat containing f. It is worth noting that, if f is a coloop of M, then f is vacuously freer than g for all $g \in E(M) - \{f\}$. Likewise, if g is a loop of M, then f is freer than g for all $f \in E(M) - \{g\}$. Consequently, our discussion of relative freedom is primarily concerned with elements of M that are neither loops nor coloops.

Duke showed that relative freedom extends nicely to both duals and minors. If f is freer than g in M, then g is freer than f in M^*. Moreover, f is freer than g in $M\backslash X/Y$ for all disjoint subsets X and Y of $E(M)$.

This section explores the notion of relative freedom of elements of a matroid and its connection to weak maps and the Tutte polynomial. The following result provides our first direct link between relative freedom and rank-preserving weak maps.

Define the map $\varphi_{gf} : E(M/f) \to E(M/g)$ by taking $\varphi_{gf}(g) = f$ and $\varphi_{gf}(e) = e$ for all $e \neq g$.

Lemma 5. *If f is freer than g in a matroid M and g is not a loop of M, then φ_{gf} is a rank-preserving weak map from M/f to M/g.*

Proof. Let I be independent in M/g. Then $I \cup g$ is independent in M. Suppose $f \notin I$. Then $\varphi_{gf}^{-1}(I) = I$. If I is dependent in M/f, then M has a circuit C such that $C \subseteq I \cup f$. Moreover, $f \in C$ since I is independent in M. As f is freer than g in M, we see that $g \in \operatorname{cl}_M(C)$. Then $I \cup g$ contains a circuit of M, a contradiction. Therefore I is independent in M/f.

Suppose $f \in I$. Then $f \in I \cup g$ and $I \cup g$ is independent in M. Therefore $\varphi_{gf}^{-1}(I)$, which equals $(I \cup g) - f$, is independent in M/f. $\square$

The next result follows immediately from Theorem 2 and Lemma 5. The straightforward proof is omitted.

Corollary 6. *Let $x > 0$ and $y > 0$. If f is freer than g in M and g is not a loop of M, then $T(M/f; x, y) \geq T(M/g; x, y)$ if and only if $x + y \geq xy$ or $M/f \cong M/g$.*

Corollary 7. *Let $x > 0$ and $y > 0$. If f is freer than g in M and g is not a coloop of M, then $T(M\backslash f; x, y) \leq T(M\backslash g; x, y)$ if and only if $x + y \geq xy$ or $M\backslash f \cong M\backslash g$.*

Proof. Since g is freer than f in M^*, it follows by Corollary 6 that $T(M^*/f; y, x) \leq T(M^*/g; y, x)$ if and only if $x + y \geq xy$ or $M^*/f \cong M^*/g$. Thus, by duality, we have $T(M\backslash f; x, y) \leq T(M\backslash g; x, y)$ if and only if $x + y \geq xy$ or $M\backslash f \cong M\backslash g$. $\square$

The following result lists several consequences of Corollary 6. We use $b(M)$, $W_k(M)$, $h(M)$, and $\gamma(M)$ to represent the numbers of bases, rank-k flats, hyperplanes, and circuits of M, respectively. To specify the numbers of such sets containing some element e of M, we write, for example, $b(e; M)$ and $W_k(e; M)$.

Corollary 8. *If f is freer than g in M, then*

(i) $b(f; M) \geq b(g; M)$;
(ii) $W_k(f; M) \geq W_k(g; M)$ *for all* $k \geq 0$, *provided g is not a loop of M;*
(iii) $h(f; M) \geq h(g; M)$, *provided g is not a loop of M;*
(iv) $\gamma(f; M) \geq \gamma(g; M)$, *provided f is not a coloop of M.*

Proof. For (i), note that $b(e; M) = b(M/e)$ for $e \in E(M)$ as long as e is not a loop. If g is a loop of M, then $b(g; M) = 0$, so (i) holds. Assume g is not a loop of M. As f is freer than g and g is not a loop, f is not a loop. Thus the map φ_{gf} is a rank-preserving weak map from M/g to M/f. By Theorem 2, we have $b(M/f) = T(M/f; 1, 1) \geq T(M/g; 1, 1) = b(M/g)$. Thus (i) holds.

