

# LOCAL ENERGY BOUNDS AND $\epsilon$ -REGULARITY CRITERIA FOR THE 3D NAVIER-STOKES SYSTEM

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ABSTRACT. The system of three dimensional Navier-Stokes equations is considered. We obtain some new local energy bounds that enable us to improve several  $\epsilon$ -regularity criteria. The key idea here is to view the ‘head pressure’ as a signed distribution belonging to certain fractional Sobolev space of negative order. This allows us to capture the oscillation of the pressure in our criteria.

## 1. INTRODUCTION

We are concerned with the three dimensional Navier-Stokes system

$$(1.1) \quad \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0,$$

where  $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  is the velocity of the fluid and the scalar function  $p = p(x, t)$  is its pressure. The system (1.1) also comes with certain boundary and initial conditions but we shall not specify them here.

Since the seminal work of Leray [10] and Hopf [6], it is known that there exist global in time weak solutions with finite energy to the initial-boundary value problem associated to (1.1). Such solutions are now called Leray-Hopf weak solutions. However, the questions of regularity and uniqueness of Leray-Hopf weak solutions are still unresolved.

To investigate the regularity of system (1.1), in the fundamental paper [1], Caffarelli-Kohn-Nirenberg introduced the notion of suitable weak solutions. They obtained existence as well as partial regularity for suitable weak solutions. Their fundamental result states that the one-dimensional parabolic Hausdorff measure of the possible singular set of suitable weak solutions is zero (see also [2]). The proof of this partial regularity result is based on the following  $\epsilon$ -regularity criterion: there is an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1 = B_1(0) \times (-1, 0)$  and satisfies

$$\limsup_{r \rightarrow 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 dy ds \leq \epsilon,$$

then  $u$  is regular at the point  $(0, 0)$ , i.e.,  $u \in L^\infty(Q_r)$  for some  $r > 0$ . Here we write  $Q_r = B_r(0) \times (-r^2, 0)$ .

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In turn the proof of this  $\epsilon$ -regularity criterion is based on another one that involves both  $u$  and  $p$  but requires the smallness at only one scale:

**Theorem 1.1** ([1]). *There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$(1.2) \quad \int_{Q_1} (|u|^3 + |p|^{3/2}) dy ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

Theorem 1.1 was first proved in [1, Proposition 1] in a slightly more general form, namely, the smallness condition (1.2) is replaced by the condition

$$(1.3) \quad \int_{Q_1} (|u|^3 + |p||u|) dy ds + \int_{-1}^0 \|p\|_{L^1(B_1(0))}^{\frac{5}{4}} ds \leq \epsilon.$$

The proof presented in [1] is based on an inductive argument that goes back to Scheffer [15]. Later Lin [11, Theorem 3.1] gave a new proof based on a compactness argument. In fact, he showed that under (1.2) the solution is Hölder continuous with respect to the space-time parabolic metric on the closure of  $B_{1/2}(0) \times (-1/4, 0)$ . See also [9, Lemma 3.1]. We mention that Theorem 1.1 has also been used as an important tool in many other papers such as [12, 5, 3, 18, 13], etc.

A more constructive approach to Theorem 1.1 can be found in [17] in which Vasseur used De Giorgi iteration technique to obtain it in the following form.

**Theorem 1.2** ([17]). *For each  $p > 1$  there exists an  $\epsilon(p) > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$\sup_{t \in [-1, 0]} \int_{B_1(0)} |u(x, t)|^2 dx + \int_{Q_1} |\nabla u|^2 dy ds + \int_{-1}^0 \|p\|_{L^1(B_1(0))}^p ds \leq \epsilon(p),$$

then  $u \in L^\infty(Q_{1/2})$ .

It is not hard to see from the generalized energy inequality (see Definition 2.1 below) and a simple covering argument that Theorem 1.2 indeed implies Theorem 1.1.

We now state another related  $\epsilon$ -regularity criterion that was obtained and used in [18, Proposition 5.1].

**Theorem 1.3** ([18]). *There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$(1.4) \quad \sup_{t \in [-1, 0]} \int_{B_1(0)} |u(x, t)|^2 dx + \int_{-1}^0 \|u\|_{L^4(B_1(0))}^2 ds + \int_{-1}^0 \|p\|_{L^2(B_1(0))} ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

Finally, we mention yet another  $\epsilon$ -regularity result that was obtained by the second named author in [13, Proposition 3.2].

**Theorem 1.4** ([13]). *There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$\int_{-1}^0 \|u\|_{L^{\frac{12}{5}}(B_1(0))}^4 ds + \int_{-1}^0 \|p\|_{L^{\frac{6}{5}}(B_1(0))}^2 ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

The goal of this paper is to sharpen and unify the results obtained in Theorems 1.1-1.4. Our first result says that in fact one can take  $p = 1$  in Theorem 1.2, i.e., we prove

**Theorem 1.5.** *There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$\sup_{t \in [-1, 0]} \int_{B_1(0)} |u(x, t)|^2 dx + \int_{Q_1} |\nabla u|^2 dy ds + \int_{-1}^0 \|p\|_{L^1(B_1(0))} ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

This theorem implies that in the condition (1.3) of Caffarelli, Kohn, and Nirenberg one can replace the power  $\frac{5}{4}$  in the pressure term by 1. Our next result reads as follows.

