CHARACTERIZATIONS OF SIGNED MEASURES IN THE DUAL OF BV AND RELATED ISOMETRIC ISOMORPHISMS

NGUYEN CONG PHUC AND MONICA TORRES

Abstract. We characterize all (signed) measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, where $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ is defined as the space of all functions u in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that Du is a finite vector-valued measure. We also show that $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ and $BV(\mathbb{R}^n)^*$ are isometrically isomorphic, where $BV(\mathbb{R}^n)$ is defined as the space of all functions u in $L^1(\mathbb{R}^n)$ such that Du is a finite vector-valued measure. As a consequence of our characterizations, an old issue raised in Meyers-Ziemer [16] is resolved by constructing a locally integrable function f such that f belongs to $BV(\mathbb{R}^n)^*$ but $|f|$ does not. Moreover, we show that the measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ coincide with the measures in $\dot{W}^{1,1}(\mathbb{R}^n)^*$, the dual of the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^n)$, in the sense of isometric isomorphism. For a bounded open set Ω with Lipschitz boundary, we characterize the measures in the dual space $BV_0(\Omega)^*$. One of the goals of this paper is to make precise the definition of $BV_0(\Omega)$, which is the space of functions of bounded variation with zero trace on the boundary of Ω. We show that the measures in $BV_0(\Omega)^*$ coincide with the measures in $W_0^{1,1}(\Omega)^*$. Finally, the class of finite measures in $BV(\Omega)^*$ is also characterized.

1. Introduction

It is a challenging problem in geometric measure theory to give a full characterization of the dual of BV, the space of functions of bounded variation. Meyers and Ziemer characterized in [16] the positive measures in $\mathbb{R}^n$ that belong to the dual of $BV(\mathbb{R}^n)$. They defined $BV(\mathbb{R}^n)$ as the space of all functions in $L^1(\mathbb{R}^n)$ whose distributional gradient is a finite vector-measure in $\mathbb{R}^n$ with norm given by

$$\|u\|_{BV(\mathbb{R}^n)} = \|Du\|(\mathbb{R}^n).$$

They showed that the positive measure μ belongs to $BV(\mathbb{R}^n)^*$ if and only if μ satisfies the condition

$$\mu(B(x,r)) \leq Cr^{n-1}$$

for every open ball $B(x,r) \subset \mathbb{R}^n$ and $C = C(n)$. Besides the classical paper by Meyers and Ziemer, we refer the interested reader to the paper by De Pauw [9], where the author analyzes SBV^*, the dual of the space of special functions of bounded variation.

In Phuc-Torres [17] we showed that there is a connection between the problem of characterizing BV^* and the study of the solvability of the equation $div\,\boldsymbol{F} = T$. Indeed, we showed that the (signed) measure μ belongs to $BV(\mathbb{R}^n)^*$ if and only if there exists a bounded vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $div\,\boldsymbol{F} = \mu$. Also, we showed that μ belongs to $BV(\mathbb{R}^n)^*$ if and only if

$$|\mu(U)| \leq C\,\mathcal{H}^{n-1}(\partial U) \tag{1.1}$$

for any open (or closed) set $U \subset \mathbb{R}^n$ with smooth boundary. The solvability of the equation $div\,\boldsymbol{F} = T$, in various spaces of functions, has been studied in Bourgain-Brezis [5], De Pauw-Pfeffer [10], De Pauw-Torres [11] and Phuc-Torres [17] (see also Tadmor [19]).

In De Pauw-Torres [11], another BV-type space was considered, the space $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, defined as the space of all functions $u \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that Du, the distributional gradient of u, is a finite vector-measure in $\mathbb{R}^n$. A closed subspace of $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, which is a Banach space denoted as CH_0,

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was characterized in [11] and it was proven that $T \in CH_0$ if and only if $T = div\,\boldsymbol{F}$, for a continuous vector field $\boldsymbol{F} \in C(\mathbb{R}^n, \mathbb{R}^n)$ vanishing at infinity.

In this paper we continue the analysis of $BV(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. We show that $BV(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ are isometrically isomorphic (see Corollary 3.3). We also show that the measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ coincide with the measures in $\dot{W}^{1,1}(\mathbb{R}^n)^*$, the dual of the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^n)$ (see Theorem 4.7), in the sense of isometric isomorphism. We remark that the space $\dot{W}^{1,1}(\mathbb{R}^n)^*$ is denoted as the G space in image processing (see Meyer [15]), and that it plays a key role in modeling the structured component of an image.

It is obvious that if μ is a locally finite signed Radon measure then $\|\mu\| \in BV(\mathbb{R}^n)^*$ implies that $\mu \in BV(\mathbb{R}^n)^*$. The converse was unknown to Meyers and Ziemer as they raised this issue in their classical paper [16, page 1356]. In Section 5, we show that the converse does not hold true in general by constructing a locally integrable function f such that $f \in BV(\mathbb{R}^n)^*$ but $|f| \notin BV(\mathbb{R}^n)^*$.

In this paper we also study these characterizations in bounded domains. Given a bounded open set Ω with Lipschitz boundary, we consider the space $BV_0(\Omega)$ defined as the space of functions of bounded variation with zero trace on $\partial\Omega$. One of the goals of this paper is to make precise the definition of this space (see Theorem 6.10) . We then characterize all (signed) measures in Ω that belong to $BV_0(\Omega)^*$. We show that a locally finite signed measure μ belongs to $BV_0(\Omega)^*$ if and only if (1.1) holds for any smooth open (or closed) set $U \subset\subset \Omega$, and if and only if $\mu = div\,\boldsymbol{F}$ for a vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ (see Theorem 7.4). Moreover, we show that the measures in $BV_0(\Omega)^*$ coincide with the measures in $W_0^{1,1}(\Omega)^*$ (see Theorem 7.6), in the sense of isometric isomorphism.

In the case of $BV(\Omega)$, the space of functions of bounded variation in a bounded open set Ω with Lipschitz boundary (but without the condition of having zero trace on $\partial\Omega$), we shall restrict our attention only to measures in $BV(\Omega)^*$ with bounded total variation in Ω, i.e., finite measures. This is in a sense natural since any *positive* measure that belongs to $BV(\Omega)^*$ must be finite due to the fact that the function $1 \in BV(\Omega)$. We show that a finite measure μ belongs to $BV(\Omega)^*$ if and only if (1.1) holds for every smooth open set $U \subset\subset \mathbb{R}^n$, where μ is extended by zero to $\mathbb{R}^n \setminus \Omega$ (see Theorem 8.2).

2. Functions of bounded variation

In this section we define all the spaces that will be relevant in this paper.

2.1. **Definition.** *Let Ω be any open set. The space $\mathcal{M}(\Omega)$ consists of all finite (signed) Radon measures μ in Ω; that is, the total variation of μ, denoted as $\|\mu\|$, satisfies $\|\mu\|(\Omega) < \infty$. The space $\mathcal{M}_{loc}(\Omega)$ consists of all locally finite Radon measures μ in Ω; that is, $\|\mu\|(K) < \infty$ for every compact set $K \subset \Omega$.*

Note here that $\mathcal{M}_{loc}(\Omega)$ is identified with the dual of the locally convex space $C_c(\Omega)$ (the space of continuous real-valued functions with compact support in Ω) (see [7]), and thus it is a real vector space. For $\mu \in \mathcal{M}_{loc}(\Omega)$, it is not required that either the positive part or the negative part of μ has finite total variation in Ω.

2.2. **Definition.** *Let Ω be any open set. The space of functions of bounded variation, denoted as $BV(\Omega)$, is defined as the space of all functions $u \in L^1(\Omega)$ such that the distributional gradient Du is a finite vector-valued measure in Ω. The space $BV(\Omega)$ is a Banach space with the norm*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega), \tag{2.1}$$

where $\|Du\|(\Omega)$ denotes the total variation of the vector-valued measure Du over Ω. For the case when $\Omega = \mathbb{R}^n$ we will equip $BV(\mathbb{R}^n)$ with the homogeneous norm given by

$$\|u\|_{BV(\mathbb{R}^n)} = \|Du\|(\mathbb{R}^n). \tag{2.2}$$

Another BV-like space is $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, defined as the space of all functions in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that Du is a finite vector-valued measure. The space $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ is a Banach space when equipped with *the norm*

$$\|u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = \|Du\|(\mathbb{R}^n).$$

2.3. **Remark.** *$BV(\mathbb{R}^n)$ is not a Banach space under the norm* (2.2)*. Also, we have*

$$\|Du\|(\Omega) = \sup\left\{\int_\Omega u \,div\, \varphi dx : \varphi \in C_c^1(\Omega) \text{ and } |\varphi(x)| \le 1\, \forall x \in \Omega\right\},$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)$ and $|\varphi(x)| = (\varphi_1(x)^2 + \varphi_2(x)^2 + \cdots + \varphi_n(x)^2)^{1/2}$. In what follows, we shall also write $\int_\Omega |Du|$ instead of $\|Du\|(\Omega)$.

We will use the following Sobolev's inequality for functions in $BV(\mathbb{R}^n)$ whose proof can be found in [3, Theorem 3.47]:

2.4. **Theorem.** *Let $u \in BV(\mathbb{R}^n)$. Then*

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C(n)\|Du\|(\mathbb{R}^n). \tag{2.3}$$

Inequality (2.3) immediately implies the following continuous embedding

$$BV(\mathbb{R}^n) \hookrightarrow BV_{\frac{n}{n-1}}(\mathbb{R}^n). \tag{2.4}$$

We recall that the standard Sobolev space $W^{1,1}(\Omega)$ is defined as the space of all functions $u \in L^1(\Omega)$ such that $Du \in L^1(\Omega)$. The Sobolev space $W^{1,1}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,1}(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|_{L^1(\Omega)} = \int_\Omega \left[|u| + (|D_1 u|^2 + |D_2 u|^2 + \cdots + |D_n u|^2)^{\frac{1}{2}}\right] dx. \tag{2.5}$$

However, we will often refer to the following homogeneous Sobolev space. Hereafter, we let $C_c^\infty(\Omega)$ denote the space of smooth functions with compact support in a general open set Ω.

2.5. **Definition.** *Let $\dot{W}^{1,1}(\mathbb{R}^n)$ denote the space of all functions $u \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ such that $Du \in L^1(\mathbb{R}^n)$. Equivalently, the space $\dot{W}^{1,1}(\mathbb{R}^n)$ can also be defined as the closure of $C_c^\infty(\mathbb{R}^n)$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ (i.e., in the norm $\|Du\|_{L^1(\mathbb{R}^n)}$). Thus, $u \in \dot{W}^{1,1}(\mathbb{R}^n)$ if and only if there exists a sequence $u_k \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |D(u_k - u)|dx = 0$, and moreover,*

$$\dot{W}^{1,1}(\mathbb{R}^n) \hookrightarrow BV_{\frac{n}{n-1}}(\mathbb{R}^n).$$

2.6. **Definition.** *Given a bounded open set Ω, we say that the boundary $\partial\Omega$ is Lipschitz if for each $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz mapping $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes if necessary, we have*

$$\Omega \cap B(x, r) = \{y = (y_1, \ldots, y_{n-1}, y_n) : h(y_1, \ldots, y_{n-1}) < y_n\} \cap B(x, r).$$

2.7. **Remark.** *Let Ω be a bounded open set with Lipschitz boundary. We denote by $W_0^{1,1}(\Omega)$ the Sobolev space consisting of all functions in $W^{1,1}(\Omega)$ with zero trace on $\partial\Omega$. Then it is well-known that $C_c^\infty(\Omega)$ is dense in $W_0^{1,1}(\Omega)$. One of the goals of this paper is to make precise the definition of $BV_0(\Omega)$, the space of all functions in $BV(\Omega)$ with zero trace on $\partial\Omega$ (see Theorem 6.10). In this paper we equip the two spaces, $BV_0(\Omega)$ and $W_0^{1,1}(\Omega)$, with the equivalent norms (see Theorem 6.11) to* (2.1) *and* (2.5)*, respectively, given by*

$$\|u\|_{BV_0(\Omega)} = \|Du\|(\Omega), \quad \text{and} \quad \|u\|_{W_0^{1,1}(\Omega)} = \int_\Omega |Du| dx.$$

2.8. **Definition.** *For any open set Ω, we let $BV_c(\Omega)$ denote the space of functions in $BV(\Omega)$ with compact support in Ω. Also, $BV^\infty(\Omega)$ and $BV_0^\infty(\Omega)$ denote the space of bounded functions in $BV(\Omega)$ and $BV_0(\Omega)$, respectively. Finally, $BV_c^\infty(\Omega)$ is the space of all bounded functions in $BV(\Omega)$ with compact support in Ω.*

We will use the following result (see [13, Proposition 1.13]). We include the proof here for the sake of completeness.

