

**LERAY'S SELF-SIMILAR SOLUTIONS TO THE
NAVIER-STOKES EQUATIONS WITH PROFILES IN
MARCINKIEWICZ AND MORREY SPACES**

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ABSTRACT. We rule out the existence of Leray's backward self-similar solutions to the Navier-Stokes equations with profiles in $L^{12/5}(\mathbb{R}^3)$ or in the Marcinkiewicz space $L^{q,\infty}(\mathbb{R}^3)$ for $q \in (12/5, 6)$. This follows from a more general result formulated in terms of Morrey spaces and the first order Riesz's potential.

1. INTRODUCTION

The motion of an incompressible fluid in three spatial dimensions, \mathbb{R}^3 , with viscosity $\nu > 0$ and zero external force is described by the Navier-Stokes equations

$$(1.1) \quad \begin{cases} u_t - \nu \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases}$$

with an initial condition $u(x, 0) = u_0(x)$. Here the unknown velocity $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$ and the unknown pressure $p = p(x, t) \in \mathbb{R}$ are defined for each position $x \in \mathbb{R}^3$ and time $t \geq 0$.

Since Leray's work [10] in 1934, there has been a long-standing question whether solutions to (1.1) develop a singularity in finite time, or whether (1.1) admits a time-global smooth solution for any given smooth and compactly supported initial datum u_0 . To look for a singular solution, Leray [10] suggested to consider the backward self-similar solutions of (1.1), i.e., those of the form

$$(1.2) \quad u(x, t) = \lambda(t)U(\lambda(t)x), \quad p(x, t) = \lambda^2(t)P(\lambda(t)x),$$

where

$$\lambda(t) = \frac{1}{\sqrt{2a(T-t)}}, \quad a > 0, \quad T > 0,$$

and $U = (U_1, U_2, U_3)$ and P are defined in the whole \mathbb{R}^3 . Note then that u is defined in $\mathbb{R}^3 \times (-\infty, T)$ and if the profile U is not identically zero then u given by (1.2) develops a singularity at time $t = T$. Here certain natural energy norms of u should be required to be finite. For otherwise the profile $U = \nabla \Phi$ and $P = -\frac{1}{2}|U|^2 - ay \cdot U$, for any non-zero harmonic function Φ , would immediately yield a non-trivial self-similar solution.

By direct calculations, one finds that (u, p) of the form (1.2) is a solution of (1.1) if and only if (U, P) solves the following nonlinear time-independent system in \mathbb{R}^3 :

$$(1.3) \quad \begin{cases} -\nu\Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla P = 0, \\ \operatorname{div} U = 0. \end{cases}$$

Leray's question was open until 1996 when Nečas, Růžička, and Šverák [12] showed that there does not exist a non-trivial solution of the form (1.2) with finite kinetic energy and satisfies the natural *global* energy inequality

$$(1.4) \quad \int_{\mathbb{R}^3} \frac{1}{2}|u(x, t)|^2 dx + \int_{t_1}^t \int_{\mathbb{R}^3} \nu |\nabla u(x, t)|^2 dx dt \leq \int_{\mathbb{R}^3} \frac{1}{2}|u(x, t_1)|^2 dx$$

for all $t \in (t_1, t_2)$. More generally, they proved that if $U \in W_{\operatorname{loc}}^{1,2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ is a weak solution of (1.3) then $U \equiv 0$. Note that the L^3 *global* integrability condition of U holds if the corresponding self-similar solution u satisfies the global energy estimates (1.4).

On the other hand, [12] left open the question of existence of self-similar singularities which satisfy only the *local* energy inequality

$$(1.5) \quad \operatorname{ess\,sup}_{t_3 < t < T} \int_{B_r(x_0)} \frac{1}{2}|u(x, t)|^2 dx + \int_{t_3}^T \int_{B_r(x_0)} \nu |\nabla u(x, t)|^2 dx dt < +\infty,$$

for some ball $B_r(x_0)$ and some $t_3 < T$. This question was later answered by Tsai in [18], where he showed that backward self-similar solutions to (1.1) satisfying (1.5) must also be zero:

Theorem 1.1 (Tsai [18]). *Suppose u is a weak solution of (1.1) satisfying the finite local energy condition (1.5) for some ball $B_r(x_0)$ and some $t_3 < T$. If u is of the form (1.2), then $u \equiv 0$.*

In the same paper Tsai also extended the result of [12] to a super critical range of the integrability condition on the profile U :

Theorem 1.2 (Tsai [18]). *If a weak solution $U \in W_{\operatorname{loc}}^{1,2}(\mathbb{R}^3)$ of (1.3) belongs $L^q(\mathbb{R}^3)$ for some $q \in (3, \infty]$ then it must be zero provided $q \neq \infty$ and constant provided $q = \infty$.*

The proofs of Theorems 1.1 and 1.2 use the important fact that the scalar function

$$\Pi(y) = \frac{1}{2}|U(y)|^2 + P(y) + ay \cdot U(y)$$

satisfies the maximum principle. This idea had been employed earlier in the work of Nečas, Růžička, and Šverák [12] to treat the critical case $q = 3$ mentioned above. See also the earlier work [3, 4, 19] in the context of stationary Navier–Stokes equations in higher dimensions.

We mention that the proof of Theorem 1.1 in [18] also makes use of a celebrated ϵ -regularity criterion due to Caffarelli, Kohn, and Nirenberg [2] to show that if (u, p) of the form (1.2) is a *suitable weak solution* of (1.1) in $B_1(0) \times (T-1, T)$ then $U(y) = O(|y|^{-1})$ as $y \rightarrow \infty$ (see [18, Corollary 4.3]).

On the other hand, one important step in Tsai's proof of Theorem 1.2 is to show that if U is a weak solution of (1.3) and $U \in L^q(\mathbb{R}^3)$ for some $3 < q < \infty$ then $U = o(|y|)$ as $y \rightarrow \infty$ (see [18, Lemma 3.3]). He also remarked that his approach to this pointwise asymptotic estimate fails at the end-point case $q = 3$ and suggested that the sub-critical case $q < 3$ would require a different idea (see [18, Remark 3.2]).

