ON PROPERTIES OF ALMOST ALL MATROIDS

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Dedicated to our friend and colleague Geoff Whittle in recognition of his many profound contributions to matroid theory.

Abstract. We give several results about the asymptotic behaviour of matroids. Specifically, almost all matroids are simple and cosimple and, indeed, are 3-connected. This verifies a strengthening of a conjecture of Mayhew, Newman, Welsh, and Whittle. We prove several quantitative results including giving bounds on the rank, a bound on the number of bases, the number of circuits, and the maximum circuit size of almost all matroids.

1. Introduction

The structure of the random labelled graph is a much-studied and very well-understood area of probabilistic combinatorics. However, the corresponding question about matroids is largely unexplored although Kelly and Oxley [2, 3, 4, 13] and Kordecki and Łuczak [7, 8, 9, 10] established some properties of random $GF(q)$-representable matroids. This is partly due to the lack of a simple model of a random matroid, combined with the fact that the large number of matroids on n elements makes simple sampling virtually impossible. Mayhew, Newman, Welsh, and Whittle [11] initiated a study of the asymptotic properties of matroids, and this paper is a continuation of that study.

The matroid terminology used here will follow Oxley [14]. Throughout this paper, we will be dealing with n-element labelled matroids. For a positive integer n, let $m(n)$ be the number of matroids on the ground set $\{e_1, e_2, \dots, e_n\}$. A matroid *property* π is a class of matroids that is closed under isomorphism. Let π_n consist of those matroids in π that have exactly n elements. We say that *almost all matroids have property* π or, equivalently, that π is *large*, if the limit $\lim_{n\to\infty} \frac{|\pi_n|}{m(n)}$ exists and is equal to 1. Similarly, we say that the class π is *small* if $\lim_{n\to\infty} \frac{|\pi_n|}{m(n)}$ exists and is equal to 0. Clearly, asymptotically almost all matroids have the property π if and only if the class of matroids without π is small.

In [11], it is shown that almost all matroids have no loop or coloop. It is also shown that the proportion of n-element matroids that are connected is asymptotically at least 1/2 and it is conjectured that almost all matroids are connected and, indeed, are k-connected for all fixed k exceeding one. In what follows, we will prove that asymptotically almost all matroids are simple and cosimple, and, more strongly, that they are 3-connected. We also give quantitative results about the rank, the number of bases, and the number and size of circuits.

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2. A Theorem about Rank

Before proving our first result about the rank of a typical matroid, we state some preliminary results which we make use of several times later. We begin with three inequalities for binomial coefficients that follow from Stirling's formula. The first is Lemma 1 of [15]. Mayhew and Welsh [12, Lemma 2.1] sketch a proof of the third but observe that they view the result as known. A straightforward modification of their proof yields the second inequality. For positive integers k and n with $k \leq n$,

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k \tag{1}$$

and

$$\frac{2^{n-1}}{n^{1/2}} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \frac{2^n}{n^{1/2}}\sqrt{\frac{2}{\pi}}. \tag{2}$$

By combining the bounds of Knuth [6] and Piff [15], one gets the following bounds on $m(n)$, the number of n-element labelled matroids. Here, and throughout the paper, all logarithms will be taken to the base 2.

$$n - (3/2)\log n + O(\log\log n) \leq \log\log m(n) \leq n - \log n + O(\log\log n). \tag{3}$$

Consider the equation $x2^{1/x} - e = 0$. Let μ be the smallest positive root of this equation. Then $0.3275 < \mu < 0.3276$. Moreover, $x2^{1/x} - e > 0$ for all x in $(0, \mu)$. The constant μ will appear in several results in this paper beginning with the next theorem, which will enable us to show that almost every matroid has rank in a specific range. This provides leverage when proving properties of almost all matroids since we only need to consider matroids having rank in the given range. We shall use the following elementary result.

