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Typical Subgraphs of 3- and 4-Connected Graphs

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We prove that, for every positive integer k, there is an integer N such that every 3-connected graph with at least N vertices has a minor isomorphic to the k-spoke wheel or $K_{3,k}$; and that every internally 4-connected graph with at least N vertices has a minor isomorphic to the $2k$-spoke double wheel, the k-rung circular ladder, the k-rung Möbius ladder, or $K_{4,k}$. We also prove an analogous result for infinite graphs. © 1993 Academic Press, Inc.

1. INTRODUCTION

In this paper, graphs are finite unless stated otherwise, and they may have loops or multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting connected (possibly infinite) subgraphs. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends. We begin by stating the following two simple results.

(1.1) *For every positive integer k, there is an integer N such that every connected graph with at least N vertices contains a subgraph isomorphic either to the path on k vertices or to the star with k vertices.*

We denote by C_k the cycle on k vertices.

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(1.2) *For every positive integer k, there is an integer N such that every 2-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of either C_k or $K_{2,k}$.*

This paper is concerned with generalizing these two results to 3- and 4-connected graphs. A *separation* of a graph is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, and there is no edge between $A - B$ and $B - A$. The *order* of (A, B) is $|A \cap B|$. A graph G is said to be *internally 4-connected* if it is 3-connected and, for every separation (A, B) of G of order 3, one of $A - B$, $B - A$ contains at most one vertex. Clearly every 4-connected graph is internally 4-connected.

We need to introduce several families of graphs (see Fig. 1). Let $k \geqslant 3$ be an integer. The *k-spoke wheel*, denoted by W_k, has vertices $v_0, v_1, ..., v_k$, where $v_1, v_2, ..., v_k$ form a cycle, and v_0 is adjacent to all of $v_1, v_2, ..., v_k$. The *$2k$-spoke double wheel*, denoted by D_k, has vertices $v_0, v_0', v_1, v_2, ..., v_k$, where $v_1, v_2, ..., v_k$ form a cycle, and both v_0 and v_0' are adjacent to all of $v_1, v_2, ..., v_k$. The *$2k$-spoke alternating double wheel*, denoted by A_k, has vertices $v_0, v_0', v_1, v_2, ..., v_{2k}$, where $v_1, v_2, ..., v_{2k}$ form a cycle in this order, v_0 is adjacent to $v_1, v_3, ..., v_{2k-1}$, and v_0' is adjacent to $v_2, v_4, ..., v_{2k}$. The *k-rung ladder*, denoted by L_k, has vertices $v_1, v_2, ..., v_k, u_1, u_2, ..., u_k$, where $v_1, v_2, ..., v_k$ and $u_1, u_2, ..., u_k$ form paths in the order listed, and v_i is adjacent to u_i for $i = 1, 2, ..., k$. The graph V_k is obtained from L_k by adding an edge between v_1 and v_k, and contracting the edges joining u_1 to v_1 and u_k to v_k. The graph O_k, called the *k-rung circular ladder*, is obtained from L_k by adding edges between v_1 and v_k and between u_1 and u_k; and the *k-rung Möbius ladder*, denoted by M_k, is obtained from L_k by adding edges between v_1 and u_k and between u_1 and v_k. The graph $K_{4,k}'$ has vertices x, y, x', y', $v_1, v_2, ..., v_k, v_1', v_2', ..., v_k'$, where v_i is adjacent to v_i', x, and y, and v_i' is adjacent to v_i, x', and y' $(i = 1, 2, ..., k)$. We remark that W_k, V_k, and $K_{3,k}$ are 3-connected, and that if $k \geqslant 4$, then A_k, D_k, O_k, M_k, $K_{4,k}$, and $K_{4,k}'$ are internally 4-connected.

The following are the main results of this paper.

(1.3) *For every integer $k \geqslant 3$, there is an integer N such that every 3-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of W_k, V_k, and $K_{3,k}$.*

(1.4) *For every integer $k \geqslant 4$, there is an integer N such that every internally 4-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of A_k, O_k, M_k, $K_{4,k}$, and $K_{4,k}'$.*

We leave it to the reader to verify that (1.3) and (1.4) imply the results stated in the abstract and that (1.3) implies that, for every integer $k \geqslant 3$, there are only finitely many simple 3-connected planar graphs having no

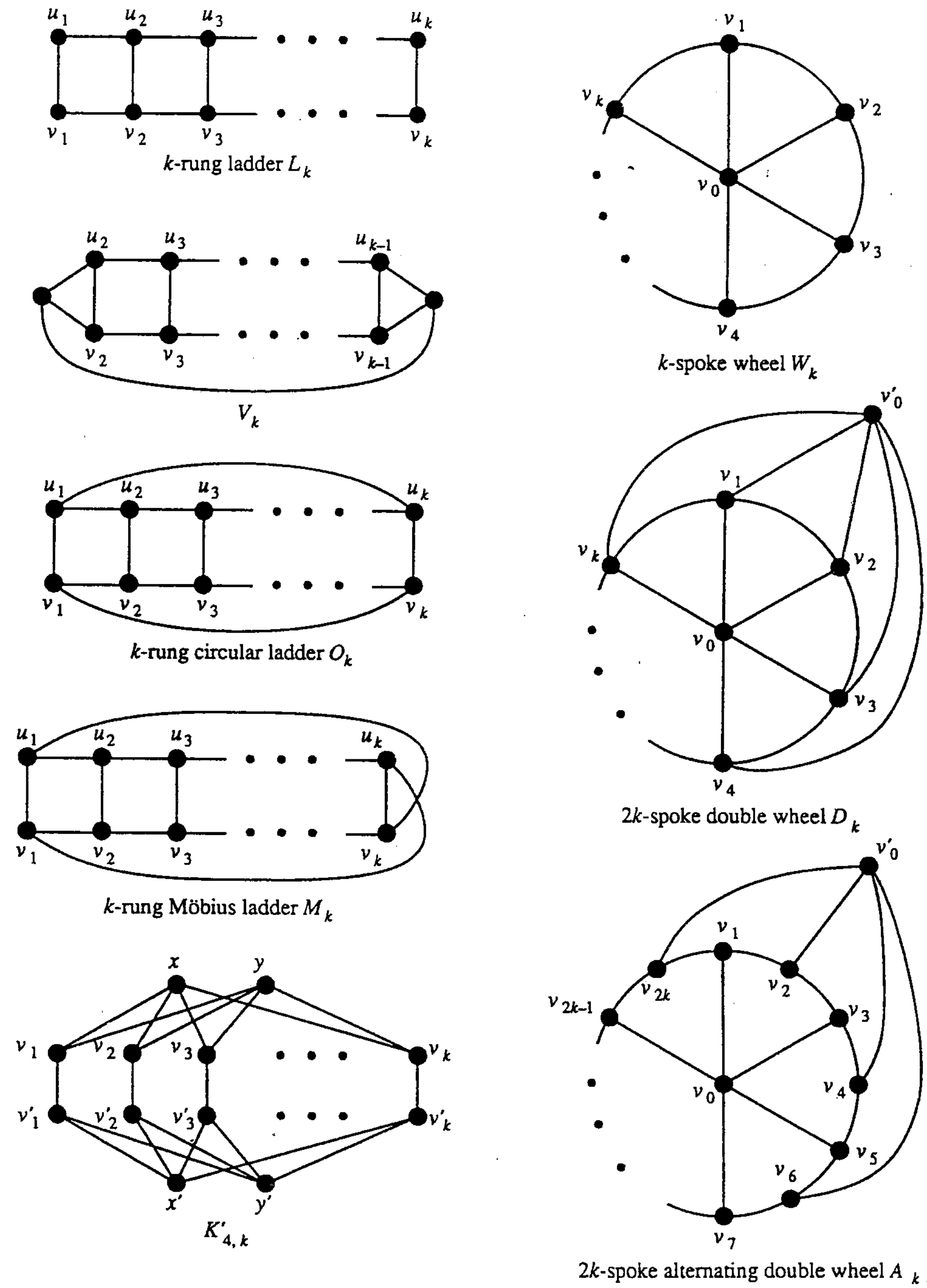

FIGURE 1

minor isomorphic to W_k. It is easy to see that (1.4) implies (1.3), as follows.

