

# AN END-POINT GLOBAL GRADIENT WEIGHTED ESTIMATE FOR QUASILINEAR EQUATIONS IN NON-SMOOTH DOMAINS

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ABSTRACT. A weighted norm inequality involving  $A_1$  weights is obtained at the natural exponent for gradients of solutions to quasilinear elliptic equations in Reifenberg flat domains. Certain gradient estimates in Lorentz-Morrey spaces below the natural exponent are also obtained as a consequence of our analysis.

## 1. INTRODUCTION

One of the main goals of this paper is to obtain global gradient weighted estimates of the form

$$(1.1) \quad \int_{\Omega} |\nabla u|^p w dx \leq C \int_{\Omega} |\mathbf{f}|^p w dx$$

for weights  $w$  in the Muckenhoupt class  $A_1$  and for solutions  $u$  to the non-homogeneous nonlinear boundary value problem

$$(1.2) \quad \begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2} \mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here  $p > 1$  and  $\operatorname{div} \mathcal{A}(x, \nabla u)$  is modelled after the standard  $p$ -Laplacian  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . Also,  $\mathbf{f}$  is a given vector field defined in a bounded domain  $\Omega$  that may have a non-smooth boundary.

More specifically, in (1.2) the nonlinearity  $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a Carathéodory vector valued function, i.e.,  $\mathcal{A}(x, \xi)$  is measurable in  $x$  for every  $\xi$  and continuous in  $\xi$  for a.e.  $x \in \mathbb{R}^n$ . We also assume that  $\mathcal{A}(x, 0) = 0$  and  $\mathcal{A}(x, \xi)$  is continuously differentiable in  $\xi$  away from the origin for a.e.  $x \in \mathbb{R}^n$ . For our purpose, we require that  $\mathcal{A}$  satisfy the following monotonicity and Lipschitz type conditions: for some  $p > 1$ , there holds

$$(1.3) \quad \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \geq \Lambda_0 (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2$$

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and

$$(1.4) \quad |\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq \Lambda_1 |\xi - \eta| (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for every  $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus (0, 0)$  and a.e.  $x \in \mathbb{R}^n$ . Here  $\Lambda_0$  and  $\Lambda_1$  are positive constants. Note that (1.4) and the assumption  $\mathcal{A}(x, 0) = 0$  for a.e.  $x \in \mathbb{R}^n$  imply the following growth condition

$$|\mathcal{A}(x, \xi)| \leq \Lambda_1 |\xi|^{p-1}.$$

Our additional regularity assumption on the nonlinearity  $\mathcal{A}$  is the following  $(\gamma, R_0)$ -BMO condition. To formulate it, for each ball  $B$ , we let

$$\bar{\mathcal{A}}_B(\xi) = \int_B \mathcal{A}(x, \xi) dx = \frac{1}{|B|} \int_B \mathcal{A}(x, \xi) dx,$$

and define the following function that measures the oscillation of  $\mathcal{A}(\cdot, \xi)$  over  $B$  by

$$\Upsilon(\mathcal{A}, B)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathcal{A}(x, \xi) - \bar{\mathcal{A}}_B(\xi)|}{|\xi|^{p-1}}.$$

**Definition 1.1.** *Given two positive numbers  $\gamma$  and  $R_0$ , we say that  $\mathcal{A}(x, \xi)$  satisfies a  $(\gamma, R_0)$ -BMO condition with exponent  $\tau > 0$  if*

$$[\mathcal{A}]_\tau^{R_0} := \sup_{y \in \mathbb{R}^n, 0 < r \leq R_0} \left( \int_{B_r(y)} \Upsilon(\mathcal{A}, B_r(y))(x)^\tau dx \right)^{\frac{1}{\tau}} \leq \gamma.$$

In the linear case, where  $\mathcal{A}(x, \xi) = A(x)\xi$  for an elliptic matrix  $A$ , we see that

$$\Upsilon(\mathcal{A}, B)(x) \leq |A(x) - \bar{A}_B|$$

for a.e.  $x \in \mathbb{R}^n$ , and thus Definition 1.1 can be viewed as a natural extension of the standard small BMO condition to the nonlinear setting. For general nonlinearities  $\mathcal{A}(x, \xi)$  of at most linear growth, i.e.,  $p = 2$ , the above  $(\gamma, R_0)$ -BMO condition was introduced in [4], whereas such a condition for general  $p > 1$  appears first in [26]. We remark that the  $(\gamma, R_0)$ -BMO condition allows the nonlinearity  $\mathcal{A}(x, \xi)$  to have certain discontinuity in  $x$ , and it can be used as an appropriate substitute for the Sarason VMO condition (vanishing mean oscillation [28], see also [2, 4, 9, 13, 24, 29, 33]).

The domain over which we solve our equations may be non-smooth but should satisfy some flatness condition. Essentially, at each boundary point and every scale, we require the boundary of the domain to be between two hyperplanes separated by a distance proportional to the scale. Absence of such flatness may result in a limited regularity of the solutions, as demonstrated in the counterexample given in [21] (see also [14]).

**Definition 1.2.** Given  $\gamma \in (0, 1)$  and  $R_0 > 0$ , we say that  $\Omega$  is a  $(\gamma, R_0)$ -Reifenberg flat domain if for every  $x_0 \in \partial\Omega$  and every  $r \in (0, R_0]$ , there exists a system of coordinates  $\{y_1, y_2, \dots, y_n\}$ , which may depend on  $r$  and  $x_0$ , so that in this coordinate system  $x_0 = 0$  and that

$$B_r(0) \cap \{y_n > \gamma r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\gamma r\}.$$

For more on Reifenberg flat domains and their many applications, we refer to the papers [10, 15, 16, 17, 27, 32]. We mention here that Reifenberg flat domains can be very rough. They include Lipschitz domains with sufficiently small Lipschitz constants (see [32]) and even some domains with fractal boundaries. In particular, all  $C^1$  domains are included in this paper.

**Remark 1.3.** If  $\Omega$  is a  $(\gamma, R_0)$ -Reifenberg flat domain with  $\gamma < 1$ , then for any point  $x \in \partial\Omega$  and  $0 < \rho < R_0(1 - \gamma)$  there exists a coordinate system  $\{z_1, z_2, \dots, z_n\}$  with the origin 0 at some point in the interior of  $\Omega$  such that in this coordinate system  $x = (0, \dots, 0, -\gamma'\rho)$  and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{(z_1, z_2, \dots, z_n) : z_n > -2\rho\gamma'\},$$

where  $\gamma' = \gamma/(1 - \gamma)$  and  $B_\rho^+(0) := B_\rho(0) \cap \{(z_1, \dots, z_n) : z_n > 0\}$ . Thus, if  $\gamma < 1/2$  then

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{(z_1, z_2, \dots, z_n) : z_n > -4\rho\gamma\}.$$

Now we shall collect some properties of weights. In this paper, we shall only be concerned with Muckenhoupt weights. By an  $A_s$  weight,  $1 < s < \infty$ , we mean a nonnegative function  $w \in L_{\text{loc}}^1(\mathbb{R}^n)$  such that the quantity

$$[w]_s := \sup_B \left( \int_B w(x) dx \right) \left( \int_B w(x)^{\frac{-1}{s-1}} dx \right)^{s-1} < +\infty,$$

where the supremum is taken over all balls  $B \subset \mathbb{R}^n$ . For  $s = 1$ , we say that  $w$  is an  $A_1$  weight if

$$[w]_1 := \sup_B \left( \int_B w(x) dx \right) \|w^{-1}\|_{L^\infty(B)} < +\infty.$$

The quantity  $[w]_s$ ,  $1 \leq s < \infty$ , will be referred to as the  $A_s$  constant of  $w$ . The  $A_s$  classes are increasing, i.e.,  $A_{s_1} \subset A_{s_2}$  whenever  $1 \leq s_1 < s_2 < \infty$ . A broader class of weights is the  $A_\infty$  weights which, by definition, is the union of all  $A_s$  weights for  $1 \leq s < \infty$ . The following characterization of  $A_\infty$  weights will be needed later (see [8, Theorem 9.3.3]).

**Lemma 1.4.** A weight  $w \in A_\infty$  if and only if there are constants  $\Xi_0, \Xi_1 > 0$  such that for every ball  $B \subset \mathbb{R}^n$  and every measurable subsets  $E$  of  $B$

$$(1.5) \quad w(E) \leq \Xi_0 \left( \frac{|E|}{|B|} \right)^{\Xi_1} w(B).$$

Moreover, if  $w$  is an  $A_s$  weight with  $[w]_s \leq \bar{w}$  then the constants  $\Xi_0$  and  $\Xi_1$  above can be chosen so that  $\max\{\Xi_0, 1/\Xi_1\} \leq c(\bar{w}, n)$ .

In (1.4), the notation  $w(E)$  stands for the integral  $\int_E w(x) dx$ , and likewise for  $w(B)$ , etc. Henceforth, we will use this notation without further explanation. Also, we will refer to  $(\Xi_0, \Xi_1)$  as a pair of  $A_\infty$  constants of  $w$  provided they satisfy (1.5).

We now recall the definition of weighted Lorentz spaces. For a general weight  $w$ , the weighted Lorentz space  $L_w(s, t)(\Omega)$  with  $0 < s < \infty$ ,  $0 < t \leq \infty$ , is the set of measurable functions  $g$  on  $\Omega$  such that

$$\|g\|_{L_w(s, t)(\Omega)} := \left[ s \int_0^\infty \alpha^t (w(\{x \in \Omega : |g(x)| > \alpha\}))^{\frac{t}{s}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}} < +\infty$$

when  $t \neq \infty$ ; for  $t = \infty$  the space  $L_w(s, \infty)(\Omega)$  is set to be the usual Marcinkiewicz space with quasinorm

$$\|g\|_{L_w(s, \infty)(\Omega)} := \sup_{\alpha > 0} \alpha w(\{x \in \Omega : |g(x)| > \alpha\})^{\frac{1}{s}}.$$

It is easy to see that when  $t = s$  the weighted Lorentz space  $L_w(s, s)(\Omega)$  is nothing but the weighted Lebesgue space  $L_w^s(\Omega)$ , which is equivalently defined as

$$g \in L_w^s(\Omega) \iff \int_\Omega |g(x)|^s w(x) dx < +\infty.$$

As usual, when  $w \equiv 1$  we simply write  $L(s, t)(\Omega)$  instead of  $L_w(s, t)(\Omega)$ .