Lemma 2.1. *Let n be a positive integer and suppose that $0 < d < 1$. For $k = \lfloor dn \rfloor$,*

$$\binom{n}{k} \leq \left(\frac{e}{d}\right)^{dn} e.$$

Proof. By (1),

$$\begin{aligned}
\binom{n}{k} &\leq \left(\frac{en}{\lfloor dn \rfloor}\right)^{\lfloor dn \rfloor} \\
&= \left(\frac{en}{dn}\right)^{\lfloor dn \rfloor}\left(\frac{dn}{\lfloor dn \rfloor}\right)^{\lfloor dn \rfloor} \\
&\leq \left(\frac{en}{dn}\right)^{dn}\left(1 + \frac{1}{\lfloor dn \rfloor}\right)^{\lfloor dn \rfloor} \leq \left(\frac{e}{d}\right)^{dn} e.
\end{aligned}$$

□

Theorem 2.2. *Let $\mathcal{C}(d)$ denote the class of n-element matroids with maximum circuit size $\lfloor dn \rfloor$, where $0 < d < 1$. Then $\mathcal{C}(d)$ is small for $d < \mu$.*

Proof. Since there are exactly 2^n matroids on a labelled n-element set in which every element is a loop or a coloop, the class of such matroids is small. Hence we may assume that all the matroids we are considering have a component with at least two elements. Let $\mathcal{C}_k$ denote the class of n-element matroids with maximum circuit size k where $k \geq 2$. Each $M \in \mathcal{C}_k$ is defined by its list of circuits all of which come from $\mathcal{F} = \{A \subseteq E(M) : |A| \leq k\}$. Also circuits form a clutter. Thus, by Kleitman, Edelberg, and Lubell's [5] extension of Sperner's Theorem [16], the maximum number of circuits is $\binom{n}{k}$ provided $k \leq \lfloor n/2 \rfloor$. Clearly $|\mathcal{F}| = \sum_{j=1}^{k} \binom{n}{j} \leq k\binom{n}{k}$. Hence

$$|\mathcal{C}_k| \leq \sum_{j=1}^{\binom{n}{k}} \binom{|\mathcal{F}|}{j} \leq \binom{n}{k}\binom{k\binom{n}{k}}{\binom{n}{k}} \text{ as } k \geq 2.$$

Therefore, using these gross overcounts, it follows by (1) that

$$|\mathcal{C}_k| \leq \binom{n}{k}\left(\frac{ek\binom{n}{k}}{\binom{n}{k}}\right)^{\binom{n}{k}} = \binom{n}{k}(ek)^{\binom{n}{k}}.$$

Suppose $k = \lfloor dn \rfloor$, where $0 < d \leq 1/2$. Then, by Lemma 2.1, $\binom{n}{k} \leq \left(\frac{e}{d}\right)^{dn} e$. Thus

$$|\mathcal{C}_k| \leq \left(\frac{e}{d}\right)^{dn} e\,(edn)^{\left(\frac{e}{d}\right)^{dn} e}.$$

Hence

$$\begin{aligned}
\log|\mathcal{C}_k| &\leq dn\log\left(\frac{e}{d}\right) + \log e + \left(\frac{e}{d}\right)^{dn} e\log(edn)\\
&\leq \left(\frac{e}{d}\right)^{dn} 2e\log n \quad \text{for } n \text{ sufficiently large;}\\
&= 2^{n\log\left(\frac{e}{d}\right)^{d} + O(\log\log n)}.
\end{aligned}$$

But

$$\log m(n) \geq 2^{n - \frac{3}{2}\log n + O(\log\log n)}.$$

Now $\mathcal{C}_k$ is small if $\lim_{n\to\infty} \frac{|\mathcal{C}_k|}{m(n)} = 0$ and this holds whenever $\log\left(\frac{e}{d}\right)^{d} < 1$ or, equivalently, whenever $\left(\frac{e}{d}\right)^{d} < 2$. Solving this for d gives $d < \mu$. $\square$

The following consequence of the last theorem adds support to the conjecture of Mayhew, Newman, Welsh, and Whittle [11, Conjecture 1.10] that almost all n-element matroids have rank in $\{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$.