Proof of (1.3) (assuming (1.4)). For $k \geqslant 4$ let $N(k)$ be the number "N" from (1.4), and let $k \geqslant 3$ be given. We claim that $N(k+1)-1$ satisfies the conclusion of (1.3). Indeed, let G be a 3-connected graph with at least

$N(k+1)-1$ vertices, and let H be the graph obtained from G by adding a new vertex v, and joining v by an edge to every vertex of G. Then H is internally 4-connected, and so, by (1.4), it contains a subgraph isomorphic to a subdivision of one of A_{k+1}, O_{k+1}, M_{k+1}, $K_{4,k+1}$, and $K'_{4,k+1}$. Now it is routine to verify that G contains a subgraph isomorphic to a subdivision of one of W_k, V_k, and $K_{3,k}$, as desired. ∎

Next we shall reformulate (1.3) and (1.4) as excluded-minor theorems which hold for every graph regardless of its connectivity. A *tree-decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a multiset $(Y_t : t \in V(T))$ such that

(W1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t; and

(W2) if t, t', $t'' \in V(T)$ and t' lies on the path between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree-decomposition (T, Y) is $\max_{t \in V(T)} (|Y_t| - 1)$, and the *adhesion* of (T, Y) is $\max_{\{t, t'\} \in E(T)} |Y_t \cap Y_{t'}|$. (If $E(T) = \varnothing$, we define the adhesion of (T, Y) to be 0.) As we shall see, (1.3) and (1.4) can be restated as follows.

(1.5) *For every integer $k \geqslant 3$, there is an integer N such that every graph G with no minor isomorphic to either W_k or $K_{3,k}$ admits a tree-decomposition of width at most N and adhesion at most* 2.

(1.6) *For every integer $k \geqslant 4$, there is an integer N such that every graph G with no minor isomorphic to any of D_k, O_k, M_k, and $K_{4,k}$ admits a tree-decomposition of width at most N and adhesion at most* 3.

The conditions of (1.5) and (1.6) are necessary and sufficient in the sense that, for instance, every graph which admits a tree-decomposition of width at most N and adhesion at most 2 has no subgraph isomorphic to a subdivision of any of W_{N+1}, V_{N+1}, and $K_{3,N+1}$. We remark that, by a result of [2], since (1.5) and (1.6) hold for finite graphs, they also hold for infinite graphs.

We wish to present yet another reformulation of our results to emphasize the general paradigm of these theorems. A *lower ideal* is a set $\mathscr{I}$ of graphs with the property that if $G \in \mathscr{I}$ and H is isomorphic to a minor of G, then $H \in \mathscr{I}$. Let $\mathscr{W}$, $\mathscr{K}_3$, $\mathscr{K}_4$, $\mathscr{D}$, $\mathscr{O}$, and $\mathscr{M}$ be the lower ideals consisting of all minors of W_k, $K_{3,k}$, $K_{4,k}$, D_k, O_k, and M_k $(k = 3, 4, \ldots)$, respectively.

(1.7) *Let $\mathscr{I}$ be a lower ideal. Then $\mathscr{I}$ contains neither $\mathscr{W}$ nor $\mathscr{K}_3$ if and* *only if there is an integer N such that every graph G in $\mathscr{I}$ admits a tree-decomposition of width at most N and adhesion at most* 2.

Proof. If $\mathscr{W} \nsubseteq \mathscr{I}$ and $\mathscr{K}_3 \nsubseteq \mathscr{I}$, then there is an integer k such that $W_k \notin \mathscr{I}$ and $K_{3,k} \notin \mathscr{I}$. Hence every G in $\mathscr{I}$ admits the desired tree-decomposition by (1.5). Conversely, if every G in $\mathscr{I}$ admits a tree-decomposition of width at most N and adhesion at most 2, then $W_{N+1} \notin \mathscr{I}$ and $K_{3,N+1} \notin \mathscr{I}$, and hence $\mathscr{W} \nsubseteq \mathscr{I}$ and $\mathscr{K}_3 \nsubseteq \mathscr{I}$. ∎

(1.8) *Let $\mathscr{I}$ be a lower ideal. Then $\mathscr{I}$ contains none of $\mathscr{O}$, $\mathscr{K}_4$, $\mathscr{D}$, and $\mathscr{M}$ if and only if there is an integer N such that every graph G in $\mathscr{I}$ admits a tree-decomposition of width at most N and adhesion at most* 3.

The proof of (1.8) is almost identical to that of (1.7) and so is omitted.

We would like to clarify the relation of (1.7) and (1.8) to well-quasi-ordering. It follows from Wagner's conjecture recently proved by Robertson and Seymour [4] that, for every lower ideal $\mathscr{I}$, there is a finite set S of graphs such that an arbitrary graph G belongs to $\mathscr{I}$ if and only if it has no minor isomorphic to any member of S. Let $\mathscr{N}$ be a set of lower ideals closed under taking subideals. It is not known whether there exists a finite set $\mathscr{F}$ of lower ideals such that an arbitrary lower ideal $\mathscr{I}$ belongs to $\mathscr{N}$ if and only if $\mathscr{I}$ has no member of $\mathscr{F}$ as a subideal. (This would be a generalization of Robertson and Seymour's theorem.) Note that each of (1.7) and (1.8) can be viewed as a characterization of a set of lower ideals in terms of a finite set of excluded ideals.

A word about the smallest N for which the conclusions of (1.3) and (1.4) hold. Although we find an explicit bound, it is far from optimal. In fact we can improve upon this number, but only at the expense of a more complicated proof. The extra effort did not seem justified.

The paper is organized as follows. In Section 2 we review and improve earlier results about tree-decompositions, which are our main tool. In Section 3 we prove (1.4), and in Section 4 we derive (1.6). We omit the proof of (1.5), because it is similar though slightly easier than the proof of (1.6). In Section 5 we prove a result analogous to (1.4) for infinite graphs.

In the rest of this section we introduce some terminology. If G is a graph, we denote its vertex-set and edge-set by $V(G)$ and $E(G)$. The union of the graphs G_1 and G_2 is the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$ and the obvious incidences. When we say that two graphs are *disjoint*, we always mean they are vertex disjoint. Let G be a graph and X be a vertex or a set of vertices of G. By $G \backslash X$ we denote the graph obtained from G by deleting X. Every path P in a graph is a non-null subgraph and has two *ends* (which are equal for the one-vertex path), and we say that P is a path *between* its ends. A vertex of P which is not an end of P is called an *internal* vertex of P. If X, Y are subsets of $V(G)$ or subgraphs of G, we say that P is a path *between* X and Y if P has one end in X and the other end in Y.

2. Lemmas about Tree-Decompositions

An important tool in the proofs of our main theorems is the following result of Robertson, Seymour, and Thomas [5], an improvement of a result in [3].

(2.1) *Let H be a planar graph, and let $n = 2\,|V(H)| + 4\,|E(H)|$. Then every graph with no minor isomorphic to H has a tree-decomposition of width less than 20^{2n^5}.*

Let G be a graph of maximum degree at most 3. Since a graph has a minor isomorphic to G if and only if it contains a subgraph isomorphic to a subdivision of G, we deduce the following:

(2.2) *Every graph with no subgraph isomorphic to a subdivision of O_k, the k-rung circular ladder, has a tree-decomposition of width less than $20^{2(16k)^5}$.*

The proof of the following easy lemma can be found, for instance, in [6].