**Theorem 1.6.** *Let  $\alpha \in [6/5, 2]$  and  $\beta = \frac{4\alpha}{7\alpha-6} \in [1, 2]$ . There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$\int_{-1}^0 \|u\|_{L^{2\alpha}(B_1(0))}^{2\beta} ds + \int_{-1}^0 \|p\|_{L^\alpha(B_1(0))}^\beta ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

The case  $(\alpha, \beta) = (18/13, 3/2)$  gives a spatial improvement of Theorem 1.1, whereas the case  $(\alpha, \beta) = (3/2, 4/3)$  gives a time improvement. Kukavica [8, p. 2845] mentioned the issue whether the number 3 in (1.2) can be replaced by some  $q < 3$ . Indeed, this is the case if we take  $q = 2\alpha = 2\beta = 20/7$ . This gives both space and time improvement of Theorem 1.1. Moreover, Theorems 1.3 and 1.4 are special end-point cases of Theorem 1.6, with  $\alpha = 2$  and  $\alpha = 6/5$ , respectively. In fact, it also implies that the first term in condition (1.4) can be dropped.

Theorem 1.6 is a consequence of the following result.

**Theorem 1.7.** *Let  $\sigma \in [0, 1]$ . There exists an  $\epsilon > 0$  such that if  $u$  is a suitable weak solution in  $Q_1$  and satisfies*

$$\int_{-1}^0 \| |u|^2 \|_{L^{-\sigma, 2}(B_1(0))}^{\frac{2}{2-\sigma}} ds + \int_{-1}^0 \|p\|_{L^{-\sigma, 2}(B_1(0))}^{\frac{2}{2-\sigma}} ds \leq \epsilon,$$

then  $u \in L^\infty(Q_{1/2})$ .

The space  $L^{-\sigma,2}(B_1(0))$  is the dual of the space of functions  $f$  in the homogeneous Sobolev space  $\dot{H}^\sigma(\mathbb{R}^3)$  such that  $\text{supp} f \subset \overline{B_1(0)}$ . We have  $L^{0,2}(B_1(0)) = L^2(B_1(0))$ . Interestingly, unlike the norm  $\|p\|_{L^\alpha(B_1(0))}$ , for  $\sigma \in (0, 1]$  the norm  $\|p\|_{L^{-\sigma,2}(B_1(0))}$  can ‘capture’ the oscillation of  $p$ . Namely, it may happen that there exists  $f \in L^{-\sigma,2}(B_1(0)) \cap L^1(B_1(0))$  but  $|f| \notin L^{-\sigma,2}(B_1(0))$ . In the case  $\sigma = 1$ , one can take for example the function  $f(x) = |x|^{-\epsilon-s} \sin(|x|^{-\epsilon})$  with  $s = 2.4$  and  $\epsilon = 0.2$ . See also the recent paper [14] for this kind of example in the context of  $(BV)^*$ , the dual of space of functions of bounded variation. We mention that by Lemma 2.3 below, this theorem implies Theorem 1.6.

The proof of Theorem 1.7 is based on Theorem 1.5 and the following new local energy bounds for suitable weak solutions.

**Theorem 1.8.** *Let  $\sigma \in [0, 1]$ . There exists a constant  $C > 0$  such that for any suitable weak solution  $(u, p)$  in  $Q_1$  we have*

$$\begin{aligned} & \sup_{-\frac{1}{4} \leq t \leq 0} 2 \int_{B_{1/2}(0)} |u(x, t)|^2 dx + 2 \int_{Q_{1/2}} |\nabla u(x, s)|^2 dx ds \\ & \leq C \left( \int_{-1}^0 \| |u^2| \|_{L^{-\sigma,2}(B_1(0))}^{\frac{2}{2-\sigma}} ds \right)^{\frac{2-\sigma}{2}} + \\ & \quad + C \left( \int_{-1}^0 \| |u^2| + 2p \|_{L^{-\sigma,2}(B_1(0))}^{\frac{2}{2-\sigma}} ds \right)^{2-\sigma}. \end{aligned}$$

For this result at every point and every scale we refer to Proposition 3.1 below. See also Proposition 3.2.

## 2. PRELIMINARIES

Throughout the paper we use the following notations for balls and parabolic cylinders:

$$B_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}, \quad x \in \mathbb{R}^3, r > 0,$$

and

$$Q_r(z) = B_r(x) \times (t - r^2, t) \quad \text{with } z = (x, t).$$

The homogeneous Sobolev space  $\dot{H}^\sigma(\mathbb{R}^3)$ ,  $\sigma \in \mathbb{R}$ , is the space of temper distributions  $f$  for which  $\|f\|_{\dot{H}^\sigma(\mathbb{R}^3)} < +\infty$ . Here we define

$$\|f\|_{\dot{H}^\sigma(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} |\xi|^{2\sigma} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad \sigma \in \mathbb{R}.$$

The space  $L^{\sigma,2}(B_r(x)) := \left\{ f \in \dot{H}^\sigma(\mathbb{R}^3) : \text{supp } f \subset \overline{B_r(x)} \right\}$ , and its corresponding the dual space is denoted by  $L^{-\sigma,2}(B_r(x))$ .