A function  $g \in L(s, t)(\Omega)$ ,  $0 < s < \infty$ ,  $0 < t \leq \infty$  is said to belong to the Lorentz-Morrey function space  $\mathcal{L}^\theta(s, t)(\Omega)$  for some  $0 < \theta \leq n$ , if

$$\|g\|_{\mathcal{L}^\theta(s, t)(\Omega)} := \sup_{\substack{0 < r \leq \text{diam}(\Omega), \\ z \in \Omega}} r^{\frac{\theta-n}{s}} \|g\|_{L(s, t)(B_r(z) \cap \Omega)} < +\infty.$$

When  $\theta = n$ , we have  $\mathcal{L}^\theta(s, t)(\Omega) = L(s, t)(\Omega)$ . Moreover, when  $s = t$  the space  $\mathcal{L}^\theta(s, t)(\Omega)$  becomes the usual Morrey space based on  $L^s$  space.

A basic use of Lorentz spaces is to improve the classical Sobolev Embedding Theorem. For example, if  $f \in W^{1, q}$  for some  $q \in (1, n)$  then

$$f \in L(nq/(n-q), q)$$

(see, e.g., [35]), which is better than the classical result

$$f \in L^{nq/(n-q)} = L(nq/(n-q), nq/(n-q))$$

since  $L(s, t_1) \subset L(s, t_2)$  whenever  $t_1 \leq t_2$ . Another use of Lorentz spaces is to capture logarithmic singularities. For example, for any  $\beta > 0$  we have

$$\frac{1}{|x|^{n/s} (\log|x|)^\beta} \in L(s, t)(B_1(0)) \quad \text{if and only if } t > \frac{1}{\beta}.$$

Lorentz spaces have also been used successfully in improving regularity criteria for the full 3D Navier-Stokes system of equations (see, e.g., [30]).

On the other hand, Lorentz-Morrey spaces are neither rearrangement invariant spaces, nor interpolation spaces. They often show up in the analysis of Schrödinger operators via the so-called Fefferman-Phong condition (see [6]), or in the regularity theory of nonlinear equations of fluid dynamics.

We are now able to state the main result of the paper.

**Theorem 1.5.** *Suppose that  $\mathcal{A}$  satisfies (1.3)-(1.4). Let  $t \in (0, \infty]$ ,  $q \geq p$ , and let  $w$  be an  $A_{q/p}$  weight. There exist constants  $\tau = \tau(n, p, \Lambda_0, \Lambda_1) > 1$  and  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, q, [w]_\infty) > 0$  such that the following holds. If  $u \in W_0^{1,p}(\Omega)$  is a solution of (1.2) in a  $(\gamma, R_0)$ -Reifenberg flat domain  $\Omega$  with  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , then one has the estimate*

$$\|\nabla u\|_{L_w(q,t)(\Omega)} \leq C \|\mathbf{f}\|_{L_w(q,t)(\Omega)},$$

where the constant  $C = C(n, p, \Lambda_0, \Lambda_1, q, t, [w]_{q/p}, \text{diam}(\Omega)/R_0)$ .

**Remark 1.6.** By Remark 3.8 below and Lemma 1.4, it follows that if  $\bar{w}$  is an upper bound for  $[w]_{q/p}$ , i.e.,  $[w]_{q/p} \leq \bar{w}$ , then the constants  $C$  and  $\gamma$  above can be chosen to depend on  $\bar{w}$  instead of  $[w]_{q/p}$  or  $[w]_\infty$ .

Theorem 1.5 follows from Theorem 3.6 below (applied with  $M = q$ ) and the boundedness property of the Hardy-Littlewood maximal function on weighted spaces. Its main contribution is the end-point case  $q = p$ , which yields inequality (1.1) for all  $A_1$  weights  $w$  as proposed earlier. The case  $q > p$  has been obtained in [22, 23] but the proofs in those papers can only yield a weak-type bound at the end-point  $q = p$ .

Theorem 3.6 also yields the following gradient estimate below the natural exponent for *very weak* solutions, i.e., distributional solutions that may not have finite  $L^p$  energy.

**Theorem 1.7.** *Suppose that  $\mathcal{A}$  satisfies (1.3)-(1.4) and let  $\theta_0$  be a fixed number in  $(0, n]$ . There exist  $\tau = \tau(n, p, \Lambda_0, \Lambda_1) > 1$ ,  $\delta = \delta(n, p, \theta_0, \Lambda_0, \Lambda_1) > 1$ , and  $\gamma = \gamma(n, p, \theta_0, \Lambda_0, \Lambda_1) > 0$  such that the following holds. If  $u \in W_0^{1,p-\delta}(\Omega)$  is a very weak solution of (1.2) in a  $(\gamma, R_0)$ -Reifenberg flat domain  $\Omega$  with  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$  and  $\mathbf{f} \in \mathcal{L}^\theta(q, t)(\Omega, \mathbb{R}^n)$ , then there holds:*

$$(1.6) \quad \|\nabla u\|_{\mathcal{L}^\theta(q,t)(\Omega)} \leq C \|\mathbf{f}\|_{\mathcal{L}^\theta(q,t)(\Omega)}$$

for all  $q \in (p - \delta, p]$ ,  $0 < t \leq \infty$  and  $\theta_0 \leq \theta \leq n$ . Here the constant  $C = C(n, p, q, t, \theta_0, \Lambda_0, \Lambda_1, \text{diam}(\Omega)/R_0)$ .

The proof of Theorem 1.7 follows by first applying Theorem 3.6 with  $M = p$  and the weight functions

$$w(x) = \min\{|x - z|^{-n+\theta-\rho}, r^{-n+\theta-\rho}\},$$

for any  $z \in \Omega$  and  $r \in (0, \text{diam}(\Omega)]$  and a fixed  $\rho \in (0, \theta)$ . Note that  $w$  is an  $A_1$  weight with its  $A_1$  constant  $[w]_1$  being bounded from above by a constant independent of  $z$  and  $r$ . See also Remark 3.8. The rest of the proof then follows verbatim as in that of [22, Theorem 2.3]. We mention that the sub-natural bound (1.6) was also obtained in our earlier work [1] but with the restriction  $\theta \in [p - 2\delta, n]$ , and in [12] with  $\theta = n$ , i.e., for pure Lebesgue spaces only. Note also that the super-natural case  $q > p$  has been obtained in [22, 23].

Unweighted estimate of the form

$$(1.7) \quad \int_{\Omega} |\nabla u|^q dx \leq C \int_{\Omega} |\mathbf{f}|^q dx$$

for solutions  $u$  to (1.2) in the full sub-natural range  $q \in (p-1, p)$  is currently a wide open problem (even for smooth domains and the standard  $p$ -Laplacian). This is essentially known as a conjecture of T. Iwaniec who originally stated it for  $\Omega = \mathbb{R}^n$  and  $q \in (\max\{1, p-1\}, p)$  in [11]. For the super-natural case  $q \geq p$ , we refer to the papers [11, 18, 19] and [3, 5]. For  $q \in [p - \delta, p)$  with a small  $\delta > 0$ , see [1, 12].

This conjecture is another motivation for us to consider weighted estimates of the form (1.1) at the natural exponent  $p$ . In fact, using the extrapolation theory of García-Cuerva and Rubio de Francia (see [7] and [8, Chapter 9]) we see that if the weighted bound (1.1) holds for all weights  $w \in A_{\frac{p}{p-1}}$  then the unweighted bound (1.7) will follow for all  $q \in (p-1, p)$ . More precisely, we have the following more general result, whose complete proof will be given in the Appendix.

**Theorem 1.8.** *For  $p > 1$ , let  $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$  be a given vector field and denote  $u \in W_0^{1,p}(\Omega)$  to be the unique weak solution to (1.2). Suppose we have that*

$$(1.8) \quad \int_{\Omega} |\nabla u|^p v(x) dx \leq C([v]_{\frac{p}{p-1}}) \int_{\Omega} |\mathbf{f}|^p v(x) dx$$

*holds for all weights  $v \in A_{\frac{p}{p-1}}$ . Then for any  $p-1 < q < \infty$ , there holds*

$$(1.9) \quad \int_{\Omega} |\nabla u|^q w(x) dx \leq C([w]_{\frac{q}{p-1}}) \int_{\Omega} |\mathbf{f}(x)|^q w(x) dx$$

*for all weights  $w \in A_{\frac{q}{p-1}}$ .*

What we obtain in this paper is the weighted bound (1.1) for all weights  $w \in A_1$  which unfortunately is not enough for us to apply the above extrapolation theorem. However, it provides us with an alternative view on the conjecture of T. Iwaniec and gives us a different sense of how far we are from completely resolving this conjecture. Of course, one can also generalize this conjecture by proposing the bound (1.8) for all weights  $v \in A_{\frac{p}{p-1}}$ .

**Notation:** Throughout the paper, we shall write  $A \lesssim B$  to denote  $A \leq cB$  for a positive constant  $c$  independent of the parameters involved. Basically,  $c$  is allowed to depend only on  $n, p, \Lambda_0, \Lambda_1, \gamma$  and  $R_0$ .