2.9. **Lemma.** *Suppose* $\{u_k\}$ *is a sequence in* $BV(\Omega)$ *such that* $u_k \to u$ *in* $L^1_{loc}(\Omega)$ *and*

$$\lim_{k\to\infty}\int_\Omega |Du_k| = \int_\Omega |Du|. \tag{2.6}$$

Then for every open set $A \subset \Omega$,

$$\int_{\overline{A}\cap\Omega} |Du| \geq \limsup_{k\to\infty}\int_{\overline{A}\cap\Omega} |Du_k|.$$

In particular, if $\int_{\partial A\cap\Omega} |Du| = 0$, *then*

$$\int_A |Du| = \lim_{k\to\infty}\int_A |Du_k|. \tag{2.7}$$

Proof. Consider the open set $B = \Omega \setminus \overline{A}$. Since $u_k \to u$ in $L^1_{loc}(\Omega)$, by the lower semicontinuity property we have

$$\int_A |Du| \leq \liminf_{k\to\infty}\int_A |Du_k|, \text{ and } \int_B |Du| \leq \liminf_{k\to\infty}\int_B |Du_k|. \tag{2.8}$$

On the other hand,

$$\begin{aligned}
\int_{\overline{A}\cap\Omega} |Du| + \int_B |Du| &= \int_\Omega |Du| \\
&= \lim_{k\to\infty}\int_\Omega |Du_k| = \lim_{k\to\infty}\left(\int_{\overline{A}\cap\Omega} |Du_k| + \int_B |Du_k|\right), \text{ by (2.6)} \\
&= \limsup_{k\to\infty}\left(\int_{\overline{A}\cap\Omega} |Du_k| + \int_B |Du_k|\right) \\
&\geq \limsup_{k\to\infty}\int_{\overline{A}\cap\Omega} |Du_k| + \liminf_{k\to\infty}\int_B |Du_k| \\
&\geq \limsup_{k\to\infty}\int_{\overline{A}\cap\Omega} |Du_k| + \int_B |Du|, \text{ by (2.8)},
\end{aligned}$$

and hence

$$\int_{\overline{A}\cap\Omega} |Du| \geq \limsup_{k\to\infty}\int_{\overline{A}\cap\Omega} |Du_k|.$$

In particular, if $\int_{\partial A\cap\Omega} |Du| = 0$ then we obtain from the last inequality

$$\int_A |Du| = \int_{\overline{A}\cap\Omega} |Du| \geq \limsup_{k\to\infty}\int_{\overline{A}\cap\Omega} |Du_k| \geq \limsup_{k\to\infty}\int_A |Du_k| \geq \liminf_{k\to\infty}\int_A |Du_k|,$$

and since, by (2.8),

$$\int_A |Du| \leq \liminf_{k\to\infty}\int_A |Du_k| \leq \limsup_{k\to\infty}\int_A |Du_k|,$$

clearly (2.7) follows. □

The following theorem from functional analysis (see [18, Theorem 1.7]) will be fundamental in this paper:

2.10. **Theorem.** *Let* X *be a normed linear space and* Y *be a Banach space. Suppose* $T : D \to Y$ *is a bounded linear transformation, where* $D \subset X$ *is a dense linear subspace. Then* T *can be uniquely extended to a bounded linear transformation* $\hat{T}$ *from* X *to* Y. *In addition, the operator norm of* T *is* c *if and only if the norm of* $\hat{T}$ *is* c.

The following formula will be important in this paper.

2.11. **Lemma.** *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n)$ and f be a function such that $\int_{\mathbb{R}^n} |f| d\,\|\mu\| < +\infty$. Then*

$$\int_{\mathbb{R}^n} f d\mu = \int_0^\infty \mu(\{f \geq t\}) dt - \int_{-\infty}^0 \mu(\{f \leq t\}) dt.$$

The same equality also holds if we replace the sets $\{f \geq t\}$ and $\{f \leq t\}$ by $\{f > t\}$ and $\{f < t\}$, respectively.

Proof. We write $f = f^+ - f^-$, where $f^+ \geq 0$ and $f^- \geq 0$ are the positive and negative parts of f. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} f d\mu &= \int_{\mathbb{R}^n} (f^+ - f^-) d\mu \\
&= \int_0^\infty \mu(\{f^+ \geq t\}) dt - \int_0^\infty \mu(\{f^- \geq t\}) dt \\
&= \int_0^\infty \mu(\{f \geq t\}) dt - \int_0^\infty \mu(\{-f \geq t\}) dt \\
&= \int_0^\infty \mu(\{f \geq t\}) dt - \int_0^\infty \mu(\{f \leq -t\}) dt \\
&= \int_0^\infty \mu(\{f \geq t\}) dt - \int_{-\infty}^0 \mu(\{f \leq s\}) ds, \text{ by making the change of variables } t = -s,
\end{aligned}$$

which is the desired result. □

3. $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$

3.1. **Theorem.** *Let $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, $u \geq 0$, and $\phi_k \in C_c^\infty(\mathbb{R}^n)$ be a nondecreasing sequence of smooth functions satisfying:*

$$0 \leq \phi_k \leq 1,\ \phi_k \equiv 1 \text{ on } B_k(0),\ \phi_k \equiv 0 \text{ on } \mathbb{R}^n \setminus B_{2k}(0) \text{ and } |D\phi_k| \leq c/k. \tag{3.1}$$

Then

$$\lim_{k\to\infty} \|(\phi_k u) - u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = 0, \tag{3.2}$$

and for each fixed $k > 0$ we have

$$\lim_{j\to\infty} \|(\phi_k u) \wedge j - \phi_k u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} = 0. \tag{3.3}$$

In particular, $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$.

Proof. As $BV_{\frac{n}{n-1}}(\mathbb{R}^n) \subset BV_{loc}(\mathbb{R}^n)$, the product rule for BV_{loc} functions gives that $D(\phi_k u) = \phi_k Du + u D\phi_k$ (as measures) (see [3, Proposition 3.1]) and hence $\phi_k u \in BV(\mathbb{R}^n) \subset BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} |D(u\phi_k - u)| = \int_{\mathbb{R}^n} |\phi_k Du - Du + u D\phi_k| \\
&\leq \int_{\mathbb{R}^n} |\phi_k - 1||Du| + \int_{\mathbb{R}^n \cap \operatorname{supp}(D\phi_k)} |u||D\phi_k| \\
&\leq \int_{\mathbb{R}^n} |\phi_k - 1||Du| + \frac{c}{k} \int_{B_{2k}\setminus B_k} |u| \\
&\leq \int_{\mathbb{R}^n} |\phi_k - 1||Du| + \frac{c}{k} \left(\int_{B_{2k}\setminus B_k} |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} |B_{2k} \setminus B_k|^{\frac{1}{n}}
\end{aligned}$$

$$\leq \int_{\mathbb{R}^n} |\phi_k - 1||Du| + c \left(\int_{B_{2k}\setminus B_k} |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}. \tag{3.4}$$

We let $k \to \infty$ in (3.4) and use (3.1) and the dominated convergence theorem together with the fact that $u \in L^{\frac{n}{n-1}}$ to obtain (3.2).

On the other hand, the coarea formula for BV functions yields

$$\begin{aligned}\int_{\mathbb{R}^n} |D(\phi_k u - (\phi_k u) \wedge j)| &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{\phi_k u - (\phi_k u) \wedge j > t\}) dt \\ &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{\phi_k u - j > t\}) dt \\ &= \int_0^\infty \mathcal{H}^{n-1}(\partial^* \{\phi_k u > j + t\}) dt \\ &= \int_j^\infty \mathcal{H}^{n-1}(\partial^* \{\phi_k u > s\}) ds.\end{aligned}$$

Here $\partial^* E$ stands for the reduced boundary of a set E. Since $\int_0^\infty \mathcal{H}^{n-1}(\partial^* \{\phi_k u > s\}) ds < \infty$, the Lebesgue dominated convergence theorem yields the limit (3.3) for each fixed $k > 0$.

By the triangle inequality and (3.2)-(3.3), each nonnegative $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ can be approximated by a function in $BV_c^\infty(\mathbb{R}^n)$. For a general $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, let u^+ be the positive part of u. From the proof of [3, Theorem 3.96], we have $u^+ \in BV_{loc}(\mathbb{R}^n)$ and $\|Du^+\|(A) \leq \|Du\|(A)$ for any open set $A \Subset \mathbb{R}^n$. Thus $\|Du^+\|(\mathbb{R}^n) \leq \|Du\|(\mathbb{R}^n) < +\infty$ and u^+ belongs to $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. Likewise, we have $u^- \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. Now by considering separately the positive and negative parts of a function $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, it is then easy to see the density of $BV_c^\infty(\mathbb{R}^n)$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. □

We have the following corollaries of Theorem 3.1:

3.2. **Corollary.** *$BV_c^\infty(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$.*

Proof. This follow immediately from (2.4) and Theorem 3.1. □

3.3. **Corollary.** *The spaces $BV(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ are isometrically isomorphic.*

Proof. We define the map

$$S : BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* \to BV(\mathbb{R}^n)^*$$

as

$$S(T) = T \llcorner BV(\mathbb{R}^n).$$

First, we note the S is injective since $S(T) = 0$ implies that $T \llcorner BV(\mathbb{R}^n) \equiv 0$. In particular, $T \llcorner BV_c^\infty(\mathbb{R}^n) \equiv 0$. Since $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ and T is continuous on $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, it is easy to see that $T \llcorner BV_{\frac{n}{n-1}}(\mathbb{R}^n) \equiv 0$. We now proceed to show that S is surjective. Let $T \in BV(\mathbb{R}^n)^*$. Then $T \llcorner BV_c^\infty(\mathbb{R}^n)$ is a continuous linear functional. Using again that $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$, $T \llcorner BV_c^\infty(\mathbb{R}^n)$ has a unique continuous extension $\hat{T} \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ and clearly $S(\hat{T}) = T$. Moreover, for any $T \in BV(\mathbb{R}^n)^*$, the unique extension $\hat{T}$ to $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ has the same norm (see Theorem 2.10), that is,

$$\|T\|_{BV(\mathbb{R}^n)^*} = \left\|\hat{T}\right\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*},$$

and hence

$$\left\|S(\hat{T})\right\|_{BV(\mathbb{R}^n)^*} = \left\|\hat{T}\right\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*},$$

which implies that S is an isometry. □

We now proceed to make precise our definitions of measures in $\dot{W}^{1,1}(\mathbb{R}^n)^*$ and $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$.

3.4. **Definition.** *We let*

$$\mathcal{M}_{loc} \cap \dot{W}^{1,1}(\mathbb{R}^n)^* := \{T \in \dot{W}^{1,1}(\mathbb{R}^n)^* : T(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\mathbb{R}^n), \forall \varphi \in C_c^\infty(\mathbb{R}^n)\}.$$

Therefore, if $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n) \cap \dot{W}^{1,1}(\mathbb{R}^n)^*$, then the action $<\mu, u>$ can be uniquely defined for all $u \in \dot{W}^{1,1}(\mathbb{R}^n)$ (because of the density of $C_c^\infty(\mathbb{R}^n)$ in $\dot{W}^{1,1}(\mathbb{R}^n)$).