A main goal of this paper is to improve the result of [12] by allowing the profile U to be in spaces strictly larger than $L^3(\mathbb{R}^3)$. Furthermore, by that way we also extend the result of Theorem 1.2 to the sub-critical range $q \in [\frac{12}{5}, 3)$. Indeed, we prove

Theorem 1.3. *Let $U \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$ be a weak solution of (1.3). If $U \in L^{q,\infty}(\mathbb{R}^3)$ for some $q \in (\frac{12}{5}, 6)$ or if $U \in L^{\frac{12}{5}}(\mathbb{R}^3)$ then it must be identically zero.*

In the above theorem, the space $L^{q,\infty}(\mathbb{R}^3)$ is the Marcinkiewicz space (or weak L^q space) defined as the set of measurable functions g in \mathbb{R}^3 such that the quasinorm

$$\|g\|_{L^{q,\infty}(\mathbb{R}^3)} := \sup_{\alpha>0} \alpha |\{x \in \mathbb{R}^3 : |g(x)| > \alpha\}|^{\frac{1}{q}} < +\infty.$$

It is well-known that $L^q(\mathbb{R}^3) \subset L^{q,\infty}(\mathbb{R}^3)$, or more generally $L^{q,s}(\mathbb{R}^3) \subset L^{q,\infty}(\mathbb{R}^3)$ for any $s > 0$, where $L^{q,s}(\mathbb{R}^3)$, $q > 0, s \in (0, \infty)$, is the Lorentz space with quasinorm

$$\|g\|_{L^{q,s}(\mathbb{R}^3)} := \left(q \int_0^\infty \alpha^s |\{x \in \mathbb{R}^3 : |g(x)| > \alpha\}|^{\frac{s}{q}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{s}}.$$

Note that $L^q(\mathbb{R}^3) = L^{q,q}(\mathbb{R}^3)$ and $L^{q,s_1}(\mathbb{R}^3) \subset L^{q,s_2}(\mathbb{R}^3)$ for $0 < s_1 \leq s_2$.

Recently in Phuc [14], it is shown that locally finite energy solutions to the Navier–Stokes equation (1.1) belonging to $L_t^\infty(L_x^{3,s})$ are regular provided $s \neq \infty$. That result strengthens the above mentioned result of [12] as it rules out the existence of self-similar solutions of the form (1.2) with profiles $U \in L^{3,s}(\mathbb{R}^3)$ provided $s \neq \infty$. Thus, in the case $q = 3$ Theorem 1.3 provides the answer to the end-point case $s = \infty$. Note on the other hand that it is still unknown whether $L_t^\infty(L_x^{3,\infty})$ solutions to the Navier–Stokes equations are regular.

In fact, we shall prove a more general result than Theorem 1.3 which allows the profile U to have a very modest decay at infinity. To describe it, recall that the Riesz potential \mathbf{I}_α , $\alpha \in (0, 3)$, on \mathbb{R}^3 is defined by

$$\mathbf{I}_\alpha f(x) = c(\alpha) \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-\alpha}} dy, \quad x \in \mathbb{R}^3,$$

for $f \in L_{\text{loc}}^1(\mathbb{R}^3)$ such that $\int_{|x| \geq 1} |x|^{\alpha-3} |f(x)| dx < +\infty$. Here the normalizing constant

$$c(\alpha) = \frac{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}{\pi^{3/2} 2^\alpha \Gamma(\alpha/2)}.$$

Additionally, we define the Morrey space $\mathcal{M}^{p,\gamma}(\mathbb{R}^3)$, $p \geq 1$, $0 < \gamma \leq 3$, to be the set of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^3)$ such that

$$\int_{B_r(x)} |f(y)|^p dy \leq C r^{3-\gamma},$$

for all $x \in \mathbb{R}^3$ and $r > 0$ with a constant C independent of x and r . The norm $\|f\|_{\mathcal{M}^{p,\gamma}(\mathbb{R}^3)}$ is given by

$$\|f\|_{\mathcal{M}^{p,\gamma}(\mathbb{R}^3)} := \sup_{x \in \mathbb{R}^3, r > 0} r^{\frac{\gamma-3}{p}} \|f\|_{L^p(B_r(x))}.$$

Obviously, when $\gamma = 3$ we have $\mathcal{M}^{p,\gamma}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$. The interest of using such a notation for Morrey spaces is to emphasize that the second index γ acts like the dimension in the Sobolev type embedding theorem. Indeed, it is now well-known from the work of D. R. Adams [1] that \mathbf{I}_α continuously maps $\mathcal{M}^{p,\gamma}(\mathbb{R}^3)$ into $\mathcal{M}^{\frac{\gamma p}{\gamma-\alpha p},\gamma}(\mathbb{R}^3)$ provided $1 < p < \gamma/\alpha$. Thus when $\gamma = 3$ the classical Sobolev embedding theorem is recovered.

We are now ready to state the next result of the paper:

Theorem 1.4. *Let $U \in W^{1,2}_{\text{loc}}(\mathbb{R}^3)$ be a weak solution of (1.3). If for some $\gamma \in (0, 3]$ it holds that*

$$(1.6) \quad \int_{B_r} \mathbf{I}_1(\chi_{B_r}|U|^2) dx \leq C r^{3-\gamma} \quad \forall \text{ balls } B_r \subset \mathbb{R}^3,$$

where χ_{B_r} is the characteristic function of B_r , then $U \equiv 0$. In particular, if $\mathbf{I}_1(|U|^2) \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ for some $\gamma \in (0, 3]$ then U must be identically zero.