Corollary 2.3. *For all $\varepsilon > 0$, asymptotically almost all n-element matroids have rank r in the range $(\mu - \varepsilon)n < r < (1 - \mu + \varepsilon)n$. Hence $0.3275n < r < 0.6725n$.*

Proof. By duality, it is enough to show that almost all matroids have rank greater than $(\mu - \varepsilon)n$. Partition the n-element rank-r matroids into those for which $(\mu - \varepsilon)n < r$ and those for which $(\mu - \varepsilon)n \geq r$. For a matroid M in the latter class, since the maximum circuit size is at most $r + 1$, we have $r + 1 \leq (\mu - \varepsilon)n + 1 = (\mu - \frac{\varepsilon}{2})n - \frac{\varepsilon}{2}n + 1$. Hence, for n sufficiently large, M has maximum circuit size less than dn for some $d < \mu$. Thus, by Theorem 2.2, the class of such matroids M is small. We conclude that asymptotically almost all n-element rank-r matroids satisfy $(\mu - \varepsilon)n < r$. $\square$

Corollary 2.4. *For all $\varepsilon > 0$, asymptotically almost all matroids on n elements have all circuits of size at most $(1 - \mu + \varepsilon)n$.*

Proof. By the last corollary, almost all n-element matroids have rank at most $(1 - \mu + \frac{\varepsilon}{2})n$ and so have all circuits of size at most $(1 - \mu + \frac{\varepsilon}{2})n + 1$. For sufficiently large n, this is at most $(1 - \mu + \varepsilon)n$. $\square$

3. Most Matroids are Simple and Cosimple

In this section, we show that almost all matroids are simple thereby extending a result of Mayhew, Newman, Welsh, and Whittle [11, Theorem 2.3] that the class of matroids with a loop is small. Both these results are special cases of the next theorem for which we shall need some preliminaries. An element e of a matroid M is *free on a flat* F of M provided that, for all flats X of M, the element e is in $\mathrm{cl}(X-e)$ if and only if $X-e \supseteq F-e$. When e is free on the flat $E(M)$, we say that e is *free* in M. Observe that e is a loop of M if and only if e is free on $\mathrm{cl}(\emptyset)$, while e is in a non-trivial parallel class if and only if e is free on some flat of rank one. Clearly if e is free on the flat F, then $r(F-e)=r(F)$. Our theorem will use the constant ν, the smallest positive root of the equation $x2^{1/2x}-e=0$. One easily checks that $0.1071<\nu<0.1072$. Our proof of the next theorem will use the following result [11, Proposition 2.2].

Lemma 3.1. *For all $n \geq 2$,*

$$\frac{m(n-1)}{m(n)} \leq 2^{-(n-3)/2}.$$

Theorem 3.2. *The class of n-element matroids having an element that is free on some flat of rank at most dn is small for all d in $[0,\nu)$.*

Proof. Since the class of matroids with a loop is small [11, Theorem 2.3], it suffices to consider loopless matroids. Let M be such a matroid having ground set $\{e_1,e_2,\ldots,e_n\}$. Suppose that M has an element that is free on some flat of rank at most k, and let e_t be the lowest-indexed such element. Delete e_t from M and replace it as a loop to give the matroid N. As M is loopless, $k>0$ and the matroid N has a unique loop.

Given N, we now consider the number of choices for the matroid M that could have produced N. We delete e_t from N and add it freely to some flat of $N\backslash e_t$ of rank at most k. The number of such flats is at most $\sum_{j=0}^{k}\binom{n-1}{j}$. Since there are $m(n)$ labelled matroids on an n-element set, there are at most $n\,m(n-1)$ choices for N. Letting $f(n)$ be the number of choices for M, we have

$$\begin{aligned} f(n) &\leq n\sum_{j=0}^{k}\binom{n-1}{j}m(n-1) \\ &\leq nk\binom{n}{k}m(n-1) \end{aligned}$$

provided $k<\lfloor n/2\rfloor$. We shall make this assumption from now on.