The following scaling invariant quantities will be employed:

$$A(z_0, r) = A(u, z_0, r) = \sup_{t_0 - r^2 \leq t \leq t_0} r^{-1} \int_{B_r(x_0)} |u(x, t)|^2 dx,$$

$$\begin{aligned}
 B(z_0, r) &= B(u, z_0, r) = r^{-1} \int_{Q_r(z_0)} |\nabla u(x, t)|^2 dx dt, \\
 C_\sigma(z_0, r) &= C_\sigma(u, z_0, r) = r^{-\frac{3}{2-\sigma}} \int_{t_0-r^2}^{t_0} \| |u|^2 \|_{L^{-\sigma, 2}(B_r(x_0))}^{\frac{2}{2-\sigma}} dt, \\
 C_{\alpha, \beta}(z_0, r) &= C_{\alpha, \beta}(u, z_0, r) = r^{-\frac{3\beta}{2}} \int_{t_0-r^2}^{t_0} \| u \|_{L^{2\alpha}(B_r(x_0))}^{2\beta} dt, \\
 D_\sigma(z_0, r) &= D_\sigma(u, z_0, r) = r^{-\frac{3}{2-\sigma}} \int_{t_0-r^2}^{t_0} \| p \|_{L^{-\sigma, 2}(B_r(x_0))}^{\frac{2}{2-\sigma}} dt, \\
 D_{\alpha, \beta}(z_0, r) &= D_{\alpha, \beta}(p, z_0, r) = r^{-\frac{3\beta}{2}} \int_{t_0-r^2}^{t_0} \| p \|_{L^\alpha(B_r(x_0))}^\beta dt.
 \end{aligned}$$

We now recall the the notion of suitable weak solutions that was first introduced in Caffarelli-Kohn-Nirenberg [1]. Here we use the version of F.-H. Lin [11] that imposes the  $3/2$  space-time integrability condition on the pressure.

**Definition 2.1.** *Let  $\omega$  be an open set in  $\mathbb{R}^3$  and let  $-\infty < a < b < \infty$ . We say that a pair  $(u, p)$  is a suitable weak solution to the Navier-Stokes equations in  $Q = \omega \times (a, b)$  if the following conditions hold:*

- (i)  $u \in L^\infty(a, b; L^2(\omega)) \cap L^2(a, b; W^{1, 2}(\omega))$  and  $p \in L^{3/2}(\omega \times (a, b))$ ;
- (ii)  $(u, p)$  satisfies the Navier-Stokes equations in the sense of distributions. That is,

$$\int_a^b \int_\omega \{-u \psi_t + \nabla u : \nabla \psi - (u \otimes u) : \nabla \psi - p \operatorname{div} \psi\} dx dt = 0$$

for all vector fields  $\psi \in C_0^\infty(\omega \times (a, b); \mathbb{R}^3)$ , and

$$\int_{\omega \times \{t\}} u(x, t) \cdot \nabla \phi(x) dx = 0$$

for a.e.  $t \in (a, b)$  and all real valued functions  $\phi \in C_0^\infty(\omega)$ ;

- (iii)  $(u, p)$  satisfies the local generalized energy inequality

$$\begin{aligned}
 & \int_\omega |u(x, t)|^2 \phi(x, t) dx + 2 \int_a^t \int_\omega |\nabla u|^2 \phi(x, s) dx ds \\
 & \leq \int_a^t \int_\omega |u|^2 (\phi_t + \Delta \phi) dx ds + \int_a^t \int_\omega (|u|^2 + 2p) u \cdot \nabla \phi dx ds
 \end{aligned}$$

for a.e.  $t \in (a, b)$  and any nonnegative function  $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$  vanishing in a neighborhood of the parabolic boundary  $\partial' Q = \omega \times \{t = a\} \cup \partial \omega \times [a, b]$ .

We next state several lemmas that are needed in this paper.

**Lemma 2.2.** *Given  $f \in \dot{H}^{s_0} \cap \dot{H}^{s_1}$ ,  $s_0, s_1 \in \mathbb{R}$  and  $0 < \theta < 1$ , the following Gagliardo-Nirenberg type inequality holds*

$$\|f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^{s_0}}^{1-\theta} \|f\|_{\dot{H}^{s_1}}^\theta$$

with  $s = (1 - \theta)s_0 + \theta s_1$ .

The proof of this lemma simply follows from Hölder's inequality.