To prove (ii), observe that, when e is not a loop of M, a set X is a rank-k flat of M if and only if $X - e$ is a rank-$(k-1)$ flat of M/e. Suppose g is not a loop of M. Then, since φ_{gf} is a rank-preserving weak map from M/f to M/g, it follows by [8, Proposition 9.3.3], that $W_{k-1}(M/f) \geq W_{k-1}(M/g)$ for all $k \geq 1$. Thus $W_k(f; M) \geq W_k(g; M)$ for all $k \geq 1$. Also $W_0(f; M) = 0 = W_0(g; M)$ since neither f nor g is a loop of M. Thus (ii) holds. Hence so does (iii).

For (iv), observe that $\gamma(e;M) = h(M^*) - h(e;M^*)$. Assume f is not a coloop of M. Since g is freer than f in M^*, we have, by (iii), that $h(f;M^*) \leq h(g;M^*)$. Therefore

$$h(M^*) - h(f;M^*) \geq h(M^*) - h(g;M^*)$$

and (iv) holds. □

To illustrate (ii) of the last corollary, consider the matroid M in Figure 2, where x_1 is freer than x_3. Evidently, $W_2(x_1;M) = 4$ and $W_2(x_3;M) = 3$, so $W_2(x_1;M) > W_2(x_3;M)$. By contrast, since every cyclic flat containing x_1 also contains x_3, the number of rank-2 cyclic flats containing x_3 is at least the number of rank-2 cyclic flats containing x_1. Indeed, the former is 2 and the latter 1.

Let $\gamma'(e;M)$ be the number of spanning circuits of M containing an element e of M.

Corollary 9. *If f is freer than g in M and f is not a coloop of M, then $\gamma'(f;M) \geq \gamma'(g;M)$.*

Proof. Take a spanning circuit D of M containing g but not f. Then $D = B \cup g$ for some basis B of M. Suppose $B \cup f$ is not a circuit of M. Then $B \cup f$ properly contains a circuit C of M and $f \in C$. Hence $\mathrm{cl}(C)$ contains g. The set $C - f$ spans C, so $g \in \mathrm{cl}(C - f)$. Thus $(C - f) \cup g$ is a dependent set that is a proper subset of the circuit D, a contradiction. □

Lemma 10. *The following are equivalent for elements f and g in a matroid M.*

(i) f is freer than g in M;
(ii) $b(f;N) \geq b(g;N)$ for all restrictions N of M containing $\{f,g\}$.

Proof. Suppose (i) holds. Then f is freer than g in all restrictions N of M containing $\{f,g\}$, so (ii) holds by Corollary 8(i).

Suppose (ii) holds and suppose f is not freer than g. Then M has a cyclic flat F containing f and avoiding g. Let $N' = M|(F \cup g)$. Note that g is a coloop of N'. Then $b(g;N') = b(N')$. As $b(f;N') \geq b(g;N')$, it follows that $b(f;N') = b(N')$. Thus f is a coloop of N', a contradiction. □

In the remaining results of this paper, we investigate several instances of equality holding between the number of distinguished sets of M containing f and the number of such sets containing g. Let x and y be elements of M. Then x and y are *clones* in M if and only if the bijection from $E(M)$ to $E(M)$ that interchanges x and y and fixes every other element is an isomorphism. It was shown in [4, Proposition 4.9] that x and y are clones if and only if the set of cyclic flats containing x is equal to the set of cyclic flats containing y. Thus x and y are clones if and only if x is freer than y, and y is freer than x in M. As an example, in Figure 2, the elements x_3 and x_3' are clones, as are x_1 and x_2, and x_4 and x_5, but there are no other pairs of clones in M. Two elements that are parallel are clones as are two elements that are in series.