(2.3) *Let (T, Y) be a tree-decomposition of a graph G, and let H be a connected subgraph of G such that $V(H) \cap Y_{t_1} \neq \emptyset \neq V(H) \cap Y_{t_2}$, where $t_1, t_2 \in V(T)$. Then $V(H) \cap Y_t \neq \emptyset$ for every $t \in V(T)$ on the path between t_1 and t_2 in T.*

A tree-decomposition (T, Y) of a graph G is said to be *linked* if

(W3) for every two vertices t_1, t_2 of T and every positive integer k, either there are k disjoint paths in G between Y_{t_1} and Y_{t_2}, or there is a vertex t of T on the path between t_1 and t_2 such that $|Y_t| < k$.

It is worth noting that, by (2.3), the two alternatives in (W3) are mutually exclusive. The following is proved in [6].

(2.4) *If a graph G admits a tree-decomposition of width at most w, where w is some integer, then G admits a linked tree-decomposition of width at most w.*

We need to strengthen this result a little. Let (T, Y) be a tree-decomposition of a graph G, let $t_0 \in V(T)$, and let B be a component of $T \backslash t_0$. We say that a vertex $v \in Y_{t_0}$ is *B-tied* if $v \in Y_t$ for some $t \in V(B)$. We say that a path P in G is *B-confined* if $|V(P)| \geqslant 3$ and every internal vertex of P belongs to

$\bigcup_{t\in V(B)} Y_t - Y_{t_0}$. We wish to consider the following three properties of (T, Y):

(W4) if t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$;

(W5) if $t_0 \in V(T)$ and B is a component of $T\backslash t_0$, then $\bigcup_{t\in V(B)} Y_t - Y_{t_0} \neq \varnothing$;

(W6) if $t_0 \in V(T)$, B is a component of $T\backslash t_0$, and u, v are B-tied vertices in Y_{t_0}, then there is a B-confined path in G between u and v.

(2.5) *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)–(W6).

Proof. Let (T, Y) be a tree-decomposition of G. By an (n, d)-*cell* in (T, Y) we mean any component of the restriction of T to $\{t \in V(T)\colon |Y_t| \geqslant n\}$ that has at least d vertices. Let us remark that if K is an (n, d)-cell in (T, Y) and $d \geqslant d'$, then K is an (n, d')-cell as well. The set of (n, d)-cells in (T, Y) will be denoted by $C(T, Y, n, d)$. The *size* of a tree-decomposition (T, Y) is the family of numbers

(1) $(a_{n,d}\colon n \geqslant 0,\ d \geqslant 1)$,

where $a_{n,d}$ is the number of (n, d)-cells in (T, Y). Sizes are ordered lexicographically, that is, if

(2) $(b_{n,d}\colon n \geqslant 0, d \geqslant 1)$

is the size of another tree-decomposition (R, Z) of the graph G, we say that (1) is greater than (2) if there are integers $n \geqslant 0$, $d \geqslant 1$ such that $a_{n,d} > b_{n,d}$ and $a_{m,l} = b_{m,l}$ whenever $m - 1/2^l > n - 1/2^d$.

(3) *The relation "to be greater than" is a well-ordering on the set of sizes of tree-decompositions of G.*

Since this ordering is clearly linear, it is enough to show that it is well-founded. Suppose for a contradiction that $\{(a^{(i)}_{n,d}\colon n \geqslant 0,\ d \geqslant 1)\}_{i=1}^{\infty}$ is a decreasing sequence of sizes, and for $i = 1, 2, ...$, let n_i, d_i be such that $a^{(i)}_{n_i,d_i} > a^{(i+1)}_{n_i,d_i}$ and $a^{(i)}_{n,d} = a^{(i+1)}_{n,d}$ for (n, d) such that $n - 1/2^d > n_i - 1/2^{d_i}$. Since $a^{(1)}_{n,d} = 0$ for all $d \geqslant 1$ and all sufficiently large n, we may assume (by taking a suitable subsequence) that $n_1 = n_2 = \cdots$, and that $d_1 \leqslant d_2 \leqslant d_3 \leqslant \cdots$. Since clearly $a^{(i)}_{n,d} \geqslant a^{(i)}_{n,d'}$ for all $n \geqslant 0$, all $d \leqslant d'$, and all $i = 1, 2, ...$, we have

$$a^{(1)}_{n_1,d_1} > a^{(2)}_{n_1,d_1} \geqslant a^{(2)}_{n_2,d_2} > a^{(3)}_{n_2,d_2} \geqslant a^{(3)}_{n_3,d_3} > \cdots,$$

a contradiction, which proves (3).

Let (T, Y) be a tree-decomposition of G of width at most w having minimal size among such tree-decompositions. The following is shown in [6].

(4) *(T, Y) is linked.*

Our aim is to show that (T, Y) satisfies (W4)–(W6) as well.

(5) *(T, Y) satisfies* (W4).

Suppose that $Y_{t_1} = Y_{t_2}$ for distinct vertices $t_1, t_2 \in V(T)$. Let f be the edge of the path between t_1 and t_2 in T incident with t_2. We let T' be the tree obtained from T by deleting t_2 and all edges incident with it, and adding an edge $\{t, t_1\}$ for all t such that $\{t, t_2\}$ is an edge of T different from f. Let $Y' = (Y_t : t \in V(T'))$. It is easy to see that (T', Y') is a tree-decomposition of G of the same width but smaller size than (T, Y); a contradiction to the choice of (T, Y).

(6) *(T, Y) satisfies* (W5).

Suppose that $t_0 \in V(T)$ and that B is a component of $T \backslash t_0$ with $\bigcup_{t \in V(B)} Y_t \subseteq Y_{t_0}$. Let T' be the tree obtained from T by deleting B, and let $Y' = (Y_t : t \in V(T'))$. Then (T', Y') is a tree-decomposition of G of the same width but smaller size than (T, Y); a contradiction to the choice of (T, Y).

It remains to prove that (T, Y) satisfies (W6). Suppose not. Then there is a vertex $t_0 \in V(T)$, a component B of $T \backslash t_0$, and B-tied vertices $u, v \in Y_{t_0}$ such that there is no B-confined path between u and v. It is now not difficult to construct sets C, D such that $C \cup D = V(G)$, $C \cap D \cap (\bigcup_{t \in V(B)} Y_t) \subseteq Y_{t_0}$, $u \in C - D$, $v \in D - C$, and such that every edge of G with one end in $C - D$ and the other in $D - C$ has both ends in some Y_t for $t \in V(T) - V(B)$. Let t_1 be the vertex of B that is adjacent to t_0 in T. Since u, v are B-tied we deduce from (2.3) that

(7) $Y_{t_1} \cap (C - D) \neq \emptyset \neq Y_{t_1} \cap (D - C)$.

Let B_1, B_2 be two isomorphic copies of B, and let $\xi_i : V(B) \to V(B_i)$ be the corresponding isomorphisms. Let R be the tree obtained from T by deleting B, adding B_1 and B_2, and adding edges joining t_0 to $\xi_1(t_1)$ and t_0 to $\xi_2(t_1)$. Let $X_t = Y_t$ if $t \in V(T) - V(B)$, and let $X_{\xi_1(t)} = Y_t \cap C$ and $X_{\xi_2(t)} = Y_t \cap D$ if $t \in V(B)$. Then, for $X = (X_t : t \in V(R))$, we have

(8) *(R, X) is a tree-decomposition of G.*

The pair (R, X) clearly satisfies (W1) by the choice of C, D. To verify (W2), let $r, r', r'' \in V(R)$, let r' lie on the path between r and r'' in R, and let $x \in X_r \cap X_{r''}$. We may assume that $r \in V(B_1)$ and $r'' \in V(B_2)$, for otherwise (W2) is clearly satisfied. Then $x \in C \cap D \cap (\bigcup_{t \in V(B)} Y_t) \subseteq Y_{t_0}$, and hence $x \in X_{r'}$ by (W2) applied to (T, Y).

