GLOBAL LORENTZ AND LORENTZ-MORREY ESTIMATES BELOW THE NATURAL EXPONENT FOR QUASILINEAR EQUATIONS

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Abstract. Lorentz and Lorentz-Morrey estimates are obtained for gradients of very weak solutions to quasilinear equations of the form

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f},$$

where $\operatorname{div} \mathcal{A}(x, \nabla u)$ is modelled after the p-Laplacian, $p > 1$. The estimates are global over bounded domains that satisfy a mild exterior uniform thickness condition that involves the p-capacity. The vector field datum $\mathbf{f}$ is allowed to have low degrees of integrability and thus solutions may not have finite L^p energy. A higher integrability result at the boundary of the ground domain is also obtained for infinite energy solutions to the associated homogeneous equations.

1. Introduction

We address the question of global regularity of *very weak solutions* to the nonhomogeneous nonlinear boundary value problems of the form

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) &= \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ potentially with a non-smooth boundary.

In (1.1), the operator $\operatorname{div} \mathcal{A}(x, \nabla u)$ is modelled after the p-Laplacian $\Delta_p u = \operatorname{div} |\nabla u|^{p-2}\nabla u$, with $p \in (1, n]$. Our main goal in this paper is to find minimal conditions on the non-linearity $\mathcal{A}$ and on the boundary of the domain so that the gradient, ∇u, of a very weak solution to (1.1) is as regular as the data $\mathbf{f}$. Here by very weak solutions we mean distributional solutions that may not have finite L^p energy. That is, solutions u are required only that $\mathcal{A}(x, \nabla u) \in L^1(\Omega)$ with a certain zero boundary condition such that

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx = \int_\Omega |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla \varphi dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

In particular, we will give various function spaces $\mathcal{S}$ such that $\mathbf{f} \in \mathcal{S}$ implies $\nabla u \in \mathcal{S}$. The function spaces we will present include the standard Lebesgue spaces, Lorentz spaces, and Lorentz-Morrey spaces that are based on L^q spaces for q in a neighborhood of p, i.e., q is allowed to lie below the natural exponent p.

More specifically, the non-linearity $\mathcal{A}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathédory vector valued function, i.e., $\mathcal{A}(x,\xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We always assume that $\mathcal{A}(x,0)=0$ for a.e. $x\in\mathbb{R}^n$. For our purpose, we also require that $\mathcal{A}$ satisfy the following monotonicity and Hölder type conditions: for some $1<p\leq n$ and $\gamma\in(0,1)$ there hold

$$\langle\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta),\xi-\zeta\rangle\geq\Lambda_0(|\xi|^2+|\zeta|^2)^{\frac{p-2}{2}}|\xi-\zeta|^2 \tag{1.2}$$

and

$$|\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta)|\leq\Lambda_1|\xi-\zeta|^{\gamma}(|\xi|^2+|\zeta|^2)^{\frac{p-1-\gamma}{2}} \tag{1.3}$$

for every $(\xi,\zeta)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{(0,0)\}$ and a.e. $x\in\mathbb{R}^n$. Here Λ_0 and Λ_1 are positive constants. Note that (1.3) and the assumption $\mathcal{A}(x,0)=0$ for a.e. $x\in\mathbb{R}^n$ imply the following condition

$$|\mathcal{A}(x,\xi)|\leq\Lambda_1\,|\xi|^{p-1}\,. \tag{1.4}$$

Moreover, assumption (1.3) for the structure of the nonlinearity is weaker than that considered in the earlier work [20], in which a Lipschitz type condition, i.e., $\gamma=1$, was used.

With regard to the domain Ω, in this paper we shall assume that Ω is a bounded domain whose complement $\Omega^c:=\mathbb{R}^n\setminus\Omega$ uniformly thick with respect to the p-capacity. Let $1<p\leq n$ and $O\subset\mathbb{R}^n$ be an open set. Recall that for a compact set $K\Subset O$, the p-capacity of K is defined by

$$\mathrm{cap}_p(K,O):=\inf\left\{\int_O|\nabla u|^p\,dx:0\leq u\in C_c^\infty(O),u\geq1\,\mathrm{on}\,K\right\}.$$

It is easy to see that for $1<p\leq n$, there holds

$$\mathrm{cap}_p(\overline{B_r(x)},B_{2r}(x))=c\,r^{n-p},$$

where $c=c(n,p)>0$ (see [18, Chapter 2]). Henceforth, the notation $B_r(x)$ denotes the Euclidean ball centered at x with radius $r>0$, and $\overline{B_r(x)}$ is its closure.

Definition 1.1 (Uniform p-thickness)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. We say that the complement $\Omega^c:=\mathbb{R}^n\setminus\Omega$ is uniformly p-thick for some $1<p\leq n$ with constants $r_0,b>0$, if the inequality*

$$\mathrm{cap}_p(\overline{B_r(x)}\cap\Omega^c,B_{2r}(x))\geq b\,\mathrm{cap}_p(\overline{B_r(x)},B_{2r}(x))$$

holds for any $x\in\partial\Omega$ and $r\in(0,r_0]$.

It is well-known that the class of domains with uniform p-thick complements is very large. They include all domains with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants $c_0,r_0>0$ such that for all $0<t\leq r_0$ and all $x\in\mathbb{R}^n\setminus\Omega$, there is $y\in B_t(x)$ such that $B_{t/c_0}(y)\subset\mathbb{R}^n\setminus\Omega$.

We now recall the definition of Lorentz and Lorentz-Morrey spaces. The Lorentz space $L(s,t)(\Omega)$, with $0<s<\infty$, $0<t\leq\infty$, is the set of measurable functions g on Ω such that

$$\|g\|_{L(s,t)(\Omega)} := \left[s\int_0^\infty \alpha^t |\{x\in\Omega : |g(x)|>\alpha\}|^{\frac{t}{s}} \frac{d\alpha}{\alpha}\right]^{\frac{1}{t}} < +\infty$$

when $t\neq\infty$; for $t=\infty$ the space $L(s,\infty)(\Omega)$ is set to be the usual Marcinkiewicz space with quasinorm

$$\|g\|_{L(s,\infty)(\Omega)} := \sup_{\alpha>0} \alpha|\{x\in\Omega : |g(x)|>\alpha\}|^{\frac{1}{s}}.$$

It is easy to see that when $t=s$ the Lorentz space $L(s,s)(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$, which is equivalently defined as

$$g\in L^s(\Omega) \Longleftrightarrow \int_\Omega |g(x)|^s\,dx<\infty.$$

A function $g\in L(s,t)(\Omega)$, $0<s<\infty$, $0<t\leq\infty$ is said to belong to the Lorentz-Morrey function space $\mathcal{L}^\theta(s,t)(\Omega)$ for some $0<\theta\leq n$, if

$$\|g\|_{\mathcal{L}^\theta(s,t)(\Omega)} := \sup_{\substack{0<r\leq \mathrm{diam}(\Omega),\\ z\in\Omega}} r^{\frac{\theta-n}{s}}\,\|g\|_{L(s,t)(B_r(z)\cap\Omega)} < +\infty.$$

When $\theta=n$, we have $\mathcal{L}^\theta(s,t)(\Omega)=L(s,t)(\Omega)$. Moreover, when $s=t$ the space $\mathcal{L}^\theta(s,t)(\Omega)$ becomes the usual Morrey space based on L^s space.

A basic use of Lorentz spaces is to improve the classical Sobolev Embedding Theorem. For example, if $f\in W^{1,q}$ for some $q\in(1,n)$ then

$$f\in L(nq/(n-q),q)$$

(see, e.g., [42]), which is better than the classical result

$$f\in L^{nq/(n-q)} = L(nq/(n-q), nq/(n-q))$$

since $L(s,t_1)\subset L(s,t_2)$ whenever $t_1\leq t_2$. Another use of Lorentz spaces is to capture logarithmic singularities. For example, for any $\beta>0$ we have

$$\frac{1}{|x|^{n/s}(-\log|x|)^\beta}\in L(s,t)(B_1(0)) \quad \text{if and only if } t>\frac{1}{\beta}.$$

Lorentz spaces have also been used successfully in improving regularity criteria for the full 3D Navier-Stokes system of equations (see, e.g., [39]).

On the other hand, Lorentz-Morrey spaces are neither rearrangement invariant spaces, nor interpolation spaces. They often show up in the analysis of Schrödinger operators (see [10]) or in the regularity theory of nonlinear equations of fluid dynamics. Moreover, estimates in Morrey spaces have been used as an indispensable tool in the recent papers [31, 38] to obtain sharp existence results for a quasilinear Riccati type equation. In fact, that is one of the main motivations in obtaining bounds in Lorentz-Morrey spaces in this paper.

We are now ready to state the first main result of the paper.

Theorem 1.2. *Let $\mathcal{A}$ satisfy (1.2)-(1.3), and let Ω be a bounded domain whose complement uniformly p-thick with constants $r_0, b > 0$. Then there exists a small $\delta = \delta(n, p, \Lambda_0, \Lambda_1, \gamma, b) > 0$ such that for any very weak solution $u \in W_0^{1,p-2\delta}(\Omega)$ to the boundary value problem (1.1) there holds*

$$\|\nabla u\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \leq C\, \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \tag{1.5}$$

for all $q \in [p-\delta, p+\delta]$, $0 < t \leq \infty$, and $\theta \in [p-2\delta, n]$. Here the constant $C = C(n, p, t, \Lambda_0, \Lambda_1, \gamma, b, \mathrm{diam}(\Omega)/r_0)$.

In the simplest case where $\theta = n$ and $t = q$, Theorem 1.2 yields the following basic Calderón-Zygmund type estimate for solutions of (1.1):

$$\|\nabla u\|_{L^q(\Omega)} \leq C\, \|\mathbf{f}\|_{L^q(\Omega)} \tag{1.6}$$

for all $q \in [p-\delta, p+\delta]$, provided $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick. We observe that inequality (1.6) has been obtained in [20] under stronger conditions on $\mathcal{A}$ and Ω. Namely, on the one hand, a Lipschitz type condition, i.e., $\gamma = 1$ in (1.3), was assumed in [20]. On the other hand, the domain Ω considered [20] was assumed to be regular in the sense that the Calderón-Zygmund type bound

$$\|\nabla v\|_{L^r(\Omega)} \leq C\, \|\mathbf{f}\|_{L^r(\Omega)} \tag{1.7}$$

holds for *all* $r \in (1, \infty)$ and all solutions to the linear equation

$$\begin{cases} \Delta v = \mathrm{div}\,\mathbf{f} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

As demonstrated by a counterexample in [29] (see also [23]), estimate (1.7), say for large r, generally fails for solutions of (1.8) even for (non-convex) Lipschitz domains. Thus the result of [20] concerning the bound (1.6) does not cover all Lipschitz domains. In this respect, the bound (1.6) for domains with thick complements is new, and in fact it is new even for linear equations, where the principal operator is replaced by just the standard Laplacian Δ.

Another new aspect of this paper is the following *boundary* higher integrability result for very weak solutions to the associated homogeneous equations.

Theorem 1.3. *Suppose that $\mathcal{A}$ satisfies (1.2) and (1.4), and that Ω is a bounded domain whose complement uniformly p-thick with constants $r_0, b > 0$. Then there exists a positive number $\overline{\delta} = \overline{\delta}(n, p, \Lambda_0, \Lambda_1, b)$ such that the following holds. For any $x_0 \in \partial\Omega$ and $R \in (0, r_0/2)$, if $w \in W^{1,p-\overline{\delta}}(\Omega \cap B_{2R}(x_0))$ is a very weak solution to the Dirichlet problem*

$$\begin{cases} \mathrm{div}\mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega \cap B_{2R}(x_0), \\ w = 0 & \text{on } \partial\Omega \cap B_{2R}(x_0), \end{cases}$$

then there holds $w \in W^{1,p+\overline{\delta}}(\Omega \cap B_R(x_0))$.

A quantitative statement of Theorem 1.3 can be found in Theorem 3.7 below. We notice that whereas *interior* higher integrability of very weak solutions to the equation $\operatorname{div}\mathcal{A}(x,\nabla w)=0$ is well-known (see [20, 27]), the boundary higher integrability result has been obtained only for *finite energy solutions* $w\in W^{1,p}(\Omega\cap B_{2R}(x_0))$ in the paper [24] (see also [32]). The fact that $|\nabla w|$ is allowed to be in $L^{p-\bar{\delta}}$ to begin with plays a crucial role in our proof of Theorem 1.2 above.

Remark 1.4. The Hölder type condition (1.3) with $\gamma\in(0,1)$ is not needed in Theorem 1.3, while this condition is assumed in Theorem 1.2. As a matter of fact, the proof of Theorem 1.2 requires (1.3) only through the use of Corollaries 2.5 and 3.5. Thus by Remark 3.6 below, making use of only (1.2), (1.4) and the p-thickness condition as in Theorem 1.2, we still obtain inequality (1.5) with a constant

$$C=C(n,p,q,t,\Lambda_0,\Lambda_1,b,\operatorname{diam}(\Omega)/r_0)$$

for any finite energy solution $u\in W_0^{1,p}(\Omega)$ provided $q\in(p,p+\delta]$.