3.5. **Definition.** *We let*

$$\mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* := \{T \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^* : T(\varphi) = \int_{\mathbb{R}^n} \varphi^* d\mu \text{ for some } \mu \in \mathcal{M}_{loc}, \forall \varphi \in BV_c^\infty(\mathbb{R}^n)\},$$

where φ^* is the precise representative of φ in $BV_c^\infty(\mathbb{R}^n)$. Thus, if $\mu \in \mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$, then the action $<\mu, u>$ can be uniquely defined for all $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ (because of the density of $BV_c^\infty(\mathbb{R}^n)$ in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$).

We will study the normed linear spaces $\mathcal{M}_{loc} \cap \dot{W}^{1,1}(\mathbb{R}^n)^*$ and $\mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ in the next section. In particular, we will show in Theorem 4.7 below that these spaces are isometrically isomorphic. In Definition 3.5, if we use $C_c^\infty(\mathbb{R}^n)$ instead of $BV_c^\infty(\mathbb{R}^n)$, then by the Hahn-Banach Theorem there exist a non-zero $T \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ that is represented by the zero measure, which would cause a problem of injectivity in Theorem 4.7.

4. Characterizations of measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$

The following lemma characterizes all the distributions in $\dot{W}^{1,1}(\mathbb{R}^n)^*$. We recall that $\dot{W}^{1,1}(\mathbb{R}^n)$ is the homogeneous Sobolev space introduced in Definition 2.5.

4.1. **Lemma.** *The distribution T belongs to $\dot{W}^{1,1}(\mathbb{R}^n)^*$ if and only if $T = \operatorname{div} \boldsymbol{F}$ for some vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Moreover,*

$$\|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} = \min\{\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\},$$

where the minimum is taken over all $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = T$. Here we use the norm

$$\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} := \left\|(F_1^2 + F_2^2 + \cdots + F_n^2)^{1/2}\right\|_{L^\infty(\mathbb{R}^n)} \text{ for } \boldsymbol{F} = (F_1, \ldots, F_n).$$

Proof. It is easy to see that if $T = div\, \boldsymbol{F}$ where $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ then $T \in \dot{W}^{1,1}(\mathbb{R}^n)^*$ with

$$\|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} \leq \|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}.$$

Conversely, let $T \in \dot{W}^{1,1}(\mathbb{R}^n)^*$. Define

$$A : \dot{W}^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n, \mathbb{R}^n), \quad A(u) = Du,$$

and note that the range of A is a closed subspace of $L^1(\mathbb{R}^n, \mathbb{R}^n)$ since $\dot{W}^{1,1}(\mathbb{R}^n)$ is complete. We denote the range of A by $R(A)$ and we define

$$T_1 : R(A) \to \mathbb{R}$$

as

$$T_1(Du) = T(u), \text{ for each } Du \in R(A).$$

Then we have

$$\|T_1\|_{R(A)^*} = \|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*}.$$

By Hahn-Banach Theorem there exists a norm-preserving extension T_2 of T_1 to all $L^1(\mathbb{R}^n, \mathbb{R}^n)$. On the other hand, by the Riesz Representation Theorem for vector valued functions (see [8, pp. 98–100]) there exists a vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$T_2(v) = \int_{\mathbb{R}^n} \boldsymbol{F} \cdot v, \text{ for every } v \in L^1(\mathbb{R}^n, \mathbb{R}^n),$$

and

$$\|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} = \|T_2\|_{L^1(\mathbb{R}^n,\mathbb{R}^n)^*} = \|T_1\|_{R(A)^*} = \|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*}.$$

In particular, for each $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$T(\varphi) = T_1(D\varphi) = T_2(D\varphi) = \int_{\mathbb{R}^n} \boldsymbol{F} \cdot D\varphi,$$

which yields

$$T = div(-\boldsymbol{F}),$$

with

$$\|-\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} = \|T\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*}.$$

□

4.2. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be any open set and suppose $\mu \in \mathcal{M}_{loc}(\Omega)$ such that*

$$|\mu(U)| \leq C\,\mathcal{H}^{n-1}(\partial U) \tag{4.1}$$

for any smooth open and bounded set $U \subset\subset \Omega$. Let A be a compact set of Ω. If $\mathcal{H}^{n-1}(A) = 0$, then $\mu(A) = 0$.

Proof. As $\mathcal{H}^{n-1}(A) = 0$, for any $0 < \varepsilon < \frac{1}{2}\text{dist}(A, \partial\Omega)$ (or for any $\varepsilon > 0$, if $\Omega = \mathbb{R}^n$), we can find a finite number of balls $B(x_i, r_i)$, $i \in I$, with $2r_i < \varepsilon$ such that $A \subset \bigcup_{i\in I} B(x_i, r_i) \subset \Omega$ and

$$\sum_{i\in I} r_i^{n-1} < \varepsilon. \tag{4.2}$$

Let $W_\varepsilon = \bigcup_{i\in I} B(x_i, r_i)$. Then

$$A \subset\subset W_\varepsilon \subset A_\varepsilon :\, = \{x \in \mathbb{R}^n : \text{dist}(x, A) < \varepsilon\}.$$

The first inclusion follows since A is compact and W_ε is open; the second one follows since $2r_i < \varepsilon$ and since we may assume that $B(x_i, r_i) \cap A \neq \emptyset$ for any $i \in I$.

We now claim that for each $\varepsilon > 0$ there exists an open set W'_ε such that W'_ε has smooth boundary and

$$\begin{cases} A \subset\subset W'_\varepsilon \subset A_{2\varepsilon} \\ \mathcal{H}^{n-1}(\partial W'_\varepsilon) \leq P(W_\varepsilon, \Omega), \end{cases} \tag{4.3}$$

where $P(E, \Omega)$ denotes the perimeter of a set E in Ω. Assume for now that (4.3) holds. Then, since A is compact,

$$\chi_{W'_\varepsilon} \to \chi_A \text{ pointwise as } \varepsilon \to 0,$$

and

$$\begin{aligned} |\mu(W'_\varepsilon)| &\leq C\mathcal{H}^{n-1}(\partial W'_\varepsilon), \text{ by our hypothesis (4.1)} \\ &\leq CP(W_\varepsilon, \Omega) \\ &\leq C\sum_{i\in I} r_i^{n-1} \leq \varepsilon C, \text{ by (4.2)}. \end{aligned}$$

Thus, the Lebesgue dominated convergence theorem yields, after letting $\varepsilon \to 0$, the desired result:

$$|\mu(A)| = 0.$$

We now proceed to prove (4.3). Let ρ be a standard symmetric mollifier:

$$\rho \geq 0,\ \rho \in C_0^\infty(B(0,1)),\ \int_{\mathbb{R}^n} \rho(x)dx = 1, \text{ and } \rho(x) = \rho(-x).$$

Define $\rho_{1/k}(x) = k^n\rho(kx)$ and

$$u_k(x) = \chi_{W_\varepsilon} * \rho_{1/k}(x) = k^n \int \rho(k(x-y))\chi_{W_\varepsilon}(y)dy$$

for $k = 1, 2, \ldots$ For k large enough, say for $k \geq k_0 = k_0(\epsilon)$, it follows that

$$u_k \equiv 1 \text{ on } A, \text{ since } A \subset\subset W_\varepsilon, \tag{4.4}$$

$$u_k \equiv 0 \text{ on } \Omega \backslash A_{2\varepsilon}, \text{ since } W_\varepsilon \subset A_\varepsilon. \tag{4.5}$$

We have

$$\begin{aligned} P(W_\varepsilon, \Omega) &= |D\chi_{W_\varepsilon}|(\Omega) \\ &\geq |Du_k|(\Omega) \\ &= \int_0^1 P(F_t^k, \Omega)dt, \text{ since } 0 \leq u_k \leq 1, \end{aligned}$$

where

$$F_t^k := \{x \in \Omega \colon u_k(x) > t\}.$$

Note that for $k \geq k_0$, and $t \in (0,1)$ we have, by (4.4) and (4.5),

$$A \subset\subset F_t^k \subset A_{2\varepsilon}.$$

For a.e. $t \in (0,1)$ the sets F_t^k have smooth boundaries. Thus we can choose $t_0 \in (0,1)$ with this property and such that

$$P(F_{t_0}^k, \Omega) \leq P(W_\varepsilon, \Omega),$$

which is

$$\mathcal{H}^{n-1}(\partial F_{t_0}^k) \leq P(W_\varepsilon, \Omega).$$

Finally, we choose $W'_\varepsilon = F_{t_0}^k$ for any fixed $k \geq k_0$. □

4.3. **Corollary.** *If $\mu \in \mathcal{M}_{loc}(\Omega)$ satisfies the hypothesis of Theorem 4.2, then $\|\mu\| << \mathcal{H}^{n-1}$ in Ω; that is, if $A \subset \Omega$ is any Borel measurable set such that $\mathcal{H}^{n-1}(A) = 0$ then $\|\mu\|(A) = 0$.*

Proof. The domain Ω can be decomposed as $\Omega = \Omega^+ \cup \Omega^-$, such that $\mu^+ = \mu \llcorner \Omega^+$ and $\mu^- = \mu \llcorner \Omega^-$, where μ^+ and μ^- are the positive and negative parts of μ, respectively. Let $A \subset \Omega$ be a Borel set satisfying $\mathcal{H}^{n-1}(A) = 0$. By writing $A = (A \cap \Omega^+) \cup (A \cap \Omega^-)$, we may assume that $A \subset \Omega^+$ and hence $\|\mu\|(A) = \mu^+(A)$. Moreover, since μ^+ is a Radon measure we can assume that A is compact. Hence, Theorem 4.2 yields $\|\mu\|(A) = \mu^+(A) = \mu(A) = 0$. □

The following theorem characterizes all the signed measures in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. This result was first proven in Phuc-Torres [17] for the space $BV(\mathbb{R}^n)^*$ with no sharp control on the involving constants. In this paper we offer a new and direct proof of (**i**) ⇒ (**ii**). We also clarify the first part of (**iii**). Moreover, our proof of (**ii**) ⇒ (**iii**) yields a sharp constant that will be needed for the proof of Theorem 4.7 below.

4.4. **Theorem.** *Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n)$ be a locally finite signed measure. The following are equivalent:*

(i) *There exists a vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = \mu$ in the sense of distributions.*

(ii) *There is a constant C such that*

$$|\mu(U)| \leq C\, \mathcal{H}^{n-1}(\partial U)$$

for any smooth bounded open (or closed) set U with $\mathcal{H}^{n-1}(\partial U) < +\infty$.

(iii) *$\mathcal{H}^{n-1}(A) = 0$ implies $\|\mu\|(A) = 0$ for all Borel sets A and there is a constant C such that, for all $u \in BV_c^\infty(\mathbb{R}^n)$,*

$$| < \mu, u > | := \left| \int_{\mathbb{R}^n} u^* d\mu \right| \leq C \int_{\mathbb{R}^n} |Du|,$$

where u^* is the representative in the class of u that is defined $\mathcal{H}^{n-1}$-almost everywhere.