By Lemma 3.1, we get

$$\frac{f(n)}{m(n)} \leq nk\binom{n}{k}2^{-(n-3)/2}.$$

Now let $k=\lfloor dn\rfloor$. Then, by Lemma 2.1,

$$\frac{f(n)}{m(n)} \leq dn^2\left(\frac{e}{d}\right)^{dn}e2^{-(n-3)/2} = dn^2e2^{3/2}\left(\frac{1}{\sqrt{2}}\left(\frac{e}{d}\right)^{d}\right)^{n}.$$

As $d < \nu$ and the function $x2^{1/2x} - e$ is strictly decreasing on the interval $(0, \nu]$ taking the value 0 at $x = \nu$, it is not difficult to check that

$$\frac{1}{\sqrt{2}}\left(\frac{e}{d}\right)^d < 1 - \varepsilon$$

for some fixed positive ε. Hence

$$\lim_{n\to\infty} \frac{f(n)}{m(n)} = 0.$$

□

The following is an immediate consequence of the last theorem and duality.

Corollary 3.3. *Almost all matroids are simple and cosimple.*

4. Most Matroids are 3-connected

By Theorem 2.2, almost all n-element matroids have a connected component of size at least $0.32n$. Next we give a lower bound on the size of all components in almost all matroids. This is rather a weak bound in view of Mayhew, Newman, Welsh, and Whittle's conjecture [11] that almost all matroids have a single component. Nevertheless, we can use this bound to prove a strengthening of that conjecture.

Lemma 4.1. *For almost all n-element matroids M, if (X, Y) is a j-separation of M for some j in $\{1, 2\}$, then* $\min\{|X|, |Y|\} \geq \log n$.

Proof. By Corollary 3.3, it suffices to consider n-element matroids M that are both simple and cosimple. Fix j in $\{1, 2\}$ and suppose that M has an exact j-separation (X, Y). If $j = 1$, then $M = M_1 \oplus M_2$ where $M_1 = M|X$ and $M_2 = M|Y$; if $j = 2$, let M_1 and M_2 be single-element extensions of $M|X$ and $M|Y$, respectively, such that $M = M_1 \oplus_2 M_2$. Let $k = |X|$ where $|X| \leq |Y|$. Then $k \geq 3$. Assume that $k \leq \log n$. There are $\binom{n}{k}$ choices for X. Hence the number of choices for M is at most

$$\sum_{k=3}^{\lfloor \log n \rfloor} \binom{n}{k} m(k+1)m(n-k+1).$$

The class of such matroids M is small if

$$\lim_{n\to\infty} \frac{(\log n)\binom{n}{\lfloor \log n \rfloor} m(\lfloor \log n \rfloor + 1)m(n-3)}{m(n)} = 0$$

and

$$\lim_{n\to\infty} \frac{\binom{n}{3} m(4)m(n-2)}{m(n)} = 0.$$

By iterating Lemma 3.1, we get that the second of these equations holds and that

$$\frac{m(n-3)}{m(n)} \leq 2^{-(3n-12)/2}.$$

Hence the theorem holds provided that

$$\lim_{n\to\infty} \frac{(\log n)\binom{n}{\lfloor \log n \rfloor} m(\lfloor \log n \rfloor + 1)2^6}{2^{3n/2}} = 0.$$

Now $(\log n)\binom{n}{\lfloor \log n\rfloor} \leq n^{\log n}$ for n sufficiently large. Thus, by (3),

$$(\log n)\binom{n}{\lfloor \log n\rfloor} m(\lfloor \log n\rfloor + 1) \leq n^{\log n} 2^{2^{\log n - \log\log n + O(\log\log\log n)}}$$
$$\leq 2^{(\log n)^2+n} \text{ for } n \text{ sufficiently large.}$$

The result follows immediately. □

Oliver Riordan (private communication) showed us how to use the case when $j = 1$ in the last result to prove that almost all matroids are connected. This prompted us to extend that lemma to the case $j = 2$ so that we could use his argument to prove the following stronger result.