We now show that Theorem 1.4 actually implies Theorem 1.3. Indeed, if $U \in L^{12/5}(\mathbb{R}^3)$ then the Sobolev embedding theorem implies that $\mathbf{I}_1(|U|^2) \in L^2(\mathbb{R}^3) = \mathcal{M}^{2,3}(\mathbb{R}^3)$. Thus by Theorem 1.4, if U is also a weak solution of (1.3) then $U \equiv 0$. Similarly, if $U \in L^{q,\infty}(\mathbb{R}^3)$ with $q \in (\frac{12}{5}, 6)$ then $\mathbf{I}_1(|U|^2) \in L^{\frac{3q}{6-q},\infty}(\mathbb{R}^3)$. Since $q > 12/5$ it follows that $\frac{3q}{6-q} > 2$ and thus by Hölder's inequality we have

$$\int_{B_r(x)} \mathbf{I}_1(|U|^2) dy \leq C \|\mathbf{I}_1(|U|^2)\|_{L^{3q/(6-q),\infty}(\mathbb{R}^3)}^2 r^{3-\frac{2(6-q)}{q}}.$$

This yields $\mathbf{I}_1(|U|^2) \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ with $\gamma = \frac{2(6-q)}{q} \in (0, 3)$, and thus Theorem 1.3 is a consequence of Theorem 1.4.

On the other hand, using Adams Embedding Theorem [1] we have another corollary of Theorem 1.4.

Corollary 1.5. *If $U \in W^{1,2}_{\text{loc}}(\mathbb{R}^3)$ is a weak solution of (1.3) such that if $U \in \mathcal{M}^{\frac{4\gamma}{2+\gamma},\gamma}(\mathbb{R}^3)$ for some $\gamma \in (2, 3]$ (or equivalently if $U \in \mathcal{M}^{p,\frac{2p}{4-p}}(\mathbb{R}^3)$ for some $p \in (2, 12/5]$), then $U \equiv 0$.*

The proof of Theorem 1.4 will be given in Section 4. Surprisingly, it is based on an application of Theorem 1.1 above. For that a pressure profile

P is built from U so that the norm of P in a Sobolev space of *negative* order (localized in each ball) is well controlled. Here one has to treat P as a *signed* distribution in \mathbb{R}^3 as no control of $|P|$ is available. This suggests a natural way to control the nonlinear and the pressure terms in the *energy equality*, and a sort of bootstrapping argument based on the energy equality eventually completes the proof.

2. PRELIMINARIES

Throughout the paper we denote by $B_r(x)$ the open ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$, i.e.,

$$B_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}.$$

We write $\partial_j u_i = \frac{\partial u_i}{\partial x_j}$ and use the letters C or c to denote generic constants that could be different from line to line.

For each bounded open set $O \subset \mathbb{R}^3$, we denote by $L^{-1,2}(O)$ the dual of the Sobolev space $W_0^{1,2}(O)$. The latter is defined as the completion of $C_0^\infty(O)$ (the space of smooth functions with compact support in O) under the Dirichlet integral

$$\|\varphi\|_{W_0^{1,2}(O)} = \left(\int_O |\nabla \varphi|^2 dx \right)^{\frac{1}{2}}.$$

We shall use the following well-known representation of a function $\varphi \in C_0^\infty(\mathbb{R}^3)$

$$(2.1) \quad \varphi(x) = \frac{1}{|S^2|} \int_{\mathbb{R}^3} \frac{(x-y) \cdot \nabla \varphi(y)}{|x-y|^3} dy,$$

where $|S^2|$ is the area of the unit sphere (see, e.g., [15, p. 125]). Identity (2.1) can be used to show that

$$(2.2) \quad \|f\|_{L^{-1,2}(O)} \leq C \|\mathbf{I}_1(\chi_O |f|)\|_{L^2(O)}$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^3)$ and any bounded open set $O \subset \mathbb{R}^3$. Indeed, for any $\varphi \in C_0^\infty(O)$ by (2.1) it holds that

$$\begin{aligned} \left| \int_O \varphi(x) f(x) dx \right| &\leq C \int_O \left[\int_O \frac{|\nabla \varphi(y)|}{|x-y|^2} dy \right] |f(x)| dx \\ &= C \int_O |\nabla \varphi(y)| \left[\int_O \frac{|f(x)| dx}{|x-y|^2} \right] dy \\ &\leq C \|\nabla \varphi\|_{L^2(O)} \|\mathbf{I}_1(\chi_O |f|)\|_{L^2(O)}, \end{aligned}$$

as desired.

The following lemma will be needed later. Its proof is based on a simple iteration and can be found in [7, Lemma 6.1].

Lemma 2.1. *Let $I(s)$ be a bounded nonnegative function in the interval $[R_1, R_2]$. Assume that for every $s, \rho \in [R_1, R_2]$ and $s < \rho$ we have*

$$I(s) \leq [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho)$$

with $A, B, C \geq 0$, $\alpha > \beta > 0$ and $\theta \in [0, 1)$. Then it holds that

$$I(R_1) \leq c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C].$$

We now make precise the definition of a weak solution to the system (1.3).

Definition 2.2. A divergence-free vector field $U = (U_1, U_2, U_3)$ is called a weak solution of (1.3) if $U \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$ and if for all divergence-free vector field $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\mathbb{R}^3)$ one has

$$(2.3) \quad \int_{\mathbb{R}^3} (\nu \nabla U \cdot \nabla \phi + [aU + a(y \cdot \nabla)U + (U \cdot \nabla)U] \cdot \phi) dy = 0.$$

3. THE PRESSURE FORMULATION

It is known that every weak solution U of (1.3) is smooth (see [5, 6, 9, 17]). Note that Definition 2.2 does not include a pressure P . However, taking the divergence of (1.3) we formally obtain the pressure equation¹

$$(3.1) \quad -\Delta P = \partial_i \partial_j (U_i U_j).$$

The main goal of this section is to recover a P (with a useful control) from a weak solution U of (1.3) for which (1.6) holds. As P is generally a signed distribution, an estimate of the form (1.6) should not hold if $|U|^2$ is replaced by $|P|$. On the other hand, we observe that (1.6) is equivalent to the condition

$$\| |U|^2 \|_{L^{-1.2}(B_r)}^2 \leq C r^{3-\gamma} \quad \forall \text{ balls } B_r \subset \mathbb{R}^3,$$

for some $\gamma \in (0, 3]$. Thus it is natural to expect that the pressure P should also satisfy a similar condition in which $|U|^2$ is replaced by P .