(9) *If* $|X_{\xi_1(t)}|$, $|X_{\xi_2(t)}| \geqslant |Y_{t_1}|$ *for some* $t \in V(B)$, *then* $|X_{\xi_1(t)}|$, $|X_{\xi_2(t)}| < |Y_t|$.

For if, say, $|X_{\xi_1(t)}| = |Y_t|$, then $Y_t \subseteq C$, and thus

$$X_{\xi_2(t)} = Y_t \cap D \subseteq Y_t \cap C \cap D \subseteq Y_t \cap Y_{t_0} \cap C \cap D \subset Y_{t_1},$$

where the last inclusion follows from (W2) and is strict because of (7). Hence $|X_{\xi_2(t)}| < |Y_{t_1}|$, contrary to our assumption.

(10) *(R, X) has smaller size than (T, Y).*

By (7), $|X_{\xi_1(t_1)}|$, $|X_{\xi_2(t_1)}| < |Y_{t_1}|$. Choose $t_2 \in V(B)$ with $|X_{\xi_1(t_2)}|$, $|X_{\xi_2(t_2)}| < |Y_{t_2}|$ such that $|Y_{t_2}|$ is maximum, and let $n = |Y_{t_2}|$. Let K be the component of T restricted to $\{t \in V(T)\colon |Y_t| \geqslant n\}$ containing t_2, and let $d = |V(K)|$. For m, l with $m \geqslant n$, we shall define a mapping $\Phi = \Phi_{m,l}$ from $C(R, X, m, l)$ into $C(T, Y, m, l)$ as follows. Let $L \in C(R, X, m, l)$. Since $|X_{\xi_1(t_1)}|$, $|X_{\xi_2(t_1)}| < n$, it follows that $V(L)$ is a subset of one of $V(R) - V(B_1 \cup B_2)$, $V(B_1)$, and $V(B_2)$. If $V(L) \subseteq V(R) - V(B_1 \cup B_2)$, we define $\Phi(L) = L$, and if $V(L) \subseteq V(B_i)$ for $i = 1$ or 2, we define $\Phi(L)$ to be the (m, l)-cell of (T, Y) containing $\{\xi_i^{-1}(r)\colon r \in V(L)\}$. Such a cell exists because $|Y_{\xi_i^{-1}(r)}| \geqslant |X_r| \geqslant m$ for every $r \in V(L)$. The mapping $\Phi_{n,d}$ is not a bijection, because no (n, d)-cell is mapped onto K. Let us choose m_0, l_0 such that Φ_{m_0, l_0} is not a bijection, but $\Phi_{m,l}$ is for every m, l with $m - 1/2^l > m_0 - 1/2^{l_0}$. Clearly, $m_0 - 1/2^{l_0} \geqslant n - 1/2^d$.

Let $\Phi = \Phi_{m_0, l_0}$. We claim that Φ is 1–1. Assume, to the contrary, that for some distinct K_1 and K_2 in $C(R, X, m_0, l_0)$, $\Phi(K_1) = \Phi(K_2)$. Choose such a pair $\{K_1, K_2\}$ so that $\max(|V(K_1)|, |V(K_2)|)$ is as large as possible. Let this maximum be $|V(K_1)|$. Suppose that $|V(K_1)| \neq |V(\Phi(K_1))|$. Let $|V(\Phi(K_1))| = l'$. Then $l' > |V(K_1)|$ and $\Phi(K_1)$ is an (m_0, l')-cell of (T, Y). Because $m_0 - 1/2^{l'} > m_0 - 1/2^{l_0}$, $\Phi_{m_0, l'}$ is a bijection from $C(R, X, m_0, l')$ to $C(T, Y, m_0, l')$. Thus, for some K in $C(R, X, m_0, l')$, $\Phi_{m_0, l'}(K) = \Phi(K_1)$. As $|V(K)| \geqslant l' > |V(K_1)|$, $K \neq K_1$. But $\Phi(K) = \Phi_{m_0, l'}(K)$, so $\Phi(K) = \Phi(K_1)$ and the maximality of $|V(K_1)|$ is contradicted. We conclude that $|V(K_1)| = |V(\Phi(K_1))|$. Thus, by the definition of Φ, one of $V(K_1)$ and $V(K_2)$ is contained in $V(B_1)$ and the other is contained in $V(B_2)$. Hence there is a vertex t in B such that one of $\xi_1(t)$ and $\xi_2(t)$ is in $V(K_1)$ and the other is in $V(K_2)$. Thus $\min(|X_{\xi_1(t)}|, |X_{\xi_2(t)}|) \geqslant m_0 \geqslant n \geqslant |Y_{t_1}|$. Therefore, by (9), $|X_{\xi_1(t)}|$, $|X_{\xi_2(t)}| < |Y_t|$. Hence $n < |Y_t|$, contrary to the choice of t_2. It follows that Φ is 1–1. Thus, as Φ is not a bijection, $|C(R, X, m_0, l_0)| < |C(T, Y, m_0, l_0)|$. But $\Phi_{m,l}$ is a bijection for all m, l such that $m - 1/2^l > m_0 - 1/2^{l_0}$. Hence, for such m and l, $|C(R, X, m, l)| = |C(T, Y, m, l)|$; (10) now follows immediately.

Claims (8) and (10) contradict the choice of (T, Y). Thus our assumption that (T, Y) does not satisfy (W6) was incorrect, and hence (T, Y) satisfies (W1)–(W6), as desired. ∎

3. Proof of (1.4)

Let $k \geqslant 4$ be an integer. The proof of the existence of the integer N having the properties claimed in (1.4) will introduce several intermediate numbers. In particular, the following integers will appear in the proof:

$$n_1 = 20^{2(16k)^5},$$

$$n_2 = (2n_1 - 3)k,$$

$$n_3 = (n_2 - 1)^2 (2k - 1),$$

$$n_4 = n_3 \binom{n_1 - 1}{2} + 1,$$

$$n_5 = n_4 + (n_2 - 1)(n_1 - 1),$$

$$n_6 = (4n_5 - 1)n_1 + 4,$$

$$n_7 = n_6^{n_1 + 1},$$

$$n_8 = \binom{n_1}{3} + 4\binom{n_1}{4}(k - 1),$$

$$n_9 = 2 + n_8 + n_8(n_8 - 1) + \cdots + n_8(n_8 - 1)^{\lceil n_7/2 \rceil - 2},$$

and

$$N = n_1 n_9.$$

We restate (1.4) as follows.

(3.1) *Every internally* 4*-connected graph with at least N vertices contains a subgraph isomorphic to a subdivision of one of A_k, O_k, M_k, $K_{4,k}$, and $K'_{4,k}$.*

Proof. Let G be an internally 4-connected graph with at least N vertices and with no subgraph isomorphic to a subdivision of any of O_k, M_k, $K_{4,k}$, and $K'_{4,k}$. We must show that G contains a subgraph isomorphic to a subdivision of A_k. From (2.2), G has a tree-decomposition of width less than n_1. Moreover, by (2.5), G has a tree-decomposition (T, Y) which satisfies (W1)–(W6) and has width less than n_1.