Theorem 4.2. *Almost all n-element matroids are 3-connected.*

Proof. Let M be an n-element matroid. We may assume that M is simple and cosimple. Assume that M is not 3-connected. Then M has an exact j-separation (X, Y) for some j in $\{1, 2\}$. Let $k = |X| \leq |Y|$. By Lemma 4.1, we may assume that $k \geq \log n$. The number $D(n)$ of choices for M is at most

$$\sum_{k=\lceil \log n\rceil}^{\lfloor n/2\rfloor} \binom{n}{k} \left(m(k)m(n-k) + m(k+1)m(n-k+1)\right).$$

Thus

$$D(n) \leq (\tfrac{n}{2})2^{n+1}\left(m(n - \lceil \log n\rceil + 1)\right)^2.$$

Hence

$$\log D(n) \leq \log n + n + 2^{n-\lceil \log n\rceil+2-\log(n-\lceil \log n\rceil+1)+O(\log\log(n-\lceil \log n\rceil+1))}.$$

Now $\log(n - \lceil \log n\rceil + 1) \geq \log n - \log 2$. Thus, for sufficiently large n,

$$\log D(n) \leq 2^{n-2\log n+3+O(\log\log n)}.$$

Hence

$$\log D(n) - \log m(n) \leq 2^{n-2\log n+3+O(\log\log n)} - 2^{n-\frac{3}{2}\log n+O(\log\log n)}.$$

Thus $\lim_{n\to\infty} \frac{D(n)}{m(n)} = 0$, that is, the theorem holds. □

The last result verifies Mayhew, Newman, Welsh, and Whittle's [11] conjecture that almost all matroids are connected. But it stops short of proving their stronger conjecture that almost all matroids are k-connected for all fixed k exceeding one. When a matroid M has a 1- or 2-separation, it breaks up as a direct sum or 2-sum. For $j \geq 3$, there is no corresponding result for j-separations in general matroids, and we cannot even see how to prove that almost all matroids are 4-connected.

The fact that almost all matroids are simple and cosimple follows immediately from the last theorem. We did use this fact in our proof of the theorem but it is not difficult to modify our argument to avoid using this fact thereby giving us an alternative proof of simplicity.

5. On Bases, Circuits, and Free Elements

Clearly, the maximum number of bases which an n-element matroid can have is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ and we know, by (2), that $\frac{2^{n-1}}{n^{1/2}} \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \le \frac{2^n}{n^{1/2}}\sqrt{\frac{2}{\pi}} \le \frac{2^n}{n^{1/2}}$. Cloteaux [1] has recently proved that, for all $\gamma > \frac{5}{2}$, the number of bases of almost all rank-r matroids on n elements is at least $\frac{\binom{n}{r}}{n^\gamma}$.

We now show that most n-element matroids have at least $\frac{2^n}{n^{3/2}}$ bases. More precisely, we prove the following result.

Theorem 5.1. *For all $\varepsilon > 0$, the class of matroids with n elements and fewer than $\frac{2^n}{n^\alpha}$ bases is small for all $\alpha \ge \frac{3}{2} + \varepsilon$.*

Proof. By Corollary 2.3, it is enough to consider only the case where the rank r is in the range $cn < r < n - cn$, where $c \approx 0.32$. Clearly, the number of rank-r matroids with at most b bases is at most $\sum_{k=1}^{b} \binom{\binom{n}{r}}{k}$. As $\lceil cn \rceil \le r \le n - \lceil cn \rceil$, the number of matroids with at most b bases is at most

$$\begin{aligned}
\sum_{r=\lceil cn \rceil}^{n-\lceil cn \rceil} \sum_{k=1}^{b} \binom{\binom{n}{r}}{k} &= \sum_{k=1}^{b} \sum_{r=\lceil cn \rceil}^{n-\lceil cn \rceil} \binom{\binom{n}{r}}{k} \\
&\le \sum_{k=1}^{b} \binom{\binom{n}{\lfloor n/2 \rfloor}}{k} (n - 2\lceil cn \rceil + 1) \\
&\le nb \binom{\binom{n}{\lfloor n/2 \rfloor}}{b} \text{ since } b \le \binom{n}{\lfloor n/2 \rfloor}; \\
&\le nb \left(\frac{e\binom{n}{\lfloor n/2 \rfloor}}{b} \right)^b \text{ by (1).}
\end{aligned}$$