(iv) *$\mu \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. The action of μ on any $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ is defined (uniquely) as*

$$< \mu, u > := \lim_{k\to\infty} < \mu, u_k > = \lim_{k\to\infty} \int_{\mathbb{R}^n} u_k^* d\mu,$$

where $u_k \in BV_c^\infty(\mathbb{R}^n)$ *converges to* u *in* $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$. *In particular, if* $u \in BV_c^\infty(\mathbb{R}^n)$ *then*

$$< \mu, u >= \int_{\mathbb{R}^n} u^* d\mu,$$

and moreover, if μ *is a non-negative measure then, for all* $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$,

$$< \mu, u >= \int_{\mathbb{R}^n} u^* d\mu.$$

Proof. Suppose **(i)** holds. Then for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \boldsymbol{F} \cdot D\varphi dx = -\int_{\mathbb{R}^n} \varphi d\mu. \tag{4.6}$$

Let $U \subset\subset \mathbb{R}^n$ be any open set (or closed set) with smooth boundary satisfying $\mathcal{H}^{n-1}(\partial U) < \infty$. Consider the characteristic function χ_U and a sequence of mollifications

$$u_k := \chi_U * \rho_{1/k},$$

where $\{\rho_{1/k}\}$ is as in the proof of Theorem 4.2. Then, since U has a smooth boundary, we have

$$u_k(x) \to \chi_U^*(x) \text{ pointwise everywhere,} \tag{4.7}$$

where $\chi_U^*(x)$ is the precise representative of χ_U (see [3, Corollary 3.80] given by

$$\chi_U^*(x) = \begin{cases} 1 & x \in \text{Int}(U), \\ \frac{1}{2} & x \in \partial U, \\ 0 & x \in \mathbb{R}^n \setminus \overline{U}. \end{cases}$$

We note that χ_U^* is the same for U open or closed, since both are the same set of finite perimeter (they differ only on ∂U, which is a set of Lebesgue measure zero). From (4.6), (4.7), and the dominated convergence theorem we obtain

$$\begin{aligned} \left|\mu(\text{Int}(U)) + \frac{1}{2}\mu(\partial U)\right| &= \left|\int_{\mathbb{R}^n} \chi_U^* d\mu\right| = \lim_{k\to\infty} \left|\int_{\mathbb{R}^n} u_k d\mu\right| \\ &= \lim_{k\to\infty} \left|\int_{\mathbb{R}^n} \boldsymbol{F} \cdot Du_k dx\right| \\ &\leq \lim_{k\to\infty} \|\boldsymbol{F}\|_\infty \int_{\mathbb{R}^n} |Du_k| dx \\ &= \|\boldsymbol{F}\|_\infty \int_{\mathbb{R}^n} |D\chi_U| = \|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial U). \end{aligned} \tag{4.8}$$

We now let

$$K := \overline{U}.$$

For each $h > 0$ we define the function

$$F_h(x) = 1 - \frac{\min\{d_K(x), h\}}{h}, \qquad x \in \mathbb{R}^n,$$

where $d_K(x)$ denotes the distance from x to K, i.e., $d_K(x) = \inf\{|x - y| : y \in K\}$. Note that F_h is a Lipschitz function such that $F_h(x) \leq 1$, $F_h(x) = 1$ if $x \in K$ and $F_h(x) = 0$ if $d_K(x) \geq h$. Moreover, F_h is differentiable $\mathcal{L}^n$-almost everywhere and

$$|DF_h(x)| \leq \frac{1}{h} \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n.$$

By standard smoothing techniques, (4.6) holds for the Lipschitz function F_h. Therefore,

$$\left|\int_{\mathbb{R}^n} F_h d\mu\right| = \left|\int_{\mathbb{R}^n} \boldsymbol{F} \cdot DF_h dx\right|. \tag{4.9}$$

Since $F_h \to \chi_K$ pointwise, it follows from the dominated convergence theorem that

$$|\mu(K)| = \left|\int_{\mathbb{R}^n} \chi_K d\mu\right| = \lim_{h\to 0}\left|\int_{\mathbb{R}^n} F_h d\mu\right|. \tag{4.10}$$

On the other hand, using the coarea formula for Lipschitz maps, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \boldsymbol{F}\cdot DF_h dx\right| &\leq \|\boldsymbol{F}\|_\infty \int_{\mathbb{R}^n} |DF_h| dx\\ &= \|\boldsymbol{F}\|_\infty \frac{1}{h}\int_{\{0<d_K<h\}} |Dd_K| dx\\ &= \|\boldsymbol{F}\|_\infty \frac{1}{h}\int_0^h \mathcal{H}^{n-1}(d_K^{-1}(t))dt\\ &= \|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(d_K^{-1}(t_e^h)),\end{aligned} \tag{4.11}$$

where $0 < t_e^h < h$, and $d_K^{-1}(t_e^h) \subset (\mathbb{R}^n \setminus K)$. Because K is smoothly bounded, it follows that

$$\mathcal{H}^{n-1}(d_K^{-1}(t_e^h)) \to \mathcal{H}^{n-1}(\partial K) \text{ as } h \to 0. \tag{4.12}$$

Since $K = \overline{U}$ and $\partial K = \partial U$, it follows from (4.9)-(4.12) that

$$|\mu(\overline{U})| \leq \|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial U). \tag{4.13}$$

From (4.8) and (4.13) we conclude that, for any open set (or closed) $U \subset\subset \mathbb{R}^n$ with smooth boundary and finite perimeter,

$$\frac{1}{2}|\mu(\partial U)| = \left|\mu(\overline{U}) - [\mu(\mathrm{Int}(U)) + \frac{1}{2}\mu(\partial U)]\right| \leq 2\|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial U),$$

and hence

$$|\mu(\mathrm{Int}(U))| \leq 3\|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial U).$$

This completes the proof of **(i)** $\Rightarrow$ **(ii)** with $C = \|\boldsymbol{F}\|_\infty$ for closed sets and $C = 3\|\boldsymbol{F}\|_\infty$ for open sets.

We proceed now to show that **(ii)** $\Rightarrow$ **(iii)**. Corollary 4.3 says that $\|\mu\| << \mathcal{H}^{n-1}$, which proves the first part of **(iii)**. We let $u \in BV_c^\infty(\mathbb{R}^n)$ and we consider the convolutions $\rho_\varepsilon * u$ and define

$$A_t^\varepsilon := \{\rho_\varepsilon * u \geq t\} \text{ for } t > 0, \text{ and } B_t^\varepsilon := \{\rho_\varepsilon * u \leq t\} \text{ for } t < 0.$$

Since $\rho_\varepsilon * u \in C_c^\infty(\mathbb{R}^n)$ it follows that ∂A_t^ε and ∂B_t^ε are smooth for a.e. t. Applying Lemma 2.11 we compute

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \rho_\varepsilon * u d\mu\right| &= \left|\int_0^\infty \mu(A_t^\varepsilon)dt - \int_{-\infty}^0 \mu(B_t^\varepsilon)dt\right|\\ &\leq \int_0^\infty |\mu(A_t^\varepsilon)|dt + \int_{-\infty}^0 |\mu(B_t^\varepsilon)|dt\\ &\leq C\int_0^\infty \mathcal{H}^{n-1}(\partial A_t^\varepsilon)dt + C\int_{-\infty}^0 \mathcal{H}^{n-1}(\partial B_t^\varepsilon)\,dt, \text{ by } \textbf{(ii)}\\ &= C\int_{\mathbb{R}^n} |D(\rho_\varepsilon * u)|\,dx, \text{ by the Coarea Formula}\\ &\leq C\int_{\mathbb{R}^n} |Du|.\end{aligned} \tag{4.14}$$

We let u^* denote the precise representative of u. We have that (see Ambrosio-Fusco-Pallara [3], Chapter 3, Corollary 3.80):

$$\rho_\varepsilon * u \to u^* \quad \mathcal{H}^{n-1}\text{-almost everywhere.} \tag{4.15}$$

We now let $\varepsilon \to 0$ in (4.14). Since u is bounded and $\|\mu\| << \mathcal{H}^{n-1}$, (4.15) and the dominated convergence theorem yield

$$\left|\int_{\mathbb{R}^n} u^* d\mu\right| \leq C \int_{\mathbb{R}^n} |Du|,$$

which completes the proof of **(ii)** $\Rightarrow$ **(iii)** with the same constant C as given in **(ii)**.

From **(iii)** we obtain that the linear operator

$$T(u) := < \mu, u > = \int_{\mathbb{R}^n} u^* d\mu, \quad u \in BV_c^\infty(\mathbb{R}^n) \tag{4.16}$$

is continuous and hence it can be uniquely extended, since $BV_c^\infty(\mathbb{R}^n)$ is dense in $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ (Lemma 3.1), to the space $BV_{\frac{n}{n-1}}(\mathbb{R}^n)$.

Assume now that μ is non-negative. We take $u \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ and consider the positive and negative parts $(u^*)^+$ and $(u^*)^-$ of the representative u^*. With ϕ_k as in Lemma 3.1, using (4.16) we have

$$T([\phi_k(u^*)^+] \wedge j) = \int_{\mathbb{R}^n} [\phi_k(u^*)^+] \wedge j \, d\mu, \quad j = 1, 2, \ldots$$

We first let $j \to \infty$ and then $k \to \infty$. Using Lemma 3.1, the continuity of T, and the monotone convergence theorem we find

$$T((u^*)^+) = \int_{\mathbb{R}^n} (u^*)^+ d\mu.$$

We proceed in the same way for $(u^*)^-$ and thus by linearity we conclude

$$T(u) = T((u^*)^+) - T((u^*)^-) = \int_{\mathbb{R}^n} (u^*)^+ - (u^*)^- d\mu = \int_{\mathbb{R}^n} u^* d\mu.$$

To prove that **(iv)** implies **(i)** we take $\mu \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$. Since $\dot{W}^{1,1}(\mathbb{R}^n) \subset BV_{\frac{n}{n-1}}(\mathbb{R}^n)$ then

$$\tilde{\mu} := \mu \llcorner \dot{W}^{1,1}(\mathbb{R}^n) \in \dot{W}^{1,1}(\mathbb{R}^n)^*,$$

and therefore Lemma 4.1 implies that there exists $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $div\, \boldsymbol{F} = \tilde{\mu}$ and thus, since $C_c^\infty \subset \dot{W}^{1,1}(\mathbb{R}^n)$, we conclude that $div\, \boldsymbol{F} = \mu$ in the sense of distributions. □

4.5. **Remark.** *Inequality* (4.13) *can also be obtained be means of the (one-sided)* outer Minskowski content. *Indeed, since* $|Dd_K| = 1$ *a.e., we find*

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \boldsymbol{F} \cdot DF_h dx\right| &\leq \|\boldsymbol{F}\|_\infty \int_{\mathbb{R}^n} |DF_h| dx \\ &= \|\boldsymbol{F}\|_\infty \frac{1}{h} |\{0 < d_K < h\}|. \end{aligned}$$

Now sending $h \to 0^+$ *and using* (4.9)-(4.10) *we have*

$$|\mu(K)| \leq \|\boldsymbol{F}\|_\infty \mathcal{SM}(K) = \|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial K),$$

where $\mathcal{SM}(K)$ *is the outer Minskowski content of* K *(see* [2, Definition 5]*), and the last equality follows from* [2, Corollary 1]. *This argument also holds in the case* U *only has a Lipschitz boundary. Note that in this case we can only say that the limit in* (4.7) *holds* $\mathcal{H}^{n-1}$*-a.e., but this is enough for* (4.8) *since* $\|\mu\| << \mathcal{H}^{n-1}$ *by* (4.6) *and* [6, Lemma 2.25].

4.6. **Remark.** *If* $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ *satisfies* $\operatorname{div} \boldsymbol{F} = \mu$ *then, for any bounded set of finite perimeter* E*, the Gauss-Green formula proved in Chen-Torres-Ziemer* [6] *yields,*

$$\mu(E^1 \cup \partial^* E) = \int_{E^1 \cup \partial^* E} \operatorname{div} \boldsymbol{F} = \int_{\partial^* E} (\mathscr{F}_e \cdot \boldsymbol{\nu})(y) d\mathcal{H}^{n-1}(y)$$

and

$$\mu(E^1) = \int_{E^1} \operatorname{div} \boldsymbol{F} = \int_{\partial^* E} (\mathscr{F}_i \cdot \boldsymbol{\nu})(y) d\mathcal{H}^{n-1}(y).$$

Here E^1 is the measure-theoretic interior of E and $\partial^* E$ is the reduced boundary of E. The estimates

$$\|\mathscr{F}_e \cdot \boldsymbol{\nu}\|_{L^\infty(\partial^* E)} \le \|\boldsymbol{F}\|_{L^\infty} \text{ and } \|\mathscr{F}_i \cdot \boldsymbol{\nu}\|_{L^\infty(\partial^* E)} \le \|\boldsymbol{F}\|_{L^\infty}$$

give

$$|\mu(E^1 \cup \partial^* E)| = |\mu(E^1) + \mu(\partial^* E)| \le \|\boldsymbol{F}\|_{L^\infty} \mathcal{H}^{n-1}(\partial^* E)$$

and

$$|\mu(E^1)| \le \|\boldsymbol{F}\|_{L^\infty} \mathcal{H}^{n-1}(\partial^* E).$$

Therefore,

$$|\mu(\partial^* E)| \le \|\boldsymbol{F}\|_{L^\infty} \mathcal{H}^{n-1}(\partial^* E) + |\mu(E^1)| \le 2 \|\boldsymbol{F}\|_{L^\infty} \mathcal{H}^{n-1}(\partial^* E).$$

We note that this provides another proof of **(i)** $\Rightarrow$ **(ii)** *(with $C = \|\boldsymbol{F}\|_\infty$ for both open and closed smooth sets) since for any bounded open (resp. closed) set U with smooth boundary we have $U = U^1$ (resp. $U = U^1 \cup \partial^* U$).*

We recall the spaces defined in Definitions 3.4 and 3.5. We now show the following new result.