(1) $|V(T)| \geqslant n_9$.

This follows immediately from (W1).

(2) *Every vertex of T has degree at most n_8.*

Suppose that $t_0 \in V(T)$ has degree at least $n_8 + 1 = \binom{n_1}{3} + 4\binom{n_1}{4}(k-1) + 1$. Let $\mathscr{C}$ be the set of components of $G \backslash Y_{t_0}$. From (W5) and (2.3) we deduce that $|\mathscr{C}| \geqslant \binom{n_1}{3} + 4\binom{n_1}{4}(k-1)+1$. For $H \in \mathscr{C}$, let $X(H)$ be the set of all vertices v of Y_{t_0} that are adjacent to some vertex of H. Since G is 3-connected, it follows that $|X(H)| \geqslant 3$ for every $H \in \mathscr{C}$. Moreover, since G is internally 4-connected, $\mathscr{C}$ does not contain distinct members H and H' with $X(H) = X(H')$ and $|X(H)| = 3$. Thus there is a subset $\mathscr{C}'$ of $\mathscr{C}$ such that $|\mathscr{C}'| \geqslant 4\binom{n_1}{4}(k-1)+1$ and $|X(H)| \geqslant 4$ for every $H \in \mathscr{C}'$. Hence, for some four-element subset of Z of Y_{t_0}, there is a subset $\mathscr{C}''$ of $\mathscr{C}'$ such that $|\mathscr{C}''| \geqslant 4(k-1)+1$ and $Z \subseteq X(H)$ for every $H \in \mathscr{C}''$. It now follows by an elementary argument that G contains a subdivision of either $K_{4,k}$ or $K'_{4,k}$, a contradiction. Hence no such vertex t_0 exists and (2) follows.

(3) *T contains a path R_1 with $|V(R_1)| \geqslant n_7$.*

This follows from (1) and (2).

(4) *$V(R_1)$ has a subset $\{r_1, r_2, ..., r_{n_6}\}$ of vertices occuring on $V(R_1)$ in the order listed such that, for some non-negative integer s, $|Y_{r_i}| = s$ for all $i = 1, 2, ..., n_6$ and $|Y_r| > s$ for every r in $V(R_1)$ between r_1 and r_{n_6}. Moreover, $4 \leqslant s \leqslant n_1$.*

We begin by showing the existence of an integer s for which the first sentence holds. We shall then verify the stated bounds on s. As $n_7 = n_6^{n_1+1}$, we can find n_6 disjoint subpaths of R_1 each containing at least $n_6^{n_1}$ vertices. Now either every one of these subpaths contains a vertex r such that $|Y_r| = 0$, or there is such a subpath R'_1 for which $|Y_r| \geqslant 1$ for all r in $V(R'_1)$. In the first case, the claim of the first sentence holds with $s = 0$. In the second case, we find n_6 disjoint subpaths of R'_1 each containing at least $n_6^{n_1-1}$ vertices. In these subpaths of R'_1, we look for vertices r such that $|Y_r| = 1$. Continuing to argue in this way, we conclude that, since $|Y_t| \leqslant n_1$ for all t in $V(R_1)$, there must be some integer s for which the claim of the first sentence holds. Moreover, $s \leqslant n_1$. To prove the lower bound on s, we note that, by (2.3), every path between $Y_{r_1} \cup Y_{r_2}$ and $Y_{r_4} \cup Y_{r_5}$ uses a vertex from Y_{r_3}. But $|(Y_{r_1} \cup Y_{r_2}) - Y_{r_3}|$, $|(Y_{r_4} \cup Y_{r_5}) - Y_{r_3}| \geqslant 2$ by (W2) and (W4), and so $s = |Y_{r_3}| \geqslant 4$ by the internal 4-connectivity of G. This completes the proof of (4).

Let R be the subpath of R_1 with ends r_2 and r_{n_6-1}. By (W3), G has s disjoint paths $P_1, P_2, ..., P_s$ between Y_{r_1} and $Y_{r_{n_6}}$. For the rest of the proof we fix s and these s disjoint paths. If $t, t' \in V(R)$ are such that $|Y_t| = |Y_{t'}| = s$, then, by (2.3), $|V(P_j) \cap Y_t| = 1$ for every $j = 1, 2, ..., s$. It follows that, for every $j \in \{1, 2, ..., s\}$, there is a unique subpath of P_j which has one end in Y_t and the other end in $Y_{t'}$. We denote this subpath by $P_j(t, t')$. Moreover, if $t_1, t_2, ..., t_p$ lie on R in this order, and $|Y_{t_i}| = s$ for $i = 1, 2, ..., p$, then $P_j(t_1, t_p)$ is obtained by pasting together $P_j(t_1, t_2)$, $P_j(t_2, t_3)$, ..., $P_j(t_{p-1}, t_p)$ in this order.

(5) *Let t_0 be a vertex of R such that $|Y_{t_0}|=s$, let B be the component of $T\backslash t_0$ containing either r_1 or r_{n_6}, and let $u, v \in Y_{t_0}$. Then G has a B-confined path with ends u, v.*

Let, say, $u \in V(P_i)$ and $v \in V(P_j)$. Since both P_i and P_j meet both Y_{r_1} and $Y_{r_{n_6}}$, it follows that u, v are B-tied. Hence the existence of the desired path follows from (W6).

Let $j \in \{1, 2, ..., s\}$. A *j-segmentation* is a sequence $t_0, t_1, ..., t_p$ of vertices of R occurring on R in this order such that $|Y_{t_i}|=s$ $(i=0, 1, ..., p)$ and $|V(P_j(t_{i-1}, t_i))| \geqslant 5$ for $i=1, 2, ..., p$.

(6) *For some $j \in \{1, 2, ..., s\}$, there is a j-segmentation $t_0, t_1, ..., t_{n_5}$.*

Let $X=\bigcup\{Y_{r_i}: 2 \leqslant i \leqslant n_6-1\}$. It follows from (W2) and (W4) that $|X| \geqslant s+n_6-3 \geqslant 4n_5 s+1$. Thus $|V(P_j) \cap X| \geqslant 4n_5+1$ for some $j \in \{1, 2, ..., s\}$. Let $\{x_0, x_1, ..., x_{4n_5}\} \subseteq V(P_j) \cap X$ and assume that $x_0, x_1, ..., x_{4n_5}$ occur on P_j in this order. For each i let t_i' be a vertex of R such that $x_i \in Y_{t_i'}$ and $|Y_{t_i'}|=s$. Then $t_0', t_4', ..., t_{4i}', ..., t_{4n_5}'$ is as desired.

We may assume that $j=1$ satisfies (6). From now on by a *segmentation* we shall mean a 1-segmentation. Thus (6) becomes:

(7) *There exists a segmentation $t_0, t_1, ..., t_{n_5}$.*

If $t, t' \in V(T)$, we denote by $Y(t, t')$ the union of the sets $Y_{t''}$ over all t'' such that $t'' \in \{t, t'\}$, or, for each member x of $\{t, t'\}$, there is a path in T joining x to t'' that avoids $\{t, t'\}-\{x\}$.

(8) *Let t, t' be vertices of R such that $|Y_t|=|Y_{t'}|=s$, let j, j' be distinct integers in $\{1, 2, ..., s\}$, and let Q be a path between $P_j(t, t')\backslash(Y_t \cup Y_{t'})$ and $P_{j'}(t, t')$ with no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$. Then $V(Q) \subseteq Y(t, t')$.*

Suppose that $v \in V(Q)-Y(t, t')$. Then $v \in Y_{t''}$ for some $t'' \in V(T)$ by (W1), and, from the symmetry, we may assume that $t'' \neq t$ and that t lies on the path between t' and t''. The path Q contains a subpath between v and $P_j(t, t')\backslash Y_t$ and, by (2.3), this subpath meets Y_t. But this subpath meets Y_t in an internal vertex of Q and $Y_t \subseteq V(P_1 \cup P_2 \cup \cdots \cup P_s)$. Thus we have a contradiction which proves (8).