There are numerous papers devoted to the L^q bound (1.6) for solutions of (1.1) in the *super-natural range* $q>p$. The pioneer work [19] dealt with the case $\Omega=\mathbb{R}^n$, and the paper [21] obtained a local interior bound. In [6], using a perturbation technique developed for fully nonlinear PDEs [5], the authors proved certain local $W^{1,q}$ regularity for an associated homogeneous quasilinear equation. Global estimates upto the boundary of a bounded domains were obtained in [22] (for $C^{1,\alpha}$ domains) and in [3, 4] (for Lipschitz domains with small Lipschitz constants or for domains that are sufficiently flat in the sense of Reifenberg). Global weighted analogues of those results that can be used to deduce the associated Morrey type bounds can be found in [30, 31, 35]. We notice that, due to the lack of duality, the results in those papers, which treat only the case $q>p$, could not be apply to the case $q\leq p$ even for good domains and for nonlinear operators with continuous coefficients. In fact, even the basic Calderón-Zygmund type bound (1.6) for *all* $p-1<q<p$ has been a long standing open problem known as a conjecture of T. Iwaniec (see [19, 33]). On the other hand, we remark that if the divergence form datum div $|\mathbf{f}|^{p-2}\mathbf{f}$ on the right-hand side of (1.1) is replaced by a finite measure μ then gradient estimnates below the natural exponent p can be obtained as demonstrated, e.g., in [7, 8, 25, 36, 37, 38] at least for $2-1/n<p\leq n$.

In this paper, to treat the sub-natural case $q\geq p-\delta$ for equation (1.1), we have to come up with some new ingredients. One such ingredient is the local interior and boundary comparison estimates below the natural exponent p (see Lemmas 2.8 and 3.10 below). Those important comparison estimates enable some of the techniques developed for the super-natural case mentioned above to be effectively employed here.

Finally, following the approach of [31], the Lorentz-Morrey bound obtained in Theorem 1.2 can be used to obtain a sharp existence result the quasilinear Riccati type equation

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) & = & |\nabla u|^q + \sigma \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega, \end{cases}$$

with a distributional datum σ in the sub-natural range $q \in (p - \delta, p]$. As this seems to be out of the scope of this paper, we choose to pursue that study elsewhere in our future work.

Notation. Throughout the paper, we shall write $A \lesssim B$ to denote $A \leq c\, B$ for a positive constant c independent of the parameters involved. Basically, c is allowed to depend only on $n, p, \gamma, \Lambda_0, \Lambda_1$, and b. Likewise, $A \gtrsim B$ means $A \geq c\, B$, and $A \simeq B$ means $c_1 B \leq A \leq c_2 B$ for some positive constants c_1 and c_2.

2. Local interior estimates

In this section, we obtain certain local interior estimates for very weak solutions of (1.1). These include the important comparison estimates below the natural exponent p. We shall make use of the following nonlinear Hodge decomposition of [20].

Theorem 2.1 (Nonlinear Hodge Decomposition [20])**.** *Let $s > 1$, $\epsilon \in (-1, s-1)$, and let $w \in W_0^{1,s}(B)$ where $B \subset \mathbb{R}^n$ is a ball. Then there exist $\phi \in W_0^{1,\frac{s}{1+\epsilon}}(B)$ and a divergence free vector field $\mathcal{H} \in L^{\frac{s}{1+\epsilon}}(B, \mathbb{R}^n)$ such that*

$$|\nabla w|^{\epsilon}\, \nabla w = \nabla \phi + \mathcal{H}.$$

Moreover, the following estimate holds:

$$\|\mathcal{H}\|_{L^{\frac{s}{1+\epsilon}}(B)} \leq C(s,n)\, |\epsilon|\, \|\nabla w\|_{L^s(B)}^{1+\epsilon}.$$

Using the above Hodge decomposition, the authors of [20] obtained gradient L^q regularity below the natural exponent for very weak solutions to certain quasilinear elliptic equations.

Theorem 2.2 ([20])**.** *Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). There exists a constant $\tilde{\delta}_1 = \tilde{\delta}_1(n, p, \Lambda_0, \Lambda_1, \gamma)$ with $0 < \tilde{\delta}_1 < \min\{1, p-1\}$ sufficiently small such that the following holds for any $\delta \in (0, \tilde{\delta}_1)$. Let B be a ball and let the vector fields $\mathbf{h}$, $\mathbf{f} \in L^{p-\delta}(B, \mathbb{R}^n)$. Then for any very weak solution $w \in W_0^{1,p-\delta}(B)$ to the equation*

$$\operatorname{div} \mathcal{A}(x, \mathbf{h} + \nabla w) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f} \quad \text{in } B,$$

there holds

$$\int_B |\nabla w(x)|^{p-\delta}\, dx \leq C \int_B \left(|\mathbf{h}(x)|^q + |\mathbf{f}(x)|^{p-\delta} \right) dx, \tag{2.1}$$

where $C = C(n, p, \Lambda_0, \Lambda_1, \gamma)$.

It is worth mentioning that inequality (2.1) was obtained in [20, Theorem 5.1] under a Lipschitz type condition on $\mathcal{A}(x,\cdot)$, i.e., (1.3) was assumed to hold with $\gamma = 1$. We observe that the proof of [20, Theorem 5.1] can easily be modified to obtain (2.1) under the weaker Hölder type condition (1.3) with any $\gamma \in (0,1)$; see also the proof of Theorem 3.4 below.

We next state a well-known *interior* higher integrability result that was originally obtained in [20] and [27] (see also [32]).

Theorem 2.3 ([20], [27])**.** *Suppose that $\mathcal{A}$ satisfies (1.2) and (1.4). There exists a constant $\tilde{\delta}_2 = \tilde{\delta}_2(n,p,\Lambda_0,\Lambda_1) \in (0,1/2)$ such that every very weak solution $w \in W^{1,p-\tilde{\delta}_2}_{\rm loc}(\tilde{\Omega})$ to the equation $\operatorname{div} \mathcal{A}(x,\nabla w) = 0$ in an open set $\tilde{\Omega}$ belongs to $W^{1,p+\tilde{\delta}_2}_{\rm loc}(\tilde{\Omega})$. Moreover, the inequality*

$$\left(\fint_{B_{r/2}(x)} |\nabla w(x)|^{p+\tilde{\delta}_2}\,dx\right)^{\frac{1}{p+\tilde{\delta}_2}} \leq C\left(\fint_{B_r(x)} |\nabla w(x)|^{p-\tilde{\delta}_2}\,dx\right)^{\frac{1}{p-\tilde{\delta}_2}} \tag{2.2}$$

holds for any ball $B_r(x) \subset \tilde{\Omega}$ with a constant $C = C(n,p,\Lambda_0,\Lambda_1)$.

Remark 2.4. We notice that Theorem 2.3 was obtained in [20] under a homogeneity condition on $\mathcal{A}(x,\cdot)$, i.e., $\mathcal{A}(x,\lambda\xi) = |\lambda|^{p-2}\lambda\mathcal{A}(x,\xi)$ for all $x,\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. This condition has been removed in [12]. Moreover, the proof of Theorem 2.3 in [20] uses inequality (2.1) and thus requires the Hölder type condition (1.3). As a matter of fact, following the method of [27], one can prove interior higher integrability under only conditions (1.2) and (1.4). For details see, e.g., [32, Theorem 9.4].

A consequence of Theorems 2.2 and 2.3 is the following important existence result.

Corollary 2.5 ([20])**.** *Under (1.2)-(1.3), let $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are as in Theorems 2.2 and 2.3, respectively. Let $B \subset \mathbb{R}^n$ be a ball. For any function $w_0 \in W^{1,p-\delta}(B)$, with $\delta \in (0, \min\{\tilde{\delta}_1,\tilde{\delta}_2\})$, there exists a very weak solution $w \in w_0 + W^{1,p-\delta}(B)$ to the equation $\operatorname{div} \mathcal{A}(x,\nabla w) = 0$ such that*

$$\int_B |\nabla w(x)|^{p-\delta}\,dx \leq C(n,p,\Lambda_0,\Lambda_1,\gamma)\int_B |\nabla w_0(x)|^{p-\delta}\,dx.$$

We shall need to prove versions of Theorems 2.2 and Corollary 2.5 for domains whose complements are uniformly p-thick. These new results will be obtained later in Theorem 3.4 and Corollary 3.5. A version of Theorem 2.3 upto the boundary of a domain whose complement is uniformly p-thick will also be obtained in Theorem 3.7 below.

Next, for each ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$ and for any $\delta \in (0,\min\{\tilde{\delta}_1,\tilde{\delta}_2\})$ with $\tilde{\delta}_1$ and $\tilde{\delta}_2$ as in Theorems 2.2 and 2.3, respectively, we define $w \in u + W^{1,p-\delta}_0(B_{2R})$ as a very weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x,\nabla w) &= 0 \quad \text{in } B_{2R}, \\ w &= u \quad \text{on } \partial B_{2R}. \end{cases} \tag{2.3}$$

The existence of w is ensured by Corollary 2.5. We mention that the uniqueness of w is still unknown, but that is not important for the purpose of this paper. Moreover, by Theorem 2.3 we have that $w \in W^{1,p}_{\rm loc}(B_{2R})$. Thus it follows from the standard interior Hölder continuity of solutions that we have the following decay estimates. The proof of such estimates can be found in [14, Theorem 7.7]. Henceforth, for $f \in L^1(B)$ we write

$$\overline{f}_B = \fint_B f(x)dx = \frac{1}{|B|}\int_B f(x)dx.$$

Lemma 2.6. *Let w be as in (2.3). There exists a $\beta_0 = \beta_0(n, p, \Lambda_0, \Lambda_1) \in (0, 1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w - \overline{w}_{B_\rho(z)}|^p\, dx\right)^{\frac{1}{p}} \leq C\,(\rho/r)^{\beta_0} \left(\fint_{B_r(z)} |w - \overline{w}_{B_r(z)}|^p\, dx\right)^{\frac{1}{p}}$$

for any $z \in B_{2R}(x_0)$ with $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p\, dx\right)^{\frac{1}{p}} \leq C\,(\rho/r)^{\beta_0-1} \left(\fint_{B_r(z)} |\nabla w|^p\, dx\right)^{\frac{1}{p}} \tag{2.4}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$.

Using the higher integrability result of Theorem 2.3, inequality (2.4) can be further ameliorated as in the following lemma. We notice that this kind of result can be proved by means of a covering/interpolation argument as demonstrated in [14, Remark 6.12].

Lemma 2.7. *Let w be as in (2.3). There exists a $\beta_0 = \beta_0(n, p, \Lambda_0, \Lambda_1) \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t\, dx\right)^{\frac{1}{t}} \leq C\,(\rho/r)^{\beta_0-1} \left(\fint_{B_r(z)} |\nabla w|^t\, dx\right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$ with the constant depending only on $n, p, \Lambda_0, \Lambda_1$, and t.

We shall now prove the following comparison estimate with exponents below the natural exponent p.

Lemma 2.8. *Under (1.2)-(1.3), let $\delta \in (0, \min\{\tilde{\delta}_1, \tilde{\delta}_2\})$, where $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are as in Theorems 2.2 and 2.3, respectively. With $\mathbf{f} \in L^{p-\delta}(\Omega)$, for any $u \in W^{1,p-\delta}_0(\Omega)$ solving*

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2}\mathbf{f}, \tag{2.5}$$

and any w as in (2.3), we have the following inequalities:

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx$$

if $p \geq 2$ *and*

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx \lesssim \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta}\,dx + \\ + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\,dx\right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\,dx\right)^{2-p}$$

if $1 < p < 2$.

Proof. Let δ be as in the hypothesis. Applying Theorem 2.1 with $s = p - \delta$ and $\epsilon = -\delta$, we have

$$|\nabla u - \nabla w|^{-\delta}(\nabla w - \nabla u) = \nabla \phi + \mathcal{H}$$

in B_{2R}. Here $\phi \in W_0^{1,\frac{p-\delta}{1-\delta}}(B_{2R})$ and $\mathcal{H}$ is a divergence free vector field with

$$\|\mathcal{H}\|_{L^{\frac{p-\delta}{1-\delta}}(B_{2R})} \lesssim \delta\, \|\nabla u - \nabla w\|_{L^{p-\delta}(B_{2R})}^{1-\delta}. \tag{2.6}$$

Using ϕ as a test function in (2.5) and (2.3), we have

$$\begin{aligned} I &:= \fint_{B_{2R}} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w)) \cdot (\nabla w - \nabla u)|\nabla w - \nabla u|^{-\delta}\,dx, \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{2.7}$$

where we have set

$$\begin{aligned} I_1 &:= \fint_{B_{2R}} (\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w)) \cdot \mathcal{H}\,dx, \\ I_2 &:= \fint_{B_{2R}} |\mathbf{f}|^{p-2}\mathbf{f} \cdot (\nabla w - \nabla u)|\nabla w - \nabla u|^{-\delta}\,dx, \\ I_3 &:= -\fint_{B_{2R}} |\mathbf{f}|^{p-2}\mathbf{f} \cdot \mathcal{H}\,dx. \end{aligned}$$

Applying the monotonicity condition (1.2), we have

$$I \gtrsim \fint_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^{2-\delta}\,dx.$$

Thus when $p \geq 2$ we can bound I from below using the triangle inequality

$$I \gtrsim \fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx. \tag{2.8}$$

For $1 < p < 2$, we have by Hölder's inequality with exponents $\frac{2-\delta}{p-\delta}$ and $\frac{2-\delta}{2-p}$, and Corollary 2.5 that

$$\begin{aligned}
&\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \\
&\quad = \fint_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{(p-\delta)(p-2)}{(2-\delta)2} + \frac{(\delta-p)(p-2)}{(2-\delta)2}} |\nabla u - \nabla w|^{p-\delta}\, dx, \\
&\quad \lesssim \left(\fint_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^{2-\delta}\, dx\right)^{\frac{p-\delta}{2-\delta}} \times \\
&\qquad \times \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx\right)^{\frac{2-p}{2-\delta}}.
\end{aligned}$$

This gives, when $1 < p < 2$, that

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim I^{\frac{p-\delta}{2-\delta}} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx\right)^{\frac{2-p}{2-\delta}}. \tag{2.9}$$