**Lemma 2.3.** *For any ball  $B_r(x) \subset \mathbb{R}^3$  and any number  $\sigma \in [0, \frac{3}{2})$  one has that  $L^{\frac{6}{3+2\sigma}}(B_r(x)) \subset L^{-\sigma,2}(B_r(x))$  and*

$$\|f\|_{L^{-\sigma,2}(B_r(x))} \leq C \|f\|_{L^{\frac{6}{3+2\sigma}}(B_r(x))}.$$

*Proof.* Observe that

$$\|f\|_{L^{-\sigma,2}(B_r(x))} = \sup_{\varphi} \int_{B_r(x)} f(y)\varphi(y)dy,$$

where the sup is taken over  $\varphi \in L^{\sigma,2}(B_r(x))$  such that  $\|\varphi\|_{\dot{H}^\sigma(\mathbb{R}^3)} \leq 1$ . Thus by Hölder and Sobolev's inequalities we find

$$\begin{aligned} \|f\|_{L^{-\sigma,2}(B_r(x))} &\leq \\ &\leq \sup_{\varphi} \left( \int_{B_r(x)} |f(x)|^{\frac{6}{3+2\sigma}} dx \right)^{\frac{3+2\sigma}{6}} \left( \int_{B_r(x)} |\varphi(y)|^{\frac{6}{3-2\sigma}} dy \right)^{\frac{3-2\sigma}{6}} \\ &\leq C \|f\|_{L^{\frac{6}{3+2\sigma}}(B_r(x))} \sup_{\varphi} \|\varphi\|_{\dot{H}^\sigma(\mathbb{R}^3)} \leq C \|f\|_{L^{\frac{6}{3+2\sigma}}(B_r(x))}. \end{aligned}$$

□

A proof of the following lemma can be found in [4, Lemma 6.1].

**Lemma 2.4.** *Let  $I(s)$  be a bounded nonnegative function in the interval  $[R_1, R_2]$ . Assume that for every  $s, \rho \in [R_1, R_2]$  and  $s < \rho$  we have*

$$I(s) \leq [A(\rho - s)^{-\alpha} + B(\rho - s)^{-\beta} + C] + \theta I(\rho)$$

*with  $A, B, C \geq 0$ ,  $\alpha > \beta > 0$  and  $\theta \in [0, 1)$ . Then there holds*

$$I(R_1) \leq c(\alpha, \theta)[A(R_2 - R_1)^{-\alpha} + B(R_2 - R_1)^{-\beta} + C].$$

We shall also need the following Sobolev interpolation inequality (see, e.g., (1.2) of [9]).

**Lemma 2.5.** *Let  $B_r \subset \mathbb{R}^3$ . For any function  $u \in W^{1,2}(B_r)$  such that  $\int_{B_r} u dx = 0$  and any  $q \in [2, 6]$ , it holds that*

$$\int_{B_r} |u|^q dx \leq C(q) \left( \int_{B_r} |\nabla u|^2 dx \right)^{3q/4-3/2} \left( \int_{B_r} |u|^2 dx \right)^{-q/4+3/2}.$$

Lemma 2.5 implies the following well-known result (see, e.g., [9, Lemma 5.1]).

**Lemma 2.6.** *Let  $u(x, t)$  be a function in  $Q_\rho(z_0)$  for some  $\rho > 0$ . Then for any  $r \in (0, \rho]$  we have*

$$r^{-2} \int_{Q_r(z_0)} |u|^3 dx dt \leq C \left( \frac{\rho}{r} \right)^3 A(z_0, \rho)^{3/4} B(z_0, \rho)^{3/4} + C \left( \frac{r}{\rho} \right)^3 A(z_0, \rho)^{3/2}.$$

## 3. LOCAL ENERGY ESTIMATES

We prove Theorem 1.8 in this section. We will do it at every point and every scale. The proof employs the idea of viewing the ‘head pressure’  $\frac{1}{2}|u|^2 + p$  as a signed distribution in  $L^{-\sigma,2}$ .

**Proposition 3.1.** *Suppose that  $(u, p)$  is a suitable weak solution to the Navier-Stokes equations in  $Q_r(z_0)$ . Then it holds that*

$$\begin{aligned} A(z_0, r/2) + B(z_0, r/2) &\leq C C_\sigma(z_0, r)^{\frac{2-\sigma}{2}} + \\ &+ C \left[ r^{\frac{-3}{2-\sigma}} \int_{t_0-r^2}^{t_0} \left( \| |u|^2 + 2p \|_{L^{-\sigma,2}(B_r(x_0))}^{\frac{2}{2-\sigma}} dt \right)^{2-\sigma} \right] \end{aligned}$$

for any  $\sigma \in [0, 1]$ .

*Proof.* For  $z_0 = (x_0, t_0)$  and  $r > 0$ , we consider the cylinders

$$Q_s(z_0) = B_s(x_0) \times (t_0 - s^2, t_0) \subset Q_\rho(z_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0),$$

where  $r/2 \leq s < \rho \leq r$ .