4.7. **Theorem.** *Let $\mathcal{E} := \mathcal{M}_{loc} \cap BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ and $\mathcal{F} := \mathcal{M}_{loc} \cap \dot{W}^{1,1}(\mathbb{R}^n)^*$. Then $\mathcal{E}$ and $\mathcal{F}$ are isometrically isomorphic.*

Proof. We define a map $S : \mathcal{E} \to \mathcal{F}$ as

$$S(T) = T \llcorner \dot{W}^{1,1}.$$

Clearly, S is a linear map. We need to show that S is 1-1 and on-to, and $\|S(T)\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} = \|T\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*}$ for all $T \in \mathcal{E}$. In order to show the injectivity we assume that $S(T) = 0 \in \mathcal{F}$ for some $T \in \mathcal{E}$. Then

$$T(u) = 0 \text{ for all } u \in \dot{W}^{1,1}(\mathbb{R}^n).$$

Thus, if μ is the measure associated to $T \in \mathcal{E}$, then

$$\int_{\mathbb{R}^n} \varphi d\mu = T(\varphi) = 0 \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n),$$

which implies that $\mu = 0$. Now, by definition of $\mathcal{E}$, we have

$$T(u) = \int_{\mathbb{R}^n} u^* d\mu = 0 \text{ for all } u \in BV_c^\infty(\mathbb{R}^n),$$

which implies, by Theorem 2.10 and Theorem 3.1, that

$$T \equiv 0 \text{ on } BV_{\frac{n}{n-1}(\mathbb{R}^n)}.$$

We now proceed to show the surjectivity and take $H \in \mathcal{F}$. Thus, there exists $\mu \in \mathcal{M}_{loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \varphi d\mu = H(\varphi) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

From Lemma 4.1, since $H \in \dot{W}^{1,1}(\mathbb{R}^n)^*$, there exists a bounded vector field $\boldsymbol{F} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\text{div}\, \boldsymbol{F} = \mu \text{ in the distributional sense and } \|H\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} = \|\mu\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*} = \|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)}. \tag{4.17}$$

Now, from the proof of Theorem 4.4, **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)**, it follows that

$$\|\mu\| << \mathcal{H}^{n-1},$$

$$|\mu(U)| \le \|\boldsymbol{F}\|_\infty \mathcal{H}^{n-1}(\partial U)$$

for all closed and smooth sets $U \subset\subset \mathbb{R}^n$, and

$$\left| \int_{\mathbb{R}^n} u^* d\mu \right| \le \|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^n)} \|u\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)} \text{ for all } u \in BV_c^\infty(\mathbb{R}^n).$$

Hence, $\mu \in BV_c^\infty(\mathbb{R}^n)^*$ and from (4.17) we obtain

$$\|\mu\|_{BV_c^\infty(\mathbb{R}^n)^*} = \|\boldsymbol{F}\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} = \|\mu\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*}.$$

From Theorem 2.10, it follows that μ can be uniquely extended to a continuous linear functional $\hat{\mu} \in BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*$ and clearly,

$$S(\hat{\mu}) = \mu,$$

which implies that S is surjective. According to Theorem 2.10, this extension preserves the operator norm and thus

$$\left\|S^{-1}(\mu)\right\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*} = \|\hat{\mu}\|_{BV_{\frac{n}{n-1}}(\mathbb{R}^n)^*} = \|\mu\|_{BV_c^\infty(\mathbb{R}^n)^*} = \|\mu\|_{\dot{W}^{1,1}(\mathbb{R}^n)^*},$$

which shows that $\mathcal{E}$ and $\mathcal{F}$ are isometrically isomorphic. □

5. On an issue raised by Meyers and Ziemer

In this section, using the result of Theorem 4.4, we construct a locally integrable function f such that $f \in BV(\mathbb{R}^n)^*$ but $|f| \notin BV(\mathbb{R}^n)^*$. This example settles an issue raised by Meyers and Ziemer in [16, page 1356]. We mention that this kind of highly oscillatory function appeared in [14] in a different context.

5.1. **Proposition.** *Let $f(x) = \epsilon|x|^{-1-\epsilon}\sin(|x|^{-\epsilon}) + (n-1)|x|^{-1}\cos(|x|^{-\epsilon})$, where $0 < \epsilon < n-1$ is fixed. Then*

$$f(x) = \operatorname{div}[x|x|^{-1}\cos(|x|^{-\epsilon})]. \tag{5.1}$$

Moreover, there exists a sequence $\{r_k\}$ decreasing to zero such that

$$\int_{B_{r_k}(0)} f^+(x)dx \geq c\, r_k^{n-1-\epsilon} \tag{5.2}$$

for a constant $c = c(n,\epsilon) > 0$ independent of k. Here f^+ is the positive part of f. Thus by Theorem 4.4 we see that f belongs to $BV(\mathbb{R}^n)^*$, whereas $|f|$ does not.

Proof. The equality (5.1) follows by a straightforward computation. To show (5.2), we let $r_k = (\pi/6 + 2k\pi)^{\frac{-1}{\epsilon}}$ for $k = 1, 2, 3, \ldots$ Then we have

$$\begin{aligned}
\int_{B_{r_k}(0)} f^+(x)dx &= s(n)\int_0^{r_k} t^n[\epsilon\, t^{-1-\epsilon}\sin(t^{-\epsilon}) + (n-1)t^{-1}\cos(t^{-\epsilon})]^+\frac{dt}{t}\\
&= \frac{s(n)}{\epsilon}\int_{r_k^{-\epsilon}}^\infty x^{\frac{-n}{\epsilon}}[\epsilon\, x^{\frac{\epsilon+1}{\epsilon}}\sin(x) + (n-1)x^{\frac{1}{\epsilon}}\cos(x)]^+\frac{dx}{x}\\
&\geq \frac{s(n)}{2}\sum_{i=0}^\infty \int_{\pi/6+2k\pi+2i\pi}^{\pi/2+2k\pi+2i\pi} x^{\frac{-n+1}{\epsilon}}dx,
\end{aligned}$$

where $s(n)$ is the area of the unit sphere in $\mathbb{R}^n$. Thus using the elementary observation

$$\int_{\pi/2+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{-n+1}{\epsilon}}dx \leq 6\int_{\pi/6+2k\pi+2i\pi}^{\pi/2+2k\pi+2i\pi} x^{\frac{-n+1}{\epsilon}}dx,$$

we find that

$$\begin{aligned}
\int_{B_{r_k}(0)} f^+(x)dx &\geq \frac{s(n)}{14}\sum_{i=0}^{\infty} 7\int_{\pi/6+2k\pi+2i\pi}^{\pi/2+2k\pi+2i\pi} x^{\frac{-n+1}{\epsilon}}dx \\
&\geq \frac{s(n)}{14}\sum_{i=0}^{\infty}\left(\int_{\pi/6+2k\pi+2i\pi}^{\pi/2+2k\pi+2i\pi} x^{\frac{-n+1}{\epsilon}}dx + \int_{\pi/2+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{-n+1}{\epsilon}}dx\right) \\
&\geq \frac{s(n)}{14}\sum_{i=0}^{\infty}\int_{\pi/6+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{-n+1}{\epsilon}}dx \\
&= \frac{s(n)}{14}\int_{\pi/6+2k\pi}^{\infty} x^{\frac{-n+1}{\epsilon}}dx = \frac{s(n)\,\epsilon}{14(n-1-\epsilon)}r_k^{n-1-\epsilon}.
\end{aligned}$$

This completes the proof of the proposition. □

6. The space $BV_0(\Omega)$

In this section we let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We have the following well known result concerning the existence of traces of functions in $BV(\Omega)$ (see for example [13, Theorem 2.10] and [4, Theorem 10.2.1]):

6.1. **Theorem.** *Let Ω be a bounded open set with Lipschitz continuous boundary $\partial\Omega$ and let $u \in BV(\Omega)$. Then, there exists a function $\varphi \in L^1(\partial\Omega)$ such that, for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$,*

$$\lim_{r\to 0} r^{-n}\int_{B(x,r)\cap\Omega} |u(y)-\varphi(x)|dy = 0.$$

From the construction of the trace φ (see [13, Lemma 2.4], we see that φ is uniquely determined. Therefore, we have a well defined operator

$$\gamma_0 : BV(\Omega) \to L^1(\partial\Omega).$$

We now define the space $BV_0(\Omega)$ as follows:

6.2. **Definition.** *Let*

$$BV_0(\Omega) = \ker(\gamma_0).$$

We also define another BV function space with a zero boundary condition.

6.3. **Definition.** *Let*

$$\mathbb{BV}_0(\Omega) := \overline{C_c^\infty(\Omega)},$$

where the closure is taken with respect to the intermediate convergence of $BV(\Omega)$.

By the intermediate convergence of $BV(\Omega)$, we mean the following

6.4. **Definition.** *Let $\{u_k\} \in BV(\Omega)$ and $u \in BV(\Omega)$. We say that u_k converges to u in the sense of intermediate (or strict) convergence if*

$$u_k \to u \text{ strongly in } L^1(\Omega) \text{ and } \int_\Omega |Du_k| \to \int_\Omega |Du|.$$

The following theorem can be found in [4, Theorem 10.2.2]:

6.5. **Theorem.** *The trace operator γ_0 is continuous from $BV(\Omega)$ equipped with the intermediate convergence onto $L^1(\partial\Omega)$ equipped with the strong convergence.*

The following theorem is well known and can be found in many standard references including [4, 12, 20, 13], but for completeness we will include the proof here.

6.6. **Theorem.** *The space $C^\infty(\Omega)\cap BV(\Omega)$ is dense in $BV(\Omega)$ equipped with the intermediate convergence. Moreover, if Ω is a Lipschitz domain then $C^\infty(\overline{\Omega})$ is also dense in $BV(\Omega)$ for the intermediate convergence.*

Proof. We note that $C^\infty(\Omega) \cap BV(\Omega) = C^\infty(\Omega) \cap W^{1,1}(\Omega)$. For Lipschitz domains, it is proved, e.g., in [12, page 127] that $C^\infty(\overline{\Omega})$ is dense in $W^{1,1}(\Omega)$, equipped with the strong convergence. This actually holds even for domains that possess the so-called *segment property* (see [1, Theorem 3.18]). Thus, since the strong convergence implies the intermediate convergence it follows that $C^\infty(\overline{\Omega})$ is dense in $C^\infty(\Omega) \cap BV(\Omega)$ in the intermediate convergence. Therefore, if $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ for the intermediate convergence, the second statement of the theorem holds. Let $\varepsilon > 0$ and $u \in BV(\Omega)$. We decompose Ω as follows:

$$\Omega = \bigcup_{i=0}^{\infty} \Omega_i, \int_{\Omega \setminus \Omega_0} |Du| < \varepsilon \text{ and } \Omega_i \subset\subset \Omega_{i+1}.$$

We consider the open cover $\{C_i\}$ defined as follows:

$$\begin{aligned} C_1 &:= \Omega_2 \\ C_i &:= \Omega_{i+1} \setminus \overline{\Omega}_{i-1}, \quad i \geq 2. \end{aligned}$$

Let $\{\varphi_i\}$ be a partition of unity associated to $\{C_i\}$; that is,

$$\varphi_i \in C_c^\infty(C_i), \quad 0 \leq \varphi_i \leq 1, \quad \sum_{i=1}^{\infty} \varphi_i = 1.$$