We denote by H the (disconnected) graph $P_2 \cup P_3 \cup \cdots \cup P_s$.

(9) *Let t, t' be vertices of R such that $|Y_t|=|Y_{t'}|=s$, let $P=P_1(t, t')$, and let u, v be the ends of P. Suppose that $|V(P)| \geqslant 5$. Then there are at least two disjoint paths between P and H contained in $Y(t, t')-\{u, v\}$.*

We first prove that there are at least two disjoint paths between P and H in $G\backslash\{u, v\}$. Suppose that two such paths do not exist. Then, by Menger's theorem, there is a vertex x of G such that every path in G between

P and H uses a vertex from $\{u, v, x\}$. Let A' be the set of all vertices w of G such that there is a path that is disjoint from $\{u, v, x\}$ and has one end equal to w and the other end in $V(P)$. Let $A = A' \cup \{u, v, x\}$ and let $B = V(G) - A'$. Then (A, B) is a separation of G of order 3. Since $|V(P)| \geqslant 5$, we deduce that $|A - B| \geqslant 2$. On the other hand, $Y_t \cap V(H) \subseteq B - \{u, v\}$ and $|Y_t \cap V(H)| = s - 1$, so $|B - A| \geqslant 2$, because $s \geqslant 4$. Thus the existence of (A, B) contradicts the internal 4-connectivity of G and hence proves that there are two disjoint paths between P and H in $G \backslash \{u, v\}$. Now if we take two such paths with no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$, it follows from (8) that they are contained in $Y(t, t')$. This completes the proof of (9).

Let t, t', P, u, v satisfy the hypotheses of (9), and let Q, Q' be two disjoint paths between P and H that are contained in $Y(t, t') - \{u, v\}$ and have no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$. Let w and w' be the ends of Q and Q', respectively, that are not in $V(P)$ and suppose $w \in V(P_l)$ and $w' \in V(P_{l'})$. If $l = l'$ for every such pair Q, Q', then we say that the pair t, t' is *singular*; otherwise we say that it is *regular*. The pair of vertices w, w' is called a *destination pair* for (t, t'), and each of w, w' is called a *destination vertex* for (t, t'). A *p-ladder* is a segmentation $t_0, t_1, \ldots, t_p$ such that, for some $j \in \{2, 3, \ldots, s\}$, there are distinct vertices $v_1, v_2, \ldots, v_p \in V(P_j)$ such that v_i is a destination vertex for (t_{i-1}, t_i) for all $i = 1, 2, \ldots, p$. A p-ladder $t_0, t_1, \ldots, t_p$ is said to be *free* if for every j and $v_1, v_2, \ldots, v_p$ as above and every $j' \in \{2, 3, \ldots, j-1, j+1, \ldots, s\}$, no path between $P_1(t_0, t_p) \cup P_j(t_0, t_p)$ and $P_{j'}(t_0, t_p)$ is contained in $Y(t_0, t_p)$.

(10) *There is no free k-ladder.*

Suppose that $t_0, t_1, \ldots, t_k$ is a free k-ladder, and let $j, v_1, v_2, \ldots, v_k$ be as in the definition of a k-ladder. We may assume without loss of generality that $j = 2$. For $i = 1, 2, \ldots, k$, let P_i' be a path between $P_1 \backslash (Y_{t_{i-1}} \cup Y_{t_i})$ and v_i that has no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$ and is contained in $Y(t_{i-1}, t_i)$. Let $L = P_1(t_0, t_k) \cup P_2(t_0, t_k) \cup P_1' \cup P_2' \cup \cdots \cup P_k'$. By (2.3), the paths $P_1', P_2', \ldots, P_k'$ are pairwise disjoint, and hence L contains a subgraph isomorphic to a subdivision of L_k. Let $Z = V(P_1(t_0, t_k) \cup P_2(t_0, t_k))$, and let $A = Y_{t_0} \cap Z$, $X = Y_{t_2} \cap Z$, and $B = Y_{t_k} \cap Z$. Then $|A| = |B| = |X| = 2$. We wish to apply (5). To do so, we first note that $r_1 \neq t_0 \neq r_{n_6}$, because r_1, $r_{n_6} \notin V(R)$. By (5) there are two paths, each of which has at least three vertices and is internally disjoint from $Y(t_0, t_k)$, such that one has both ends in A and the other has both ends in B. We may now apply the internal 4-connectivity of G to deduce that there are two disjoint paths Q_1, Q_2 between A and B in $G \backslash X$. We claim that no internal vertex of either of Q_1, Q_2 belongs to L. Assume the contrary. Then there is a path between $P_1(t_0, t_2) \cup P_2(t_0, t_2)$ and B in $G \backslash (A \cup X)$, or a path between $P_1(t_2, t_k) \cup P_2(t_2, t_k)$ and A in $G \backslash (X \cup B)$. From the symmetry, we may assume the former. Then, by (2.3), there is a path between $(P_1(t_0, t_2) \cup P_2(t_0, t_2)) \backslash (A \cup X)$ and $Y_{t_2} - X$. Since $Y_{t_2} - X \subseteq \bigcup_{3 \leqslant j' \leqslant s} V(P_{j'}(t_0, t_2))$, it follows that there is a path Q between $(P_1(t_0, t_2) \cup P_2(t_0, t_2)) \backslash (A \cup X)$ and $P_{j'}(t_0, t_2)$ for some $j' \in \{3, 4, ..., s\}$ such that no internal vertex of Q is in $P_1 \cup P_2 \cup \cdots \cup P_s$. But then $V(Q) \subseteq Y(t_0, t_2) \subseteq Y(t_0, t_k)$ by (8), contrary to the freedom of $t_0, t_1, ..., t_k$. This proves our claim that no internal vertex of either of Q_1, Q_2 belongs to L. Thus $L \cup Q_1 \cup Q_2$ is isomorphic to a subdivision of either M_k or O_k, a contradiction which proves (10).

(11) *There is no n_2-ladder.*

Suppose that $t_0, t_1, ..., t_{n_2}$ is an n_2-ladder. Let $j, v_1, v_2, ..., v_{n_2}$ be as in the definition of an n_2-ladder. We may assume that $j = 2$. For $i = 1, 2, ..., n_2$, let P'_i be a path between $P_1 \backslash (Y_{t_{i-1}} \cup Y_{t_i})$ and v_i that has no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$ and is contained in $Y(t_{i-1}, t_i)$. Let $i \in \{0, 1, ..., n_1 - 2\}$. Since $t_{2ik}, t_{2ik+1}, ..., t_{(2i+1)k}$ is a k-ladder, it is not free by (10), and so, for some $l_i \in \{1, 2\}$ and some $j_i \in \{3, 4, ..., s\}$, there is a path Q_i between $P_{l_i}(t_{2ik}, t_{(2i+1)k})$ and $P_{j_i}(t_{2ik}, t_{(2i+1)k})$ that is contained in $Y(t_{2ik}, t_{(2i+1)k})$ and has no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$. Let $i, i' \in \{0, 1, ..., n_1 - 2\}$ be such that $j_i = j_{i'}$. We may assume that $j_i = j_{i'} = 3$ and that $i < i'$. By (5), there are paths Q'_i and $Q'_{i'}$, the first between $Y_{t_{2ik}} \cap V(P_{3-l_i})$ and $Y_{t_{2ik}} \cap V(P_4)$, and the second between $Y_{t_{(2i'+1)k}} \cap V(P_{3-l_{i'}})$ and $Y_{t_{(2i'+1)k}} \cap V(P_4)$, such that neither path has any internal vertices in $Y(t_{2ik}, t_{(2i'+1)k})$. It follows from (2.3) that the union of the paths $Q_i, Q'_i, P_1, P_2, P_3, P_4, Q_{i'}, Q'_{i'}, P'_{(2i+1)k+1}, P'_{(2i+1)k+2}, ..., P'_{2(i+1)k}$ contains a subgraph isomorphic to a subdivision of either O_k or M_k, a contradiction which proves (11).