## 2. LOCAL DIFFERENCE ESTIMATES

In this section, we obtain certain local interior and boundary difference estimates that are essential to our global estimates later.

**2.1. Interior estimates.** Let  $u \in W_0^{1,p-\delta}(\Omega)$  for some  $\delta \in (0, \min\{1, p-1\})$  be a very weak solution to the equation

$$(2.1) \quad \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} |\mathbf{f}|^{p-2} \mathbf{f}$$

in a domain  $\Omega$ . For each ball  $B_{2R} = B_{2R}(x_0) \Subset \Omega$ , we let  $w \in u + W_0^{1,p-\delta}(B_{2R})$  be a very weak solution to the problem

$$(2.2) \quad \begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } B_{2R} \\ w = u & \text{on } \partial B_{2R}. \end{cases}$$

For sufficiently small  $\delta$ , the existence of such  $w$  follows from the result of [12, Theorem 2]. The following theorem tells more on the integrability property of  $w$  and its relation to  $u$  by means of a comparison estimate.

**Theorem 2.1.** *Under (1.3) and (1.4), there exists a small number  $\delta_0 = \delta_0(n, p, \Lambda_0, \Lambda_1) > 0$  such that the following holds for any  $\delta \in (0, \delta_0)$ . Given any  $u \in W_{\text{loc}}^{1,p-\delta}(\Omega)$  solving (2.1) and any  $w$  as in (2.2), we have the following comparison estimate*

$$\int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} dx \leq C \delta^{\frac{p-\delta}{p-1}} \int_{B_{2R}} |\nabla u|^{p-\delta} dx + \int_{B_{2R}} |\mathbf{f}|^{p-\delta} dx$$

if  $p \geq 2$ , and

$$\begin{aligned} \int_{B_{2R}} |\nabla u - \nabla w|^{p-\delta} dx &\leq C \delta^{p-\delta} \int_{B_{2R}} |\nabla u|^{p-\delta} dx + \\ &\quad + C \left( \int_{B_{2R}} |\mathbf{f}|^{p-\delta} dx \right)^{p-1} \left( \int_{B_{2R}} |\nabla u|^{p-\delta} dx \right)^{2-p} \end{aligned}$$

if  $1 < p < 2$ . Moreover,

$$(2.3) \quad \int_{B_{2R}} |\nabla w|^{p-\delta} dx \leq C \int_{B_{2R}} |\nabla u|^{p-\delta} dx,$$

and for any ball  $B_r(y) \subset B_{2R}$

$$(2.4) \quad \left( \int_{B_{r/2}(y)} |\nabla w|^{p+\delta_0} dx \right)^{\frac{1}{p+\delta_0}} \leq C \left( \int_{B_r(y)} |\nabla w|^{p-\delta_0} dx \right)^{\frac{1}{p-\delta_0}}$$

Here the constants  $C$  depend only on  $n, p, \Lambda_0$  and  $\Lambda_1$ .

The bound (2.3) was obtained in [12, Theorem 2]. The higher integrability result, inequality (2.4), was proved in [12, Theorem 1] (see also [20]). On the other hand, the comparison estimate above has been obtained just recently in [1, Lemma 2.8].

Now with  $u$  as in (1.2) and  $w$  as in (2.2), we further define another function  $v \in w + W_0^{1,p}(B_R)$  as the unique solution to the Dirichlet problem

$$(2.5) \quad \begin{cases} \operatorname{div} \bar{\mathcal{A}}_{B_R}(\nabla v) = 0 & \text{in } B_R, \\ v = w & \text{on } \partial B_R, \end{cases}$$

where  $B_R = B_R(x_0)$ . This equation makes sense since we have good regularity for  $w$  as a consequence of Theorem 2.1. We shall now prove another useful interior difference estimate.

**Lemma 2.2.** *Under (1.3)-(1.4), let  $\delta \in (0, \delta_0)$ , where  $\delta_0$  is as in Theorem 2.1. Let  $w$  and  $v$  be as in (2.2) and (2.5). For  $\tau = \frac{p}{\delta_0} \frac{(p+\delta_0)}{(p-1)}$ , there exists a constant  $C = C(n, p, \Lambda_0, \Lambda_1)$  such that*

$$\begin{aligned} \int_{B_R} |\nabla v - \nabla w|^{p-\delta} dx &\leq C \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^\tau dx \right)^{\min\{p-\delta, \frac{p-\delta}{p-1}\}/\tau} \times \\ &\quad \times \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right). \end{aligned}$$

*Proof.* Using (1.3) and the fact that both  $v$  and  $w$  are solutions, we have

$$\begin{aligned} &\int_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla w - \nabla v|^2 dx \\ &\quad \leq \int_{B_R} \langle \bar{\mathcal{A}}_{B_R}(\nabla w) - \bar{\mathcal{A}}_{B_R}(\nabla v), \nabla w - \nabla v \rangle dx \\ &\quad = C \int_{B_R} \langle \bar{\mathcal{A}}_{B_R}(\nabla w) - \mathcal{A}(x, \nabla w), \nabla w - \nabla v \rangle dx \\ &\quad \leq \int_{B_R} \Upsilon(\mathcal{A}, B_R)(x) |\nabla w|^{p-1} |\nabla w - \nabla v| dx. \end{aligned}$$

Using Hölder's inequality with exponents  $p$ ,  $\frac{p+\delta_0}{p-1}$ , and  $\tau$  we get

$$\begin{aligned}
 & \int_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla w - \nabla v|^2 dx \\
 & \leq \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^\tau dx \right)^{\frac{1}{\tau}} \left( \int_{B_R} |\nabla w|^{p+\delta_0} dx \right)^{\frac{p-1}{p+\delta_0}} \times \\
 (2.6) \quad & \quad \times \left( \int_{B_R} |\nabla w - \nabla v|^p dx \right)^{\frac{1}{p}} \\
 & \leq \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^\tau dx \right)^{\frac{1}{\tau}} \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{p-1}{p-\delta}} \times \\
 & \quad \times \left( \int_{B_R} |\nabla w - \nabla v|^p dx \right)^{\frac{1}{p}},
 \end{aligned}$$

where the last inequality follows from (2.4) of Theorem 2.1.

Thus for  $p \geq 2$ , using pointwise estimate

$$|\nabla w - \nabla v|^p \leq (|\nabla v|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla w - \nabla v|^2,$$

we find

$$\left( \int_{B_R} |\nabla w - \nabla v|^p dx \right)^{\frac{p-1}{p}} \lesssim \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau dx \right)^{\frac{1}{\tau}} \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{p-1}{p-\delta}}.$$

By Hölder's inequality this yields the desired estimate in the case  $p \geq 2$ .

For  $1 < p < 2$  we write

$$|\nabla v - \nabla w|^p = (|\nabla v|^2 + |\nabla w|^2)^{\frac{(p-2)p}{4}} |\nabla w - \nabla v|^p (|\nabla v|^2 + |\nabla w|^2)^{\frac{(2-p)p}{4}},$$

and apply Hölder's inequality with exponents  $\frac{2}{p}$  and  $\frac{2}{2-p}$  to obtain

$$\begin{aligned}
 \int_{B_R} |\nabla w - \nabla v|^p dx & \leq \left( \int_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla w - \nabla v|^2 dx \right)^{\frac{p}{2}} \times \\
 & \quad \times \left( \int_{B_R} (|\nabla v|^2 + |\nabla w|^2)^{\frac{p}{2}} dx \right)^{\frac{2-p}{2}} \\
 & \lesssim \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau dx \right)^{\frac{p}{2\tau}} \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{(p-1)p}{(p-\delta)^2}} \times \\
 & \quad \times \left( \int_{B_R} |\nabla w - \nabla v|^p dx \right)^{\frac{1}{2}} \left( \int_{B_R} |\nabla w|^p dx \right)^{\frac{2-p}{2}}.
 \end{aligned}$$

Here we used (2.6) and the easy energy bound  $\int_{B_R} |\nabla v|^p dx \leq c \int_{B_R} |\nabla w|^p dx$  in the last inequality. Using (2.4) of Theorem 2.1, this yields

$$\int_{B_R} |\nabla w - \nabla v|^p dx \lesssim \left( \int_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau dx \right)^{\frac{p}{\tau}} \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{p}{p-\delta}}.$$

Now an application of Hölder's inequality gives the desired estimate in the case  $1 < p < 2$ .  $\square$

**Corollary 2.3.** *Under (1.3)-(1.4), let  $\tau = \frac{p}{\delta_0} \frac{(p+\delta_0)}{(p-1)}$  and  $\delta \in (0, \delta_0)$ , where  $\delta_0$  is as in Theorem 2.1. Then for any  $\epsilon > 0$ , there exists  $\gamma = \gamma(\epsilon) > 0$  such that if  $u \in W_0^{1,p-\delta}(\Omega)$  is a very weak solution of (1.2) satisfying*

$$\int_{B_{2R}} |\nabla u|^{p-\delta} dx \leq 1, \int_{B_{2R}} |\mathbf{f}|^{p-\delta} dx \leq \gamma^{p-\delta}, \text{ and } \int_{B_R} \Upsilon(\mathcal{A}, B_R)^\tau dx \leq \gamma^\tau,$$

for a ball  $B_{2R} \Subset \Omega$ , then there exists  $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$  such that

$$\int_{B_R} |\nabla u - \nabla v|^{p-\delta} dx \leq \epsilon^{p-\delta}, \text{ and } \|\nabla v\|_{L^\infty(B_{R/2})} \leq C_0 = C_0(n, p, \Lambda_0, \Lambda_1).$$