We shall estimate I_1 from above by making use of Hölder's inequality along with (1.4), (2.6), and Corollary 2.5 to obtain

$$\begin{aligned}
|I_1| &\leq \Lambda_1 \fint_{B_{2R}} (|\nabla u|^{p-1} + |\nabla w|^{p-1}) |\mathcal{H}|\, dx, \\
&\lesssim \delta \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx\right)^{\frac{1-\delta}{p-\delta}} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx\right)^{\frac{p-1}{p-\delta}}.
\end{aligned} \tag{2.10}$$

We estimate I_2 from above by using Hölder's inequality to obtain

$$|I_2| \leq \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx\right)^{\frac{p-1}{p-\delta}} \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx\right)^{\frac{1-\delta}{p-\delta}}. \tag{2.11}$$

Finally, for I_3, we combine Hölder's inequality with (2.6) and obtain

$$\begin{aligned}
|I_3| &\leq \fint_{B_{2R}} |\mathbf{f}|^{p-1} |\mathcal{H}|\, dx, \\
&\lesssim \delta \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx\right)^{\frac{1-\delta}{p-\delta}} \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx\right)^{\frac{p-1}{p-\delta}}.
\end{aligned} \tag{2.12}$$

At this point, combining estimates (2.10), (2.11), (2.12) with (2.7) and (2.8) we get the desired estimate when $p \geq 2$:

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx.$$

Likewise, for $1 < p < 2$, combining the estimates (2.10), (2.11), (2.12) with (2.7) and (2.9), we have

$$\begin{aligned}
&\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \\
&\qquad\lesssim \Bigg\{ \delta \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \\
&\qquad + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \\
&\qquad + \delta \left(\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{\frac{p-1}{p-\delta}} \Bigg\}^{\frac{p-\delta}{2-\delta}} \times \\
&\qquad\qquad \times \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{\frac{2-p}{2-\delta}}.
\end{aligned}$$

Simplifying the above inequality, we get the desired estimate for the case $1 < p < 2$:

$$\begin{aligned}
\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta}\, dx \lesssim\ & \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx + \\
& + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta}\, dx \right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta}\, dx \right)^{2-p}.
\end{aligned}$$

This completes the proof of Lemma 2.8. □

3. Local boundary estimates

We now extend the results of the previous section upto the boundary of a domain whose complement is uniformly p-thick. While the approach of [20] via nonlinear Hodge decomposition could be used upto the boundary of the domain, it requires that the boundary be sufficiently regular. To overcome the roughness of the domain boundary, we shall employ the Lipschitz truncation method introduced in [27]. Here some of the ideas of [41] and the pointwise Hardy inequality obtained in [15] will be useful for our purpose. On the other hand, it should be noted that the approach of this section could be modified to obtain, e.g., the local interior comparison estimate (Lemma 2.8) that was previously derived by means of the nonlinear Hodge decomposition.

We start with some preliminary results. First we recall that an A_s weight, $1 < s < \infty$, is a non-negative function $w \in L^1_{\rm loc}(\mathbb{R}^n)$ such that the quantity

$$[w]_s := \sup \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{\frac{-1}{s-1}}\, dx \right)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. The quantity $[w]_s$ is referred to as the A_s constant of w.

A nonnegative function $w \in L^1_{\rm loc}(\mathbb{R}^n)$ is called an A_1 weight if there exists a constant $A > 0$ such that

$$\mathcal{M}(w)(x) \leq A\, w(x)$$

holds for a.e. $x \in \mathbb{R}^n$. In this case A is called an A_1 constant of w. Here $\mathcal{M}$ is the Hardy-Littlewood maximal function defined for each $f \in L^1_{\rm loc}(\mathbb{R}^n)$ by

$$\mathcal{M}(f)(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)| dy, \qquad x \in \mathbb{R}^n.$$

Beside the standard boundedness property of $\mathcal{M}$ on L^s spaces, we also use the following property. Given a non-zero function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and a number $\beta \in (0,1)$, there holds $\mathcal{M}(f)^\beta \in A_1$ with an A_1 constant depending only on n and β. Moreover, if β is away from 1, say $\beta \leq 0.9$, then an A_1 constant can be chosen to be independent of β (see, e.g., [40] p. 229).

Lemma 3.1. *Let $\tilde{\Omega}$ is a bounded domain whose complement is uniformly p-thick with constants r_0 and $b > 0$. There exists a $\delta_0 = \delta_0(n,p,b) \in (0,1/2)$ such that the following holds for any $\delta \in (0, \delta_0/2)$. Let $v \in W^{1,p-\delta}_0(\tilde{\Omega})$, $v \not\equiv 0$, and extend v by zero outside $\tilde{\Omega}$. Define*

$$g(x) = \max\left\{ \mathcal{M}(|\nabla v|^q)^{1/q}(x), \frac{|v(x)|}{d(x, \partial\tilde{\Omega})} \right\},$$

where $q \in (p - \delta_0, p - 2\delta]$ and $d(x, \partial\tilde{\Omega})$ is the distance of x from $\partial\tilde{\Omega}$. Then we have $g \simeq \mathcal{M}(|\nabla v|^q)^{1/q}$ a.e. in $\mathbb{R}^n$ and

$$\int_{\tilde{\Omega}} g^{p-\delta}\, dx \lesssim \int_{\tilde{\Omega}} |\nabla v|^{p-\delta}\, dx. \tag{3.1}$$

Moreover, the function $g^{-\delta}$ is in the Muckenhoupt class $A_{p/q}$ with $[g^{-\delta}]_{A_{p/q}} \leq C = C(n,p,b)$.

Proof. As $\tilde{\Omega}^c$ is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$ with $p_0 = p_0(n,p,b)$ (see [26]). Moreover, there exists a constant $\delta_0 = \delta_0(n,p,b) \in (0,1/2)$ with $p - \delta_0 \geq p_0$ such that for $q \in (p - \delta_0, p - 2\delta]$, where $\delta \in (0, \delta_0/2)$, the pointwise Hardy inequality

$$\frac{|v(x)|}{d(x, \partial\tilde{\Omega})} \lesssim \mathcal{M}(|\nabla v|^q)^{1/q}(x)$$

holds for a.e. $x \in \tilde{\Omega}$ (see [15]). It follows that $g(x) \simeq \mathcal{M}(|\nabla v|^q)^{1/q}(x)$ for a.e. $x \in \mathbb{R}^n$. Thus by the boundedness of the Hardy-Littlewood maximal function $\mathcal{M}$ we obtain inequality (3.1). Moreover, for any ball $B \subset \mathbb{R}^n$ we have

$$\begin{aligned}
&\fint_B g^{-\delta}\,dx\left(\fint_B g^{\frac{\delta q}{p-q}}\,dx\right)^{\frac{p-q}{q}}\\
&\quad\lesssim \fint_B \mathcal{M}(|\nabla v|^q)^{-\delta/q}\,dx\left(\fint_B \mathcal{M}(|\nabla v|^q)^{\frac{\delta}{p-q}}\,dx\right)^{\frac{p-q}{q}}\\
&\quad\lesssim \left\{\inf_{y\in B}\mathcal{M}(|\nabla v|^q)(y)\right\}^{-\delta/q}\left\{\inf_{y\in B}\mathcal{M}(|\nabla v|^q)(y)\right\}^{\delta/q}\\
&\quad\le C.
\end{aligned}$$

Here we used that the function $\mathcal{M}(|\nabla v|^q)^{\frac{\delta}{p-q}}$ is an A_1 weight since $\delta/(p-q)\le 1/2<1$ (see, e.g., [40] p. 229). □

We now present an extension lemma which can be found in [41]. For the sake of completeness, we give the proof.

Lemma 3.2. *Let $v\in W_0^{1,s}(\tilde\Omega)$, $s\ge 1$, where $\tilde\Omega$ is a bounded domain and let $\lambda>0$. Extend v by zero outside $\tilde\Omega$ and set*

$$F_\lambda(v,\tilde\Omega)=\left\{x\in\tilde\Omega:\mathcal{M}(|\nabla v|^s)^{1/s}(x)\le\lambda,\,|v(x)|\le\lambda d(x,\partial\tilde\Omega)\right\},\tag{3.2}$$

where $d(x,\partial\tilde\Omega)$ is the distance of x from $\partial\tilde\Omega$. Then there exists a $c\lambda$-Lipschitz function v_λ defined on $\mathbb{R}^n$ with $c=c(n)\ge 1$ and the following properties:

- *$v_\lambda(x)=v(x)$ and $\nabla v_\lambda(x)=\nabla v(x)$ for a.e. $x\in F_\lambda$;*
- *$v_\lambda(x)=0$ for every $x\in\tilde\Omega^c$; and*
- *$|\nabla v_\lambda(x)|\le c(n)\lambda$ for a.e. $x\in\mathbb{R}^n$.*

Proof. Given the hypothesis of the lemma, there exists a set $N\subset\mathbb{R}^n$ with $|N|=0$ such that

$$|v(x)-v(y)|\le c\,|x-y|[\mathcal{M}(|\nabla v|^s)^{1/s}(x)+\mathcal{M}(|\nabla v|^s)^{1/s}(y)]\tag{3.3}$$

holds for every x, $y\in\mathbb{R}^n\setminus N$. The proof of inequality (3.3) is due to L. I. Hedberg which can be found in [17]. It is then easy to show that $v|_{(F_\lambda\setminus N)\cup\tilde\Omega^c}$ is a $c\lambda$-Lipschitz continuous function for some $c(n)\ge 1$. Indeed, in the case when $x,y\in F_\lambda\setminus N$, then by using (3.2) in (3.3), we see that

$$\begin{aligned}
|v(x)-v(y)|&\le c\,|x-y|[\mathcal{M}(|\nabla v|^q)^{1/q}(x)+\mathcal{M}(|\nabla v|^q)^{1/q}(y)]\\
&\le 2c\,\lambda|x-y|.
\end{aligned}$$

On the other hand, if $x\in F_\lambda\setminus N$ but $y\in\tilde\Omega^c$, by making use of (3.2), we observe that

$$|v(x)-v(y)|=|v(x)|\le\lambda d(x,\partial\tilde\Omega)\le\lambda|x-y|.$$

We can now extend $v|_{F_\lambda\setminus N)\cup\tilde\Omega^c}$ to a Lipschitz continuous function v_λ on the whole $\mathbb{R}^n$ with the same Lipschitz constant by the classical Kirszbraun-McShane extension theorem (see, e.g., [9, p. 80]). This extension satisfies all the properties highlighted in this lemma. □

We next state a generalized Sobolev-Poincaré's inequality which was originally obtained by V. Maz'ya [28, Sec. 10.1.2]. See also [24, Sec. 3.1] and [1, Corollary 8.2.7].

Theorem 3.3. *Let B be a ball and $\phi \in W^{1,s}(B)$ be s-quasicontinuous, with $s > 1$. Let $\kappa = n/(n-s)$ if $1 < s < n$ and $\kappa = 2$ if $s = n$. Then there exists a constant $c = c(n,s) > 0$ such that*

$$\left(\fint_B |\phi|^{\kappa s}\, dx\right)^{\frac{1}{\kappa s}} \leq c \left(\frac{1}{\mathrm{cap}_s(N(\phi), 2B)} \int_B |\nabla \phi|^s\, dx\right)^{\frac{1}{s}},$$

where $N(\phi) = \{x \in B : \phi(x) = 0\}$.

The following estimate with exponents below the natural one has been known only for regular domains (see [20]). Here, for the first time, it is obtained for domains with p-thick complements.

Theorem 3.4. *Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Let $\tilde{\Omega}$ be a bounded domain whose complement is uniformly p-thick with constants r_0 and $b > 0$. Then there exists a constant $\delta_1 = \delta_1(n, p, b, \Lambda_0, \Lambda_1, \gamma) \in (0, \delta_0/2]$, with δ_0 as in Lemma 3.1, such that the following holds for any $\delta \in (0, \delta_1)$. Given any $\mathbf{h}, \mathbf{f} \in L^{p-\delta}(\tilde{\Omega})$ and any very weak solution $w \in W_0^{1,p-\delta}(\tilde{\Omega})$ to equation*

$$\mathrm{div}\, \mathcal{A}(x, \mathbf{h} + \nabla w) = \mathrm{div}\, |\mathbf{f}|^{p-2}\mathbf{f}, \tag{3.4}$$

there holds

$$\int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx \leq C \int_{\tilde{\Omega}} \left(|\mathbf{h}(x)|^{p-\delta} + |\mathbf{f}(x)|^{p-\delta}\right) dx, \tag{3.5}$$

with a constant $C = C(n, p, b, \Lambda_0, \Lambda_1, \gamma)$.