To construct such a P we start with the following lemma.

Lemma 3.1. *Let $U = (U_1, U_2, U_3)$ satisfy (1.6) for some $\gamma \in (0, 3]$. For each $i, j \in \{1, 2, 3\}$ there exists a vector field $G_{ij} \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ such that*

$$(3.2) \quad -\operatorname{div} G_{ij} = U_i U_j$$

in the sense of distributions, i.e.,

$$\int_{\mathbb{R}^3} G_{ij} \cdot \nabla \varphi dx = \int_{\mathbb{R}^3} U_i U_j \varphi dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$.

¹Here and in what follows we use the usual convention and sum over the repeated indices.

Proof. By hypothesis, we see that for every $i, j \in \{1, 2, 3\}$,

$$\int_{B_r} \mathbf{I}_1(\chi_{B_r} |U_i U_j|)^2 dx \leq C r^{3-\gamma}, \quad \forall B_r \subset \mathbb{R}^3.$$

Let $\{\Psi_N(x)\}$ be a sequence of smooth functions in \mathbb{R}^3 such that $0 \leq \Psi_N \leq 1$, $\Psi_N(x) = 1$ for $|x| \leq N/2$, $\Psi_N(x) = 0$ for $|x| \geq N$, and $|\nabla \Psi_N(x)| \leq c/N$. Thus we also have

$$\int_{B_r} \mathbf{I}_1(\chi_{B_r} \Psi_N |U_i U_j|)^2 dx \leq C r^{3-\gamma}, \quad \forall B_r \subset \mathbb{R}^3,$$

which by (2.2) yields

$$(3.3) \quad \|\Psi_N U_i U_j\|_{L^{-1,2}(B_r)} \leq C r^{\frac{3-\gamma}{2}}, \quad \forall B_r \subset \mathbb{R}^3,$$

where C is independent of N .

For each $N \geq 1$, we claim that there exists a vector field $G_{ij}^N \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ such that

$$(3.4) \quad -\operatorname{div} G_{ij}^N = \Psi_N U_i U_j$$

and

$$(3.5) \quad G_{ij}^N \rightarrow G_{ij} \text{ strongly in } \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$$

as $N \rightarrow +\infty$ for some vector field G_{ij} . In particular, $\lim_{N \rightarrow \infty} \langle G_{ij}^N, \phi \rangle = \langle G_{ij}, \phi \rangle$ for every vector field $\phi \in C_0^\infty(\mathbb{R}^3)$. In view of (3.4), this gives (3.2) as as desired.

Thus it is left to show (3.4) and (3.5). To that end, we define

$$G_{ij}^N(x) = \frac{1}{|S^2|} \int_{\mathbb{R}^3} \frac{y-x}{|y-x|^3} \Psi_N(y) U_i(y) U_j(y) dy.$$

Note that $G_{i,j}^N = \nabla(-\Delta)^{-1}(\Psi_N U_i U_j)$ in the sense of distributions, i.e.,

$$\langle G_{i,j}^N, \phi \rangle := \langle \Psi_N U_i U_j, -\operatorname{div} \mathbf{I}_2(\phi) \rangle = \langle U_i U_j, -\Psi_N \operatorname{div} \mathbf{I}_2(\phi) \rangle$$

for $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\mathbb{R}^3)$. Then obviously, (3.4) holds in the sense of distributions.

On the other hand, for every $k \in \{1, 2, 3\}$, $N, M \geq 1$, and every scalar function $\varphi \in C_0^\infty(B_r)$ we have

$$(3.6) \quad \langle (\Psi_N - \Psi_M) U_i U_j, \partial_k \mathbf{I}_2(\varphi) \rangle = \sum_{\nu=0}^{\infty} \langle (\Psi_N - \Psi_M) U_i U_j, \eta_\nu \partial_k \mathbf{I}_2(\varphi) \rangle,$$

where $\{\eta_\nu\}_{\nu=0}^{\infty}$ is a smooth partition of unity associated to the ball B_r . That is, $\eta_0 \in C_0^\infty(B_{2r})$, $\eta_\nu \in C_0^\infty(B_{2^{\nu+1}r} \setminus B_{2^\nu r})$, $\nu = 1, 2, 3, \dots$, such that

$$0 \leq \eta_\nu \leq 1, \quad |\nabla \eta_\nu| \leq c(2^\nu r)^{-1}, \quad \nu = 0, 1, 2, \dots,$$

and

$$\sum_{\nu=0}^{\infty} \eta_\nu(x) = 1, \quad x \in \mathbb{R}^3.$$

Note that the sum in (3.6) has only a finite number of non-zero terms. Moreover, by the property of $\{\Psi_N\}$ we have

$$\langle (\Psi_N - \Psi_M)U_i U_j, \partial_k \mathbf{I}_2(\varphi) \rangle = \sum_{\nu=N_0}^{\infty} \langle (\Psi_N - \Psi_M)U_i U_j, \eta_\nu \partial_k \mathbf{I}_2(\varphi) \rangle,$$

where $N_0 \rightarrow +\infty$ as $M, N \rightarrow +\infty$. Using this and (3.3), we have

$$\begin{aligned} & |\langle (\Psi_N - \Psi_M)U_i U_j, \partial_k \mathbf{I}_2(\varphi) \rangle| \\ & \leq c \sum_{\nu=N_0}^{\infty} (2^\nu r)^{\frac{3-\gamma}{2}} \|\nabla[\eta_\nu \partial_k \mathbf{I}_2(\varphi)]\|_{L^2(B_{2^{\nu+1}r})} \\ (3.7) \quad & \leq c \sum_{\nu=N_0}^{\infty} (2^\nu r)^{\frac{3-\gamma}{2}} 2^{-\nu \frac{3}{2}} \|\varphi\|_{L^2(B_r)} \\ & \leq c r^{\frac{3-\gamma}{2}} \|\varphi\|_{L^2(B_r)} \sum_{\nu=N_0}^{\infty} 2^{-\frac{\nu\gamma}{2}}. \end{aligned}$$

Here in (3.7) we used the bound

$$\|\nabla[\eta_\nu \partial_k \mathbf{I}_2(\varphi)]\|_{L^2(B_{2^{\nu+1}r})} \leq c 2^{-\nu \frac{3}{2}} \|\varphi\|_{L^2(B_r)},$$

which holds for all $\nu = 0, 1, 2, \dots$, and $\varphi \in C_0^\infty(B_r)$ (see [11, Proposition 4.2(ii)]).