Theorem 11. *Let f be freer than g in M. Then $b(f;M) = b(g;M)$ if and only if f and g are clones in M.*

Proof. If f and g are clones in M, then clearly $b(f;M) = b(g;M)$. To prove the converse, suppose $b(f;M) = b(g;M)$. First assume that g is a loop of M. Then $b(g;M) = 0 = b(f;M)$. Thus f is a loop of M. Therefore f and g are clones in M. Similarly, if g is a coloop of M, then f and g are clones in M. We may assume that f and g are neither loops nor coloops. Thus $b(f;M) = b(M/f)$ and $b(g;M) = b(M/g)$.

Let $|E(M)| \in \{2, 3\}$. Since f and g are not loops or coloops in M, we have that $M \in \{U_{1,2}, U_{2,3}, U_{1,3}, U_{1,2} \oplus U_{0,1}, U_{1,2} \oplus U_{1,1}\}$. It is straightforward to check that, in these cases, f and g are clones.

Assume the result holds for $|E(M)| < n$ and let $|E(M)| = n \geq 4$. Suppose f and g are not clones in M. Take an element $e \in E(M) - \{f, g\}$. If e is a loop or a coloop in M, then $b(f;M\backslash e) = b(f;M)$ and $b(g;M\backslash e) = b(g;M)$. Thus $b(f;M\backslash e) = b(g;M\backslash e)$. By the induction assumption, f and g are clones in $M\backslash e$. Hence f and g are clones in M, a contradiction. Thus e is neither a loop nor a coloop of M.

Suppose $\{e, f\}$ is a circuit of M. Then $\{e, f, g\}$ is contained in a parallel class since f is freer than g and g is not a loop in M, so f and g are clones of M. Thus we may assume that e is not a loop of M/f. If $\{e, g\}$ is a circuit of M, then e is a loop in M/g, so $b(M/g) = b(M/g\backslash e)$. Therefore $b(M/f\backslash e) + b(M/f/e) = b(M/g\backslash e)$. By Lemma 10, $b(M/f\backslash e) \geq b(M/g\backslash e)$, so $b(M/f) > b(M/g)$, a contradiction.

Now e is not a loop or a coloop of M, M/f, or M/g and it follows that $b(M/f) = b(M/f\backslash e) + b(M/f/e)$ and $b(M/g) = b(M/g\backslash e) + b(M/g/e)$. By assumption,

$$b(M/f\backslash e) + b(M/f/e) = b(M/g\backslash e) + b(M/g/e).$$

By Lemma 10, $b(M/f\backslash e) \geq b(M/g\backslash e)$. Thus $b(M/f/e) \leq b(M/g/e)$. Since f is freer than g in M/e, it follows, by Corollary 8(i), that $b(M/f/e) = b(M/g/e)$. Consequently, $b(M/f\backslash e) = b(M/g\backslash e)$. Therefore, by the induction assumption, f and g are clones in $M\backslash e$.

Since f and g are not clones in M, there is a circuit C of M containing g such that $f \notin \mathrm{cl}_M(C)$. Assume there is an element $e \in E(M) - (C \cup$

f). Then C is a circuit of $M\backslash e$ containing g such that $f \notin \mathrm{cl}_{M\backslash e}(C)$. Hence f and g are not clones in $M\backslash e$, a contradiction. It follows that $C = E(M) - \{f\}$. Thus $r_M(C) = r(M\backslash f) = r(M) - 1$. Therefore f is a coloop of M, a contradiction. □

Proposition 12. *Let f be freer than g in M. Let L be obtained from M by deleting every element of $E(M) - \{f, g\}$ that is parallel to g. Then $h(f;M) = h(g;M)$ if and only if f and g are clones in L.*

Proof. Suppose f and g are clones in L. As $h(f;M) = h(f;L)$ and $h(g;M) = h(g;L)$, we have $h(f;M) = h(g;M)$. To prove the converse, suppose that $h(f;M) = h(g;M)$. Let $\mathcal{H}(M;f;\overline{g})$ be the set of hyperplanes of M containing f but not g. Then $|\mathcal{H}(M;f;\overline{g})| = |\mathcal{H}(M;g;\overline{f})|$. Hence $|\mathcal{H}(L;f;\overline{g})| = |\mathcal{H}(L;g;\overline{f})|$. Clearly, if $\{f,g\}$ is a 2-circuit of L, then f and g are clones in L. Thus, we may assume that no 2-circuit of L contains g.