Proof. As $\tilde{\Omega}^c$ is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$. Let $\delta_0 \in (0, 1/2)$, with $p - \delta_0 \geq p_0$, be as in Lemma 3.1. Let $\delta \in (0, \delta_0/2)$ and q be such that $p - \delta_0 < q \leq p - 2\delta < p - \delta$. Defining

$$g(x) := \max\left\{\mathcal{M}(|\nabla w|^q)^{1/q}(x), \frac{|w(x)|}{d(x, \partial\tilde{\Omega})}\right\},$$

then it follows from Lemma 3.1 that

$$\int_{\tilde{\Omega}} g(x)^{p-\delta}\, dx \lesssim \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx. \tag{3.6}$$

We now apply Lemma 3.2 with $s = q$ and $v = w$, to get a global $c\lambda$-Lipschitz function v_λ such that $v_\lambda \in W_0^{1, \frac{p-\delta}{1-\delta}}(\tilde{\Omega})$. Using v_λ as a test function in (3.4) together with (1.4) we have

$$
\begin{aligned}
&\int_{\tilde{\Omega}\cap F_\lambda} \mathcal{A}(x,\nabla w)\cdot\nabla v_\lambda\,dx - \int_{\tilde{\Omega}\cap F_\lambda} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v_\lambda\,dx \\
&- \int_{\tilde{\Omega}\cap F_\lambda} (\mathcal{A}(x,\nabla w) - \mathcal{A}(x,\mathbf{h}+\nabla w))\cdot\nabla v_\lambda\,dx \\
&\qquad = -\int_{\tilde{\Omega}\cap F_\lambda^c} \mathcal{A}(x,\mathbf{h}+\nabla w)\cdot\nabla v_\lambda\,dx + \int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v_\lambda\,dx \\
&\qquad \lesssim \lambda\int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{h}+\nabla w|^{p-1}\,dx + \lambda\int_{\tilde{\Omega}\cap F_\lambda^c} |\mathbf{f}|^{p-1}\,dx,
\end{aligned}
\tag{3.7}
$$

where $F_\lambda := F_\lambda(w,\tilde{\Omega}) = \{x\in\tilde{\Omega} : g(x)\le\lambda\}$. Multiplying equation (3.7) by $\lambda^{-(1+\delta)}$ and integrating from 0 to ∞ with respect to λ, we get

$$
\begin{aligned}
&I_1 - I_2 - I_3 := \\
&\int_0^\infty \lambda^{-(1+\delta)} \int_{\tilde{\Omega}\cap F_\lambda} \mathcal{A}(x,\nabla w)\cdot\nabla v_\lambda\,dx\,d\lambda \\
&- \int_0^\infty \lambda^{-(1+\delta)} \int_{\tilde{\Omega}\cap F_\lambda} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v_\lambda\,dx\,d\lambda \\
&- \int_0^\infty \lambda^{-(1+\delta)} \int_{\tilde{\Omega}\cap F_\lambda} (\mathcal{A}(x,\nabla w) - \mathcal{A}(x,\mathbf{h}+\nabla w))\cdot\nabla v_\lambda\,dx\,d\lambda \\
&\lesssim \int_0^\infty \lambda^{-\delta} \int_{\tilde{\Omega}\cap F_\lambda^c} \left(|\mathbf{h}+\nabla w|^{p-1} + |\mathbf{f}|^{p-1}\right)\,dx\,d\lambda =: I_4.
\end{aligned}
$$

We now continue with the following estimates for I_j, $j = 1,2,3,4$.
Estimate for I_1 from below: Note that we have $\nabla v_\lambda = \nabla w$ a.e. on F_λ. Thus by changing the order of integration and using (1.2), we get

$$
\begin{aligned}
I_1 &= \int_{\tilde{\Omega}} \int_{g(x)}^\infty \lambda^{-(1+\delta)} \mathcal{A}(x,\nabla w)\cdot\nabla w\,d\lambda\,dx \\
&= \frac{1}{\delta}\int_{\tilde{\Omega}} g(x)^{-\delta}\mathcal{A}(x,\nabla w)\cdot\nabla w\,dx \\
&\gtrsim \frac{1}{\delta}\int_{\tilde{\Omega}} g(x)^{-\delta}|\nabla w|^p\,dx.
\end{aligned}
\tag{3.8}
$$

By Hölder's inequality, we have

$$
\int_{\Omega_{2\rho}} |\nabla w|^{p-\delta}\,dx \lesssim \left(\int_{\Omega_{2\rho}} |\nabla w|^p g(x)^{-\delta}\,dx\right)^{\frac{p-\delta}{p}} \left(\int_{\Omega_{2\rho}} g(x)^{p-\delta}\,dx\right)^{\frac{\delta}{p}},
$$

and then by making use of (3.6), we obtain the estimate

$$
\int_{\Omega_{2\rho}} |\nabla w|^{p-\delta}\,dx \lesssim \int_{\Omega_{2\rho}} |\nabla w|^p g(x)^{-\delta}\,dx. \tag{3.9}
$$

Now we combine (3.8) with (3.9) and get

$$I_1 \gtrsim \frac{1}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx. \tag{3.10}$$

Estimate for I_2 from above: Again by changing the order of integration and making use of Young's inequality, we get

$$\begin{aligned} I_2 &= \int_{\tilde{\Omega}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla w \, d\lambda \, dx \\ &= \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} |\mathbf{f}|^{p-2}\mathbf{f} \cdot \nabla w \, dx \\ &\leq \frac{1}{\delta} \int_{\tilde{\Omega}} |\mathbf{f}|^{p-1}\, |\nabla w|^{1-\delta}\, dx \\ &\leq \frac{c(\epsilon)}{\delta} \int_{\tilde{\Omega}} |\mathbf{f}|^{p-\delta}\, dx + \frac{\epsilon}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx \end{aligned}$$

for any $\epsilon > 0$. Here we used that $g^{-\delta} \leq |\nabla w|^{-\delta}$ a.e. in $\tilde{\Omega}$ in the first inequality.

Estimate for I_3 from above: Likewise, changing the order of integration and making use of Young's inequality along with the Hölder type condition (1.3), we get

$$\begin{aligned} I_3 &= \int_{\tilde{\Omega}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} (\mathcal{A}(x, \nabla w) - \mathcal{A}(x, \mathbf{h} + \nabla w)) \cdot \nabla w \, d\lambda \, dx \\ &\lesssim \frac{1}{\delta} \int_{\tilde{\Omega}} g(x)^{-\delta} |\mathbf{h}|^{\gamma} \left(|\mathbf{h}|^{p-1-\gamma} + |\nabla w|^{p-1-\gamma}\right) |\nabla w|\, dx \\ &\lesssim \frac{c(\epsilon)}{\delta} \int_{\tilde{\Omega}} |\mathbf{h}|^{p-\delta}\, dx + \frac{\epsilon}{\delta} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx \end{aligned}$$

for any $\epsilon > 0$.

Estimate for I_4 from above: Changing the order of integration and applying Young's inequality along with estimate (3.6), we get

$$\begin{aligned} I_4 &= \int_{\tilde{\Omega}} \int_0^{g(x)} \lambda^{-\delta} (|\mathbf{h} + \nabla w|^{p-1} + |\mathbf{f}|^{p-1})\, d\lambda \, dx \\ &= \frac{1}{1-\delta} \int_{\tilde{\Omega}} g(x)^{1-\delta} (|\mathbf{h} + \nabla w|^{p-1} + |\mathbf{f}|^{p-1})\, dx \\ &\lesssim \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx + \int_{\tilde{\Omega}} (|\mathbf{h}|^{p-\delta} + |\mathbf{f}|^{p-\delta})\, dx. \end{aligned} \tag{3.11}$$

Combining estimates (3.10)-(3.11) and recalling that $I_1 - I_2 - I_3 \lesssim I_4$, we have

$$\begin{aligned} \int_{\tilde{\Omega}} |\nabla w|^{p-\delta} dx \quad &\leq \quad c_1(c(\epsilon) + \delta) \int_{\tilde{\Omega}} |\mathbf{f}|^{p-\delta} dx + c_1(2\epsilon + \delta) \int_{\tilde{\Omega}} |\nabla w|^{p-\delta}\, dx + \\ &\quad + c_1(c(\epsilon) + \delta) \int_{\tilde{\Omega}} |\mathbf{h}|^{p-\delta}\, dx \end{aligned}$$

for a constant c_1 independent of ϵ and δ.

We now choose $\epsilon = 1/(4c_1)$ and $\delta_1 = \min\{1/(4c_1), \delta_0/2\}$ in the last inequality to obtain estimate (3.5) for any $\delta \in (0, \delta_1)$. □

Once we have the a priori estimate (3.5) and the interior higher integrability result from Theorem 2.3, the following existence result follows by using techniques employed in the proof of [20, Theorem 2].

Corollary 3.5. *Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Let $\tilde{\Omega}$ be a bounded domain whose complement is uniformly p-thick with constants r_0 and $b > 0$. Let $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$, with δ_1 as in Theorem 3.4 and $\tilde{\delta}_2$ as in Theorem 2.3. Then given any $w_0 \in W^{1,p-\delta}(\tilde{\Omega})$, there exists a very weak solution $w \in w_0 + W_0^{1,p-\delta}(\tilde{\Omega})$ to the equation $\mathrm{div}\mathcal{A}(x, \nabla w) = 0$ such that*

$$\int_{\tilde{\Omega}} |\nabla w|^{p-\delta} dx \leq C \int_{\tilde{\Omega}} |\nabla w_0|^{p-\delta} dx,$$

where $C = C(n, p, b, \Lambda_0, \Lambda_1, \gamma)$

Remark 3.6. It is well-known that in the case $\delta = 0$ Corollary 3.5 and Corollary 2.5 hold as long as $\mathcal{A}$ satisfies (1.2) and (1.4), i.e., the condition (1.3) with $\gamma \in (0, 1)$ is not needed. Moreover, the so-obtained solution w is unique in this case, whereas uniqueness remains unknown in the case $\delta > 0$. We also notice that Corollary 3.5 has been known earlier but only for more regular domains (see [20]).

In what follows, we shall only consider Ω to be a bounded domain whose complement is uniformly p-thick with constants r_0 and b. Fix $x_0 \in \partial\Omega$ and choose $R > 0$ such that $2R \leq r_0$. Let $\Omega_{2R} = \Omega_{2R}(x_0) = \Omega \cap B_{2R}(x_0)$. With some $\delta \in (0, \min\{1, p-1\})$, we consider the following Dirichlet problem:

$$\begin{cases} \mathrm{div}\mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{2R}, \\ w = 0 & \text{on } \partial\Omega \cap B_{2R}(x_0). \end{cases} \tag{3.12}$$

A function $w \in W^{1,\,p-\delta}(\Omega_{2R})$ is called a very weak solution to (3.12) if its zero extension from $\Omega_{2R}(x_0)$ to $B_{2R}(x_0)$ belongs to $W^{1,\,p-\delta}(B_{2R}(x_0))$ and for all $\varphi \in W_0^{1,\,\frac{p-\delta}{1-\delta}}(\Omega_{2R})$, we have

$$\int_{\Omega_{2R}} \mathcal{A}(x, \nabla w) \cdot \nabla \varphi \, dx = 0.$$

In the following theorem we obtain a higher integrability result for equation (3.12), which gives a boundary analogue of Theorem 2.3, and hence Theorem 1.3. We shall follow the Lipschitz truncation method of [27] that was used to treat the interior case; see also [32, Theorem 9.4]. Here to deal with the boundary case we use an idea of [41].

Theorem 3.7. *Suppose that $\mathcal{A}$ satisfies (1.2) and (1.4), and that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants r_0 and b.. There exists a constant $\delta_2 = \delta_2(n, p, b, \Lambda_0, \Lambda_1) > 0$ sufficiently small such that if $w \in W^{1,p-\delta_2}(\Omega_{2R})$ is a* *very weak solution to equation* (3.12)*, then* $w \in W^{1,p+\delta_2}(\Omega_R)$*. Moreover, if we extend* w *by zero from* Ω_{2R} *to* B_{2R}*, then the estimate*

$$\left(\fint_{\frac{1}{2}B} |\nabla w|^{p+\delta_2}\, dx\right)^{\frac{1}{p+\delta_2}} \leq C\left(\fint_{7B} |\nabla w|^{p-\delta_2}\, dx\right)^{\frac{1}{p-\delta_2}}$$

holds for all balls B *such that* $7B \subset B_{2R}$*. Here* $C = C(n, p, b, \Lambda_0, \Lambda_1)$*.*

Proof. Let $z \in \partial\Omega \cap B_{2R}(x_0)$ be a boundary point and let $\rho > 0$ be such that $B_{2\rho}(z) \subset B_{2R}(x_0)$. We now set $\Omega_{2\rho} = \Omega_{2\rho}(z) = \Omega \cap B_{2\rho}(z)$. Note then that $\Omega_{2\rho} \subset \Omega_{2R}(x_0)$.

As Ω^c is uniformly p-thick, it is also uniformly p_0-thick for some $1 < p_0 < p$. The same is also true for $\Omega^c_{2\rho}$. Let $\delta_0 \in (0, 1/2)$, with $p - \delta_0 \geq p_0$, be as in Lemma 3.1 with $\tilde{\Omega} = \Omega_{2\rho}$. Let $\delta \in (0, \delta_0/2)$ and q be such that $p - \delta_0 < q \leq p - 2\delta < p - \delta$.