*Proof.* Let  $w$  and  $v$  be as in (2.2) and (2.5) respectively. Since we have  $v \in W^{1,p}(B_R)$ , standard regularity theory gives (see, e.g., [31])

$$\begin{aligned} \|\nabla v\|_{L^\infty(B_{R/2})}^p &\lesssim \int_{B_R} |\nabla v|^p dx \lesssim \int_{B_R} |\nabla w|^p dx \\ &\lesssim \left( \int_{B_{2R}} |\nabla w|^{p-\delta} dx \right)^{\frac{p}{p-\delta}} \lesssim \left( \int_{B_{2R}} |\nabla u|^{p-\delta} dx \right)^{\frac{p}{p-\delta}} \leq C_0. \end{aligned}$$

Here we applied Theorem 2.1. The proof of the corollary now follows from the comparison estimate in Theorem 2.1 and Lemma 2.2.  $\square$

**2.2. Boundary estimates.** We now consider the corresponding local estimates near the boundary. Suppose that the domain  $\Omega$  is  $(\gamma, R_0)$ -Reifenberg flat with  $\gamma < 1/2$ . Let  $x_0 \in \partial\Omega$ ,  $R \in (0, R_0/20)$ , and let  $u \in W_0^{1,p-\delta}(\Omega)$  be a very weak solution to (1.2) for some  $\delta \in (0, \min\{1, p-1\})$ . On  $\Omega_{20R} = \Omega_{20R}(x_0) = B_{20R}(x_0) \cap \Omega$ , we let  $w \in u + W_0^{1,p-\delta}(\Omega_{20R}(x_0))$  be a very weak solution to the problem:

$$(2.7) \quad \begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{20R}, \\ w = u & \text{on } \partial\Omega_{20R}. \end{cases}$$

We now extend  $u$  by zero to  $\mathbb{R}^n \setminus \Omega$  and then extend  $w$  by  $u$  to  $\mathbb{R}^n \setminus \Omega_{20R}(x_0)$ . Analogous to Theorem 2.1, we have the following boundary counterpart.

**Theorem 2.4.** *Under (1.3) and (1.4), there exists a small number  $\tilde{\delta}_0 = \tilde{\delta}_0(n, p, \Lambda_0, \Lambda_1, \gamma) > 0$  such that the following holds for any  $\delta \in (0, \tilde{\delta}_0)$ . For any  $u \in W_0^{1,p-\delta}(\Omega)$  solving (1.2) and any  $w$  as in (2.7), after extending  $\mathbf{f}$*

and  $u$  outside  $\Omega$  by zero and  $w$  by  $u$  outside  $\Omega_{20R}$ , we have the following comparison estimate

$$\int_{B_{20R}} |\nabla u - \nabla w|^{p-\delta} dx \leq C \delta^{\frac{p-\delta}{p-1}} \int_{B_{20R}} |\nabla u|^{p-\delta} dx + \int_{B_{20R}} |\mathbf{f}|^{p-\delta} dx$$

if  $p \geq 2$ , and

$$\begin{aligned} \int_{B_{20R}} |\nabla u - \nabla w|^{p-\delta} dx &\leq C \delta^{p-\delta} \int_{B_{20R}} |\nabla u|^{p-\delta} dx + \\ &+ C \left( \int_{B_{20R}} |\mathbf{f}|^{p-\delta} dx \right)^{p-1} \left( \int_{B_{20R}} |\nabla u|^{p-\delta} dx \right)^{2-p} \end{aligned}$$

if  $1 < p < 2$ . Moreover,

$$(2.8) \quad \int_{B_{20R}} |\nabla w|^{p-\delta} dx \leq C \int_{B_{20R}} |\nabla u|^{p-\delta} dx,$$

and for any ball  $B_r(y)$  such that  $B_{7r}(y) \subset B_{20R}$

$$(2.9) \quad \left( \int_{B_{r/2}(y)} |\nabla w|^{p+\tilde{\delta}_0} dx \right)^{\frac{1}{p+\tilde{\delta}_0}} \leq C \left( \int_{B_{7r}(y)} |\nabla w|^{p-\tilde{\delta}_0} dx \right)^{\frac{1}{p-\tilde{\delta}_0}}.$$

Here the constants  $C = C(n, p, \Lambda_0, \Lambda_1, \gamma)$ .

Theorem 2.4 was actually proved for a much larger class of domains and more general nonlinearities in [1]. More explicitly, the existence of  $w$  and the bound (2.8) are contained in [1, Corollary 3.5]; the higher integrability estimate (2.9) is obtained in [1, Theorem 3.7]; and the comparison estimate is the result of [1, Lemma 3.10].

With  $x_0 \in \partial\Omega$  and  $0 < R < R_0/20$  as above, we now set  $\rho = R(1 - \gamma)$ . By Remark (1.3), there exists a coordinate system  $\{z_1, z_2, \dots, z_n\}$  with the origin  $0 \in \Omega$  such that in this coordinate system  $x_0 = (0, \dots, 0, -\rho\gamma/(1 - \gamma)) \in \partial\Omega$  and

$$(2.10) \quad B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{(z_1, z_2, \dots, z_n) : z_n > -4\rho\gamma\}.$$

Here recall that  $B_\rho^+(0) = B_\rho(0) \cap \{(z_1, \dots, z_n) : z_n > 0\}$  denotes an upper half ball in the corresponding coordinate system.

With this  $\rho$  and thanks to the existence and regularity of  $w$  in Theorem 2.4, we define another function  $v \in w + W_0^{1,p}(\Omega_\rho(0))$  as the unique solution to the Dirichlet problem

$$(2.11) \quad \begin{cases} \operatorname{div} \bar{\mathcal{A}}_{B_\rho}(\nabla v) = 0 & \text{in } \Omega_\rho(0), \\ v = w & \text{on } \partial\Omega_\rho(0). \end{cases}$$

We then set  $v$  to be equal to  $w$  in  $\mathbb{R}^n \setminus \Omega_\rho(0)$ . The following boundary difference estimate can be proved in a way just similar to the proof of Lemma 2.2.

**Lemma 2.5.** *Under (1.3) and (1.4), let  $\delta \in (0, \tilde{\delta}_0)$ , where  $\tilde{\delta}_0$  is in Theorem 2.4. Let  $w$  and  $v$  be as in (2.7) and (2.11). For  $\tau = \frac{p}{\delta_0} \frac{(p+\tilde{\delta}_0)}{(p-1)}$ , there exists a constant  $C = C(n, p, \Lambda_0, \Lambda_1, \gamma)$  such that*

$$\begin{aligned} \int_{B_\rho(0)} |\nabla v - \nabla w|^{p-\delta} dx &\leq C \left( \int_{B_\rho(0)} \Upsilon(\mathcal{A}, B_\rho(0))(x)^\tau dx \right)^{\min\{p-\delta, \frac{p-\delta}{p-1}\}/\tau} \times \\ &\quad \times \left( \int_{B_{14\rho}(0)} |\nabla w|^{p-\delta} dx \right). \end{aligned}$$

As the boundary of  $\Omega$  can be very irregular, the  $L^\infty$ -norm of  $\nabla v$  up to the boundary of  $\Omega$  could be unbounded. Therefore, we consider another equation:

$$(2.12) \quad \begin{cases} \operatorname{div} \bar{\mathcal{A}}_{B_\rho}(\nabla V) = 0 & \text{in } B_\rho^+(0), \\ V = 0 & \text{on } T_\rho, \end{cases}$$

where  $T_\rho$  is the flat portion of  $\partial B_\rho^+(0)$ . A function  $V \in W^{1,p}(B_\rho^+(0))$  is a weak solution of (2.12) if its zero extension to  $B_\rho(0)$  belongs to  $W^{1,p}(B_\rho(0))$  and if

$$\int_{B_\rho^+(0)} \bar{\mathcal{A}}_{B_\rho}(\nabla V) \cdot \nabla \phi dx = 0$$

for all  $\phi \in W_0^{1,p}(B_\rho^+(0))$ .

We shall need the following key perturbation result obtained earlier in [25, Theorem 2.12].

**Theorem 2.6** ([25]). *Suppose that  $\mathcal{A}$  satisfies (1.3) and (1.4). For any  $\epsilon > 0$ , there exists a small  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \epsilon) > 0$  such that if  $v \in W^{1,p}(\Omega_\rho(0))$  is a solutions of (2.11) under the geometric setting (2.10), then there exists a weak solution  $V \in W^{1,p}(B_\rho^+(0))$  of (2.12) whose zero extension to  $B_\rho(0)$  satisfies*

$$\|\nabla V\|_{L^\infty(B_{\rho/4}(0))}^p \leq C \int_{B_\rho(0)} |\nabla v|^p dx,$$

with  $C = C(n, p, \Lambda_0, \Lambda_1)$  and

$$\int_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p dx \leq \epsilon^p \int_{B_\rho(0)} |\nabla v|^p dx.$$

We now have the boundary analogue of Corollary 2.3. The proof of the following corollary follows with obvious modification as in [23, Corollary 2.10].