Let  $\phi(x, t) = \eta_1(x)\eta_2(t)$  where  $\eta_1 \in C_0^\infty(B_\rho(x_0))$ ,  $0 \leq \eta_1 \leq 1$  in  $\mathbb{R}^n$ ,  $\eta_1 \equiv 1$  on  $B_s(x_0)$ , and

$$|\nabla^\alpha \eta_1| \leq \frac{c}{(\rho - s)^{|\alpha|}}$$

for all multi-indices  $\alpha$  with  $|\alpha| \leq 3$ . The function  $\eta_2(t)$  is chosen so that  $\eta_2 \in C_0^\infty(t_0 - \rho^2, t_0 + \rho^2)$ ,  $0 \leq \eta_2 \leq 1$  in  $\mathbb{R}$ ,  $\eta_2(t) \equiv 1$  for  $t \in [t_0 - s^2, t_0 + s^2]$ , and

$$|\eta_2'(t)| \leq \frac{c}{\rho^2 - s^2} \leq \frac{c}{r(\rho - s)}.$$

Then it holds that

$$|\phi_t| \leq \frac{c}{r(\rho - s)}, \quad |\nabla \phi_t| \leq \frac{c}{r(\rho - s)^2},$$

$$|\nabla^3 \phi| \leq \frac{c}{(\rho - s)^3}, \quad |\nabla^2 \phi| \leq \frac{c}{(\rho - s)^2}, \quad |\nabla \phi| \leq \frac{c}{\rho - s}.$$

We next define

$$I(s) = I_1(s) + I_2(s),$$

where

$$I_1(s) = \sup_{t_0 - s^2 \leq t \leq t_0} \int_{B_s(x_0)} |u(x, t)|^2 dx = s A(z_0, s)$$

and

$$I_2(s) = \int_{t_0 - s^2}^{t_0} \int_{B_s(x_0)} |\nabla u(x, t)|^2 dx dt = s B(z_0, s).$$

For  $0 \leq \sigma \leq 1$ , using  $\phi$  as a test function in the generalized energy inequality we find

$$\begin{aligned}
(3.1) \quad I(s) &\leq \int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{L^{-\sigma,2}(B_\rho(x_0))} \| \phi_t + \Delta \phi \|_{L^{\sigma,2}(B_\rho(x_0))} dt + \\
&\quad + \int_{t_0-\rho^2}^{t_0} \left\{ \| |u|^2 + 2p \|_{L^{-\sigma,2}(B_\rho(x_0))} \| u \cdot \nabla \phi \|_{L^{\sigma,2}(B_\rho(x_0))} \right\} dt \\
&=: J_1 + J_2.
\end{aligned}$$

Applying the Gagliardo-Nirenberg type inequality (Lemma 2.2), properties of the test function  $\phi$ , and Hölder's inequality we have

$$\begin{aligned}
J_1 &\leq C \int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{L^{-\sigma,2}(B_\rho(x_0))} \times \\
&\quad \times \left\{ \| \phi_t + \Delta \phi \|_{L^2(B_\rho(x_0))}^{1-\sigma} \| \nabla \phi_t + \nabla \Delta \phi \|_{L^2(B_\rho(x_0))}^\sigma \right\} dt \\
&\leq C \frac{\rho^{\frac{3}{2}}}{(\rho-s)^{2+\sigma}} \int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{L^{-\sigma,2}(B_\rho(x_0))} dt \\
&\leq C \frac{\rho^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} \left( \int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{L^{-\sigma,2}(B_\rho(x_0))}^{\frac{2}{2-\sigma}} dt \right)^{\frac{2-\sigma}{2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_2 &\leq C \left( \int_{t_0-\rho^2}^{t_0} \| |u|^2 + 2p \|_{L^{-\sigma,2}(B_\rho(x_0))}^{\frac{2}{2-\sigma}} dt \right)^{\frac{2-\sigma}{2}} \times \\
&\quad \times \left( \int_{t_0-\rho^2}^{t_0} \| u \cdot \nabla \phi \|_{L^2(B_\rho(x_0))}^{\frac{2(1-\sigma)}{\sigma}} \| \nabla u \cdot \nabla \phi + u \cdot \nabla^2 \phi \|_{L^2(B_\rho(x_0))}^2 dt \right)^{\frac{\sigma}{2}}.
\end{aligned}$$

Let us set

$$X = \int_{t_0-r^2}^{t_0} \| |u|^2 \|_{L^{-\sigma,2}(B_r(x_0))}^{\frac{2}{2-\sigma}} dt,$$

$$Y = \int_{t_0-r^2}^{t_0} \| |u|^2 + 2p \|_{L^{-\sigma,2}(B_r(x_0))}^{\frac{2}{2-\sigma}} dt.$$

Then combining (3.1) with the estimates for  $J_1$  and  $J_2$ , it follows that

$$\begin{aligned}
I(s) &\leq C \frac{\rho^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} X^{\frac{2-\sigma}{2}} + Y^{\frac{2-\sigma}{2}} \sup_{t_0-\rho^2 \leq t \leq t_0} \|u \cdot \nabla \phi\|_{L^2(B_\rho(x_0))}^{1-\sigma} \times \\
&\quad \times \left( \int_{t_0-\rho^2}^{t_0} \|\nabla u \cdot \nabla \phi + u \cdot \nabla^2 \phi\|_{L^2(B_\rho(x_0))}^2 dt \right)^{\frac{\sigma}{2}} \\
&\leq C \frac{\rho^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} X^{\frac{2-\sigma}{2}} + C Y^{\frac{2-\sigma}{2}} \left( \frac{I_1(\rho)}{(\rho-s)^2} \right)^{\frac{1-\sigma}{2}} \times \\
&\quad \times \left\{ \left( \frac{\rho^2 I_1(\rho)}{(\rho-s)^4} \right)^{\frac{\sigma}{2}} + \left( \frac{I_2(\rho)}{(\rho-s)^2} \right)^{\frac{\sigma}{2}} \right\}.
\end{aligned}$$