Suppose now that $w \in W^{1,p-\delta}(\Omega_{2R}(x_0))$ is a solution of (3.12). Extending w to $B_{2\rho} = B_{2\rho}(z)$ by zero we have $w \in W^{1,p-\delta}(B_{2\rho})$. Let $\phi \in C_c^\infty(B_{2\rho})$ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on B_ρ and $|\nabla\phi| \leq 4/\rho$. Define $\bar{w} = \phi w$ and g to be the function

$$g(x) = \max\left\{\mathcal{M}(|\nabla\bar{w}|^q)^{1/q}(x), \frac{|\bar{w}(x)|}{d(x, \partial\Omega_{2\rho})}\right\}.$$

Then it follows from Lemma 3.1 that

$$\int_{\Omega_{2\rho}} g^{p-\delta}\, dx \lesssim \int_{\Omega_{2\rho}} |\nabla\bar{w}|^{p-\delta}\, dx. \tag{3.13}$$

We now apply Lemma 3.2 with $s = q$, $\tilde{\Omega} = \Omega_{2\rho}$ and $v = \bar{w}$, to get a global $c\lambda$-Lipschitz function v_λ such that $v_\lambda \in W_0^{1,\frac{p-\delta}{1-\delta}}(\Omega_{2\rho})$. Using v_λ as a test function in (3.12) together with (1.4) we have

$$\begin{aligned}\int_{\Omega_{2\rho}\cap F_\lambda} \mathcal{A}(x, \nabla w)\cdot\nabla v_\lambda\, dx &= -\int_{\Omega_{2\rho}\cap F_\lambda^c} \mathcal{A}(x, \nabla w)\cdot\nabla v_\lambda\, dx\\ &\lesssim \lambda\int_{\Omega_{2\rho}\cap F_\lambda^c} |\nabla w|^{p-1}\, dx,\end{aligned} \tag{3.14}$$

where $F_\lambda := F_\lambda(\bar{w}, \Omega_{2\rho}) = \{x \in \Omega_{2\rho} : g(x) \leq \lambda\}$. Multiply equation (3.14) by $\lambda^{-(1+\delta)}$ and integrate from 0 to ∞ with respect to λ, we then get

$$\begin{aligned} I_1 :&= \int_0^\infty \lambda^{-(1+\delta)} \int_{\Omega_{2\rho}\cap F_\lambda} \mathcal{A}(x, \nabla w)\cdot\nabla v_\lambda\, dx\, d\lambda\\ &\lesssim \int_0^\infty \lambda^{-\delta}\int_{\Omega_{2\rho}\cap F_\lambda^c} |\nabla w|^{p-1}\, dx\, d\lambda\\ &= \int_{\Omega_{2\rho}}\int_0^{g(x)} \lambda^{-\delta} d\lambda\, |\nabla w|^{p-1}\, d\lambda\, dx\\ &= \frac{1}{1-\delta}\int_{\Omega_{2\rho}} g(x)^{1-\delta}|\nabla w|^{p-1}\, dx\end{aligned}$$

where the first equality follows by Fubini's Theorem. Thus applying Young's inequality and using (3.13), we obtain

$$
\begin{aligned}
I_1 &\lesssim \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta}\,dx + \int_{\Omega_{2\rho}} |\nabla \bar{w}|^{p-\delta}\,dx \\
&\lesssim \int_{\Omega_{2\rho}} (|\nabla w|^{p-\delta} + |w/\rho|^{p-\delta})\,dx \qquad (3.15)\\
&\lesssim \int_{B_{2\rho}} |\nabla w|^{p-\delta}\,dx.
\end{aligned}
$$

Here the last inequality follows from Theorem 3.3 since $w = 0$ on $\Omega^c \cap B_{2\rho}$.

Our next goal is to estimate I_1 from below. To this end, changing the order of integration and noting that $\nabla v_\lambda = \nabla \bar{w}$ a.e. on F_λ, we can write

$$
\begin{aligned}
I_1 &= \int_{\Omega_{2\rho}} \int_{g(x)}^{\infty} \lambda^{-(1+\delta)} d\lambda\, \mathcal{A}(x, \nabla w) \cdot \nabla \bar{w}\, d\lambda\, dx \\
&= \frac{1}{\delta} \int_{\Omega_{2\rho}} g(x)^{-\delta} \mathcal{A}(x, \nabla w) \cdot \nabla \bar{w}\ dx.
\end{aligned}
$$

To continue we set

$$
D_1 = \left\{ x \in \Omega_{2\rho} \setminus \Omega_\rho : \mathcal{M}(|\nabla \bar{w}|^q)^{1/q} \le \delta \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{1/q} \right\},
$$
$$
D_2 = \Omega_{2\rho} \setminus (\Omega_\rho \cup D_1),
$$

and note that $w = \bar{w}$ on Ω_ρ. Thus it follows from (1.2) and (1.4) that

$$
\begin{aligned}
\delta\, I_1 &\ge \Lambda_0 \int_{\Omega_\rho} g^{-\delta} |\nabla w|^p\,dx + \int_{D_1} g(x)^{-\delta} \mathcal{A}(x, \nabla w) \cdot \nabla \bar{w}\ dx \\
&\quad + \int_{D_2} g(x)^{-\delta} \mathcal{A}(x, \nabla w) \cdot \nabla \phi w\ dx \\
&\ge \Lambda_0 \int_{\Omega_\rho} g^{-\delta} |\nabla w|^p\,dx - \Lambda_1 \int_{D_1} g^{-\delta} |\nabla w|^{p-1} |\nabla \bar{w}|\,dx \qquad (3.16)\\
&\quad - \frac{4\Lambda_1}{\rho} \int_{D_2} g^{-\delta} |\nabla w|^{p-1} |w|\,dx \\
&=: I_2 - I_3 - I_4.
\end{aligned}
$$

Combining (3.15) and (3.16), we obtain

$$
I_2 \lesssim I_3 + I_4 + \delta \int_{B_{2\rho}} |\nabla w|^{p-\delta}\,dx. \tag{3.17}
$$

We now consider the following estimates for I_2, I_3, and I_4.

Estimate for I_2 from below: Recall that by Lemma 3.1, $g^{-\delta} \in A_{p/q}$. Thus by the boundedness of $\mathcal{M}$ we have

$$
I_2 = \Lambda_0 \int_{\Omega_\rho} g(x)^{-\delta} |\nabla w|^p\,dx \gtrsim \int_{B_\rho} g(x)^{-\delta} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{p/q}\,dx. \tag{3.18}
$$

On the other hand, for $x \in B_{\rho/2}$, there holds

$$\begin{aligned}\mathcal{M}(|\nabla \bar{w}|^q)^{1/q}(x) &\leq \sup_{\substack{x\in B'\\ B'\subset B_\rho}} \left(\fint_{B'} |\nabla \bar{w}|^q dy\right)^{1/q} + \sup_{\substack{x\in B'\\ B'\cap B_\rho^c\neq\emptyset}} \left(\fint_{B'} |\nabla \bar{w}|^q dy\right)^{1/q}\\ &\leq \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) + c\left(\fint_{B_{2\rho}} |\nabla \bar{w}|^q dy\right)^{1/q},\end{aligned}$$

where we have used that $\bar{w} = w$ on B_ρ and $w = 0$ on $\Omega^c \cap B_\rho$. Also, recall that $\bar{w}$ is zero outside $B_{2\rho}$. By Theorem 3.3 we find

$$\fint_{B_{2\rho}} |\nabla \bar{w}|^q dy \leq \fint_{B_{2\rho}} |\nabla w|^q dy + \frac{c}{\rho^q} \fint_{B_{2\rho}} |w|^q dy \leq c \fint_{B_{2\rho}} |\nabla w|^q dy,$$

which gives

$$\begin{aligned}g(x) &\leq c\, \mathcal{M}(|\nabla \bar{w}|^q)^{1/q}(x)\\ &\leq c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) + c_2 \left(\fint_{B_{2\rho}} |\nabla w|^q dy\right)^{1/q}\end{aligned}$$

for all $x \in B_{\rho/2}$. Here recall from Lemma 3.1 that $g \simeq \mathcal{M}(|\nabla \bar{w}|^q)^{1/q}$ a.e. in $\mathbb{R}^n$.

Letting now

$$G = \Big\{ x \in B_{\rho/2} : c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x) \geq c_2 \Big(\fint_{B_{2\rho}} |\nabla w|^q dy\Big)^{1/q} \Big\},$$

then for every $x \in G$ we have

$$g(x) \leq 2c_1 \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{1/q}(x). \tag{3.19}$$

Combining (3.18) and (3.19) we can estimate I_2 from below by

$$\begin{aligned}I_2 &\geq c \int_G \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{-\delta/q} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{p/q}\, dx\\ &\geq c \int_{B_{\rho/2}} |\nabla w|^{p-\delta}\, dx - c_0 \rho^n \left(\fint_{B_{2\rho}} |\nabla w|^q\, dx\right)^{\frac{p-\delta}{q}}.\end{aligned} \tag{3.20}$$

Estimate for I_3 from above: By the definition of D_1 and the boundedness of the maximal function $\mathcal{M}$, we have

$$\begin{aligned}
I_3 &= \Lambda_1 \int_{D_1} g^{-\delta} |\nabla w|^{p-1} |\nabla \bar{w}| \, dx \\
&\lesssim \int_{D_1} \mathcal{M}(|\nabla \bar{w}|^q)^{\frac{1-\delta}{q}} |\nabla w|^{p-1} \, dx \\
&\lesssim \delta^{1-\delta} \int_{\Omega_{2\rho}} \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{\frac{1-\delta}{q}} |\nabla w|^{p-1} \, dx \\
&\lesssim \delta^{1-\delta} \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta} \, dx.
\end{aligned}$$

Estimate for I_4 from above: By the definition of D_2 we have

$$\begin{aligned}
I_4 &= \frac{4\beta}{\rho} \int_{D_2} g^{-\delta} |\nabla w|^{p-1} |w| \, dx \\
&\lesssim \frac{1}{\rho} \int_{D_2} \mathcal{M}(|\nabla \bar{w}|^q)^{-\delta/q} |\nabla w|^{p-1} |w| \, dx \\
&\lesssim \frac{\delta^{-\delta}}{\rho} \int_{D_2} \mathcal{M}(|\nabla w|^q \chi_{\Omega_\rho})^{(p-1-\delta)/q} \, |w| \, dx.
\end{aligned}$$

With this and making use of Young's inequality, we find, for any $\epsilon > 0$,

$$\begin{aligned}
I_4 &\lesssim \epsilon \int_{\Omega_{2\rho}} \mathcal{M}(|\nabla w|^q \chi_{\Omega_{2\rho}})^{\frac{p-\delta}{q}} \, dx + \frac{c(\epsilon)}{\rho^{p-\delta}} \int_{B_{2\rho}} |w|^{p-\delta} \, dx \\
&\lesssim \epsilon \int_{\Omega_{2\rho}} |\nabla w|^{p-\delta} \, dx + c(\epsilon) \rho^n \left(\fint_{B_{2\rho}} |\nabla w|^q \, dx \right)^{\frac{p-\delta}{q}}.
\end{aligned} \tag{3.21}$$

Here the last inequality follows from the boundedness of $\mathcal{M}$ and Theorem 3.3 provided δ_0 is sufficiently small so that $nq/(n-q) \geq p$.

Collecting all of the estimates in (3.17), (3.20)-(3.21) we obtain

$$\begin{aligned}
\int_{B_{\rho/2}} |\nabla w|^{p-\delta} \, dx &\lesssim (1 + c(\epsilon)) \rho^n \left(\fint_{B_{2\rho}} |\nabla w|^q \, dx \right)^{\frac{p-\delta}{q}} \\
&\quad + (\delta + \delta^{1-\delta} + \epsilon) \int_{B_{2\rho}} |\nabla w|^{p-\delta} \, dx.
\end{aligned} \tag{3.22}$$

Recall that the balls in (3.22) are centered at $z \in \partial\Omega \cap B_{2R}(x_0)$ and $B_{2\rho} = B_{2\rho}(z) \subset B_{2R}(x_0)$. Let $x_1 \in B_{2R}(x_0)$ and $\rho > 0$ be such that $B_{7\rho}(x_1) \subset B_{2R}(x_0)$ and assume for now that $B_\rho(x_1) \cap \partial\Omega \neq \emptyset$. Choosing $z \in \partial\Omega \cap B_\rho(x_1)$ such that $|x_1 - z| = d(x_1, \partial\Omega)$, we have $|x_1 - z_0| \leq \rho$ and thus

$$B_{\rho/2}(x_1) \subset B_{3\rho/2}(z) \subset B_{6\rho}(z) \subset B_{7\rho}(x_1).$$

With this, applying (3.22) we have

$$\int_{B_{\rho/2}(x_1)} |\nabla w|^{p-\delta}\, dx \lesssim (1+c(\epsilon))\rho^n \left(\fint_{B_{7\rho}(x_1)} |\nabla w|^q\, dx \right)^{\frac{p-\delta}{q}} + (\delta + \delta^{1-\delta} + \epsilon) \int_{B_{7\rho}(x_1)} |\nabla w|^{p-\delta}\, dx. \tag{3.23}$$

At this point, choosing δ and ϵ small enough in (3.23) we arrive at

$$\fint_{B_{\rho/2}(x_1)} |\nabla w|^{p-\delta}\, dx \leq c\left(\fint_{B_{7\rho}(x_1)} |\nabla w|^q\, dx \right)^{\frac{p-\delta}{q}} + \frac{1}{2} \fint_{B_{7\rho}(x_1)} |\nabla w|^{p-\delta}\, dx.$$

On the other hand, from the interior higher integrability bound (2.2) in Theorem 2.3 it follows that the last inequality also holds with any ball $B_{7\rho}(x_1) \subset B_{2R}(x_0)$ such that $B_\rho(x_1) \subset \Omega$, as long as we further restrict $\delta_0 \in (0, \tilde{\delta}_2)$ so that $q > p - \tilde{\delta}_2$. Here $\tilde{\delta}_2$ is as in Theorem 2.3.

Now using the well-known Gehring's lemma (see [13, p. 122]; see also [11] and [32]) and a simple covering argument, we get the desired higher integrability upto the boundary. □

We now set $\delta_3 = \min\{\delta_1, \tilde{\delta}_2, \delta_2\}$ with $\delta_1, \tilde{\delta}_2$, and δ_2 as in Theorems 3.4, 2.3, and 3.7, respectively. For $u \in W_0^{1,p-\delta}(\Omega)$, $\delta \in (0, \delta_3)$, we let $w \in W^{1,\,p-\delta}(\Omega_{2R}(x_0))$ be a very weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{2R}(x_0), \\ \qquad\qquad w &\in u + W_0^{1,\,p-\delta}(\Omega_{2R}(x_0)). \end{cases} \tag{3.24}$$

The existence of such a w is now ensured by Corollary 3.5. Moreover, since we have higher integrability upto the boundary from Theorem 3.7, we can now obtain the boundary versions of Lemmas 2.6 and 2.7. See Lemmas 3.7 and 3.8 in [38].