Therefore, letting $Q(b)$ be the total number of matroids with at most b bases such that $r \in [cn, n - cn]$, we have $Q(b) \le nbe^b \left(\frac{2^n}{b\sqrt{n}}\right)^b$ by (2). So if $b = \lceil \frac{2^n}{n^\alpha} \rceil$, this yields $Q(b) \le nbe^b \left(\frac{2^n}{\sqrt{n}} \frac{n^\alpha}{2^n}\right)^b = nbe^b(n^{\alpha-1/2})^b$. As n grows, Q is dominated by the term $(n^{\alpha-1/2})^b$. In other words, $\log\log Q$ is dominated by the behaviour of $\log((\alpha - 1/2)b \log n) = \log((\alpha - 1/2)\lceil \frac{2^n}{n^\alpha} \rceil \log n) = n - \alpha \log n + \log\log n + O(1)$. Comparing this with $n - \frac{3}{2}\log n + O(\log\log n)$, our lower bound for $\log\log m(n)$, we obtain the result. $\square$

The following is a result showing that almost all matroids have many circuits.

Theorem 5.2. *For all $\varepsilon > 0$, the class of matroids with n elements and fewer than $2^{n-\beta \log n}$ circuits is small for all $\beta \ge \frac{3}{2} + \varepsilon$.*

Proof. The number of matroids with k circuits and n elements is clearly at most $\binom{2^n}{k}$. Let $T(n, \beta)$ be the number of n-element matroids with fewer than $2^{n-\beta \log n}$

circuits. Thus

$$\begin{aligned}
T(n,\beta) &\leq \sum_{k=0}^{\lfloor 2^{n-\beta\log n}\rfloor} \binom{2^n}{k} \\
&\leq 2^{n-\beta\log n}\binom{2^n}{\lceil 2^{n-\beta\log n}\rceil} \text{ for } \beta > 0 \text{ and } n \text{ sufficiently large;} \\
&\leq 2^{n-\beta\log n}\left(\frac{e2^n}{\lceil 2^{n-\beta\log n}\rceil}\right)^{\lceil 2^{n-\beta\log n}\rceil} \quad \text{by (1);} \\
&\leq 2^{n-\beta\log n}\left(\frac{e2^n}{2^{n-\beta\log n}}\right)^{2^{n-\beta\log n}+1} \\
&= 2^{n-\beta\log n}\left(en^{\beta}\right)^{2^{n-\beta\log n}+1}.
\end{aligned}$$

Hence, as $\beta \geq \frac{3}{2}+\varepsilon$,

$$\begin{aligned}
\log T(n,\beta) &\leq n - \beta\log n + (2^{n-\beta\log n}+1)(\log e + \beta\log n) \\
&\leq n + \log e + \left(\frac{\log e + \beta\log n}{n^{\varepsilon}}\right)\frac{2^n}{n^{3/2}}.
\end{aligned}$$

Using (3), Knuth's lower bound for $m(n)$, we see that

$$\begin{aligned}
\log m(n) &\geq 2^{n-\frac{3}{2}\log n + O(\log\log n)} \\
&= \frac{2^n}{n^{3/2}} 2^{O(\log\log n)}.
\end{aligned}$$

Thus, for n sufficiently large,

$$\log m(n) \geq \frac{2^n}{n^{3/2}}\left(\frac{1}{(\log n)^{\delta}}\right) \text{ for some positive constant } \delta.$$