Thus, using  $r/2 \leq \rho \leq r$  and  $I_1(\rho), I_2(\rho) \leq I(\rho)$ , we get

$$\begin{aligned}
I(s) &\leq C \frac{\rho^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} X^{\frac{2-\sigma}{2}} + C Y^{\frac{2-\sigma}{2}} \frac{\rho^\sigma I_1(\rho)^{\frac{1}{2}}}{(\rho-s)^{1+\sigma}} \\
&\quad + C Y^{\frac{2-\sigma}{2}} \left( \frac{I_1(\rho)}{(\rho-s)^2} \right)^{\frac{1-\sigma}{2}} \left( \frac{I_2(\rho)}{(\rho-s)^2} \right)^{\frac{\sigma}{2}} \\
&\leq C \frac{r^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} X^{\frac{2-\sigma}{2}} + C \frac{Y^{\frac{2-\sigma}{2}} r^\sigma}{(\rho-s)^{1+\sigma}} I(\rho)^{\frac{1}{2}} + C \frac{Y^{\frac{2-\sigma}{2}}}{(\rho-s)} I(\rho)^{\frac{1}{2}}.
\end{aligned}$$

Then by Young's inequality it follows that

$$\begin{aligned}
I(s) &\leq C \frac{r^{\frac{3}{2}+\sigma}}{(\rho-s)^{2+\sigma}} X^{\frac{2-\sigma}{2}} + C \left[ \frac{r^{2\sigma}}{(\rho-s)^{2+2\sigma}} + \frac{1}{(\rho-s)^2} \right] Y^{2-\sigma} \\
&\quad + \frac{1}{2} I(\rho).
\end{aligned}$$

As this holds for all  $r/2 \leq s < \rho \leq r$  by Lemma 2.4 we obtain

$$I(r/2) \leq C \frac{X^{\frac{2-\sigma}{2}}}{r^{1/2}} + C \frac{Y^{2-\sigma}}{r^2},$$

from which the proposition follows.  $\square$

By Lemma 2.3 we have the following consequence of Proposition 3.1.

**Proposition 3.2.** *Suppose that  $(u, p)$  is a suitable weak solution to the Navier-Stokes equations in  $Q_r(z_0)$ . Then one has*

$$A(z_0, r/2) + B(z_0, r/2) \leq C [C_{\alpha, \beta}(z_0, r)^{\frac{1}{\beta}} + C_{\alpha, \beta}(z_0, r)^{\frac{2}{\beta}} + D_{\alpha, \beta}(z_0, r)^{\frac{2}{\beta}}]$$

for any  $\alpha \in [6/5, 2]$  and  $\beta = \frac{4\alpha}{7\alpha-6}$ .

4.  $\epsilon$ -REGULARITY CRITERIA

In this section we prove Theorems 1.5 and 1.7. We start with the following lemma.

**Lemma 4.1.** *Let  $h$  be a harmonic function in  $B_{2r}(x_0)$  and  $0 \leq \sigma \leq 1$ . Then we have*

$$\|h\|_{L^2(B_r(x_0))} \leq C \left\| \frac{h}{r^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))}.$$

*Proof.* The case  $\sigma = 0$  is obvious. We thus assume that  $0 < \sigma \leq 1$ . Let  $f$  be a harmonic function in  $B_2(0)$ . Let  $\varphi \in \mathcal{C}_0^\infty(B_{3/2}(0))$  be such that  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  in  $B_1(0)$  and  $|\nabla\varphi| \leq c$ . Hence,  $f\varphi \in \mathcal{C}_0^\infty(B_{3/2}(0))$  and

$$(4.1) \quad \|f\varphi\|_{L^2(B_{3/2}(0))} \leq \|f\varphi\|_{L^{-\sigma,2}(B_{3/2}(0))}^{\frac{1}{2}} \|f\varphi\|_{L^{\sigma,2}(B_{3/2}(0))}^{\frac{1}{2}}.$$

Observe that, for any  $g \in L^{\sigma,2}(B_{3/2}(0))$ , by [7, Theorem A.12] we have

$$\|g\varphi\|_{L^{\sigma,2}(B_{3/2}(0))} \leq C \|g\|_{L^{\sigma,2}(B_{3/2}(0))},$$

and thus

$$\begin{aligned} \int_{B_{3/2}(0)} f\varphi g dx &\leq \|f\|_{L^{-\sigma,2}(B_{3/2}(0))} \|g\varphi\|_{L^{\sigma,2}(B_{3/2}(0))} \\ &\leq C \|f\|_{L^{-\sigma,2}(B_{3/2}(0))} \|g\|_{L^{\sigma,2}(B_{3/2}(0))}. \end{aligned}$$