Lemma 3.8 ([38])**.** *Let $u \in W_0^{1,p-\delta}(\Omega)$, with $\delta \in (0, \delta_3)$, and let w be a very weak solution of (3.24). Then there exists a $\beta_0 = \beta_0(n, p, b, \Lambda_0, \Lambda_1) \in (0, 1/2]$ such that*

$$\left(\fint_{B_\rho(z)} |w|^p\, dx \right)^{\frac{1}{p}} \leq C\, (\rho/r)^{\beta_0} \left(\fint_{B_r(z)} |w|^p\, dx \right)^{\frac{1}{p}}$$

for any $z \in \partial\Omega$ with $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Moreover, there holds

$$\left(\fint_{B_\rho(z)} |\nabla w|^p\, dx \right)^{\frac{1}{p}} \leq C\, (\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^p\, dx \right)^{\frac{1}{p}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Here $C = C(n, p, b, \Lambda_0, \Lambda_1)$.

Lemma 3.9 ([38])**.** *Let $u \in W_0^{1,p-\delta}(\Omega)$, with $\delta \in (0, \delta_3)$, and let w be a very weak solution of (3.24). Then there exists a $\beta_0 = \beta_0(n, p, b, \Lambda_0, \Lambda_1) \in (0, 1/2]$ such that for any $t \in (0, p]$ there holds*

$$\left(\fint_{B_\rho(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}} \le C \, (\rho/r)^{\beta_0 - 1} \left(\fint_{B_r(z)} |\nabla w|^t \, dx \right)^{\frac{1}{t}}$$

for any $z \in B_{2R}(x_0)$ such that $B_\rho(z) \subset B_r(z) \Subset B_{2R}(x_0)$. Here $C = C(n, p, b, t, \Lambda_0, \Lambda_1)$.

We now prove the boundary analogue of Lemma 2.8.

Lemma 3.10. *Under (1.2)-(1.3), let $u \in W_0^{1,p-\delta}(\Omega)$, $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$, with δ_1 and $\tilde{\delta}_2$ as in Theorems 3.4 and 2.3, respectively, be a very weak solution to (2.5) with $\mathbf{f} \in L^{p-\delta}(\Omega)$. Let $w \in u + W_0^{1,p-\delta}(\Omega_{2R})$, $\Omega_{2\rho} = \Omega_{2R}(x_0)$ with $x_0 \in \partial\Omega$ and $2R \le r_0$, be a very weak solution to (3.24). Then after extending $\mathbf{f}$ and u by zero outside Ω and w by u outside Ω_{2R}, we have*

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \lesssim \delta^{\frac{p-\delta}{p-1}} \fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx + \fint_{B_{2R}} |\mathbf{f}|^{p-\delta} \, dx$$

if $p \ge 2$ and

$$\fint_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \lesssim \delta^{p-\delta} \fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx + \\ + \left(\fint_{B_{2R}} |\mathbf{f}|^{p-\delta} \, dx \right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^{p-\delta} \, dx \right)^{2-p}$$

if $1 < p < 2$.

Proof. Let $\delta \in (0, \min\{\delta_1, \tilde{\delta}_2\})$. Then $\delta \in (0, \delta_0/2)$ with δ_0 as in Lemma 3.1. Let $q \in (p - \delta_0, p - 2\delta]$ and define g to be the function

$$g(x) = \max\left\{ \mathcal{M}(|\nabla u - \nabla w|^q)^{1/q}(x), \frac{|u(x) - w(x)|}{d(x, \partial\Omega_{2R})} \right\}.$$

Then it follows from Lemma 3.1 with $\tilde{\Omega} = \Omega_{2R}$ that

$$\int_{\Omega_{2R}} g^{p-\delta} \, dx \lesssim \int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx. \tag{3.25}$$

Also, by Theorem 3.4 we have

$$\int_{\Omega_{2R}} |\nabla w|^{p-\delta} \, dx \lesssim \int_{\Omega_{2R}} |\nabla u|^{p-\delta} \, dx. \tag{3.26}$$

We now apply Lemma 3.2 with $s = q$, $\tilde{\Omega} = \Omega_{2R}$ and $v = u - w$, to get a global $c\lambda$-Lipschitz function $v_\lambda \in W_0^{1, \frac{p-\delta}{1-\delta}}(\Omega_{2R})$. Using v_λ as a test function in (2.5) and (3.24) along with (1.4), we obtain

$$\begin{aligned}
&\int_{\Omega_{2R}\cap F_\lambda} (\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla w))\cdot\nabla v_\lambda\,dx - \int_{\Omega_{2R}\cap F_\lambda} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v_\lambda\,dx\\
&\quad= \int_{\Omega_{2R}\cap F_\lambda^c} (\mathcal{A}(x,\nabla w)-\mathcal{A}(x,\nabla u))\cdot\nabla v_\lambda\,dx + \int_{\Omega_{2R}\cap F_\lambda^c} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v_\lambda\,dx\\
&\quad\lesssim \lambda\int_{\Omega_{2R}\cap F_\lambda^c} \left(|\mathbf{f}|^{p-1}+|\nabla u|^{p-1}+|\nabla w|^{p-1}\right)\,dx,
\end{aligned}$$

where $F_\lambda := F_\lambda(u-w,\Omega_{2R}) = \{x\in\Omega_{2R} : g(x)\leq\lambda\}$. Multiplying the above equation by $\lambda^{-(1+\delta)}$ and integrating from 0 to ∞ with respect to λ, we then get

$$\begin{aligned}
I_1 - I_2 :=& \int_0^\infty\int_{\Omega_{2R}\cap F_\lambda} \lambda^{-(1+\delta)}\,(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla w))\cdot(\nabla u-\nabla w)\,dx\,d\lambda\\
&- \int_0^\infty\int_{\Omega_{2R}\cap F_\lambda} \lambda^{-(1+\delta)}|\mathbf{f}|^{p-2}\mathbf{f}\cdot(\nabla u-\nabla w)\,dx\,d\lambda\\
\lesssim& \int_0^\infty\int_{\Omega_{2R}\cap F_\lambda^c} \lambda^{-\delta}\left(|\mathbf{f}|^{p-1}+|\nabla u|^{p-1}+|\nabla w|^{p-1}\right)\,dx\,d\lambda =: I_3.
\end{aligned}$$

We now proceed with the following estimates for I_1, I_2, and I_3.
Estimate for I_1 from below: By changing the order of integration and making use of (1.2), we get

$$\begin{aligned}
I_1 &= \int_{\Omega_{2R}}\int_{g(x)}^\infty \lambda^{-(1+\delta)}\,(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla w))\cdot(\nabla u-\nabla w)\,d\lambda\,dx\\
&= \frac{1}{\delta}\int_{\Omega_{2R}} g(x)^{-\delta}\,(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla w))\cdot(\nabla u-\nabla w)\,dx \qquad (3.27)\\
&\gtrsim \frac{1}{\delta}\int_{\Omega_{2R}} g(x)^{-\delta}\left(|\nabla u|^2+|\nabla w|^2\right)^{\frac{p-2}{2}}|\nabla u-\nabla w|^2\,dx.
\end{aligned}$$

We now consider separately the case $p\geq 2$ and $1<p<2$.
Case i: For $p\geq 2$, by using (3.25) along with Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_{2R}} |\nabla u-\nabla w|^{p-\delta}\,dx\\
&\qquad\leq \left(\int_{\Omega_{2R}} g^{-\delta}|\nabla u-\nabla w|^p\,dx\right)^{\frac{p-\delta}{p}}\left(\int_{\Omega_{2R}} g^{p-\delta}\,dx\right)^{\frac{\delta}{p}}\\
&\qquad\leq \left(\int_{\Omega_{2R}} g^{-\delta}|\nabla u-\nabla w|^2(|\nabla u|^2+|\nabla w|^2)^{\frac{p-2}{2}}\,dx\right)^{\frac{p-\delta}{p}}\times\\
&\qquad\qquad\times\left(\int_{\Omega_{2R}} |\nabla u-\nabla w|^{p-\delta}\,dx\right)^{\frac{\delta}{p}}.
\end{aligned}$$

Simplifying the above expression and substituting into (3.27), we get

$$I_1 \gtrsim \frac{1}{\delta}\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx. \tag{3.28}$$

Case ii: For $1 < p < 2$, we use the following equality

$$\begin{aligned}|\nabla u - \nabla w|^{p-\delta} = \left[(|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}}|\nabla u - \nabla w|^2 g^{-\delta}\right]^{\frac{p-\delta}{2}} \times \\ \times \left(|\nabla u|^2 + |\nabla w|^2\right)^{\frac{(p-\delta)(2-p)}{4}} g^{\frac{p-\delta}{2}\delta}.\end{aligned} \tag{3.29}$$

Integrating (3.29) over Ω_{2R} and making use of Hölder's inequality with exponents $2/(p-\delta), 2/(2-p)$ and $2/\delta$, we get

$$\begin{aligned}&\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx \\ &\leq \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-\delta}{2}}\,dx\right)^{\frac{2-p}{2}} \times \left(\int_{\Omega_{2R}} g(x)^{p-\delta}\,dx\right)^{\frac{\delta}{2}} \times \\ &\quad \times \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}}|\nabla u - \nabla w|^2 g(x)^{-\delta}\,dx\right)^{\frac{p-\delta}{2}}.\end{aligned} \tag{3.30}$$

Combining (3.25) and (3.26) into (3.30) and then simplifying we get

$$\begin{aligned}\left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{1-\frac{\delta}{2}} \lesssim \left(\int_{\Omega_{2R}} |\nabla u|^{p-\delta}\,dx\right)^{\frac{2-p}{2}} \times \\ \times \left(\int_{\Omega_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}}|\nabla u - \nabla w|^2 g(x)^{-\delta}\,dx\right)^{\frac{p-\delta}{2}}.\end{aligned}$$

Using this in (3.27), we arrive at

$$I_1 \gtrsim \frac{1}{\delta}\left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{\frac{2-\delta}{p-\delta}} \left(\int_{\Omega_{2R}} |\nabla u|^{p-\delta}\,dx\right)^{\frac{p-2}{p-\delta}}. \tag{3.31}$$

Estimate for I_2 from above: By changing the order of integration, we get

$$\begin{aligned}I_2 &= \int_{\Omega_{2R}}\int_{g(x)}^{\infty} \lambda^{-(1+\delta)}|\mathbf{f}|^{p-2}\mathbf{f}\cdot(\nabla u - \nabla w)\,d\lambda\,dx \\ &= \frac{1}{\delta}\int_{\Omega_{2R}} g(x)^{-\delta}|\mathbf{f}|^{p-2}\mathbf{f}\cdot(\nabla u - \nabla w)\,dx \\ &\leq \frac{1}{\delta}\int_{\Omega_{2R}} g(x)^{-\delta}|\mathbf{f}|^{p-1}|\nabla u - \nabla w|\,dx.\end{aligned} \tag{3.32}$$

Since $|\nabla u(x) - \nabla w(x)| \leq g(x)$ for a.e. x, by using Hölder's inequality in (3.32), we have

$$
\begin{aligned}
I_2 &\leq \frac{1}{\delta} \int_{\Omega_{2R}} |\nabla u - \nabla w|^{-\delta} |\mathbf{f}|^{p-1} |\nabla u - \nabla w| \, dx \\
&\leq \frac{1}{\delta} \left(\int_{\Omega_{2R}} |\mathbf{f}|^{p-\delta} \, dx \right)^{\frac{p-1}{p-\delta}} \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \right)^{\frac{1-\delta}{p-\delta}}.
\end{aligned} \tag{3.33}
$$

Estimate for I_3 from above: By changing the order of integration, we get

$$
\begin{aligned}
I_3 &= \int_{\Omega_{2R}} \int_0^{g(x)} \lambda^{-\delta} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1} \right) \, d\lambda \, dx \\
&= \frac{1}{1-\delta} \int_{\Omega_{2R}} g(x)^{1-\delta} \left(|\mathbf{f}|^{p-1} + |\nabla u|^{p-1} + |\nabla w|^{p-1} \right) \, dx.
\end{aligned}
$$

Thus Hölder's inequality along with (3.25) and Theorem 3.4 then yield

$$
I_3 \lesssim \left(\int_{\Omega_{2R}} |\nabla u - \nabla w|^{p-\delta} \, dx \right)^{\frac{1-\delta}{p-\delta}} \left(\int_{\Omega_{2R}} |\mathbf{f}|^{p-\delta} + |\nabla u|^{p-\delta} \, dx \right)^{\frac{p-1}{p-\delta}}. \tag{3.34}
$$

As $I_1 - I_2 \lesssim I_3$, we can now combine estimates (3.33) and (3.34), along with (3.28) in the case $p \geq 2$ or (3.31) in the case $1 < p < 2$ to obtain the desired bounds. □

4. Local estimates in Lorentz spaces

We now recall an elementary characterization of functions in Lorentz spaces, which can easily be proved using methods in standard measure theory.

Lemma 4.1. *Assume that $g \geq 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^n$. Let $\theta > 0$, $\Lambda > 1$ be constants. Then for $0 < s, t < \infty$, we have*

$$
g \in L(s,t)(U) \Longleftrightarrow S := \sum_{k \geq 1} \Lambda^{tk} |\{x \in U : g(x) > \theta \Lambda^k\}|^{\frac{t}{s}} < +\infty
$$

and moreover the estimate

$$
C^{-1} \, S \leq \|g\|^t_{L(s,t)(U)} \leq C \, (|U|^{\frac{t}{s}} + S),
$$

holds where $C > 0$ is a constant depending only on θ, Λ, and t. Analogously, for $0 < s < \infty$ and $t = \infty$ we have

$$
C^{-1} T \leq \|g\|_{L(s,\infty)(\Omega)} \leq C \, (|\Omega|^{\frac{1}{s}} + T),
$$

where T is the quantity

$$
T := \sup_{k \geq 1} \Lambda^k |\{x \in \Omega : |g(x)| > \theta \Lambda^k\}|^{\frac{1}{s}}.
$$

The following technical lemma is a version of the Calderón- Zygmund-Krylov-Safonov decomposition that has been used in [6, 34]. It allows one to work with balls instead of cubes. A proof of this lemma, which uses Lebesgue Differentiation Theorem and the standard Vitali covering lemma, can be found in [2] with obvious modifications to fit the setting here.