Note that $\varphi_1 \equiv 1$ on Ω_1. Let ρ be a standard mollifier as in the proof of Theorem 4.2. For each i, choose $\varepsilon_i > 0$ so that:

$$\text{spt } (\rho_{\varepsilon_i} * \varphi_i u) \subset C_i,$$

$$\int_\Omega |\rho_{\varepsilon_i} * (u\varphi_i) - u\varphi_i| < \frac{\varepsilon}{2^i}, \tag{6.1}$$

$$\int_\Omega |\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i| < \frac{\varepsilon}{2^i}, \tag{6.2}$$

$$\left| \int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)| dx - \int_\Omega |\varphi_1 Du| \right| < \varepsilon. \tag{6.3}$$

Define

$$u_\varepsilon := \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (u\varphi_i).$$

Then

$$\int_\Omega |u - u_\varepsilon| \, dx \leq \sum_{i=1}^{\infty} \int_\Omega |\rho_{\varepsilon_i} * (u\varphi_i) - u\varphi_i| \, dx < \varepsilon, \quad \text{by (6.1)}.$$

We have

$$\begin{aligned} Du_\varepsilon &= \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (\varphi_i Du) + \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (uD\varphi_i) \\ &= \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (\varphi_i Du) + \sum_{i=1}^{\infty} \big(\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i\big). \end{aligned}$$

Then, on the one hand,

$$\begin{aligned}
\left|\int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\,dx - \int_\Omega |Du_\varepsilon|\right| &\leq \sum_{i=2}^{\infty}\int_\Omega |\rho_{\varepsilon_i} * (\varphi_i Du)|\,dx + \sum_{i=1}^{\infty}\int_\Omega |\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i|\,dx \\
&\leq \sum_{i=2}^{\infty}\int_\Omega |\rho_{\varepsilon_i} * (\varphi_i Du)| + \varepsilon, \quad \text{by (6.2)} \\
&\leq \sum_{i=2}^{\infty}\int_\Omega |\varphi_i Du| + \varepsilon, \ \text{by a property of convolution} \\
&\leq \int_{\Omega\setminus\Omega_0} |Du| + \varepsilon \\
&\leq \varepsilon + \varepsilon = 2\varepsilon. \qquad (6.4)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|\int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\,dx - \int_\Omega |Du|\right| &= \left|\int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\,dx - \int_\Omega |\varphi_1 Du| - \int_\Omega (1-\varphi_1)|Du|\right| \\
&\leq \varepsilon + \int_\Omega (1-\varphi_1)|Du|, \ \text{by (6.3)} \\
&\leq \varepsilon + \int_{\Omega\setminus\Omega_0} |Du| \leq 2\varepsilon, \ \text{since } \varphi_1 \equiv 1 \text{ on } \Omega_1. \qquad (6.5)
\end{aligned}$$

From (6.4) and (6.5):

$$\left|\int_\Omega |Du_\varepsilon| - \int_\Omega |Du|\right| < 4\varepsilon.$$

□

6.7. **Theorem.** *Let Ω be any bounded open set with Lipschitz boundary. Then $BV_c(\Omega)$ is dense in $BV_0(\Omega)$ in the strong topology of $BV(\Omega)$.*

Proof. We consider first the case $u \in BV_0(C_{R,T})$, where $C_{R,T}$ is the open cylinder

$$C_{R,T} = \mathcal{B}_R \times (0,T),$$

$\mathcal{B}_R$ is an open ball of radius R in $\mathbb{R}^{n-1}$, and $\operatorname{supp}(u) \cap \partial C_{R,T} = \operatorname{supp}(u) \cap (\mathcal{B}_R \times \{0\})$. A generic point in $C_{R,T}$ will be denoted by (x', t), with $x' \in \mathcal{B}_R$ and $t \in (0,T)$. From Theorem 6.6, we can approximate u with a sequence of functions $u_k \in C^\infty(\overline{C_{R,T}})$ such that

$$u_k \to u \text{ in } L^1(C_{R,T}) \text{ and } \int_{C_{R,T}} |Du_k| \to \int_{C_{R,T}} |Du|. \qquad (6.6)$$

Notice that the condition $\operatorname{supp}(u) \cap \partial C_{R,T} = \operatorname{supp}(u) \cap (\mathcal{B}_R \times \{0\})$ implies that

$$\gamma_0(u_k) \llcorner (\partial C_{R,T} \setminus (\mathcal{B}_R \times \{0\})) \equiv 0.$$

From Theorem 6.5, $\gamma_0(u_k) \to \gamma_0(u)$ in $L^1(\partial C_{R,T})$ and hence

$$\gamma_0(u_k) \llcorner (\mathcal{B}_R \times \{0\}) = u_k\big|_{(\mathcal{B}_R \times \{0\})} \to 0 \text{ in } L^1(\mathcal{B}_R \times \{0\}). \qquad (6.7)$$

For $x' \in \mathcal{B}_R$, $0 \leq x_n \leq T$, we have

$$u_k(x', x_n) - u_k(x', 0) = \int_0^{x_n} \frac{\partial u_k}{\partial x_n}(x', t)\,dt,$$

and hence,

$$|u_k(x', x_n)| \leq |u_k(x', 0)| + \int_0^{x_n} \left|\frac{\partial u_k}{\partial x_n}(x', t)\right| dt. \qquad (6.8)$$

We integrate both sides in (6.8) to obtain:

$$\int_{\mathcal{B}_R} |u_k(x', x_n)| dx' \leq \int_{\mathcal{B}_R} |u_k(x', 0)| dx' + \int_0^{x_n} \int_{\mathcal{B}_R} |Du_k(x', t)| dx'\, dt. \tag{6.9}$$

From (6.7) we have

$$\lim_{k\to\infty} \int_{\mathcal{B}_R} |u_k(x', 0)| dx' = 0, \tag{6.10}$$

and thus, letting $k \to \infty$ in (6.9) and using (6.10), (6.6) and Lemma 2.9 (in particular, (2.7) with $A := \mathcal{B}_R \times (0, x_n)$ for a.e. $0 < x_n < T$) we obtain

$$\int_{\mathcal{B}_R} |u(x', x_n)| dx' \leq \int_0^{x_n} \int_{\mathcal{B}_R} |Du| = \|Du\| \left(\mathcal{B}_R \times (0, x_n)\right) \text{ for a.e. } 0 < x_n < T. \tag{6.11}$$

Consider a function $\varphi \in C_c^\infty(\mathbb{R})$ such that φ is decreasing in $[0, +\infty)$ and satisfies

$$\varphi \equiv 1 \text{ on } [0, 1], \varphi \equiv 0 \text{ on } \mathbb{R} \setminus [-1, 2], 0 \leq \varphi \leq 1.$$

We define

$$\begin{aligned} \varphi_k(t) &= \varphi(kt), \quad k = 1, 2, \ldots \\ v_k(x', t) &= (1 - \varphi_k(t)) u(x', t). \end{aligned} \tag{6.12}$$

Clearly, $v_k \to u$ in $L^1(C_{R,T})$. Also, if $u \geq 0$ then $v_k \uparrow u$ since φ is decreasing in $[0, +\infty)$. Moreover,

$$\begin{aligned} \frac{\partial v_k}{\partial t} &= (1 - \varphi_k) \frac{\partial u}{\partial t} - k\varphi'(kt) u, \\ D_{x'} v_k &= (1 - \varphi_k) D_{x'} u. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_{C_{R,T}} |Dv_k - Du| &= \int_{C_{R,T}} \left| \Big(D_{x'} u - \varphi_k D_{x'} u, \frac{\partial u}{\partial t} - \varphi_k \frac{\partial u}{\partial t} - k\varphi'(kt) u \Big) - \Big(D_{x'} u, \frac{\partial u}{\partial t} \Big) \right| \\ &= \int_{C_{R,T}} \left| \Big(- \varphi_k D_{x'} u, -\varphi_k \frac{\partial u}{\partial t} - k\varphi'(kt) u \Big) \right|. \end{aligned}$$

Since $\varphi_k(t) = 0$ for $t > \frac{2}{k}$ we have the following:

$$\begin{aligned} \int_{C_{R,T}} |Dv_k - Du| &\leq C \Big(\int_{C_{R,T}} \varphi_k |Du| + \int_{C_{R,T}} k |\varphi'(kx_n)| |u| \Big) \\ &\leq C \int_0^{2/k} \int_{\mathcal{B}_R} |Du| + C\, k \int_0^{2/k} \int_{\mathcal{B}_R} |u(x', t)| dx' dt \\ &\leq C \int_0^{2/k} \int_{\mathcal{B}_R} |Du| + C\, k \int_0^{2/k} \|Du\| \left(\mathcal{B}_R \times (0, t)\right) dt, \text{ by (6.11)} \\ &\leq C \int_0^{2/k} \int_{\mathcal{B}_R} |Du| + C\, k \cdot \|Du\| \left(\mathcal{B}_R \times (0, 2/k)\right) \cdot \int_0^{2/k} dt \\ &= C \int_0^{2/k} \int_{\mathcal{B}_R} |Du| + C\, k \cdot \frac{2}{k} \cdot \int_0^{2/k} \int_{\mathcal{B}_R} |Du| \\ &= C \int_0^{2/k} \int_{\mathcal{B}_R} |Du|. \end{aligned} \tag{6.13}$$

Since $\|Du\|$ is a Radon measure and $\cap_{k=1}^\infty (\mathcal{B}_R \times (0, \frac{2}{k})) = \emptyset$ we find

$$\int_0^{2/k} \int_{\mathcal{B}_R} |Du| \to 0, \text{ as } k \to \infty,$$

which by (6.13) yields

$$\lim_{k\to\infty}\int_{C_{R,T}}|Dv_k - Du| = 0.$$

Thus

$$v_k \to u \text{ in the strong topology of } BV(C_{R,T}). \tag{6.14}$$

We consider now the general case of a bounded open set Ω with Lipschitz boundary and let $u \in BV_0(\Omega)$. For each point $x_0 \in \partial\Omega$, there exists a neighborhood A and a bi-Lipschitz function $g : B(0,1) \to A$ that maps $B(0,1)^+$ onto $A\cap\Omega$ and the flat part of $\partial B(0,1)^+$ onto $A\cap\partial\Omega$. A finite number of such sets $A_1, A_2, \dots, A_n$ cover $\partial\Omega$. By adding possibly an additional open set $A_0 \subset\subset \Omega$, we get a finite covering of $\overline{\Omega}$. Let $\{\alpha_i\}$ be a partition of unity relative to that covering, and let g_i be the bi-Lipschitz map relative to the set A_i for $i = 1, 2, \dots, N$. For each $i \in \{1, 2, \dots, N\}$ the function

$$U_i = (\alpha_i u)\circ g_i$$

belongs to $BV_0(B(0,1)^+)$, and has support non-intersecting the curved part of $\partial B(0,1)^+$. Thus, we can extend U_i to the whole cylinder $C_{1,1} := \mathcal{B}_1(0)\times(0,1)$ by setting U_i equal to zero outside $B(0,1)^+$. By (6.14), for each $\varepsilon > 0$, we can find a function $W_i \in BV_c(C_{1,1})$ such that

$$\|W_i - U_i\|_{BV(C_{1,1})} \le \varepsilon, \tag{6.15}$$

for $i = 1, 2, \dots, N$. Letting now

$$w_i = W_i \circ g_i^{-1},\ i = 1, 2, \dots, N,$$

we have $w_i \in BV_c(A_i\cap\Omega)$ and

$$\begin{aligned}
\|D(w_i - \alpha_i u)\|(A_i\cap\Omega) &= \left\|D(W_i\circ g_i^{-1} - ((\alpha_i u)\circ g_i)\circ g_i^{-1})\right\|(A_i\cap\Omega)\\
&= \|D(g_{i\#}(W_i - (\alpha_i u)\circ g_i))\|(A_i\cap\Omega)\\
&\le C\, g_{i\#}\|D(W_i - (\alpha_i u)\circ g_i)\|(A_i\cap\Omega), \text{ by [3, Theorem 3.16]}\\
&= C\int_{g_i^{-1}(A_i\cap\Omega)}|D(W_i - U_i)|, \text{by definition of } g_{i\#} \text{ acting on measures}\\
&= C\int_{B(0,1)^+}|D(W_i - U_i)|\\
&\le C\,\varepsilon, \text{ by (6.15)}. \qquad (6.16)
\end{aligned}$$