**Corollary 2.7** ([23]). *For any  $\epsilon > 0$  there exist  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \epsilon) > 0$  and  $\tilde{\delta}_1 = \tilde{\delta}_1(n, p, \Lambda_0, \Lambda_1, \epsilon) \in (0, \tilde{\delta}_0)$ , where  $\tilde{\delta}_0$  is as in Theorem 2.4, such that the following holds with  $\tau = \frac{p}{\tilde{\delta}_0} \frac{(p+\tilde{\delta}_0)}{(p-1)}$ . If  $\Omega$  is  $(\gamma, R_0)$ -Reifenberg flat and if  $u \in W_0^{1, p-\delta}(\Omega)$ ,  $\delta \in (0, \tilde{\delta}_1)$ , is a very weak solution of (1.2) with*

$$\int_{B_{20R}(x_0)} |\nabla u|^{p-\delta} \chi_\Omega dx \leq 1, \int_{B_{20R}(x_0)} |\mathbf{f}|^{p-\delta} \chi_\Omega dx \leq \gamma^{p-\delta}, \text{ and } [\mathcal{A}]_\tau^{R_0} \leq \gamma,$$

where  $x_0 \in \partial\Omega$  and  $R \in (0, R_0/20)$ , then there is a function

$$V \in W^{1, \infty}(B_{R/10}(x_0))$$

such that

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))} \leq C_0 = C_0(n, p, \Lambda_0, \Lambda_1),$$

and

$$(2.13) \quad \int_{B_{R/10}(x_0)} |\nabla u - \nabla V|^{p-\delta} dx \leq \epsilon^{p-\delta}.$$

*Proof.* With  $x_0 \in \partial\Omega$  and  $R \in (0, R_0/20)$ , we set  $\rho = R(1-\gamma)$ . Also, extend both  $u$  and  $\mathbf{f}$  by zero to  $\mathbb{R}^n \setminus \Omega$ . By Remark (1.3) and by translating and rotating if necessary, we may assume that  $0 \in \Omega$ ,  $x_0 = (0, \dots, 0, -\rho\gamma/(1-\gamma))$  and the geometric setting

$$(2.14) \quad B_\rho^+(0) \subset \Omega_\rho(0) \subset B_\rho(0) \cap \{x_n > -4\gamma\rho\}.$$

Moreover, we shall further restrict  $\gamma \in (0, 1/45)$  so that we have

$$B_{R/10}(x_0) \subset B_{\rho/8}(0).$$

We now choose  $w$  and  $v$  as in (2.7) and (2.11) corresponding to these  $R$  and  $\rho$ . Then, since  $B_{14\rho}(0) \subset B_{20R}(x_0)$ , there holds

$$\int_{B_\rho(0)} |\nabla v|^p dx \leq C \int_{B_{14\rho}(0)} |\nabla w|^p dx \leq C \int_{B_{20R}(x_0)} |\nabla u|^p dx \leq C.$$

By Theorem 2.6 for any  $\eta > 0$  we can find a  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \eta) \in (0, 1/45)$  such that, under (2.14), there is a function  $V \in W^{1, p}(B_\rho(0)) \cap W^{1, \infty}(B_{\rho/4}(0))$  such that

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))}^p \leq C \|\nabla V\|_{L^\infty(B_{\rho/4}(0))}^p \leq C \int_{B_\rho(0)} |\nabla v|^p dx \leq C,$$

and

$$\int_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p dx \leq \eta^p \int_{B_\rho(0)} |\nabla v|^p dx \leq C\eta^p.$$

By Hölder's inequality, the last bound gives

$$(2.15) \quad \int_{B_{\rho/8}(0)} |\nabla v - \nabla V|^{p-\delta} dx \leq C\eta^{p-\delta}.$$

Now writing

$$\int_{B_{R/10}(x_0)} |\nabla u - \nabla V|^{p-\delta} dx = \int_{B_{\rho/8}(0)} |\nabla(u-w) + \nabla(w-v) + \nabla(v-V)|^{p-\delta} dx,$$

and using (2.15) along with Theorem 2.4 and Lemma 2.5, we obtain inequality (2.13) as desired (after choosing  $\tilde{\delta}_1 = \tilde{\delta}_1(\epsilon)$ ,  $\eta = \eta(\epsilon)$ , and  $\gamma = \gamma(\epsilon)$  appropriately for any given  $\epsilon > 0$ ).  $\square$

### 3. WEIGHTED ESTIMATES

We now use Corollaries 2.3 and 2.7 to obtain the following technical result.

**Proposition 3.1.** *Under (1.3)-(1.4), there are  $\lambda = \lambda(n, p, \Lambda_0, \Lambda_1) > 1$  and  $\tau = \tau(n, p, \Lambda_0, \Lambda_1) > 1$  such that the following holds. For any  $\epsilon > 0$ , there exist  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \epsilon) > 0$  and  $\bar{\delta} = \bar{\delta}(n, p, \Lambda_0, \Lambda_1, \epsilon) > 0$  such that if  $u \in W_0^{1, p-\delta}(\Omega)$ ,  $\delta \in (0, \bar{\delta})$ , is a very weak solution to (1.2) with  $\Omega$  being  $(\gamma, R_0)$ -Reifenberg flat,  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , and if, for some ball  $B_\rho(y)$  with  $\rho < R_0/1200$ ,*

$$(3.1) \quad \begin{aligned} & B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq 1\} \cap \\ & \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) \leq \gamma\} \neq \emptyset, \end{aligned}$$

then one has

$$(3.2) \quad |\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)| < \epsilon |B_\rho(y)|.$$

*Proof.* By (3.1), there exists an  $x_0 \in B_\rho(y)$  such that for any  $r > 0$ ,

$$(3.3) \quad \int_{B_r(x_0)} |\nabla u|^{p-\delta} dx \leq 1 \quad \text{and} \quad \int_{B_r(x_0)} \chi_\Omega |\mathbf{f}|^{p-\delta} dx \leq \gamma^{p-\delta}.$$

By the first inequality in (3.3), for any  $x \in B_\rho(y)$ , there holds

$$(3.4) \quad \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq \max \left\{ \mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x), 3^n \right\}.$$

To prove (3.2), it is enough to consider the case  $B_{4\rho}(y) \subset \Omega$  and the case  $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$ . First we consider the latter. Let  $y_0 \in B_{4\rho}(y) \cap \partial\Omega$ , we then have

$$B_{2\rho}(y) \subset B_{6\rho}(y_0) \subset B_{1200\rho}(y_0) \subset B_{1205\rho}(x_0).$$

Thus by (3.3) we obtain

$$\int_{B_{1200\rho}(y_0)} |\nabla u|^{p-\delta} dx \leq c \quad \text{and} \quad \int_{B_{1200\rho}(y_0)} \chi_\Omega |\mathbf{f}|^{p-\delta} dx \leq c\gamma^{p-\delta},$$

where  $c = (1205/1200)^n$ . Since  $60\rho < R_0/20$ , by Corollary 2.7 (with  $R = 60\rho$ ), there exists a  $\tau(n, p, \Lambda_0, \Lambda_1) > 1$  such that the following holds. For any  $\eta \in (0, 1)$ , there are  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \eta) > 0$ ,  $\bar{\delta} = \bar{\delta}(n, p, \Lambda_0, \Lambda_1, \eta) > 0$  such that if  $\Omega$  is a  $(\gamma, R_0)$ -Reifenberg flat domain and  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , then one can find a function  $V \in W^{1,\infty}(B_{6\rho}(y_0))$  with

$$(3.5) \quad \|\nabla V\|_{L^\infty(B_{2\rho}(y))} \leq \|\nabla V\|_{L^\infty(B_{6\rho}(y_0))} \leq C_0,$$

and, for  $\delta \in (0, \bar{\delta})$ ,

$$(3.6) \quad \int_{B_{2\rho}(y)} |\nabla u - \nabla V|^{p-\delta} dx \leq C \int_{B_{6\rho}(y_0)} |\nabla u - \nabla V|^{p-\delta} dx \leq C\eta^{p-\delta}.$$

In view of (3.4), (3.5) and the triangle inequality we see that, for  $\lambda = \max\{3^n, 2C_0\}$ ,

$$\begin{aligned} & \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y) \subset \\ & \subset \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y) \\ & \subset \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)} |\nabla u - \nabla V|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda/2\} \cap B_\rho(y). \end{aligned}$$

Thus by the weak-type (1, 1) inequality for the Hardy-Littlewood maximal function and (3.6), we find

$$\begin{aligned} & |\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)| \leq \\ & \leq \frac{C}{\lambda^{p-\delta}} \int_{B_{2\rho}(y)} |\nabla u - \nabla V|^{p-\delta} dx \leq \frac{C}{C_0^{p-\delta}} |B_{2\rho}(y)| \eta^{p-\delta}. \end{aligned}$$

This gives the estimate (3.2) in the case  $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$ , provided  $\eta$  is appropriately chosen. The interior case  $B_{4\rho}(y) \subset \Omega$  can be obtained in a similar way by using Corollary 2.3, instead of Corollary 2.7.  $\square$

Proposition 3.1 can now be used to obtain the following result which involves  $A_\infty$  weights.

**Proposition 3.2.** *Under (1.3)-(1.4), there exist  $\lambda = \lambda(n, p, \Lambda_0, \Lambda_1) > 1$  and  $\tau = \tau(n, p, \Lambda_0, \Lambda_1) > 1$  such that the following holds. For any weight  $w \in A_\infty$  and any  $\epsilon > 0$ , there exist  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \epsilon, [w]_\infty) > 0$  and  $\bar{\delta} = \bar{\delta}(n, p, \Lambda_0, \Lambda_1, \epsilon, [w]_\infty) > 0$  such that if  $u \in W_0^{1,p-\delta}(\Omega)$ ,  $\delta \in (0, \bar{\delta})$ , is a*

very weak solution of (1.2) with  $\Omega$  being  $(\gamma, R_0)$ -Reifenberg flat,  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , and if, for some ball  $B_\rho(y)$  with  $\rho < R_0/1200$ ,

$$w(\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)) \geq \epsilon w(B_\rho(y)),$$

then one has

$$(3.7) \quad \begin{aligned} B_\rho(y) \subset & \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\} \cup \\ & \cup \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\}. \end{aligned}$$

*Proof.* Suppose that  $(\Xi_0, \Xi_1)$  is a pair of  $A_\infty$  constants of  $w$  and let  $\lambda$  and  $\tau$  be as in Proposition 3.1. Given  $\epsilon > 0$ , we choose a  $\gamma = \gamma(\Xi_0, \Xi_1, \epsilon)$  and  $\bar{\delta} = \bar{\delta}(\Xi_0, \Xi_1, \epsilon)$  as in Proposition 3.1 with  $[\epsilon/(2\Xi_0)]^{1/\Xi_1}$  replacing  $\epsilon$ . The proof then follows by a contradiction. To that end, suppose that the inclusion in (3.7) fails for this  $\gamma$ , then we must have that

$$\begin{aligned} B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) \leq 1\} \cap \\ \cap \{x \in \mathbb{R}^n : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) \leq \gamma\} \neq \emptyset \end{aligned}$$

for some  $\delta \in (0, \bar{\delta})$ . Hence by Proposition 3.1, if  $\Omega$  is a  $(\gamma, R_0)$ -Reifenberg flat and  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , there holds

$$|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda \cap B_\rho(y)\}| \leq \left(\frac{\epsilon}{2\Xi_0}\right)^{1/\Xi_1} |B_\rho(y)|.$$

Thus using the  $A_\infty$  characterization of  $w$  (Lemma 1.4), we immediately get that

$$\begin{aligned} w(\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)) \\ \leq \Xi_0 \left[ \frac{|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\} \cap B_\rho(y)|}{|B_\rho(y)|} \right]^{\Xi_1} w(B_\rho(y)) \\ \leq \frac{\epsilon}{2} w(B_\rho(y)) < \epsilon w(B_\rho(y)). \end{aligned}$$

This yields a contradiction and thus the proof is complete.  $\square$

The following Calderón-Zygmund decomposition type lemma will allow us to iterate the result of Proposition 3.2 to obtain Theorem 3.4 below. In the unweighted case various versions of this lemma have been obtained (see, e.g., [5, 34, 2]). The proof of this weighted version was presented in [21].