Therefore, for all $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(B_r)$, we find

$$\begin{aligned} |\langle G_{ij}^N - G_{ij}^M, \phi \rangle| &= |\langle (\Psi_N - \Psi_M)U_i U_j, -\operatorname{div} \mathbf{I}_2(\phi) \rangle| \\ &\leq c r^{\frac{3-\gamma}{2}} \|\phi\|_{L^2(B_r)} \sum_{\nu=N_0}^{\infty} 2^{-\frac{\nu\gamma}{2}}. \end{aligned}$$

That is, by duality

$$\|G_{ij}^N - G_{ij}^M\|_{\mathcal{M}^{2,\gamma}(\mathbb{R}^3)} \leq c \sum_{\nu=N_0}^{\infty} 2^{-\frac{\nu\gamma}{2}} \rightarrow 0 \text{ as } M, N \rightarrow +\infty.$$

A similar (and simpler) argument also shows that $G_{i,j}^N \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ for all $N \geq 1$. Thus $\{G_{ij}^N\}$ is a Cauchy sequence in $\mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ which converges to a limit $G_{i,j}$ as claimed in (3.5). \square

We are now ready to construct the desired pressure P .

Lemma 3.2. *Suppose that U is a weak solution of (1.3) such that (1.6) holds for some $0 < \gamma \leq 3$. Let $R_j = \partial_j(-\Delta)^{\frac{1}{2}}$ be the j -th Riesz transform for $j = 1, 2, 3$. Define a distribution P by letting*

$$\langle P, \varphi \rangle = \langle R_i R_j(G_{ij}), \nabla \varphi \rangle, \quad \varphi \in C_0^\infty(\mathbb{R}^3),$$

where $G_{ij} = (G_{ij}^1, G_{ij}^2, G_{ij}^3) \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ is given by Lemma 3.1, and

$$R_i R_j(G_{ij}) = (R_i R_j(G_{ij}^1), R_i R_j(G_{ij}^2), R_i R_j(G_{ij}^3)).$$

Then P satisfies the following growth estimate

$$(3.8) \quad \|P\|_{L^{-1,2}(B_r(x))} \leq C r^{\frac{3-\gamma}{2}}, \quad \forall x \in \mathbb{R}^3, r > 0.$$

Moreover, P satisfies (3.1) and (U, P) smoothly solves

$$(3.9) \quad -\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla P = 0 \quad \text{in } \mathbb{R}^3.$$

Proof. By Lemma 3.1, the vector fields $G_{ij} \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$. Since $R_i R_j$ is bounded on $\mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ (see, e.g., [13]) this implies that P is well-defined and (3.8) holds. Indeed, for any $\varphi \in C_0^\infty(B_r(x))$ we have

$$\begin{aligned} |\langle P, \varphi \rangle| &= \left| \int_{B_r(x)} (R_i R_j(G_{ij})) \cdot \nabla \varphi dy \right| \\ &\leq \|R_i R_j(G_{ij})\|_{L^2(B_r(x))} \|\nabla \varphi\|_{L^2(B_r(x))} \\ &\leq C \|G_{ij}\|_{\mathcal{M}^{2,\gamma}(\mathbb{R}^3)} r^{\frac{3-\gamma}{2}} \|\nabla \varphi\|_{L^2(B_r(x))}, \end{aligned}$$

which obviously yields (3.8).

Using the facts that $-\operatorname{div} G_{ij} = U_i U_j$ and $R_i R_j \Delta \varphi = -\partial_i \partial_j \varphi$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$ (see [15, p. 59]) we can now calculate

$$\begin{aligned} \langle \Delta P, \varphi \rangle &= \int_{\mathbb{R}^3} (R_i R_j(G_{ij})) \cdot \nabla \Delta \varphi dx \\ (3.10) \quad &= \int_{\mathbb{R}^3} G_{ij} \cdot \nabla R_i R_j(\Delta \varphi) dx \\ &= \int_{\mathbb{R}^3} U_i U_j R_i R_j(\Delta \varphi) dx \\ &= - \int_{\mathbb{R}^3} U_i U_j \partial_i \partial_j \varphi dx. \end{aligned}$$

That is, P is a distributional solution of (3.1) and thus by Weyl's lemma it is smooth (since $U_i U_j$ is smooth). Note that the second equality in (3.10) requires an explanation as in general $|G_{ij}| \notin L^2(\mathbb{R}^3)$ unless $\gamma = 3$. But since $|G_{ij}| \in \mathcal{M}^{2,\gamma}(\mathbb{R}^3)$ we have $|G_{ij}| \in L_{\text{loc}}^2(\mathbb{R}^3)$ and moreover

$$(3.11) \quad \int_{\mathbb{R}^3} |G_{ij}|^2 (1 + |x|^2)^{-\frac{\mu}{2}} dx \leq c \|G_{ij}\|_{\mathcal{M}^{2,\gamma}}^2 < +\infty.$$

for any $\mu > 3 - \gamma$. Inequality (3.11) can be found in [8], page 132.