(12) *For every segmentation $t_0, t_1, ..., t_p$, there are at most $(n_2 - 1)(n_1 - 1)$ numbers $i \in \{1, 2, ..., p\}$ such that the pair (t_{i-1}, t_i) is singular.*

If not, then, for some $j \in \{2, 3, ..., s\}$ and some integers $l_1, l_2, ..., l_{n_2}$ with $0 < l_1 < \cdots < l_{n_2} \leqslant p$, there are distinct vertices v_i, u_i of P_j for every i in $\{1, 2, ..., n_2\}$, such that v_i, u_i is a destination pair for $(t_{l_i - 1}, t_{l_i})$ for all such i. We may assume that the notation is chosen so that $u_1, v_1, u_2, v_2, ..., u_{n_2}, v_{n_2}$ occur on P_j in this order. Hence $u_1, u_2, ..., u_{n_2}$ are all distinct, and thus $t_{l_1}, t_{l_2}, ..., t_{l_{n_2}}$ is an n_2-ladder, contrary to (11).

(13) *There is a segmentation $t_0, t_1, ..., t_{n_4}$ such that (t_{i-1}, t_i) is regular for every $i = 1, 2, ..., n_4$.*

Such a segmentation can be obtained from a segmentation as in (7) by deleting all t_{i-1} with (t_{i-1}, t_i) singular. By (12), it suffices to delete at most $(n_2 - 1)(n_1 - 1)$ such vertices.

Now we are ready to complete the proof of (3.1) by showing that G contains a subdivision of A_k. Recall that $n_4 = n_3\binom{n_1 - 1}{2} + 1$. Then, by considering the segmentation from (13), it follows that there is a segmentation $t'_0, t'_1, ..., t'_{n_3+1}$ such that, for some distinct $j, j' \in \{2, 3, ..., s\}$, there are vertices $u_1, u_2, ..., u_{n_3+1} \in V(P_j)$ and vertices $u'_1, u'_2, ..., u'_{n_3+1} \in V(P_{j'})$ such that u_i, u'_i is a destination pair for (t'_{i-1}, t'_i) for all i in $\{1, 2, ..., n_3+1\}$. At most $n_2 - 1$ of the vertices $u_1, u_2, ..., u_{n_3+1}$ are distinct because otherwise there is an n_2-ladder. Thus, as $n_3 + 1 = (n_2 - 1)^2 (2k-1) + 1$, there are at least $(n_2 - 1)(2k-1) + 1$ successive u_i's all of which are equal to some fixed vertex, say u. Consider the corresponding set of u'_i's. At most $n_2 - 1$ of these are distinct and so, among this set of u'_i's, there are at least $(2k-1)+1$ successive ones that are all equal to some fixed vertex, say u'. We conclude that there is a segmentation $t_0, t_1, ..., t_{2k}$ and vertices $u \in V(P_j)$, $u' \in V(P_{j'})$ such that u, u' is a destination pair for (t_{i-1}, t_i) for every $i = 1, 2, ..., 2k$.

For $i = 1, 2, ..., 2k$, let Q_i, Q'_i be two disjoint paths each of which is contained in $Y(t_{i-1}, t_i)$, has no internal vertex in $P_1 \cup P_2 \cup \cdots \cup P_s$, and has one end in $P_1(t_{i-1}, t_i) \backslash (Y_{t_{i-1}} \cup Y_{t_i})$; the other end of Q_i is u and the other end of Q'_i is u'. We may assume that $j = 2$ and $j' = 3$. By (5) and (2.3), there are paths Q and Q' between $V(P_1) \cap Y_{t_0}$ and $V(P_4) \cap Y_{t_0}$ and between $V(P_1) \cap Y_{t_{2k}}$ and $V(P_4) \cap Y_{t_{2k}}$, respectively, such that neither path has any internal vertices in $Y(t_0, t_{2k})$. It follows that the union of the paths Q, Q', $P_1(t_0, t_{2k})$, $P_4(t_0, t_{2k})$, Q_{2i-1}, Q'_{2i} $(i = 1, 2, ..., k)$ forms a subdivision of A_k, as desired. This completes the proof of (3.1). ∎

4. Proof of (1.6)

We require the following two easy lemmas.

(4.1) *Let G be a graph that is not internally 4-connected. Then there is a separation (J_1, J_2) of G of order at most 3 with $J_1, J_2 \neq V(G)$ and with the property that both G_1 and G_2 are isomorphic to minors of G, where G_i is obtained from G by deleting vertices not in J_i, and adding an edge joining every pair of distinct vertices in $J_1 \cap J_2$.*

(4.2) *Let (T, Y) be a tree-decomposition of a graph G, and let K be a complete subgraph of G. Then there is a vertex $t \in V(T)$ such that $V(K) \subseteq Y_t$.*

Proof of (1.6). Let $k \geqslant 4$ be given. We take N to be the "N" from (1.4) and proceed by induction. Assume first that $|V(G)| < N$. Let T be a one-vertex tree with $V(T) = \{t_0\}$ say, let $Y_{t_0} = V(G)$, and let $Y = (Y_{t_0})$. Then (T, Y) is a tree-decomposition of G of width at most N and adhesion at most 3, as desired. Now let G be a graph with $|V(G)| \geqslant N$ and with no minor isomorphic to any of D_k, O_k, M_k, and $K_{4,k}$. Then G contains no subgraph isomorphic to a subdivision of any of A_k, O_k, M_k, $K_{4,k}$, and $K'_{4,k}$, and so, by (1.4), G is not internally 4-connected. We consider the graphs G_1, G_2 as in (4.1). Clearly neither G_1 nor G_2 has a minor isomorphic to any of D_k, O_k, M_k, and $K_{4,k}$, and so, by the induction hypothesis, G_i has a tree-decomposition (T^i, Y^i) of width at most N and adhesion at most 3 $(i=1, 2)$. By (4.2), there are vertices $t_1 \in V(T^1)$, $t_2 \in V(T^2)$ such that $V(G_1) \cap V(G_2) \subseteq Y^1_{t_1} \cap Y^2_{t_2}$. Let T be the tree obtained by taking the disjoint union of T^1 and T^2 and adding an edge joining t_1 and t_2. Let Y_t be Y^1_t if $t \in V(T^1)$ and Y^2_t if $t \in V(T^2)$, and let $Y=(Y_t: t \in V(T))$. Then it is easy to see that (T, Y) is a tree-decomposition of G of width at most N and adhesion at most 3. This completes the proof of (1.6). ∎

5. Infinite Graphs

In this section we discuss an analog of (1.3) and (1.4) for infinite graphs. Halin [1] proved the following:

(5.1) *Let n be a positive integer. Every uncountable n-connected graph contains a subgraph isomorphic to a subdivision of $K_{n,\aleph_1}$.*

Our objective is to prove a similar result for countable graphs. A *ray* is a one-way infinite path. Let T be a finite tree and let R be a ray. By $T \times R$ we denote the following graph (a product of T and R): its vertices are pairs (t, r), where $t \in V(T)$ and $r \in V(R)$, and two vertices (t, r) and (t', r') are adjacent if either $t=t'$ and r, r' are adjacent in R, or $r=r'$ and t, t' are adjacent in T. A *semi-weighting* on a tree T is a function from $V(T)$ into the set of non-negative integers; a semi-weighting ϕ is a *weighting* if $\phi(t)>0$ for every $t \in V(T)$ of valency one. The *order* of a semi-weighting is $|V(T)| + \sum_{t \in V(T)} \phi(t)$. Let T be a finite tree and let ϕ be a semi-weighting on T. We define a graph $G(T, \phi)$ to be the graph obtained from $T \times R$ by adding, for each $t \in V(T)$, a set of $\phi(t)$ pairwise nonadjacent vertices and joining each of these vertices to (t, r) for every $r \in V(R)$. We say that an infinite graph G is *essentially k-connected*, where k is a positive integer, if there is a constant C such that for every separation (A, B) of G of order less than k, either $|A| \leqslant C$ or $|B| \leqslant C$. We remark that each $G(T, \phi)$ is essentially k-connected, where k is the order of ϕ. The following is our result for infinite graphs.