**Lemma 3.3.** *Let  $\Omega$  be a  $(\gamma, R_0)$ -Reifenberg flat domain with  $\gamma < 1/8$ , and let  $w$  be an  $A_\infty$  weight. Suppose that the sequence of balls  $\{B_r(y_i)\}_{i=1}^L$  with centers  $y_i \in \bar{\Omega}$  and a common radius  $r \leq R_0/4$  covers  $\Omega$ . Let  $C \subset D \subset \Omega$  be measurable sets for which there exists  $0 < \epsilon < 1$  such that*

- (1)  $w(C) < \epsilon w(B_r(y_i))$  for all  $i = 1, \dots, L$ , and  
 (2) for all  $x \in \Omega$  and  $\rho \in (0, 2r]$ , if  $w(C \cap B_\rho(x)) \geq \epsilon w(B_\rho(x))$ , then  $B_\rho(x) \cap \Omega \subset D$ .

Then we have the estimate

$$w(C) \leq B \epsilon w(D)$$

for a constant  $B$  depending only on  $n$  and the  $A_\infty$  constants of  $w$ .

**Theorem 3.4.** Under (1.3)-(1.4), let  $\lambda$  and  $\tau$  be as in Proposition 3.2. Then for any weight  $w \in A_\infty$  and any  $\epsilon > 0$ , there exist constants  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, \epsilon, [w]_\infty) > 0$  and  $\bar{\delta} = \bar{\delta}(n, p, \Lambda_0, \Lambda_1, \epsilon, [w]_\infty) > 0$  such that the following holds. Suppose that  $u \in W_0^{1, p-\delta}(\Omega)$ ,  $\delta \in (0, \bar{\delta})$ , is a very weak solution of (1.2) in a  $(\gamma, R_0)$ -Reifenberg flat domain  $\Omega$ , with  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ . Suppose also that  $\{B_r(y_i)\}_{i=1}^L$  is a sequence of balls with centers  $y_i \in \bar{\Omega}$  and a common radius  $0 < r \leq R_0/4000$  that covers  $\Omega$ . If for all  $i = 1, \dots, L$

$$(3.8) \quad w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) < \epsilon w(B_r(y_i)),$$

then for any  $s > 0$  and any integer  $k \geq 1$  there holds

$$\begin{aligned} w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^s &\leq \\ &\leq \sum_{i=1}^k (A\epsilon)^{si} w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma \lambda^{(k-i)}\})^s + \\ &\quad + (A\epsilon)^{sk} w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s, \end{aligned}$$

where the constant  $A = A(n, [w]_\infty)$ .

*Proof.* The theorem will be proved by induction on  $k$ . Given  $w \in A_\infty$  and  $\epsilon > 0$ , we take  $\gamma = \gamma(\epsilon, [w]_\infty)$  and  $\bar{\delta} = \bar{\delta}(\epsilon, [w]_\infty)$  as in Proposition 3.2. The case  $k = 1$  follows from Proposition 3.2 and Lemma 3.3. Indeed, for  $\delta \in (0, \bar{\delta})$ , let

$$C = \{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}$$

$$D = \{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\}.$$

Then from assumption (3.8), it follows that  $w(C) < \epsilon w(B_r(y_i))$  for all  $i = 1, \dots, L$ . Moreover, if  $y \in \Omega$  and  $\rho \in (0, 2r)$  such that  $w(C \cap B_\rho(y)) \geq \epsilon w(B_\rho(y))$ , then  $0 < \rho \leq R_0/1200$  and  $B_\rho(y) \cap \Omega \subset D$  by Proposition 3.2. Thus all hypotheses of Lemma 3.3 are satisfied, which yield, for a constant  $B = B(n, [w]_\infty)$ ,

$$\begin{aligned} w(C)^s &\leq B^s \epsilon^s w(D)^s \\ &\leq B^s 2^s \epsilon^s w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s + \\ &\quad + B^s 2^s \epsilon^s w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\})^s \end{aligned}$$

for any given  $s > 0$ . This proves the case  $k = 1$  with  $A = 2B$ . Suppose now that the conclusion of the lemma is true for some  $k > 1$ . Normalizing  $u$  to  $u_\lambda = u/\lambda$  and  $\mathbf{f}_\lambda = \mathbf{f}/\lambda$ , we see that for every  $i = 1, \dots, L$ ,

$$\begin{aligned} w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) &= \\ &= w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^2\}) \\ &\leq w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}} > \lambda\}) \\ &< \epsilon w(B_r(y_i)). \end{aligned}$$

Here we used the fact that  $\lambda > 1$  in the first inequality. Note that  $u_\lambda$  solves

$$\begin{cases} \operatorname{div} \tilde{\mathcal{A}}(x, \nabla u_\lambda) &= \operatorname{div} |\mathbf{f}_\lambda|^{p-2} \mathbf{f}_\lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where  $\tilde{\mathcal{A}}(x, \xi) = \mathcal{A}(x, \lambda\xi)/\lambda^{p-1}$  which obeys the same structural conditions (1.3)-(1.4). Thus by inductive hypothesis, it follows that

$$\begin{aligned} &w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^s \\ (3.9) \quad &\leq \sum_{i=1}^k (A\epsilon)^{si} w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}_\lambda|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{(k-i)}\})^s + \\ &\quad + (A\epsilon)^{sk} w(\{x \in \Omega : \mathcal{M}(|\nabla u_\lambda|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s. \end{aligned}$$

Finally, applying the case  $k = 1$  to the last term in (3.9) we conclude that

$$\begin{aligned} &w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^{k+1}\})^s \\ &\leq \sum_{i=1}^{k+1} (A\epsilon)^{si} w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma\lambda^{k+1-i}\})^s \\ &\quad + (A\epsilon)^{s(k+1)} w(\{x \in \Omega : \mathcal{M}(|\nabla u|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^s. \end{aligned}$$

This completes the proof of the theorem.  $\square$

The following result is a characterization of functions in weighted Lorentz space and can easily be proved using methods in standard measure theory.

**Lemma 3.5.** *Assume that  $g \geq 0$  is a measurable function in a bounded subset  $\Omega \subset \mathbb{R}^n$ . Let  $\theta > 0$ ,  $\Lambda > 1$  be constants, and let  $w$  be a weight in  $\mathbb{R}^n$ . Then for  $0 < q, t < \infty$ , we have*

$$g \in L_w(q, t)(\Omega) \iff S := \sum_{k \geq 1} \Lambda^{tk} w(\{x \in \Omega : g(x) > \theta\Lambda^k\})^{\frac{t}{q}} < +\infty.$$

Moreover, there exists a positive constant  $C = C(\theta, \Lambda, t) > 0$  such that

$$C^{-1} S \leq \|g\|_{L_w(q, t)(\Omega)}^t \leq C (w(\Omega)^{\frac{t}{q}} + S).$$

Analogously, for  $0 < q < \infty$  and  $t = \infty$  we have

$$C^{-1}T \leq \|g\|_{L_w(q, \infty)(\Omega)} \leq C(w(\Omega)^{\frac{1}{q}} + T),$$

where  $T$  is the quantity

$$T := \sup_{k \geq 1} \Lambda^k w(\{x \in \Omega : |g(x)| > \theta \Lambda^k\})^{\frac{1}{q}}.$$

We are now ready to obtain the main result of this section.

**Theorem 3.6.** *Suppose that  $\mathcal{A}$  satisfies (1.3)-(1.4). Let  $M > 1$  and let  $w$  be an  $A_\infty$  weight. There exist constants  $\tau = \tau(n, p, \Lambda_0, \Lambda_1) > 1$ ,  $\delta = \delta(n, p, \Lambda_0, \Lambda_1, M, [w]_\infty) > 0$  and  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, M, [w]_\infty) > 0$  such that the following holds for any  $t \in (0, \infty]$  and  $q \in (0, M]$ . If  $u \in W_0^{1, p-\delta}(\Omega)$  is a very weak solution of (1.2) in a  $(\gamma, R_0)$ -Reifenberg flat domain  $\Omega$  with  $[\mathcal{A}]_\tau^{R_0} \leq \gamma$ , then one has the estimate*

$$(3.10) \quad \|\nabla u\|_{L_w(q, t)(\Omega)} \leq C \|\mathcal{M}(|\mathbf{f}|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q, t)(\Omega)},$$

where the constant  $C = C(n, p, \Lambda_0, \Lambda_1, t, q, M, [w]_\infty, \text{diam}(\Omega)/R_0)$ .