Lemma 4.2. *Assume that $E \subset \mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1 > 0$ such that*

$$|B_t(x) \cap E| \geq c_1\, |B_t(x)|$$

holds for all $x \in E$ and $0 < t \leq r_1$. Fix $0 < r \leq r_1$ and let $C \subset D \subset E$ be measurable sets for which there exists $0 < \epsilon < 1$ such that

- $|C| < \epsilon\, r^n |B_1|$
- *for all $x \in E$ and $\rho \in (0, r]$, if $|C \cap B_\rho(x)| \geq \epsilon\, |B_\rho(x)|$, then $B_\rho(x) \cap E \subset D$.*

Then we have the estimate

$$|C| \leq (c_1)^{-1} \epsilon\, |D|.$$

Remark 4.3. Henceforth, unless otherwise stated, we shall always consider $\delta \in (0, \min\{\tilde{\delta}_1, \tilde{\delta}_2, \delta_1, \delta_2\})$, where $\tilde{\delta}_1, \tilde{\delta}_2, \delta_1$, and δ_2 are as in Theorems 2.2, 2.3, 3.4, and 3.7, respectively.

Proposition 4.4. *There exists $A = A(n, p, b, \Lambda_0, \Lambda_1, \gamma) > 1$ sufficiently large so that the following holds for any $T > 1$ and any $\lambda > 0$. Fix a ball $B_0 = B_{R_0}$ and assume that for some ball $B_\rho(y)$ with $\rho \leq \min\{r_0, 2R_0\}/26$, we have*

$$B_\rho(y) \cap B_0 \cap \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq \lambda\} \cap \{\mathcal{M}(\chi_{4B_0} |\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}} \leq \epsilon(T)\lambda\} \neq \emptyset,$$

with $\epsilon(T) = T^{\frac{-2\delta}{p-\delta} \max\left\{1, \frac{1}{p-1}\right\}}$. Then there holds

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > AT\lambda\} \cap B_\rho(y)| < H\, |B_\rho(y)|, \tag{4.1}$$

where

$$H = H(T) = T^{-(p+\delta)} + \delta^{(p-\delta) \min\left\{1, \frac{1}{p-1}\right\}}.$$

Proof. By hypothesis, there exists $x_0 \in B_\rho(y) \cap B_0$ such that for any $r > 0$, we have

$$\fint_{B_r(x_0)} \chi_{4B_0} |\nabla u|^{p-\delta}\, dx \leq \lambda^{p-\delta} \tag{4.2}$$

and

$$\fint_{B_r(x_0)} \chi_{4B_0} |\mathbf{f}|^{p-\delta}\, dx \leq [\epsilon(T)\lambda]^{p-\delta}. \tag{4.3}$$

Since $8\rho \le R_0$, we have $B_{23\rho}(y) \subset B_{24\rho}(x_0) \subset 4B_0$. We now claim that for $x \in B_\rho(y)$, there holds

$$\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})(x) \le \max\left\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})(x), 3^n\lambda^{p-\delta}\right\}. \tag{4.4}$$

Indeed, for $r \le \rho$ we have $B_r(x) \cap 4B_0 \subset B_{2\rho}(y) \cap 4B_0 = B_{2\rho}(y)$ and thus

$$\fint_{B_r(x)} \chi_{4B_0}|\nabla u|^{p-\delta}\,dz = \fint_{B_r(x)} \chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta}\,dz,$$

whereas for $r > \rho$ we have $B_r(x) \subset B_{3r}(x_0)$ from which, by making use of (4.2), yields

$$\fint_{B_r(x)} \chi_{4B_0}|\nabla u|^{p-\delta}\,dz \le 3^n \fint_{B_{3r}(x_0)} \chi_{4B_0}|\nabla u|^{p-\delta}\,dz \le 3^n\lambda^{p-\delta}.$$

We now restrict A to the range $A \ge 3^{\frac{n}{p-\delta}}$. Then in view of (4.4) we see that in order to obtain (4.1), it is enough to show that

$$|\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| < H\,|B_\rho(y)|. \tag{4.5}$$

Moreover, since $|\nabla u| = 0$ outside Ω, the later inequality trivially holds provided $B_{4\rho}(y) \subset \mathbb{R}^n \setminus \Omega$, thus it is enough to consider (4.5) for the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \ne \emptyset$.

Let us first consider the interior case, i.e., $B_{4\rho}(y) \subset \Omega$. Let $w = u + W_0^{1,p-\delta}(B_{4\rho})(y)$ be a solution, obtained from Corollary 2.5, to the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x,\nabla w) &= 0 \quad \text{in } B_{4\rho}(y), \\ w &= u \quad \text{on } \partial B_{4\rho}(y). \end{cases}$$

By the weak type $(1,1)$ estimate for the maximal function, we have

$$\begin{aligned} &|\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| \\ &\quad \le |\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla w|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda/2\} \cap B_\rho(y)| \\ &\qquad + |\{\mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u - \nabla w|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda/2\} \cap B_\rho(y)| \\ &\quad \lesssim (AT\lambda)^{-(p+\delta)} \int_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\,dx + \\ &\qquad + (AT\lambda)^{-(p-\delta)} \int_{B_{2\rho}(y)} |\nabla u - \nabla w|^{p-\delta}\,dx. \end{aligned} \tag{4.6}$$

On the other hand, applying Theorem 2.3, we get

$$\begin{aligned} \fint_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\,dx &\lesssim \left(\fint_{B_{4\rho}(y)} |\nabla u|^{p-\delta}\,dx\right)^{\frac{p+\delta}{p-\delta}} \\ &\quad + \left(\fint_{B_{4\rho}(y)} |\nabla u - \nabla w|^{p-\delta}\,dx\right)^{\frac{p+\delta}{p-\delta}}, \end{aligned} \tag{4.7}$$

whereas by (4.2)-(4.3) and Lemma 2.8 there hold

$$\fint_{B_{4\rho}(y)} |\nabla u|^{p-\delta}\, dx \lesssim \fint_{B_{5\rho}(x_0)} |\nabla u|^{p-\delta}\, dx \lesssim \lambda^{p-\delta} \tag{4.8}$$

and

$$\begin{aligned}\fint_{B_{4\rho}(y)} &|\nabla u - \nabla w|^{p-\delta}\, dx \\ &\lesssim \delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} \lambda^{p-\delta} + [\epsilon(T)^{\min\{1,p-1\}}\lambda]^{p-\delta} \\ &\lesssim \lambda^{p-\delta}\Big[\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta}\Big],\end{aligned} \tag{4.9}$$

where we used $B_{4\rho}(y) \subset B_{5\rho}(x_0)$ and the definition of $\epsilon(T)$.

Combining (4.6)-(4.9) we now obtain

$$\begin{aligned}|\{\mathcal{M}(\chi_{B_{2\rho}(y)}&|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > AT\lambda\} \cap B_\rho(y)| \\ &\lesssim |B_\rho(y)|(AT)^{-(p+\delta)}\left[1 + \delta^{(p+\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta\frac{p+\delta}{p-\delta}}\right] \\ &\quad + |B_\rho(y)|(AT)^{-(p-\delta)}\left[\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}} + T^{-2\delta}\right] \\ &\lesssim |B_\rho(y)|\, A^{-(p-\delta)}T^{-(p+\delta)} + |B_\rho(y)|\, A^{-(p-\delta)}\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}}\end{aligned}$$

since $A, T > 1$ and $\delta \in (0,1)$.

At this point, we can take A sufficiently large to get the desired estimates in the interior case $B_{4\rho}(y) \subset \Omega$.

We now look at the boundary case when $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. Recall that $u \in W_0^{1,p-\delta}(\Omega)$. Let $y_0 \in \partial\Omega$ be a boundary point such that $|y - y_0| = \mathrm{dist}(y, \partial\Omega)$. Define $w \in u + W_0^{1,p-\delta}(\Omega_{32\rho}(y_0))$ as a solution to the problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{32\rho}(y_0), \\ w = u & \text{on } \partial\Omega_{32\rho}(y_0). \end{cases}$$

Here we first extend u to be zero on $\mathbb{R}^n \setminus \Omega$ and then extend w to be u on $\mathbb{R}^n \setminus \Omega_{16\rho}(y_0)$. Since

$$B_{28\rho}(y) \subset B_{32\rho}(y_0) \subset B_{36\rho}(y) \subset B_{37\rho}(x_0) \subset 4B_0,$$

we then obtain by making use of Theorem 3.7,

$$\begin{aligned}\Big(\fint_{B_{2\rho}(y)} |\nabla w|^{p+\delta}\, dx\Big)^{\frac{p-\delta}{p+\delta}} &\lesssim \fint_{B_{28\rho}(y)} |\nabla w|^{p-\delta} dx \\ &\lesssim \fint_{B_{37\rho}(x_0)} |\nabla u|^{p-\delta}\, dx + \fint_{B_{32\rho}(y_0)} |\nabla u - \nabla w|^{p-\delta}\, dx.\end{aligned}$$

Now using (4.2)-(4.3) and Lemma 3.10 in (4.6), we obtain the desired estimate in the boundary case. The details are left to the interested reader. □

The above proposition can be restated in the following way.

Proposition 4.5. *There exists a constant* $A = A(n, p, b, \Lambda_0, \Lambda_1, \gamma) > 1$ *such that the following holds for any* $T > 1$ *and any* $\lambda > 0$*. Let* $u \in W_0^{1,p-\delta}(\Omega)$ *be a solution of* (1.1) *with* $\mathcal{A}$ *satisfying* (1.2)-(1.3)*. Fix a ball* $B_0 = B_{R_0}$*, and suppose that for some ball* $B_\rho(y)$ *with* $\rho \leq \min\{r_0, 2R_0\}/26$ *we have*

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > AT\lambda\} \cap B_\rho(y)| \geq H\,|B_\rho(y)|.$$

Then there holds

$$B_\rho(y) \cap B_0 \subset \{\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > \lambda\} \cup \{\mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}} > \epsilon(T)\lambda\}.$$

Here $\epsilon(T)$ *and* $H = H(T)$ *are as defined in Proposition 4.4.*

We can now apply Lemma 4.2 and the previous proposition to get the following result.

Lemma 4.6. *There exists a constant* $A = A(n, p, b, \Lambda_0, \Lambda_1, \gamma) > 1$ *such that the following holds for any* $T > 2$*. Let* u *be a solution of* (1.1) *with* $\mathcal{A}$ *satisfying* (1.2)-(1.3)*, and let* B_0 *be a ball of radius* R_0*. Fix a real number* $0 < r \leq \min\{r_0, 2R_0\}/26$ *and suppose that there exists* $N > 0$ *such that*

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < H\,r^n|B_1|. \tag{4.10}$$

Then for any integer $k \geq 0$ *there holds*

$$\begin{aligned}
&|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^{k+1}\}| \\
&\quad \leq c(n)\,H\,|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^k\}| \\
&\qquad + c(n)\,|\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^k\}|.
\end{aligned}$$

Here $\epsilon(T)$ *and* $H = H(T)$ *are as defined in Proposition 4.4.*

Proof. Let A be as in Proposition 4.5 and set

$$C = \{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^{k+1}\}, \quad D = D_1 \cap B_0,$$

where D_1 is the union

$$\begin{aligned}
D_1 \;=\; &\{\mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^k\} \\
&\cup \{\mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T)N(AT)^k\}.
\end{aligned}$$

with $\epsilon(T)$ and H being as defined in Proposition 4.4.

Since $AT > 1$ the assumption (4.10) implies that $|C| < H\,r^n|B_1|$. Moreover, if $x \in B_0$ and $\rho \in (0, r]$ such that $|C \cap B_\rho(x)| \geq H\,|B_\rho(x)|$, then using Proposition 4.5 with $\lambda = N(AT)^k$ we have

$$B_\rho(x) \cap B_0 \subset D.$$

Thus the hypotheses of Lemma 4.2 are satisfied with $E = B_0$ and $\epsilon = H \in (0,1)$. This yields

$$\begin{aligned}|C| &\le c(n)\, H\, |D| \\ &\le c(n)\, H\, |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N(AT)^k\}| \\ &\quad + c(n)\, |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \epsilon(T) N(AT)^k\}|\end{aligned}$$

as desired. □

Using Lemma 4.6, we can now obtain a gradient estimate in Lorentz spaces over every ball centered in the domain.

Theorem 4.7. *Suppose that $\Omega \subset \mathbb{R}^n$ be a bounded domain whose complement is uniformly p-thick with constants $r_0, b > 0$. Then, with δ as in Remark 4.3, for any $p - \delta/2 \le q \le p + \delta/2$, $0 < t \le \infty$ and for any very weak solution solution $u \in W_0^{1,p-\delta}(\Omega)$ to (1.1), there holds*

$$\|\nabla u\|_{L(q,t)(B_0)} \le C|B_0|^{\frac{1}{q}} \|\nabla u\|_{L^{p-\delta}(4B_0)} [\min\{r_0, 2R_0\}]^{\frac{-n}{p-\delta}} + C\|\mathbf{f}\|_{L(q,t)(4B_0)}.$$

Here $B_0 = B_{R_0}(z_0)$ is any ball with $z_0 \in \Omega$ and $R_0 > 0$, and the constant $C = C(n, p, t, \gamma, \Lambda_0, \Lambda_1, b)$.