Now let $\chi_{B_R(0)}$ be the characteristic function of $B_R(0)$. Using Hölder's inequality we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (R_i R_j(G_{ij}) - R_i R_j(\chi_{B_R(0)} G_{ij})) \cdot \nabla \Delta \varphi dx \right| \\ &\leq \left(\int_{\mathbb{R}^3} |R_i R_j(G_{ij} - \chi_{B_R(0)} G_{ij})|^2 w(x) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla \Delta \varphi|^2 w(x)^{-1} dx \right)^{\frac{1}{2}}, \end{aligned}$$

where we choose the weight $w(x) = (1 + |x|^2)^{-\frac{\mu}{2}}$. Note that w belongs to the Muckenhoupt A_2 class provided we choose $\mu \in (3 - \gamma, 3)$ (see, e.g., [16,

Chap. V]). Since $R_i R_j$ is bounded on the weighted space $L_w^2(\mathbb{R}^3)$ (see [16, p. 205]) we then have

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} (R_i R_j(G_{ij}) - R_i R_j(\chi_{B_R(0)} G_{ij})) \cdot \nabla \Delta \varphi dx \right| \\ & \leq C \left(\int_{\mathbb{R}^3} |G_{ij} - \chi_{B_R(0)} G_{ij}|^2 w(x) dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus using (3.11) and the Lebesgue's dominated convergence theorem, we eventually find

$$\begin{aligned} \int_{\mathbb{R}^3} (R_i R_j(G_{ij})) \cdot \nabla \Delta \varphi dx &= \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^3} (R_i R_j(\chi_{B_R(0)} G_{ij})) \cdot \nabla \Delta \varphi dx \\ &= \lim_{R \rightarrow +\infty} \int_{\mathbb{R}^3} (\chi_{B_R(0)} G_{ij}) \cdot R_i R_j(\nabla \Delta \varphi) dx \\ &= \int_{\mathbb{R}^3} G_{ij} \cdot \nabla R_i R_j(\Delta \varphi) dx, \end{aligned}$$

as desired. Here the second equality follows since $\chi_{B_R(0)} G_{ij} \in L^2(\mathbb{R}^3)$, and the last equality follows since $R_i R_j(\nabla \Delta \varphi) = \nabla R_i R_j(\Delta \varphi)$ has compact support.

Finally, to prove (3.9), we let

$$F := -\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla P$$

and show that $F \equiv 0$.

Using (2.3) with appropriate test functions ϕ we find $\operatorname{curl} F = 0$. Also, by (3.1) and the fact that $\operatorname{div} U = 0$ we have $\operatorname{div} F = 0$. These imply that $\Delta F = 0$. Thus by the mean-value property of harmonic functions we find

$$(3.12) \quad F(0) = \epsilon^3 \int_{\mathbb{R}^3} F(y) \varphi(\epsilon y) dy$$

for any $\epsilon > 0$ and every radial function $\varphi \in C_0^\infty(B_1(0))$ such that $\int_{\mathbb{R}^3} \varphi dy = 1$ (see [15, p. 275]).

Following [12], by analyticity, to show that $F \equiv 0$ it is enough to verify that $D^\alpha F(0) = 0$ for each for each multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| \geq 0$. To this end, we first apply (3.12) to the harmonic function $D^\alpha F$ and then integrate by parts to obtain

$$D^\alpha F(0) = \epsilon^3 \int_{\mathbb{R}^3} D^\alpha F(y) \varphi(\epsilon y) dy = (-1)^{|\alpha|} \epsilon^3 \int_{\mathbb{R}^3} F(y) \epsilon^{|\alpha|} (D^\alpha \varphi)(\epsilon y) dy,$$

where ϵ and φ are as above. Thus to show that $D^\alpha F(0) = 0$, it is enough to verify that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^3 \int_{\mathbb{R}^3} F(y) \theta(\epsilon y) dy = 0$$

for any function $\theta \in C_0^\infty(B_1(0))$. That is, we need to show that

$$(3.13) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^3 \int_{\mathbb{R}^3} [-\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla P] \theta(\epsilon y) dy = 0.$$

The first four terms in the above expression can be treated similarly. For example, for the term involving $a(y \cdot \nabla)U$ by integrating by parts we have

$$\begin{aligned} & \epsilon^3 \int (y \cdot \nabla)U(y) \theta(\epsilon y) dy \\ &= -\epsilon^3 \int 3U(y) \theta(\epsilon y) dy - \epsilon^3 \int U(y) (\epsilon y_i) (\partial_i \theta)(\epsilon y) dy \\ &= -3\epsilon^3 \int U(y) \theta(\epsilon y) dy - \epsilon^3 \int U(y) \tilde{\theta}(\epsilon y) dy, \end{aligned}$$

where $\tilde{\theta}(y) = (\epsilon y_i) (\partial_i \theta)(y)$.

On the other hand, by Hölder's inequality and a simple change of variables we find

$$\begin{aligned} \left| \int_{\mathbb{R}^3} U(y) \theta(\epsilon y) dy \right| &\leq \left(\int_{B_{1/\epsilon}} |U|^2(y) |\theta(\epsilon y)| dy \right)^{\frac{1}{2}} \left(\int_{B_{1/\epsilon}} |\theta(\epsilon y)| dy \right)^{\frac{1}{2}} \\ &\leq \epsilon^{-\frac{3}{2}} \left(\int_{B_{1/\epsilon}} |U|^2(y) |\theta(\epsilon y)| dy \right)^{\frac{1}{2}} \left(\int_{B_1} |\theta(z)| dz \right)^{\frac{1}{2}} \\ &\leq c \epsilon^{-\frac{3}{2}} \left(\int_{B_{1/\epsilon}} |U|^2(y) |\theta(\epsilon y)| dy \right)^{\frac{1}{2}}. \end{aligned}$$