For $J \in \mathcal{H}(L;g;\overline{f})$, let B_J be an arbitrarily chosen basis of J containing g. Then $\mathrm{cl}_L(B_J - g)$ is a rank-$(r-2)$ flat of L. Let $\mathrm{cl}_L(B_J - g) \cup f = F$. Then $r(F) = r - 1$ otherwise $f \in \mathrm{cl}_L(B_J - g)$, so $g \in \mathrm{cl}_L(B_J - g)$, a contradiction. Consider $\mathrm{cl}_L(F)$. Assume f is not a coloop of $\mathrm{cl}_L(F)$. Then $\mathrm{cl}_L(F)$ contains a circuit C containing f. Since f is freer than g, we see that $g \in \mathrm{cl}_L(F)$. Then $\mathrm{cl}_L(F) \supseteq B_J$. Thus $\mathrm{cl}_L(F) \supseteq J$. As $r(\mathrm{cl}_L(F)) = r(J)$, we deduce that $\mathrm{cl}_L(F) = J$. But $f \notin J$, a contradiction. Thus f is a coloop of $\mathrm{cl}_L(F)$, so $\mathrm{cl}_L(B_J - g) \cup f \in \mathcal{H}(L;f;\overline{g})$. Let $\psi(J) = \mathrm{cl}_L(B_J - g) \cup f$. Note that ψ depends upon the choices made for the bases B_J. Moreover, ψ maps $\mathcal{H}(L;g;\overline{f})$ to $\mathcal{H}(L;f;\overline{g})$.

12.1. *ψ is bijective.*

To see that ψ is injective, suppose that, for distinct members J_1 and J_2 of $\mathcal{H}(L;g;\overline{f})$, the hyperplanes $\psi(J_1)$ and $\psi(J_2)$ are equal. Then $\mathrm{cl}_L(B_{J_1} - g) = \mathrm{cl}_L(B_{J_2} - g)$. Now the rank-$(r-2)$ flat $J_1 \cap J_2$ contains g and so contains the rank-$(r-1)$ set B_{J_1}, a contradiction. Since $|\mathcal{H}(L;g;\overline{f})| = |\mathcal{H}(L;f;\overline{g})|$ and ψ is injective, we conclude that (12.1) holds.

12.2. *g is a coloop of $L|J$ for every $J \in \mathcal{H}(L;g;\overline{f})$.*

Suppose g is not a coloop of $L|J$. Then $\mathrm{cl}_L(B_J - g)$ is a subset of J avoiding g. As g is not a coloop, there is an element h of $J - \mathrm{cl}_L(B_J - g) - g$. Since no 2-circuit of L contains g, the elements g and h are not parallel. Thus $\{g,h\}$ is independent. Extend $\{g,h\}$ to a basis B'_J of $L|J$. Then $\mathrm{cl}_L(B'_J - g) \neq \mathrm{cl}_L(B_J - g)$ because $h \in \mathrm{cl}_L(B'_J - g) - \mathrm{cl}_L(B_J - g)$. Thus $\mathrm{cl}_L(B'_J - g) \cup f$ is a member of $\mathcal{H}(L;f;\overline{g})$

that is not in the set $\psi(\mathcal{H}(L; g; \overline{f}))$. As $|\mathcal{H}(L; g; \overline{f})| = |\mathcal{H}(L; f; \overline{g})|$ and ψ is a bijection, this is a contradiction. Thus (12.2) holds.

Suppose g is not freer than f in L. Then L has a cyclic flat K containing g and avoiding f. Take a basis B for K and consider $B \cup f$. Extend $B \cup f$ to get a basis B_L for L. Then $\mathrm{cl}_L(B_L - f)$ is a hyperplane of L containing g and avoiding f. Moreover, since $K \subseteq \mathrm{cl}_L(B_L - f)$, the hyperplane $\mathrm{cl}_L(B_L - f)$ has a circuit containing g; that is, g is not a coloop of $\mathrm{cl}_L(B_L - f)$, a contradiction to (12.2). □

The next corollary is obtained by applying Proposition 12 to M^*.