Here $C = \max_i\{[\mathrm{Lip}(g_i)]^{n-1}\}$ (see [3, Theorem 3.16]). Let $w_0 = \alpha_0 u$. Then $w_0 \in BV_c(\Omega)$. Define

$$w = \sum_{i=0}^{N} w_i.$$

We have $w \in BV_c(\Omega)$, and by (6.16)

$$\begin{aligned}
\|D(w-u)\|(\Omega) &\le \sum_{i=0}^{N}\|D(w_i - \alpha_i u)\|(A_i\cap\Omega)\\
&= \sum_{i=1}^{N}\|D(w_i - \alpha_i u)\|(A_i\cap\Omega)\\
&\le NC\,\varepsilon.
\end{aligned}$$

Likewise, by (6.15) and a change of variables we have

$$\|w-u\|_{L^1(\Omega)} \le \sum_{i=0}^{N}\|w_i - \alpha_i u\|_{L^1(A_i\cap\Omega)} \le \sum_{i=1}^{N}\|w_i - \alpha_i u\|_{L^1(A_i\cap\Omega)} \le Nc\,\varepsilon.$$

Thus $\overline{BV_c(\Omega)} = BV_0(\Omega)$ in the strong topology of $BV(\Omega)$. □

6.8. **Remark.** *By (6.12) and the construction of w in the proof of Theorem 6.7 above, we see that each $u \in BV_0(\Omega)$ can be approximated by a sequence $\{u_k\} \subset BV_c(\Omega)$ such that $u_k = u$ in $\Omega \setminus N_k$ for a set $N_k = \{x \in \Omega : d(x, \partial\Omega) \leq \delta(k)\}$ with $\delta(k) \to 0$ as $k \to +\infty$. Moreover, if $u \geq 0$ then so is u_k and $u_k \uparrow u$ as k increases to $+\infty$.*

We will also need the following density result.

6.9. **Lemma.** *$BV_0^\infty(\Omega)$ is dense in $BV_0(\Omega)$. Likewise, $BV_c^\infty(\Omega)$ is dense in $BV_c(\Omega)$, and $BV^\infty(\Omega)$ is dense in $BV(\Omega)$ in the strong topology of $BV(\Omega)$.*

Proof. We shall only prove the first statement as the others can be shown in a similar way. Let $u \in BV_0^+(\Omega)$ and define

$$u_j := u \wedge j, \quad j = 1, 2, \ldots$$

Obviously, $u_j \to u$ in $L^1(\Omega)$. We will now show that $\|D(u - u_j)\|(\Omega) \to 0$. The coarea formula yields

$$\begin{aligned}
\int_\Omega |D(u - u_j)| &= \int_0^\infty \mathcal{H}^{n-1}(\Omega \cap \partial^*\{u - u_j > t\})dt \\
&= \int_0^\infty \mathcal{H}^{n-1}(\Omega \cap \partial^*\{u - j > t\})dt \\
&= \int_0^\infty \mathcal{H}^{n-1}(\Omega \cap \partial^*\{u > j + t\})dt \\
&= \int_j^\infty \mathcal{H}^{n-1}(\Omega \cap \partial^*\{u > s\})ds.
\end{aligned}$$

Since $\int_0^\infty \mathcal{H}^{n-1}(\Omega \cap \partial^*\{u > s\})ds < \infty$, the Lebesgue dominated convergence theorem implies that

$$\int_\Omega |D(u - u_j)| \to 0 \text{ as } j \to \infty. \tag{6.17}$$

If $u \in BV_0(\Omega)$, we write $u = u^+ - u^-$ and define $f_j = u^+ \wedge j$ and $g_j = u^- \wedge j$. Thus $f_j - g_j \in BV_0(\Omega)$ and

$$\begin{aligned}
\int_\Omega |D(u - (f_j - g_j))| &= \int_\Omega |Du^+ - Du^- - Df_j + Dg_j| \\
&\leq \int_\Omega |D(u^+ - f_j)| + \int_\Omega |D(u^- - g_j)| \\
&\to 0 \text{ as } j \to \infty,
\end{aligned}$$

due to (6.17). That completes the proof of the lemma. □

We are now ready to prove the main theorem of this section that makes precise the definition of the space of functions of bounded variation in Ω with zero trace on the boundary of Ω.

6.10. **Theorem.** *$\mathbb{BV}_0(\Omega) = BV_0(\Omega)$.*

Proof. Let Let $u \in \mathbb{BV}_0(\Omega)$. Then Definition 6.3 implies the existence of a sequence $\{u_k\} \in C_c^\infty(\Omega)$ such that

$$u_k \to u \quad \text{in} \quad L^1(\Omega) \text{ and } \int_\Omega |Du_k| \to \int_\Omega |Du|.$$

Since $u_k \in C_c^\infty(\Omega)$, we have $\gamma_0(u_k) \equiv 0$. Then Theorem 6.5 yields

$$\gamma_0(u_k) \to \gamma(u) \quad \text{in } L^1(\partial\Omega),$$

and so

$$\gamma(u) = 0 \quad \text{ and } \quad u \in BV_0(\Omega).$$

In the other direction, let $u \in BV_0(\Omega)$. Then, from Theorem 6.7 there exists a sequence $u_k \in BV_c(\Omega)$ such that

$$\lim_{k\to\infty}\int_\Omega |u_k - u| = \lim_{k\to\infty}\int_\Omega |Du_k - Du| = 0. \tag{6.18}$$

Given a sequence $\varepsilon_k \to 0$, we consider the sequence of mollifications

$$w_k := u_k * \rho_{\varepsilon_k}.$$

We can choose ε_k sufficiently small to have

$$w_k \in C_c^\infty(\Omega).$$

Also, for each k,

$$\lim_{\varepsilon\to 0}\int_\Omega |D(u_k * \rho_\varepsilon)| = \int_\Omega |Du_k|,$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega |u_k * \rho_\varepsilon - u_k| = 0.$$

Thus we can choose ε_k small enough so that, for each k,

$$\left|\int_\Omega |D(u_k * \rho_{\varepsilon_k})| - \int_\Omega |Du_k|\right| \le \frac{1}{k}, \tag{6.19}$$

and

$$\int_\Omega |u_k * \rho_{\varepsilon_k} - u_k| \le \frac{1}{k}. \tag{6.20}$$

Using (6.20) and (6.18) we obtain

$$\lim_{k\to\infty}\int_\Omega |w_k - u| \le \lim_{k\to\infty}\int_\Omega |w_k - u_k| + \lim_{k\to\infty}\int_\Omega |u_k - u| = 0. \tag{6.21}$$

Also, letting $k \to \infty$ in (6.19) and using (6.18), we obtain

$$\lim_{k\to\infty}\int_\Omega |D(u_k * \rho_{\varepsilon_k})| = \int_\Omega |Du|. \tag{6.22}$$

From (6.21) and (6.22) we conclude that $w_k \to u$ in the intermediate convergence which implies that $u \in \mathbb{BV}_0(\Omega)$. □

With Theorem 6.10 we can now prove the following Sobolev's inequality for functions in $BV_0(\Omega)$:

6.11. **Theorem.** *Let $u \in BV_0(\Omega)$, where Ω is a bounded open set with Lipschitz boundary. Then*

$$\|u\|_{L^{\frac{n}{n-1}}(\Omega)} \le C\,\|Du\|\,(\Omega),$$

for a constant $C = C(n)$.

Proof. The Sobolev inequality for smooth functions states that

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le C\int_{\mathbb{R}^n} |Du| \text{ for each } u \in C_c^\infty(\mathbb{R}^n). \tag{6.23}$$

From Theorem 6.10 there exists a sequence $u_k \in C_c^\infty(\Omega)$ such that

$$u_k \to u \quad \text{in} \quad L^1(\Omega) \text{ and } \int_\Omega |Du_k| \to \int_\Omega |Du|. \tag{6.24}$$

Since $u_k \to u$ in $L^1(\Omega)$ then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that

$$u_{k_j}(x) \to u(x) \text{ for a.e. } x \in \Omega.$$

Using Fatou's Lemma and (6.23), we obtain

$$\int_\Omega |u|^{\frac{n}{n-1}} \leq \liminf_{j\to\infty} \int_\Omega |u_{k_j}|^{\frac{n}{n-1}} \leq \liminf_{j\to\infty} \left(C \int_\Omega |Du_{k_j}| \right)^{\frac{n}{n-1}}. \tag{6.25}$$

Finally, using (6.24) in (6.25) we conclude

$$\left(\int_\Omega |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq C \int_\Omega |Du|.$$

□

By Theorem 6.11, we see that $\|u\|_{BV(\Omega)}$ is equivalent to $\|Du\|(\Omega)$ whenever $u \in BV_0(\Omega)$ (or $\mathbb{BV}_0(\Omega)$) and Ω is a bounded Lipschitz domain. Thus, for the rest of the paper we will equip $BV_0(\Omega)$ with the homogeneous norm:

$$\|u\|_{BV_0(\Omega)} = \|Du\|(\Omega).$$

From Theorem 6.7 and Lemma 6.9 we obtain

6.12. **Corollary.** *Let Ω be any bounded open set with Lipschitz boundary. Then $BV_c^\infty(\Omega)$ is dense in $BV_0(\Omega)$.*

7. Characterizations of measures in $BV_0(\Omega)^*$

First, as in the case of $\mathbb{R}^n$, we make precise the definitions of measures in the spaces $W_0^{1,1}(\Omega)^*$ and $BV_0(\Omega)^*$.

7.1. **Definition.** *For a bounded open set Ω with Lipschitz boundary, we let*

$$\mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^* := \{T \in W_0^{1,1}(\Omega)^* : T(\varphi) = \int_\Omega \varphi d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\Omega), \forall \varphi \in C_c^\infty(\Omega)\}.$$

Therefore, if $\mu \in \mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^*$, then the action $< \mu, u >$ can be uniquely defined for all $u \in W_0^{1,1}(\Omega)$ (because of the density of $C_c^\infty(\Omega)$ in $W_0^{1,1}(\Omega)$).

7.2. **Definition.** *For a bounded open set Ω with Lipschitz boundary, we let*

$$\mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^* := \{T \in BV_0(\Omega)^* : T(\varphi) = \int_\Omega \varphi^* d\mu \text{ for some } \mu \in \mathcal{M}_{loc}(\Omega), \forall \varphi \in BV_c^\infty(\Omega)\},$$

where φ^* is the precise representative of φ. Thus, if $\mu \in \mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^*$, then the action $< \mu, u >$ can be uniquely defined for all $u \in BV_0(\Omega)$ (because of the density of $BV_c^\infty(\Omega)$ in $BV_0(\Omega)$ by Corollary 6.12).

We will use the following characterization of $W_0^{1,1}(\Omega)^*$ whose proof is completely analogous to that of Lemma 4.1.

7.3. **Lemma.** *Let Ω be any bounded open set with Lipschitz boundary. The distribution T belongs to $W_0^{1,1}(\Omega)^*$ if and only if $T = \operatorname{div} \boldsymbol{F}$ for some vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$. Moreover,*

$$\|T\|_{W_0^{1,1}(\Omega)^*} = \min\{\|\boldsymbol{F}\|_{L^\infty(\Omega,\mathbb{R}^n)}\},$$

where the minimum is taken over all $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = T$. Here we use the norm

$$\|\boldsymbol{F}\|_{L^\infty(\Omega,\mathbb{R}^n)} := \left\| (F_1^2 + F_2^2 + \cdots + F_n^2)^{1/2} \right\|_{L^\infty(\Omega)} \text{ for } \boldsymbol{F} = (F_1, \ldots, F_n).$$

We are now ready to state the main result of this section.