This means that

$$(4.2) \quad \|f\varphi\|_{L^{-\sigma,2}(B_{3/2}(0))} \leq C \|f\|_{L^{-\sigma,2}(B_{3/2}(0))}.$$

Also,

$$\begin{aligned} \|f\varphi\|_{L^{\sigma,2}(B_{3/2}(0))} &\leq \|f\varphi\|_{L^2(B_{3/2}(0))}^{1-\sigma} \|\nabla(f\varphi)\|_{L^2(B_{3/2}(0))}^\sigma \\ &\leq \|f\varphi\|_{L^2(B_{3/2}(0))}^{1-\sigma} \|(\nabla f)\varphi + f(\nabla\varphi)\|_{L^2(B_{3/2}(0))}^\sigma \\ &\leq c \|f\|_{L^2(B_{3/2}(0))}^{1-\sigma} \left( \|\nabla f\|_{L^2(B_{3/2}(x_0))}^\sigma + \|f\|_{L^2(B_{3/2}(0))}^\sigma \right) \\ &\leq c \|f\|_{L^2(B_{3/2}(0))}^{1-\sigma} \left( \|f\|_{L^2(B_2(0))}^\sigma + \|f\|_{L^2(B_{3/2}(0))}^\sigma \right) \\ (4.3) \quad &\leq c \|f\|_{L^2(B_2(0))}. \end{aligned}$$

Here in the 4th inequality we used the fact that  $f$  is harmonic in  $B_2(0)$ .

Hence, (4.1), (4.2) and (4.3) yield

$$(4.4) \quad \|f\|_{L^2(B_1(0))} \leq \|f\varphi\|_{L^2(B_{3/2}(0))} \leq C \|f\|_{L^{-\sigma,2}(B_{3/2}(0))}^{\frac{1}{2}} \|f\|_{L^2(B_2(0))}^{\frac{1}{2}}.$$

Now for  $r > 0$ , let  $h$  be a harmonic function in  $B_{2r}(x_0)$ . We define  $f(x) = h(rx + x_0)$  for  $x \in B_2(0)$ . Then  $f$  is harmonic in  $B_2(0)$ .

Note that for any  $\varphi \in L^{\sigma,2}(B_{3/2}(0))$  we have

$$\begin{aligned} \left\| \varphi \left( \frac{\cdot - x_0}{r} \right) \right\|_{L^{\sigma,2}(B_{3r/2}(x_0))} &= \left( \int_{\mathbb{R}^3} |\xi|^{2\sigma} \left| \widehat{\varphi} \left( \frac{\cdot - x_0}{r} \right) \right|^2 d\xi \right)^{\frac{1}{2}} \\ &= r^3 \left( \int_{\mathbb{R}^3} |\xi|^{2\sigma} |\widehat{\varphi}(r\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= r^3 \left( \int_{\mathbb{R}^3} \left| \frac{\zeta}{r} \right|^{2\sigma} |\widehat{\varphi}(\zeta)|^2 r^{-3} d\zeta \right)^{\frac{1}{2}} = r^{\frac{3}{2}-\sigma} \|\varphi\|_{L^{\sigma,2}(B_{3/2}(0))}. \end{aligned}$$

Thus for such  $\varphi$ ,

$$\begin{aligned} \int_{B_{3/2}(0)} h(rx + x_0) \varphi(x) dx &= r^{-3} \int_{B_{3r/2}(x_0)} h(y) \varphi \left( \frac{y - x_0}{r} \right) dy \\ &\leq r^{-3} \|h\|_{L^{-\sigma,2}(B_{3r/2}(x_0))} \left\| \varphi \left( \frac{\cdot - x_0}{r} \right) \right\|_{L^{\sigma,2}(B_{3r/2}(x_0))} \\ &\leq r^{-\frac{3}{2}-\sigma} \|h\|_{L^{-\sigma,2}(B_{3r/2}(x_0))} \|\varphi\|_{L^{\sigma,2}(B_{3/2}(0))}. \end{aligned}$$

This implies that

$$\|f\|_{L^{-\sigma,2}(B_{3/2}(0))} \leq r^{-\frac{3}{2}} \left\| \frac{h}{r^\sigma} \right\|_{L^{-\sigma,2}(B_{3r/2}(x_0))},$$

and by substituting into (4.4) we have

$$\left( \int_{B_r(x_0)} |h|^2 dx \right)^{\frac{1}{2}} \leq Cr^{-\frac{3}{4}} \left\| \frac{h}{r^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))}^{\frac{1}{2}} \left( \int_{B_{2r}(x_0)} |h|^2 dx \right)^{\frac{1}{4}}.$$

Or equivalently,

$$(4.5) \quad \int_{B_r(x_0)} |h|^2 dx \leq C \left\| \frac{h}{r^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))} \left( \int_{B_{2r}(x_0)} |h|^2 dx \right)^{\frac{1}{2}}.$$