Corollary 13. *Let f be freer than g in M. Let N be obtained from M by contracting every element of $E(M) - \{f, g\}$ that is in series with f. Then $\gamma(f; M) = \gamma(g; M)$ if and only if f and g are clones in N.*

Proof. Clearly $\gamma(f; N) = \gamma(g; N)$ if f and g are clones in N. Hence $\gamma(f; M) = \gamma(g; M)$. To prove the converse, suppose that $\gamma(f; M) = \gamma(g; M)$. Then $\gamma(f; N) = \gamma(g; N)$. If f and g are in series in N, then f and g are clones in N. Thus, we will assume that f and g are not a series pair in N.

Let $\mathcal{C}(M; f, \overline{g})$ be the set of circuits of M containing f and avoiding g. Then $|\mathcal{C}(N; f; \overline{g})| = |\mathcal{C}(N; g; \overline{f})|$. By duality, $|\mathcal{C}(N; f, \overline{g})| = |\mathcal{H}(N^*; g; \overline{f})|$. Therefore $|\mathcal{H}(N^*; g; \overline{f})| = |\mathcal{H}(N^*; f; \overline{g})|$, and, consequently, $h(g; M^*) = h(f; M^*)$. Since g is freer than f in M^*, by Proposition 12, we see that f and g are clones in $M^* \backslash X$ where X is the set of elements of $E(M) - \{f, g\}$ that are parallel to f in M^*. It follows that f and g are clones in N. □

Based on Theorem 11, Proposition 12, and Corollary 13, one may guess that, when f is freer than g in M and $\gamma'(f; M) = \gamma'(g; M)$, the elements f and g must be clones in M when f is not in any 2-cocircuit of M. To see that this is not so, let M be the rank-5 matroid that is obtained by taking the 2-sum across a common basepoint p of two 4-point lines M_2 and M_3 and of a 6-element rank-3 matroid M_1 that has $\{g, a, b\}$ as its only non-spanning circuit and has f, p, and c as free elements. Then f is freer than g in M and $\gamma'(f; M) = 0 = \gamma'(g; M)$. But f and g are not clones in the cosimple matroid M.

The *truncation* of M, which we will denote $\tau(M)$, is the matroid obtained from M by taking the free extension $M +_{E(M)} e$ of M by e and then contracting the free element e. In particular, when $r(M) > 0$, the independent sets of $\tau(M)$ are the independent sets of M with at most $r(M) - 1$ elements. Note that we use $\tau(M)$ rather than the more standard $T(M)$ to denote truncation in order to avoid confusion with the Tutte polynomial of M.

Corollary 14. *Let $r(M) = r \geq 2$. If f is freer than g in M, then $W_{r-2}(f;M) = W_{r-2}(g;M)$ if and only if f and g are clones in the matroid obtained from $\tau(M)$ by deleting every element of $E(M) - \{f,g\}$ that is parallel to g.*

Proof. Let F be a rank-$(r-2)$ flat of M. Then F is a rank-$(r-2)$ flat of $M +_{E(M)} e$ avoiding e. Hence F is a hyperplane of $\tau(M)$. Therefore, the rank-$(r-2)$ flats of M containing an element x are precisely the hyperplanes of $\tau(M)$ containing x, that is, $W_{r-2}(f;M) = W_{r-2}(g;M)$ if and only if $h(f;\tau(M)) = h(g;\tau(M))$. As f is freer than g in $\tau(M)$, the result follows immediately from Proposition 12. □

The following result generalizes Corollary 14 to the *i-th truncation* $\tau^i(M)$ of M, defined recursively by $\tau^i(M) = \tau(\tau^{i-1}(M))$ where $\tau^0(M) = M$.

Corollary 15. *Suppose f is freer than g in M and $r(M) = r \geq 2$. Then $W_k(f;M) = W_k(g;M)$ for some k with $1 \leq k \leq r-1$ if and only if f and g are clones in the matroid obtained from $\tau^{r-k-1}(M)$ by deleting every element of $E(M) - \{f,g\}$ that is parallel to g.*

Proof. This follows immediately from Corollary 14. □

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