Thus by (2.2) it holds that

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} U(y) \theta(\epsilon y) dy \right| \\ &\leq c \epsilon^{-\frac{3}{2}} \left(\| |U|^2 \|_{L^{-1,2}(B_{1/\epsilon})} \right)^{\frac{1}{2}} \left(\int_{B_{1/\epsilon}} |\epsilon (\nabla |\theta|)(\epsilon y)|^2 dy \right)^{\frac{1}{4}} \\ &\leq c \epsilon^{-\frac{3}{2} + \frac{1}{2}} \left(\int_{B_{1/\epsilon}} |\mathbf{I}_1(|U|^2)|^2 dy \right)^{\frac{1}{4}} \left(\int_{B_{1/\epsilon}} |(\nabla \theta)(\epsilon y)|^2 dy \right)^{\frac{1}{4}} \\ &\leq c \epsilon^{-1 - \frac{3}{4}} \left(\int_{B_{1/\epsilon}} |\mathbf{I}_1(|U|^2)|^2 dy \right)^{\frac{1}{4}} \left(\int_{B_1} |\nabla \theta(z)|^2 dz \right)^{\frac{1}{4}}, \end{aligned}$$

which by our assumption on U yields

$$\left| \int_{\mathbb{R}^3} U(y) \theta(\epsilon y) dy \right| \leq c \epsilon^{-\frac{7}{4}} (1/\epsilon)^{\frac{3-\gamma}{4}} = c \epsilon^{-\frac{5}{2} + \frac{\gamma}{4}}.$$

Of course, the same inequality also holds with $\tilde{\theta}$ in place of θ . Hence, using these in the expression for $\epsilon^3 \int (y \cdot \nabla) U(y) \theta(\epsilon y) dy$ above we obtain

$$\left| \epsilon^3 \int (y \cdot \nabla) U(y) \theta(\epsilon y) dy \right| \leq c \epsilon^{\frac{1}{2} + \frac{\gamma}{4}} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

For what concerns the term involving ∇P in (3.13), using integration by parts and the bound (3.8) we have

$$\begin{aligned} \left| \epsilon^3 \int \nabla P \theta(\epsilon y) dy \right| &= \left| -\epsilon^4 \int P(y) (\nabla \theta)(\epsilon y) dy \right| \\ &\leq c \epsilon^4 \|P\|_{L^{-1,2}(B_{1/\epsilon})} \left(\int_{B_{1/\epsilon}} |\epsilon (\nabla^2 \theta)(\epsilon y)|^2 dy \right)^{\frac{1}{2}} \\ &\leq c \epsilon^5 (1/\epsilon)^{\frac{3-\gamma}{2}} \left(\int_{B_{1/\epsilon}} |(\nabla^2 \theta)(\epsilon y)|^2 dy \right)^{\frac{1}{2}} \\ &\leq c \epsilon^{5 - \frac{3}{2} - \frac{3}{2} + \frac{\gamma}{2}} \left(\int_{B_1} |\nabla^2 \theta(z)|^2 dz \right)^{\frac{1}{2}}. \end{aligned}$$

Thus we also have

$$\left| \epsilon^3 \int \nabla P \theta(\epsilon y) dy \right| \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

In conclusion, we obtain (3.13) and that completes the proof of the lemma. \square

4. PROOF OF THEOREM 1.4

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Obviously, it is enough to prove the first statement of the theorem. Henceforth, suppose that U is a weak solution of (1.3) such that condition (1.6) holds for some $\gamma \in (0, 3]$. Let the function P be defined as in Lemma 3.2. Then (U, P) smoothly solves (1.3), and thus the functions

$$u(x, t) = \lambda(t) U(\lambda(t)x), \quad \text{and} \quad p(x, t) = \lambda^2(t) P(\lambda(t)x),$$

with $\lambda(t) = [2a(T-t)]^{-1/2}$, $T > 0$, solves the Navier–Stokes equations (1.1) in the classical sense in $\mathbb{R}^3 \times (-\infty, T)$. By Theorem 1.1, it is enough to check that

$$(4.1) \quad \text{ess sup}_{T-1 < t < T} \int_{B_1(0)} \frac{1}{2} |u(x, t)|^2 dx + \int_{T-1}^T \int_{B_1(0)} \nu |\nabla u(x, t)|^2 dx dt < +\infty.$$

To this end, we first observe that for any ball $B_r(x_0) \subset \mathbb{R}^3$ and $t < T$, it holds that

$$(4.2) \quad \left\| |u(\cdot, t)|^2 \right\|_{L^{-1,2}(B_r(x_0))} + \|p(\cdot, t)\|_{L^{-1,2}(B_r(x_0))} \leq c \lambda(t)^{1 - \frac{\gamma}{2}} r^{\frac{3-\gamma}{2}}.$$

Indeed for any $\varphi \in C_0^\infty(B_r(x_0))$ and with $\lambda = \lambda(t)$ we have

$$\begin{aligned} \left| \int_{B_r(x_0)} |u(x, t)|^2 \varphi(x) dx \right| &= \lambda^2 \left| \int_{B_r(x_0)} |U(\lambda x)|^2 \varphi(x) dx \right| \\ &= \lambda^{-1} \left| \int_{B_{r\lambda}(\lambda x_0)} |U(z)|^2 \varphi(z/\lambda) dz \right|. \end{aligned}$$

From this using inequality (2.2) we obtain

$$\begin{aligned} &\left| \int_{B_r(x_0)} |u(x, t)|^2 \varphi(x) dx \right| \\ &\leq c \lambda^{-1} \| |U|^2 \|_{L^{-1,2}(B_{r\lambda}(\lambda x_0))} \left(\int_{B_{r\lambda}(\lambda x_0)} \left| \frac{1}{\lambda} \nabla \varphi \left(\frac{z}{\lambda} \right) \right|^2 dz \right)^{\frac{1}{2}} \\ &\leq c \lambda^{-2} (r\lambda)^{\frac{3-\gamma}{2}} \left(\int_{B_r(x_0)} |\nabla \varphi(x)|^2 \lambda^3 dx \right)^{\frac{1}{2}} \\ &\leq \lambda^{1-\frac{\gamma}{2}} r^{\frac{3-\gamma}{2}} \left(\int_{B_r(x_0)} |\nabla \varphi(x)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

This gives

$$\| |u(\cdot, t)|^2 \|_{L^{-1,2}(B_r(x_0))} \leq c \lambda(t)^{1-\frac{\gamma}{2}} r^{\frac{3-\gamma}{2}}.$$

Likewise, using the bound (3.8) and an analogous argument we obtain a similar bound for p . Thus (4.2) is proved.