7.4. **Theorem.** *Let Ω be any bounded open set with Lipschitz boundary and $\mu \in \mathcal{M}_{loc}(\Omega)$. Then, the following are equivalent:*

(i) *There exists a vector field $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{div} \boldsymbol{F} = \mu$.*

(ii) $|\mu(U)| \leq C\,\mathcal{H}^{n-1}(\partial U)$ *for any smooth open (or closed) set $U \subset\subset \Omega$ with $\mathcal{H}^{n-1}(\partial U) < +\infty$.*

(iii) $\mathcal{H}^{n-1}(A) = 0$ *implies $\|\mu\|(A) = 0$ for all Borel sets $A \subset \Omega$ and there is a constant C such that, for all $u \in BV_c^\infty(\Omega)$,*

$$| < \mu, u > | := \left| \int_\Omega u^* d\mu \right| \leq C \int_\Omega |Du|,$$

where u^* is the representative in the class of u that is defined $\mathcal{H}^{n-1}$-almost everywhere.

(iv) $\mu \in BV_0(\Omega)^*$. *The action of μ on any $u \in BV_0(\Omega)$ is defined (uniquely) as*

$$< \mu, u > := \lim_{k\to\infty} < \mu, u_k > = \lim_{k\to\infty} \int_\Omega u_k^* d\mu,$$

where $u_k \in BV_c^\infty(\Omega)$ converges to u in $BV_0(\Omega)$. In particular, if $u \in BV_c^\infty(\Omega)$ then

$$< \mu, u > = \int_\Omega u^* d\mu,$$

and moreover, if μ is a non-negative measure then, for all $u \in BV_0(\Omega)$,

$$< \mu, u > = \int_\Omega u^* d\mu.$$

Proof. Suppose **(i)** holds. Then for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \boldsymbol{F} \cdot D\varphi dx = -\int_\Omega \varphi d\mu.$$

Let $U \subset\subset \Omega$ be any open (or closed) set with smooth boundary satisfying $\mathcal{H}^{n-1}(\partial U) < \infty$. We proceed as in Theorem 4.4 and consider the characteristic function χ_U and the sequence $u_k := \chi_U * \rho_{1/k}$. Since U is strictly contained in Ω, for k large enough, the support of $\{u_k\}$ are contained in Ω. We can then proceed exactly as in Theorem 4.4 to conclude that

$$|\mu(U)| \leq C\,\mathcal{H}^{n-1}(\partial U),$$

where $C = \|\boldsymbol{F}\|_{L^\infty(\Omega)}$ for closed sets U and $C = 3\,\|\boldsymbol{F}\|_{L^\infty(\Omega)}$ for open sets U.

If μ satisfies **(ii)** with a constant $C > 0$, then Corollary 4.3 implies that $\|\mu\| << \mathcal{H}^{n-1}$. We let $u \in BV_c^\infty(\Omega)$ and $\{\rho_\varepsilon\}$ be a standard sequence of mollifiers. Consider the convolution $\rho_\varepsilon * u$ and note that $\rho_\varepsilon * u \in C_c^\infty(\Omega)$, for ε small enough. Then as in the proof of Theorem 4.4 we have, for ε small enough,

$$\left| \int_\Omega \rho_\varepsilon * u d\mu \right| \leq C \int_\Omega |Du|.$$

Sending ε to zero and using the dominated convergence theorem yield

$$\left| \int_\Omega u^* d\mu \right| \leq C \int_\Omega |Du|,$$

with the same constant C as in **(ii)**. This gives **(ii)** $\Rightarrow$ **(iii)**.

From **(iii)** we obtain that the linear operator

$$T(u) := < \mu, u > = \int_\Omega u^* d\mu, \quad u \in BV_c^\infty(\Omega) \tag{7.1}$$

is continuous and hence it can be uniquely extended, since $BV_c^\infty(\Omega)$ is dense in $BV_0(\Omega)$ (Corollary 6.12), to the space $BV_0(\Omega)$.

Assume now that μ is non-negative. We take $u \in BV_0(\Omega)$ and consider the positive and negative parts $(u^*)^+$ and $(u^*)^-$ of the representative u^*. By Remark 6.8, there is an increasing sequence of nonnegative functions $\{v_k\} \subset BV_c(\Omega)$ that converges to $(u^*)^+$ pointwise and in the BV_0 norm. Therefore, using (7.1) we have

$$T(v_k \wedge j) = \int_\Omega v_k \wedge j d\mu, \quad j = 1, 2, \dots$$

We first send j to infinity and then k to infinity. Using the continuity of T, (6.17), and the monotone convergence theorem we get

$$T((u^*)^+) = \int_\Omega (u^*)^+ d\mu.$$

We proceed in the same way for $(u^*)^-$ and thus by linearity we conclude

$$T(u) = T((u^*)^+) - T((u^*)^-) = \int_\Omega (u^*)^+ - (u^*)^- d\mu = \int_\Omega u^* d\mu.$$

Finally, to prove that **(iv)** implies **(i)** we take $\mu \in BV_0(\Omega)^*$. Since $W_0^{1,1}(\Omega) \subset BV_0(\Omega)$ then

$$\tilde{\mu} := \mu \llcorner W_0^{1,1}(\Omega) \in W_0^{1,1}(\Omega)^*,$$

and therefore Lemma 7.3 implies that there exists $\boldsymbol{F} \in L^\infty(\Omega, \mathbb{R}^n)$ such that $div\, \boldsymbol{F} = \tilde{\mu}$ and thus, since $C_c^\infty \subset W_0^{1,1}(\Omega)$, we conclude that $div\, \boldsymbol{F} = \mu$ in the sense of distributions. □

7.5. **Remark.** *If Ω is a bounded domain containing the origin then the function f given in Proposition 5.1 belongs to $BV_0(\Omega)^*$ but $|f|$ does not.*

Theorem 7.4 and Lemma 7.3 immediately imply the following new result which states that the set of measures in $BV_0(\Omega)^*$ coincides with that of $W_0^{1,1}(\Omega)^*$.

7.6. **Theorem.** *The normed spaces $\mathcal{M}_{loc}(\Omega) \cap BV_0(\Omega)^*$ and $\mathcal{M}_{loc}(\Omega) \cap W_0^{1,1}(\Omega)^*$ are isometrically isomorphic.*

The proof of Theorem 7.6 is similar to that of Theorem 4.7 but this time one uses Theorem 7.4 and Corollary 6.12 in place of Theorem 4.4 and Theorem 3.1, respectively. Thus we shall omit its proof.

8. Finite measures in $BV(\Omega)^*$

In this section, we characterize all *finite* signed measures that belong to $BV(\Omega)^*$. Note that the finiteness condition here is necessary at least for *positive* measures in $BV(\Omega)^*$. By a measure $\mu \in BV(\Omega)^*$ we mean that the inequality

$$\left| \int_\Omega u^* d\mu \right| \leq C \|u\|_{BV(\Omega)}$$

holds for all $u \in BV^\infty(\Omega)$. By Lemma 6.9 we see that such a μ can be uniquely extended to be a continuous linear functional in $BV(\Omega)$.

We will use the following result, whose proof can be found in [20, Lemma 5.10.14]:

8.1. **Lemma.** *Let Ω be an open set with Lipschitz boundary and $u \in BV(\Omega)$. Then, the extension of u to $\mathbb{R}^n$ defined by*

$$u_0(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

satisfies that $u_0 \in BV(\mathbb{R}^n)$ and

$$\|u_0\|_{BV(\mathbb{R}^n)} \leq C \|u\|_{BV(\Omega)},$$

where $C = C(\Omega)$.

8.2. **Theorem.** *Let Ω be an open set with Lipschitz boundary and let μ be a finite signed measure in Ω. Extend μ by zero to $\mathbb{R}^n \setminus \Omega$ by setting $\mu(\mathbb{R}^n \setminus \Omega) = 0$. Then, $\mu \in BV(\Omega)^*$ if and only if*

$$|\mu(U)| \leq C\,\mathcal{H}^{n-1}(\partial U) \tag{8.1}$$

for every smooth open set $U \subset \mathbb{R}^n$ and a constant $C = C(\Omega, \mu)$.

Proof. Suppose that $\mu \in BV(\Omega)^*$. Let $u \in BV_c^\infty(\mathbb{R}^n)$ and assume that u is the representative that is defined $\mathcal{H}^{n-1}$-almost everywhere. Consider $v := u\chi_\Omega$ and note that $v \llcorner \Omega \in BV^\infty(\Omega)$ since Dv is a finite vector-measure in $\mathbb{R}^n$ given by

$$Dv = uD\chi_\Omega + \chi_\Omega Du,$$

and therefore,

$$\begin{aligned} \int_\Omega |Dv| &= \int_\Omega |uD\chi_\Omega + \chi_\Omega Du| \leq \int_\Omega |u||D\chi_\Omega| + \int_\Omega |Du| \\ &= \int_\Omega |Du| \leq \int_{\mathbb{R}^n} |Du| = \|u\|_{BV(\mathbb{R}^n)} < +\infty. \end{aligned} \tag{8.2}$$

Since $\mu \in BV(\Omega)^*$ there exists a constant $C = C(\Omega, \mu)$ such that

$$\left| \int_\Omega v d\mu \right| \leq C\|v\|_{BV(\Omega)}. \tag{8.3}$$

Then,

$$\begin{aligned} \left|\int_{\mathbb{R}^n} ud\mu\right| = \left|\int_\Omega ud\mu\right| &= \left|\int_\Omega vd\mu\right| \leq C\,\|v\|_{BV(\Omega)},\ \text{by (8.3)} \\ &= C\,\|v\|_{L^1(\Omega)} + C\int_\Omega |Dv| \\ &\leq C\,\|v\|_{L^1(\Omega)} + C\int_{\mathbb{R}^n} |Du|,\ \text{by (8.2)} \\ &\leq C\,\|v\|_{L^{\frac{n}{n-1}}(\Omega)} + C\int_{\mathbb{R}^n} |Du|,\ \text{since } \Omega \text{ is bounded} \\ &= C\,\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} + C\int_{\mathbb{R}^n} |Du| \\ &\leq C\int_{\mathbb{R}^n} |Du| = \|u\|_{BV(\mathbb{R}^n)}\,,\ \text{by Theorem 2.4,} \end{aligned}$$

which implies that $\mu \in BV(\mathbb{R}^n)^*$. Thus, Theorem 4.4 gives

$$|\mu(U)| \leq C\mathcal{H}^{n-1}(\partial U)$$

for every open set $U \subset \mathbb{R}^n$ with smooth boundary.

Conversely, assume that μ satisfies condition (8.1). Then Theorem 4.4 yields that $\mu \in BV(\mathbb{R}^n)^*$. Let $u \in BV^\infty(\Omega)$ and consider its extension $u_0 \in BV(\mathbb{R}^n)$ as in Lemma 8.1. Then, since $u_0 \in BV_c^\infty(\mathbb{R}^n)$, there exists a constant C such that

$$\left|\int_{\mathbb{R}^n} (u_0)^* d\mu\right| \leq C\|u_0\|_{BV(\mathbb{R}^n)}. \tag{8.4}$$

Now, Lemma 8.1 yields $\|u_0\|_{BV(\mathbb{R}^n)} \leq C\|u\|_{BV(\Omega)}$ and since $u_0 \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ and $u_0 \equiv u$ on Ω, we obtain from (8.4) the inequality

$$\left|\int_\Omega u^* d\mu\right| \leq C\|u\|_{BV(\Omega)}, \tag{8.5}$$

which means that $\mu \in BV(\Omega)^*$.

□

8.3. **Remark.** *It is easy to see that if μ is a positive measure in $BV(\Omega)^*$ then its action on $BV(\Omega)$ is given by*

$$< \mu, u >= \int_\Omega u^* d\mu$$

for all $u \in BV(\Omega)$.

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[Nguyen Cong Phuc] Department of Mathematics, Lousiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA. http://www.math.lsu.edu/˜pcnguyen

E-mail address: pcnguyen@math.lsu.edu

[Monica Torres] Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA. http://www.math.purdue.edu/˜torres

E-mail address: torres@math.purdue.edu