Therefore

$$\begin{aligned}
\log\frac{T(n,\beta)}{m(n)} &\leq n + \log e + \frac{2^n}{n^{3/2}}\left(\frac{\log e + \beta\log n}{n^{\varepsilon}} - \frac{1}{(\log n)^{\delta}}\right) \\
&= n + \log e + \frac{2^n}{n^{3/2}}\left(\frac{(\log e + \beta\log n)(\log n)^{\delta} - n^{\varepsilon}}{n^{\varepsilon}(\log n)^{\delta}}\right).
\end{aligned}$$

Hence $\log\frac{T(n,\beta)}{m(n)} \to -\infty$ as $n\to\infty$, so $\lim_{n\to\infty}\frac{T(n,\beta)}{m(n)} = 0$ as required. □

The following result about the absence of free elements in most matroids is more surprising than those presented above. The argument follows easily from the result that asymptotically almost all matroids have no loops. Recall that an element e in a matroid is free if e is not a coloop and the only circuits containing e are spanning.

Theorem 5.3. *The class of matroids having at least one free element is small.*

Proof. Since the class of matroids with a loop is small, it suffices to show that the class of loopless matroids with a free element is small. Let M be such a matroid on $\{e_1, e_2, \ldots, e_n\}$ and let e_j be the lowest-indexed element that is free in M. Let N be the matroid that is obtained from M by deleting e_j and then adding e_j back as a loop. Since N has a loop and M is uniquely recoverable from N, the theorem follows. □

6. Conclusion

By simple duality arguments, our results also yield corresponding results for hyperplanes and cocircuits.

We close by mentioning that the above results give a little support to the conjecture made in [11, Conjecture 1.6] that asymptotically almost all matroids are paving. Since Knuth's bound, $n - (3/2)\log n + O(\log\log n) \le \log\log m(n)$, which we use repeatedly, also holds for paving matroids, it is easy to check that all the theorems proved above remain true for the class of paving matroids.

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References

[1] B. Cloteaux, Approximating the number of bases for almost all matroids, Congressus Numerantium 33 (2010) 149–154.
[2] D. G. Kelly, J. G. Oxley, Threshold functions for some properties of random subsets of projective spaces, Quart. J. Math. Oxford Ser.(2) 33 (1982) 463–469.
[3] D. G. Kelly, J. G. Oxley, Asymptotic properties of random subsets of projective spaces, Math. Proc. Cambridge Philos. Soc. 91 (1982) 119–130.
[4] D. G. Kelly, J. G. Oxley, On random representable matroids, Stud. Appl. Math. 71 (1984) 181–205.
[5] D. Kleitman, M. Edelbert, D. Lubell, Maximal sized antichains in partial orders, Discrete Math. 1 (1971/72) 47–53.
[6] D. E. Knuth, The asymptotic number of geometries, J. Combin. Theory Ser. A 17 (1974) 398–401.
[7] W. Kordecki, Strictly balanced submatroids in random subsets of projective geometries, Colloq. Math. 55 (1988) 371–375.
[8] W. Kordecki, Small submatroids in random matroids, Combin. Probab. Comput. 5 (1996) 257–266.
[9] W. Kordecki, T. Łuczak, On random subsets of projective spaces, Colloq. Math. 62 (1991) 353–356.
[10] W. Kordecki, T. Łuczak, On the connectivity of random subsets of projective spaces, Discrete Math. 196 (1999) 207–217.
[11] D. Mayhew, M. Newman, D. Welsh, G. Whittle, On the asymptotic proportion of connected matroids, European J. Combin. 32 (2011) 882–890.
[12] D. Mayhew, D. Welsh, On the number of sparse paving matroids, Adv. in Appl. Math., to appear.
[13] J. G. Oxley, Threshold distribution functions for some random representable matroids, Math. Proc. Cambridge Philos. Soc. 95 (1984) 335–347.
[14] J. Oxley, Matroid Theory, Second Edition, Oxford University Press, New York, 2011.
[15] M. J. Piff, An upper bound for the number of matroids, J. Combin. Theory Ser. B 14 (1973) 241–245.
[16] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.

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