**Remark 3.7.** The introduction of  $M$  in the above theorem is just for a technical reason. It ensures that the constant  $\delta$  is independent of  $q$  as the proof of the theorem reveals.

**Remark 3.8.** It follows also from the proof of Theorem 3.6 that if  $(\Xi_0, \Xi_1)$  is pair of  $A_\infty$  constants of  $w$  such that  $\max\{\Xi_0, 1/\Xi_1\} \leq \bar{w}$  then the constants  $\delta, \gamma$  and  $C$  above can be chosen to depend just on the upper-bound  $\bar{w}$  instead of  $(\Xi_0, \Xi_1)$ .

*Proof.* Let  $\lambda = \lambda(n, p, \Lambda_0, \Lambda_1)$  and  $\tau = \tau(n, p, \Lambda_0, \Lambda_1)$  be as in Theorem 3.4. Take  $\epsilon = \lambda^{-M} A^{-1} 2^{-1}$  and choose  $\delta = \bar{\delta}(n, p, \Lambda_0, \Lambda_1, \epsilon, [w]_\infty)/2$ , where  $A = A(n, [w]_\infty)$  and  $\bar{\delta}$  are as in Theorem 3.4; thus  $\delta = \delta(n, p, \Lambda_0, \Lambda_1, M, [w]_\infty)$ , which is independent of  $q$ . Using Theorem 3.4 we also get a constant  $\gamma = \gamma(n, p, \Lambda_0, \Lambda_1, M, [w]_\infty) > 0$  for this choice of  $\epsilon$ .

We shall prove (3.10) only for  $t \in (0, \infty)$ , as for  $t = \infty$  the proof is just similar. Choose a finite number of points  $\{y_i\}_{i=1}^L \subset \Omega$  and a ball  $B_0$  of radius  $2 \text{diam}(\Omega)$  such that

$$\Omega \subset \bigcup_{i=1}^L B_r(y_i) \subset B_0,$$

where  $r = \min\{R_0/4000, \text{diam}(\Omega)\}$ . We claim that we can choose  $N$  large such that for  $u_N = u/N$  and for all  $i = 1, \dots, L$ ,

$$(3.11) \quad w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) < \epsilon w(B_r(y_i)).$$

Indeed from the weak-type  $(1, 1)$  estimate for the maximal function, there exists a constant  $C(n) > 0$  such that

$$|\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}| < \frac{C(n)}{(\lambda N)^{p-\delta}} \int_{\Omega} |\nabla u|^{p-\delta} dx.$$

If  $(\Xi_0, \Xi_1)$  is a pair of  $A_{\infty}$  constants of  $w$ , then using Lemma 1.4, we see that

$$(3.12) \quad \begin{aligned} & w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda\}) \\ & < \Xi_0 \left( \frac{C(n)}{(\lambda N)^{p-\delta} |B_0|} \int_{\Omega} |\nabla u|^{p-\delta} dx \right)^{\Xi_1} w(B_0). \end{aligned}$$

Also, there are  $C_1 = C_1(n, [w]_{\infty}) \geq 1$  and  $p_1 = p_1(n, [w]_{\infty}) \geq 1$  such that

$$(3.13) \quad w(B_0) \leq C_1 \left( \frac{|B_0|}{|B_r(y_i)|} \right)^{p_1} w(B_r(y_i))$$

for every  $i = 1, 2, \dots, L$ . This follows from the so-called *strong doubling property* of  $A_{\infty}$  weights (see, e.g., [8, Chapter 9]). In view of (3.12) and (3.13), we now choose  $N$  such that

$$\frac{C(n)}{(\lambda N)^{p-\delta} |B_0|} \int_{\Omega} |\nabla u|^{p-\delta} dx = \left( \frac{|B_r(y_i)|}{|B_0|} \right)^{p_1/\Xi_1} \left( \frac{\epsilon}{\Xi_0 C_1} \right)^{1/\Xi_1}.$$

This gives the desired estimate (3.11). Note that for this  $N$  we have

$$(3.14) \quad \begin{aligned} N & \leq C |B_0|^{\frac{-1}{p-\delta}} \|\nabla u\|_{L^{p-\delta}(\Omega)} \leq C |B_0|^{\frac{-1}{p-\delta}} \|\mathbf{f}\chi_{\Omega}\|_{L^{p-\delta}(B_0)} \\ & \leq C \mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_{\Omega})(x)^{\frac{1}{p-\delta}} \end{aligned}$$

for all  $x \in \Omega$ . Here  $C = C(n, p, \Lambda_0, \Lambda_1, M, [w]_{\infty}, \text{diam}(\Omega)/R_0)$  and the second inequality follows from Theorem [1, Theorem 1.2].

With this  $N$ , we denote by

$$S = \sum_{k=1}^{\infty} \lambda^{tk} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^{\frac{t}{q}}$$

and for  $J \geq 1$  let

$$S_J = \sum_{k=1}^J \lambda^{tk} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > \lambda^k\})^{\frac{t}{q}}$$

be its partial sum. By Lemma 3.5, we see that

$$(3.15) \quad C^{-1} S \leq \|\mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w^{(q,t)}(\Omega)}^t \leq C(w(\Omega))^{\frac{t}{q}} + S.$$

By (3.11) and Theorem 3.4, we find

$$S_J \leq \sum_{k=1}^J \lambda^{tk} \left[ \sum_{j=1}^k (A\epsilon)^{\frac{t}{q}j} w(\{x \in \Omega : \mathcal{M}(|\mathbf{f}_N|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}(x) > \gamma \lambda^{(k-j)}\})^{\frac{t}{q}} \right] \\ + \sum_{k=1}^J \lambda^{tk} (A\epsilon)^{\frac{t}{q}k} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^{\frac{t}{q}}.$$

Here recall that  $\epsilon = \lambda^{-M} A^{-1} 2^{-1}$  and  $A = A(n, [w]_\infty)$ . Now interchanging the order of summation, we get

$$S_J \leq \sum_{j=1}^J (A\epsilon \lambda^q)^{\frac{t}{q}j} \left[ \sum_{k=j}^J \lambda^{t(k-j)} w(\Omega \cap \{\mathcal{M}(|\mathbf{f}_N|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}} > \gamma \lambda^{(k-j)}\})^{\frac{t}{q}} \right] \\ + \sum_{k=1}^J (A\epsilon \lambda^q)^{\frac{t}{q}k} w(\{x \in \Omega : \mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}(x) > 1\})^{\frac{t}{q}} \\ \leq C \left[ \|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)}^t + w(\Omega)^{\frac{t}{q}} \right] \sum_{j=1}^{\infty} 2^{-\frac{t}{q}j} \\ \leq C \left[ \|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)}^t + w(\Omega)^{\frac{t}{q}} \right]$$

for a constant  $C = C(n, p, \Lambda_0, \Lambda_1, q, t, M, [w]_\infty)$ . Letting  $J \rightarrow \infty$  and making use of (3.15), we arrive at

$$\|\mathcal{M}(|\nabla u_N|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)}^t \leq C \left[ \|\mathcal{M}(|\mathbf{f}_N|^{p-\delta})^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)}^t + w(\Omega)^{\frac{t}{q}} \right].$$

This gives

$$\|\nabla u\|_{L_w(q,t)(\Omega)} \leq C \left[ \|\mathcal{M}(|\mathbf{f}|^{p-\delta} \chi_\Omega)^{\frac{1}{p-\delta}}\|_{L_w(q,t)(\Omega)} + N w(\Omega)^{\frac{1}{q}} \right],$$

which in view of (3.14) yields the desired estimate.  $\square$

## A. APPENDIX: PROOF OF THEOREM 1.8

In this appendix, we provide a complete proof of Theorem 1.8.

*Proof.* First we consider the sub-natural case  $p-1 < q < p$ . To that end, let  $w \in A_{\frac{q}{p-1}}$  and suppose that  $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n) \cap L_w^q(\Omega, \mathbb{R}^n)$  satisfying (1.8) for all  $v \in A_{\frac{p}{p-1}}$ . Extend both  $\mathbf{f}$  and  $u$  by zero to  $\mathbb{R}^n \setminus \Omega$  and define

$$\mathcal{R}(\mathbf{f})(x) := \sum_{k=0}^{\infty} \frac{\mathcal{M}^{(k)}(|\mathbf{f}|^{p-1})(x)}{2^k \|\mathcal{M}\|_{L_w^{q/(p-1)} \rightarrow L_w^{q/(p-1)}}^k}.$$

Here  $\mathcal{M}^{(k)} = \mathcal{M} \circ \mathcal{M} \circ \dots \circ \mathcal{M}$  ( $k$  times) and note that (see, e.g., [8, Chapter 9])

$$(A.1) \quad \|\mathcal{M}\|_{L_w^{q/(p-1)} \rightarrow L_w^{q/(p-1)}} \leq C(n, p, q, [w]_{\frac{q}{p-1}}).$$

Now it is easy to observe from the definition of  $\mathcal{R}(\mathbf{f})$  that

$$(A.2) \quad |\mathbf{f}(x)|^{p-1} \leq \mathcal{R}(\mathbf{f})(x), \quad \text{and} \quad \|\mathcal{R}(\mathbf{f})\|_{L_w^{q/(p-1)}} \leq 2\|\mathbf{f}\|_{L_w^q}^{p-1}.$$

An important result which we shall need is the following estimate:

$$(A.3) \quad \mathcal{R}(\mathbf{f})^{-\frac{(p-q)}{(p-1)}} w \in A_{\frac{p}{p-1}} \quad \text{with} \quad [\mathcal{R}(\mathbf{f})^{-\frac{(p-q)}{(p-1)}} w]_{\frac{p}{p-1}} \leq C([w]_{\frac{q}{p-1}}).$$