(5.2) *Let k be a positive integer, and let G be an essentially k-connected infinite graph. Then either G has a minor isomorphic to $K_{k,\aleph_0}$, or, for some finite tree T and some weighting ϕ on T of order k, the graph G has a minor isomorphic to $G(T, \phi)$.*

Proof. We begin with the following.

(1) *If there is a finite subset X of $V(G)$ such that $G\backslash X$ has infinitely many components, then G has a minor isomorphic to $K_{k,\aleph_0}$.*

Since G is essentially k-connected, $|X| \geqslant k$ and (1) follows. We may therefore assume that

(2) *for every finite subset X of $V(G)$, the graph $G\backslash X$ has only finitely many components.*

We now claim the following.

(3) *G has a ray.*

Choose $v_0 \in V(G)$ arbitrarily. Now assume that $v_0, v_1, ..., v_{n-1}$ have already been chosen in such a way that $v_0, v_1, ..., v_{n-1}$ (in this order) form the vertex-set of a path and v_{n-1} has a neighbor in an infinite component of $G\backslash\{v_0, v_1, ..., v_{n-1}\}$. We let v_n be a neighbor of v_{n-1} in an infinite component of $G\backslash\{v_0, v_1, ..., v_{n-1}\}$; it follows from (2) that v_n has a neighbor in an infinite component of $G\backslash\{v_0, v_1, ..., v_n\}$. This completes the inductive definition. Now $v_0, v_1, ...$ form the vertex-set of a ray in G, as desired.

We say that two rays P, P' are *parallel* if, for every finite set $X \subseteq V(G)$, all subrays of $P\backslash X$ and $P'\backslash X$ belong to the same component of $G\backslash X$. This defines an equivalence relation on rays. Let $\mathscr{E}$ be an equivalence class of rays, and let $P_1, P_2, ..., P_m$ be disjoint rays in $\mathscr{E}$.

(4) *G has a minor isomorphic to $T \times R$ for some tree T on m vertices.*

To see this we define an auxiliary graph K with vertex set $\{1, 2, ..., m\}$ in which i is adjacent to i' if there are infinitely many disjoint paths between P_i and $P_{i'}$, each disjoint from P_j for every $j \in \{1, 2, ..., m\} - \{i, i'\}$. We claim that K is connected. Indeed, suppose i, i' are two nonadjacent vertices of K. Then, by an infinite analog of Menger's theorem, there is a finite subset $X(i, i')$ of $V(G)$ such that no path between P_i and $P_{i'}$ in $G\backslash X(i, i')$ is disjoint from every P_j for $j \in \{1, 2, ..., m\} - \{i, i'\}$. Let $X = \bigcup X(i, i')$, where the union is taken over all nonadjacent pairs i, i' of vertices of K. Now if i, i' belong to different components of K, it follows that $P_i\backslash X$ and $P_{i'}\backslash X$ belong to different components of $G\backslash X$, contrary to the fact that they are parallel. This proves that K is connected. Let T be a spanning tree of K. A minor of G isomorphic to $T \times R$ can now be constructed by a simple greedy process. We omit the details.

For a finite subset X of $V(G)$ let βX denote the vertex-set of the component of $G\backslash X$ containing a subray of some (and hence every) ray of $\mathscr{E}$. We say that a vertex v of G is *major* if $v \in X \cup \beta X$ for every finite subset X of $V(G)$.

(5) *If no vertex is major, then, for every finite subset A of $V(G)$, there is a finite subset B of $V(G)$ with $B \cup \beta B \subseteq \beta A$.*

Let A be a finite subset of $V(G)$. Then certainly there is a finite subset B of $V(G)$ such that $B \subseteq A \cup \beta A$ and $\beta B \subseteq \beta A$. Moreover, we may assume that B is chosen so that $|A \cap B|$ is a minimum. We claim that $A \cap B = \emptyset$. Suppose to the contrary that $v \in A \cap B$. Since v is not major, there is a finite subset X of $V(G)$ with $v \notin X \cup \beta X$. Then the set $(B \cup (X \cap \beta A)) - \{v\}$ contradicts the choice of B. This proves (5).

Let $m = k$ if there are k disjoint rays in $\mathscr{E}$, otherwise let m be the maximum number of disjoint rays in $\mathscr{E}$.

(6) *There are at least* $k - m$ *major vertices.*

Let M be the set of major vertices. We may assume that $|M| < k$. Let $H = G \backslash M$. We define β in H by the same rule as in G. Then H has no major vertices. Let A_0 be an arbitrary nonempty finite subset of $V(H)$, and let $A_1, A_2, \ldots$ be finite subsets of $V(H)$ such that, for every $i \geqslant 0$,

(i) $A_{i+1} \cup \beta A_{i+1} \subseteq \beta A_i$, and

(ii) $|A_{i+1}|$ is minimal subject to (i).

It follows that $|A_1| \leqslant |A_2| \leqslant \cdots$, that for every finite subset X of $V(H)$ there is an $i \geqslant 0$ such that $X \cap \beta A_i = \emptyset$ (this follows by examining shortest paths from $X \cap \beta A_i$ to A_i), and that there are $|A_i|$ disjoint paths between A_i and A_{i+1}. By putting these paths together, we get $\lim_{i \geqslant 1} |A_i|$ disjoint rays. We claim that each such ray belongs to $\mathscr{E}$. Indeed, suppose that one of them, say P, does not belong to $\mathscr{E}$. Then there is a finite subset X of $V(G)$ such that some subray P' of P satisfies $V(P') \cap \beta X = \emptyset$. Choose i such that $X \cap \beta A_i = \emptyset$ and $A_{i+1} \cap V(P') \neq \emptyset$. But $A_{i+1} \subseteq \beta A_i \subseteq \beta X$, a contradiction which proves the claim that each of the specified rays belongs to $\mathscr{E}$. Thus $\min(k, |A_i|) \leqslant m$ for all sufficiently large i, and since, for every $i \geqslant 1$, $A_i \cup M$ separates G, and G is essentially k-connected, we deduce that $m + |M| \geqslant k$. This proves (6).

Let T be as in (4), where m is as specified prior to (6). Using the $k - m$ major vertices, it is now easy to construct a semi-weighting ϕ of order k on T and a minor of G isomorphic to $G(T, \phi)$. By contracting subgraphs of G corresponding to rays $\{(t, r)\}_{r \in V(R)}$ of $T \times R$, where t has valency 1 in T and $\phi(t) = 0$, we obtain a minor of G isomorphic to some $G(T', \phi')$, where ϕ' is a weighting on T' of order k. We omit further details. ∎

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