Next we shall make use of the well-known energy equality:

$$\begin{aligned} &\int_{B_R} |u(x, t)|^2 \phi(x, t) dx + 2\nu \int_{T_1}^t \int_{B_R} |\nabla u|^2 \phi(x, s) dx ds \\ &= \int_{T_1}^t \int_{B_R} |u|^2 (\phi_t + \nu \Delta \phi) dx ds + \int_{T_1}^t \int_{B_R} (|u|^2 + 2p) u \cdot \nabla \phi dx ds, \end{aligned}$$

which holds for every ball $B_R = B_R(0)$, $t \in (T_1, T)$, and any nonnegative function $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ vanishing in a neighborhood of the parabolic boundary $B_R \times \{t = T_1\} \cup \partial B_R \times [T_1, T]$ of the cylinder $B_R \times (T_1, T)$.

Let $T_\epsilon = T - \epsilon$ for sufficiently small $\epsilon > 0$, say, $\epsilon \in (1/2, 0)$. For any balls

$$B_s = B_s(0), \quad B_\rho = B_\rho(0),$$

with $1 \leq s < \rho \leq 2$, we consider a test function $\phi(x, t) = \eta_1(x)\eta_2(t)$ where $\eta_1 \in C_0^\infty(B_\rho)$, $0 \leq \eta_1 \leq 1$ in \mathbb{R}^n , $\eta_1 \equiv 1$ on B_s , and

$$|\nabla^\alpha \eta_1| \leq \frac{c}{(\rho - s)^{|\alpha|}}$$

for all multi-indices α with $|\alpha| \leq 3$. The function $\eta_2(t)$ is chosen so that $\eta_2 \in C_0^\infty(T_\epsilon - \rho^2, T_\epsilon + \rho^2)$, $0 \leq \eta_2 \leq 1$ in \mathbb{R} , $\eta_2(t) \equiv 1$ for $t \in [T_\epsilon - s^2, T_\epsilon + s^2]$, and

$$|\eta_2'(t)| \leq \frac{c}{\rho^2 - s^2} \leq \frac{c}{\rho - s}.$$

Thus

$$|\nabla \phi_t| + |\nabla \Delta \phi| \leq \frac{c}{(\rho - s)^3}, \quad |\nabla^2 \phi| \leq \frac{c}{(\rho - s)^2}, \quad |\nabla \phi| \leq \frac{c}{\rho - s}.$$

Let

$$I(s, \epsilon) = \sup_{T_\epsilon - s^2 \leq t \leq T_\epsilon} \int_{B_s} |u(x, t)|^2 dx + \int_{T_\epsilon - s^2}^{T_\epsilon} \int_{B_s} |\nabla u(x, t)|^2 dx dt,$$

which is a finite quantity provided $\epsilon > 0$.

Using ϕ as a test function in the energy equality above we have

$$\begin{aligned} I(s, \epsilon) &\leq c \int_{T_\epsilon - \rho^2}^{T_\epsilon} \| |u|^2 \|_{L^{-1,2}(B_\rho)} \|\nabla \phi_t + \nabla \Delta \phi\|_{L^2(B_\rho)} dt + \\ &\quad + c \int_{T_\epsilon - \rho^2}^{T_\epsilon} \| |u|^2 + 2p \|_{L^{-1,2}(B_\rho)} \|\nabla u \cdot \nabla \phi + u \cdot \nabla^2 \phi\|_{L^2(B_\rho)} dt \\ &\leq c \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} \|\nabla \phi_t + \nabla \Delta \phi\|_{L^2(B_\rho)} dt + \\ &\quad + c \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} \|\nabla u \cdot \nabla \phi + u \cdot \nabla^2 \phi\|_{L^2(B_\rho)} dt, \end{aligned}$$

where we used (4.2) and the fact that $1 \leq \rho \leq 2$. From the choice of ϕ , this gives

$$\begin{aligned} I(s, \epsilon) &\leq \frac{c}{(\rho - s)^3} \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} dt + \\ &\quad + c \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} \left(\frac{\|\nabla u\|_{L^2(B_\rho)}}{\rho - s} + \frac{\|u\|_{L^2(B_\rho)}}{(\rho - s)^2} \right) dt. \end{aligned}$$

Thus by Hölder's inequality we find

$$\begin{aligned} I(s, \epsilon) &\leq \frac{c}{(\rho - s)^3} \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} dt + \\ &\quad + \frac{c}{\rho - s} \left(\int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{2-\gamma} dt \right)^{\frac{1}{2}} \left(\int_{T_\epsilon - \rho^2}^{T_\epsilon} \|\nabla u\|_{L^2(B_\rho)}^2 dt \right)^{\frac{1}{2}} + \\ &\quad + \frac{c}{(\rho - s)^2} \left(\int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{2-\gamma} dt \right)^{\frac{1}{2}} \left(\sup_{T_\epsilon - \rho^2 \leq t \leq T_\epsilon} \|u(\cdot, t)\|_{L^2(B_\rho)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now using Young's inequality we arrive at

$$I(s, \epsilon) \leq \frac{c}{(\rho - s)^3} \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} dt + \left(\frac{c}{(\rho - s)^2} + \frac{c}{(\rho - s)^4} \right) \int_{T_\epsilon - \rho^2}^{T_\epsilon} \lambda(t)^{2-\gamma} dt + \frac{1}{2} I(\rho, \epsilon)$$

As this holds for all $1 \leq s < \rho \leq 2$ by Lemma 2.1 we find

$$\begin{aligned} I(1, \epsilon) &\leq C \int_{T_\epsilon - 4}^{T_\epsilon} \lambda(t)^{1-\frac{\gamma}{2}} dt + C \int_{T_\epsilon - 4}^{T_\epsilon} \lambda(t)^{2-\gamma} dt \\ &\leq C \int_{T-5}^T [\lambda(t)^{1-\frac{\gamma}{2}} + \lambda(t)^{2-\gamma}] dt \\ &\leq C(a, \gamma) < +\infty. \end{aligned}$$

Since this holds for every $\epsilon \in (1/2, 0)$ we deduce that (4.1) holds and thus the proof is complete. \square

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