The proof of (A.3) is obtained as follows: it follows from (A.1) and the definition of  $\mathcal{R}(\mathbf{f})$  that

$$\mathcal{M}(\mathcal{R}(\mathbf{f})) \leq C([w]_{\frac{q}{p-1}})\mathcal{R}(\mathbf{f}),$$

and thus we get that

$$\mathcal{R}(\mathbf{f})(x)^{-1} \leq C([w]_{\frac{q}{p-1}}) \left( \frac{1}{|B|} \int_B \mathcal{R}(\mathbf{f}) \, dy \right)^{-1}$$

for any ball  $B \subset \mathbb{R}^n$  containing  $x$ . Set now  $s = \frac{(p-q)q}{(p-1)p}$ . Using the last inequality, we find for any ball  $B \subset \mathbb{R}^n$ ,

$$(A.4) \quad \int_B \mathcal{R}(\mathbf{f})^{-s\frac{p}{q}} w \, dx \leq C([w]_{\frac{q}{p-1}}) \left( \int_B \mathcal{R}(\mathbf{f}) \, dy \right)^{-s\frac{p}{q}} \left( \int_B w(x) \, dx \right).$$

On the other hand, by Hölder's inequality there holds

$$(A.5) \quad \begin{aligned} \left( \int_B [\mathcal{R}(\mathbf{f})^{-s\frac{p}{q}} w(x)]^{1-p} \, dx \right)^{\frac{1}{p-1}} &= \left( \int_B \mathcal{R}(\mathbf{f})^{p-q} w(x)^{1-p} \, dx \right)^{\frac{1}{p-1}} \\ &\leq \left( \int_B \mathcal{R}(\mathbf{f}) \, dx \right)^{\frac{p-q}{p-1}} \left( \int_B w(x)^{\frac{1-p}{1-p+q}} \, dx \right)^{\frac{1-p+q}{p-1}}. \end{aligned}$$

Multiplying (A.4) by (A.5), we obtain the conclusion stated in (A.3).

We now obtain by Hölder's inequality

$$(A.6) \quad \begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^q w \, dx &= \int_{\mathbb{R}^n} |\nabla u|^q \mathcal{R}(\mathbf{f})^{-s} \mathcal{R}(\mathbf{f})^s w \, dx \\ &\leq \left( \int_{\mathbb{R}^n} |\nabla u|^p \mathcal{R}(\mathbf{f})^{-s\frac{p}{q}} w \, dx \right)^{q/p} \left( \int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{s\frac{q}{p-q}} w \, dx \right)^{(p-q)/p}. \end{aligned}$$

By making use of the hypothesis of the theorem along with (A.2), we can then estimate the right hand side of (A.6) as

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u|^q w \, dx &\leq C \left( [\mathcal{R}(\mathbf{f})^{-s \cdot \frac{p}{q}} w]_{\frac{p}{p-1}} \right) \left( \int_{\mathbb{R}^n} |\mathbf{f}|^p \mathcal{R}(\mathbf{f})^{-s \cdot \frac{p}{q}} w \, dx \right)^{q/p} \times \\ &\quad \times \left( \int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{s \cdot \frac{q}{(p-q)}} w \, dx \right)^{(p-q)/p} \\ &\leq C \left( [\mathcal{R}(\mathbf{f})^{-s \cdot \frac{p}{q}} w]_{\frac{p}{p-1}} \right) \left( \int_{\mathbb{R}^n} \mathcal{R}(\mathbf{f})^{\frac{q}{p-1}} w \, dx \right) \\ &\leq C \left( [\mathcal{R}(\mathbf{f})^{-s \cdot \frac{p}{q}} w]_{\frac{p}{p-1}} \right) 2^{\frac{q}{p-1}} \|\mathbf{f}\|_{L_w^q}^q. \end{aligned}$$

Then applying (A.3), we obtain (1.9) in the case  $p-1 < q < p$ .

We now consider the case  $p < q < \infty$  and in this regard, we fix a  $w \in A_{\frac{q}{p-1}}$  and let  $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n) \cap L_w^q(\Omega, \mathbb{R}^n)$  be as in the theorem. For any  $h \in L_w^{(q/p)'}(\mathbb{R}^n)$ , define

$$\mathcal{R}'(h)(x) := \sum_{k=0}^{\infty} \frac{(\mathcal{M}')^{(k)}(|h|^{\frac{(q/p)'}{(q/(p-1))'}})(x)}{2^k \|\mathcal{M}'\|_{L_w^{(q/(p-1))'} \rightarrow L_w^{(q/(p-1))'}}^k},$$

where  $\mathcal{M}'(h) := \frac{\mathcal{M}(hw)}{w}$  and  $(q/p)' = \frac{q}{q-p}$ ,  $(q/(p-1))' = \frac{q}{q-p+1}$  denote the conjugate Hölder exponents. Then it is easy to observe that

$$(A.7) \quad |h|^{\frac{(q/p)'}{(q/(p-1))'}}(x) \leq \mathcal{R}'(h)(x), \text{ and } \|\mathcal{R}'(h)\|_{L_w^{(q/(p-1))'}} \leq 2 \|h\|_{L_w^{(q/p)'}}.$$

We now choose an  $h \in L_w^{(q/p)'}(\mathbb{R}^n)$  with  $\|h\|_{L_w^{(q/p)'}} = 1$  such that

$$(A.8) \quad \int_{\mathbb{R}^n} |\nabla u|^q w(x) \, dx = \|\nabla u\|_{L_w^{q/p}}^{q/p} = \left( \int_{\mathbb{R}^n} |\nabla u|^p h(x) w(x) \, dx \right)^{q/p}.$$

For this choice of  $h$ , define  $H := [\mathcal{R}'(h)]^{\frac{(q/(p-1))'}{(q/p)'}}$ . It is easy to see from (A.7) that  $0 \leq h \leq H$ . We now prove the following important estimate:

$$(A.9) \quad (Hw) \in A_{\frac{p}{p-1}} \quad \text{with} \quad [Hw]_{\frac{p}{p-1}} \leq C([w]_{\frac{q}{p-1}}).$$

Analogous to (A.1), we observe that  $\mathcal{M}'(\mathcal{R}'(h)) \leq C([w]_{\frac{q}{p-1}}) \mathcal{R}'(h)$ . Thus for any ball  $B$  containing  $x$ ,

$$(Hw)(x)^{1-p} \leq C([w]_{\frac{q}{p-1}}) \left( \int_B H^{\frac{(q/p)'}{(q/(p-1))'}} w(y) \, dy \right)^{\frac{(q/(p-1))'}{(q/p)'(1-p)}} w(x)^{\frac{1-p}{q-p+1}},$$

where we have used the fact that  $\left(\frac{(q/(p-1))'}{(q/p)'} - 1\right)(p-1) = \frac{1-p}{q-p+1}$ . With this we obtain the estimate

$$(A.10) \quad \begin{aligned} & \left( \int_B (Hw)^{1-p} dx \right)^{\frac{1}{p-1}} \\ & \leq C([w]_{\frac{q}{p-1}}) \left( \int_B H^{\frac{(q/p)'}{(q/(p-1))'}} w dy \right)^{-\frac{(q/(p-1))'}{(q/p)'}} \left( \int_B w^{\frac{1-p}{q-p+1}} dx \right)^{\frac{1}{p-1}} \end{aligned}$$

for all balls  $B \subset \mathbb{R}^n$ .

On the other hand, by Hölder's inequality, we obtain

$$(A.11) \quad \int_Q Hw dx \leq \left( \int_Q H^{\frac{(q/p)'}{(q/(p-1))'}} w dx \right)^{\frac{(q/(p-1))'}{(q/p)'}} \left( \int_Q w dx \right)^{1 - \frac{(q/(p-1))'}{(q/p)'}}.$$

Multiplying (A.10) by (A.11) and observing that  $1 - \frac{(q/(p-1))'}{(q/p)'}$  =  $\frac{1}{q-p+1}$ , we get

$$[Hw]_{\frac{p}{p-1}} \leq C([w]_{\frac{q}{p-1}}) \left( \int_Q w^{\frac{1-p}{q-p+1}} dx \right)^{\frac{1}{p-1}} \left( \int_Q w dx \right)^{\frac{1}{q-p+1}} \leq C([w]_{\frac{q}{p-1}}),$$

which completes the proof of (A.9).

Using our hypothesis on  $\mathbf{f}$  and Hölder's inequality we now obtain

$$(A.12) \quad \begin{aligned} & \int_{\mathbb{R}^n} |\nabla u|^p h w dx \leq \int_{\mathbb{R}^n} |\nabla u|^p H w dx \\ & \leq C \left( [Hw]_{\frac{p}{p-1}} \right) \int_{\mathbb{R}^n} |\mathbf{f}|^p H w dx \\ & \leq C \left( [Hw]_{\frac{p}{p-1}} \right) \left( \int_{\mathbb{R}^n} |\mathbf{f}|^q w dx \right)^{p/q} \left( \int_{\mathbb{R}^n} |H|^{(q/p)'} w dx \right)^{1/(q/p)'}. \end{aligned}$$

Concerning the last term on the right, we have

$$(A.13) \quad \begin{aligned} \int_{\mathbb{R}^n} |H|^{(q/p)'} w dx &= \int_{\mathbb{R}^n} \mathcal{R}'(h)^{(q/(p-1))'} w dx \\ &= \|\mathcal{R}'(h)\|_{L_w^{(q/(p-1))'}}^{(q/(p-1))'} \leq 2^{(q/(p-1))'} \|h\|_{L_w^{(q/p)'}}^{(q/p)'}, \end{aligned}$$

where the last inequality follows from (A.7).

Substituting (A.13) into (A.12) and recalling (A.8), we obtain the desired estimate when  $p < q < \infty$ .  $\square$

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