Proof. Let B_0 be a ball of radius $R_0 > 0$ and set $r = \min\{r_0, 2R_0\}/26$. As usual we set u and $\mathbf{f}$ to be zero in $\mathbb{R}^n \setminus \Omega$. In what follows we consider only the case $t \ne \infty$ as for $t = \infty$ the proof is similar. Moreover, to prove the theorem, we may assume that

$$\|\nabla u\|_{L^{p-\delta}(B_0)} \ne 0.$$

For $T > 2$ to be determined, we claim that there exists $N > 0$ such that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < H\, r^n |B_1|.$$

with $H = H(T)$ being as in Proposition 4.4. To see this, we first use the weak type $(1,1)$ estimate for the maximal function to get

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{4B_0}|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > N\}| < \frac{C(n)}{N^{p-\delta}} \int_{4B_0} |\nabla u|^{p-\delta}\, dx.$$

Then we choose $N > 0$ so that

$$\frac{C(n)}{N^{p-\delta}} \int_{4B_0} |\nabla u|^{p-\delta}\, dx = H\, r^n\, |B_1|. \tag{4.11}$$

Let A and $\epsilon(T)$ be as in Proposition 4.4. For $0 < t < \infty$, we now consider the sum

$$S = \sum_{k=1}^{\infty} (AT)^{tk} |\{x \in B_0 : \mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > (AT)^k\}|^{\frac{t}{q}}.$$

By Lemma 4.1, we have

$$C^{-1} S \le \left\| \mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}} \right\|_{L(q,t)(B_0)}^t \le C\, (|B_0|^{\frac{t}{q}} + S).$$

We next evaluate S by making use of Lemma 4.6 as follows:

$$\begin{aligned} S &\leq c\sum_{k=1}^{\infty}(AT)^{tk}\Big\{H\,|\{x\in B_0:\mathcal{M}(\chi_{4B_0}|\nabla u/N|^{p-\delta})^{\frac{1}{p-\delta}}(x)>(AT)^{k-1}\}| \\ &\qquad + |\{x\in B_0:\mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}(x)>\epsilon(T)(AT)^{k-1}\}|\Big\}^{\frac{t}{q}} \\ &\leq c\,(AT)^t H^{\frac{t}{q}}(S+|B_0|^{\frac{t}{q}})+c\,\|\mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L(q,\,t)(B_0)}. \end{aligned}$$

At this point we choose T large enough and δ small so that

$$c\,(AT)^t H^{\frac{t}{q}} = c\,(AT)^t\left(T^{-(p+\delta)}+\delta^{(p-\delta)\min\left\{1,\frac{1}{p-1}\right\}}\right)^{\frac{t}{q}}\leq 1/2.$$

This is possible as $q\leq p+\delta/2$, and moreover, T can be chosen to be independent of q. We then obtain

$$S\lesssim |B_0|^{\frac{t}{q}}+\|\mathcal{M}(\chi_{4B_0}|\mathbf{f}/N|^{p-\delta})^{\frac{1}{p-\delta}}\|^t_{L(q,\,t)(B_0)}.$$

Now applying the boundedness property of the maximal function $\mathcal{M}$ and recalling N from (4.11), we finally get

$$\begin{aligned} \|\nabla u\|_{L(q,t)(B_0)} &\lesssim |B_0|^{\frac{1}{q}}N+\|\mathbf{f}\|_{L(q,t)(4B_0)} \\ &\lesssim |B_0|^{\frac{1}{q}}\|\nabla u\|_{L^{p-\delta}(4B_0)}r^{\frac{-n}{p-\delta}}+\|\mathbf{f}\|_{L(q,t)(4B_0)}. \end{aligned}$$

□

5. Proof of Theorem 1.2

We are now ready to prove the main result of the paper.

Proof of Theorem 1.2. Let $\delta>0$ be as in Remark 4.3, and let $B_0=B_{R_0}(z_0)$, where $z_0\in\Omega$ and $0<R_0\leq \operatorname{diam}(\Omega)$. We shall prove the theorem with $\delta/2$ in place of δ. Hence, we assume that $p-\delta/2\leq q\leq p+\delta/2$, $\theta\in[p-\delta,n]$, and $u\in W_0^{1,p-\delta}(\Omega)$. By Theorem 4.7, we have

$$\begin{aligned} \|\nabla u\|_{L(q,t)(B_0)} &\lesssim |B_0|^{\frac{1}{q}}\|\nabla u\|_{L^{p-\delta}(4B_0)}\,[\min\{r_0,2R_0\}]^{-n/(p-\delta)} \\ &\quad +\|\mathbf{f}\|_{L(q,t)(4B_0)} \\ &\lesssim |B_0|^{\frac{1}{q}}\|\nabla u\|_{L^{p-\delta}(4B_0)}\,[\min\{r_0,2R_0\}]^{-n/(p-\delta)} \\ &\quad + R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)}, \end{aligned} \tag{5.1}$$

where the second inequality follows from just the definition of Morrey spaces.

To continue we consider the following two cases.

Case (i). $\frac{r_0}{8} < R_0 \le \mathrm{diam}(\Omega)$: By using (5.1) and the inequality

$$\begin{aligned}\int_{4B_0} |\nabla u|^{p-\delta}\,dx &\le C\int_{\Omega} |\mathbf{f}|^{p-\delta}\,dx \\ &\le C\,\mathrm{diam}(\Omega)^{n-\frac{n(p-\delta)}{q}}\|\mathbf{f}\|^{p-\delta}_{L(q,t)(\Omega)} \\ &\le C\,\mathrm{diam}(\Omega)^{n-\frac{\theta(p-\delta)}{q}}\|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^{\theta}(q,t)(\Omega)},\end{aligned}$$

which follows from Theorem 3.4 and Hölder's inequality, we get

$$\begin{aligned}\|\nabla u\|_{L(q,t)(B_0)} &\lesssim R_0^{n/q}\|\nabla u\|_{L^{p-\delta}(4B_0)} r_0^{-n/(p-\delta)} + R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \\ &\lesssim R_0^{n/q}\mathrm{diam}(\Omega)^{-\theta/q}[\mathrm{diam}(\Omega)/r_0]^{\frac{n}{(p-\delta)}}\|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \\ &\quad + R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \\ &\lesssim R_0^{\frac{n-\theta}{q}}\|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)}\left\{[\mathrm{diam}(\Omega)/r_0]^{\frac{n}{(p-\delta)}}+1\right\}.\end{aligned} \tag{5.2}$$

Case (ii). $0 < R_0 \le \min\left\{\frac{r_0}{8}, \mathrm{diam}(\Omega)\right\}$: From (5.1), we have

$$\|\nabla u\|_{L(q,t)(B_0)} \lesssim R_0^{n/q}\|\nabla u\|_{L^{p-\delta}(4B_0)} R_0^{-n/(p-\delta)} + \|\mathbf{f}\|_{L(q,t)(B_0)}. \tag{5.3}$$

We next aim to estimate the first term on the right-hand side of (5.3). To that end, let $r \in (0, r_0]$. If $B_{r/4}(z_0) \subset \Omega$ we let $w \in u + W_0^{1,p-\delta}(B_{r/5}(z_0))$ solve

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x,\nabla w) = 0 & \text{in } B_{r/5}(z_0), \\ w = u & \text{on } \partial B_{r/5}(z_0). \end{cases}$$

Otherwise, i.e., $B_{r/4}(z_0) \cap \partial\Omega \ne \emptyset$, we let $w \in u + W_0^{1,p-\delta}(\Omega_{r_0/2}(x_0))$ be a solution to

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x,\nabla w) = 0 & \text{in } \Omega_{r/2}(x_0), \\ w = u & \text{on } \partial\Omega_{r/2}(x_0). \end{cases}$$

Here $x_0 \in \partial\Omega \cap B_{r/4}(z_0)$ is chosen so that $|z_0 - x_0| = \mathrm{dist}(z_0, \partial\Omega)$, and thus it follows that $B_{r_0/5}(z_0) \Subset B_{r/2}(x_0) \subset B_{3r/4}(z_0)$. The existence of w follows from Corollary 2.5 or Corollary 3.5. In any case, by Lemmas 2.7 and 3.9 for any $0 < \rho \le r/5$ we have

$$\int_{B_\rho(z_0)} |\nabla w|^{p-\delta}\,dx \lesssim (\rho/r)^{n+(p-\delta)(\beta_0-1)}\int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\,dx,$$

where $\beta_0 = \beta_0(n,p,b,\Lambda_0,\Lambda_1) \in (0,1/2]$ is the smallest of those found in Lemmas 2.7 and 3.9.

Hence, when $p \geq 2$, we get from Lemmas 2.8 and 3.10 that

$$\begin{aligned}\int_{B_\rho(z_0)} |\nabla u|^{p-\delta} &\lesssim \int_{B_\rho(z_0)} |\nabla w|^{p-\delta}\,dx + \int_{B_\rho(z_0)} |\nabla u - \nabla w|^{p-\delta}\,dx \\ &\lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\,dx \\ &\quad + \int_{B_{r/5}(z_0)} |\nabla u - \nabla w|^{p-\delta}\,dx \\ &\lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\,dx \\ &\quad + \delta^{\frac{p-\delta}{p-1}} \int_{B_{3r/4}(z_0)} |\nabla u|^{p-\delta}\,dx + \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\,dx.\end{aligned}$$

Similarly, in the case $1 < p < 2$, using Lemmas 2.8 and 3.10 and Young's inequality we find, for any $\epsilon > 0$,

$$\begin{aligned}\int_{B_\rho(z_0)} |\nabla u|^{p-\delta} &\lesssim \left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} \int_{B_{r/5}(z_0)} |\nabla w|^{p-\delta}\,dx + \\ &\quad + (\delta^{p-\delta} + \epsilon) \int_{B_{3r/4}(z_0)} |\nabla u|^{p-\delta}\,dx + C(\epsilon) \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\,dx.\end{aligned}$$

Therefore, if we denote by

$$\phi(\rho) = \int_{B_\rho(z_0)} |\nabla u|^{p-\delta}\,dx,$$

then we have

$$\begin{aligned}\phi(\rho) &\lesssim \left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1, \frac{1}{p-1}\right\}} + \epsilon \right] \phi(3r/4) \\ &\quad + C(\epsilon) \int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\,dx,\end{aligned} \tag{5.4}$$

which holds for all $\epsilon > 0$ and $\rho \in (0, r/5]$. By enlarging the constant if necessary, we see that (5.4) actually holds for all $\rho \in (0, 3r/4]$.

On the other hand, by Hölder's inequality there holds

$$\int_{B_{3r/4}(z_0)} |\mathbf{f}|^{p-\delta}\,dx \lesssim r^{n - \frac{n(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{L(q,t)(B_{3r/4}(z_0))} \lesssim r^{n - \frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)},$$

and thus (5.4) yields

$$\begin{aligned}\phi(\rho) &\lesssim \left[\left(\frac{\rho}{r}\right)^{n+(p-\delta)(\beta_0-1)} + \delta^{(p-\delta)\min\left\{1, \frac{1}{p-1}\right\}} + \epsilon \right] \phi(3r/4) \\ &\quad + C(\epsilon)\, r^{n - \frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^\theta(q,t)(\Omega)}\end{aligned} \tag{5.5}$$

for all $\rho \in (0, 3r/4]$. Since $\theta \in [p-\delta, n]$ and $q \in [p-\delta/2, p+\delta/2]$, we have

$$0 \le n - \frac{\theta(p-\delta)}{q} < n + (p-\delta)(\beta_0 - 1), \tag{5.6}$$

as long as we restrict $\delta < 2p\beta_0/(1+\beta_0)$. Note that the constant hidden in $\lesssim$ in (5.5) depends only on $n, p, \Lambda_0, \Lambda_1, \gamma$, and b. Thus using (5.5)-(5.6), we can now apply Lemma 3.4 from [16] to obtain a $\overline{\delta} = \overline{\delta}(n, p, \Lambda_0, \Lambda_1, \gamma, b) > 0$ such that

$$\phi(\rho) \lesssim \left(\frac{\rho}{r}\right)^{n - \frac{\theta(p-\delta)}{q}} \phi(3r/4) + \rho^{n - \frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^{\theta}(q,t)(\Omega)}$$

provided we further restrict $\delta < \overline{\delta}$. Since this estimate holds for all $0 < \rho \le 3r/4 \le 3r_0/4$, we can choose $\rho = 4R_0 \le \frac{r_0}{2}$ and $r = r_0$ to arrive at

$$\phi(4R_0) \lesssim \left(\frac{R_0}{r_0}\right)^{n - \frac{\theta(p-\delta)}{q}} \phi(3r_0/4) + R_0^{n - \frac{\theta(p-\delta)}{q}} \|\mathbf{f}\|^{p-\delta}_{\mathcal{L}^{\theta}(q,t)(\Omega)}. \tag{5.7}$$

Substituting (5.7) into (5.3), we find

$$\begin{aligned} \|\nabla u\|_{L(q,t)(B_0)} &\lesssim R_0^{\frac{n-\theta}{q}} r_0^{\frac{\theta}{q} - \frac{n}{p-\delta}} \|\nabla u\|_{L^{p-\delta}(\Omega)} + R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \\ &\lesssim R_0^{\frac{n-\theta}{q}} r_0^{\frac{\theta}{q} - \frac{n}{p-\delta}} \|\mathbf{f}\|_{L^{p-\delta}(\Omega)} + R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)}, \end{aligned} \tag{5.8}$$

where we used Theorem 3.4 in the last inequality. Thus using Hölder's inequality in (5.8) we get

$$\|\nabla u\|_{L(q,t)(B_0)} \lesssim R_0^{\frac{n-\theta}{q}} \|\mathbf{f}\|_{\mathcal{L}^{\theta}(q,t)(\Omega)} \left\{ \left(\mathrm{diam}(\Omega)/r_0\right)^{\frac{n}{p-\delta} - \frac{\theta}{q}} + 1 \right\}. \tag{5.9}$$

Finally, combining the decay estimates (5.2) and (5.9) for $\|\nabla u\|_{L(q,t)(B_0)}$ in both cases we arrive at the desired Morrey space estimate. ☐

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