Let  $r \leq s < t \leq 2r$ . The ball  $B_s(x_0)$  can be covered by a collection of balls  $\{B_i = B_{\frac{t-s}{2}}(x_i) : x_i \in B_s(x_0)\}$ , in such a way that each point  $y \in \mathbb{R}^n$  belongs to at most  $N = N(n)$  balls in the collection  $\{2B_i = B_{t-s}(x_i)\}$ , that is,

$$\sum_i \chi_{2B_i}(y) \leq N(n).$$

Then applying (4.5) to the balls  $B_i$ , we find

$$\begin{aligned} \int_{B_i} |h|^2 dx &\leq C \left\| \frac{h}{(t-s)^\sigma} \right\|_{L^{-\sigma,2}(2B_i)} \left( \int_{2B_i} |h|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left\| \frac{h}{(t-s)^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))} \left( \int_{2B_i} |h|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{B_s(x_0)} |h|^2 dx &\leq \sum_i \int_{B_i} |h|^2 dx \\ &\leq C \left\| \frac{h}{(t-s)^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))} \sum_i \left( \int_{2B_i} |h|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Note that

$$\sum_i \int_{2B_i} |h|^2 dx = \int_{B_t(x_0)} |h|^2 \sum_i \chi_{2B_i}(x) dx \leq N(n) \int_{B_t(x_0)} |h|^2 dx,$$

and thus

$$\int_{B_s(x_0)} |h|^2 dx \leq C(n) \left\| \frac{h}{(t-s)^\sigma} \right\|_{L^{-\sigma,2}(B_{2r}(x_0))} \left( \int_{B_t(x_0)} |h|^2 dx \right)^{\frac{1}{2}}.$$

Then by Young's inequality it follows that

$$\int_{B_s(x_0)} |h|^2 dx \leq C(t-s)^{-2\sigma} \|h\|_{L^{-\sigma,2}(B_{2r}(x_0))}^2 + \frac{1}{2} \int_{B_t(x_0)} |h|^2 dx.$$

Thus applying Lemma 2.4 we have

$$\int_{B_r(x_0)} |h|^2 dx \leq Cr^{-2\sigma} \|h\|_{L^{-\sigma,2}(B_{2r}(x_0))}^2$$

as desired.  $\square$

The next lemma provides bounds for the pressure.

**Lemma 4.2.** *Suppose that  $(u, p)$  is a suitable weak solution to the Navier-Stokes equations in  $Q_\rho(z_0)$ . For any  $r \in (0, \rho/2]$  we have the following bounds:*

$$\begin{aligned} &r^{-\frac{3}{2}} \int_{t_0-r^2}^{t_0} \|p(\cdot, t)\|_{L^2(B_r(x_0))} dt \\ (4.6) \quad &\leq C\rho^{-3} \int_{t_0-\rho^2}^{t_0} \|p\|_{L^1(B_\rho(x_0))} dt + C\left(\frac{\rho}{r}\right)^{3/2} A(z_0, \rho)^{1/4} B(z_0, \rho)^{3/4}, \end{aligned}$$

$$\begin{aligned}
& r^{-\frac{3}{2}} \int_{t_0-r^2}^{t_0} \|p(\cdot, t) - [p(\cdot, t)]_{x_0, r}\|_{L^2(B_r(x_0))} dt \\
(4.7) \quad & \leq C \left(\frac{r}{\rho}\right) \rho^{-3} \int_{t_0-\rho^2}^{t_0} \|p(\cdot, t) - [p(\cdot, t)]_{x_0, \rho}\|_{L^1(B_\rho(x_0))} dt + \\
& \quad + C \left(\frac{\rho}{r}\right)^{3/2} A(z_0, \rho)^{1/4} B(z_0, \rho)^{3/4},
\end{aligned}$$

and

$$\begin{aligned}
& r^{-3} \int_{t_0-r^2}^{t_0} \|p\|_{L^1(B_r(x_0))} dt \\
(4.8) \quad & \leq C \rho^{-\frac{3}{2}-\sigma} \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{-\sigma, 2}(B_\rho(x_0))} dt + \\
& \quad + C \left(\frac{\rho}{r}\right)^3 A(z_0, \rho)^{1/4+\sigma/2} B(z_0, \rho)^{3/4-\sigma/2}
\end{aligned}$$

for any  $\sigma \in [0, 3/2)$ .

*Proof.* Let  $h_{x_0, \rho} = h_{x_0, \rho}(\cdot, t)$  be a function on  $B_\rho(x_0)$  for a.e.  $t$  such that

$$h_{x_0, \rho} = p - \tilde{p}_{x_0, \rho} \quad \text{in } B_\rho(x_0),$$

where  $\tilde{p}_{x_0, \rho}$  is defined by

$$\tilde{p}_{x_0, \rho} = R_i R_j [(u_i - [u_i]_{x_0, \rho})(u_j - [u_j]_{x_0, \rho}) \chi_{B_\rho(x_0)}].$$

Here  $R_i = D_i(-\Delta)^{-\frac{1}{2}}$ ,  $i = 1, 2, 3$ , is the  $i$ -th Riesz transform, and we used the notation

$$[f]_{x_0, \rho} := \int_{B_\rho(x_0)} f(x) dx = \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f(x) dx.$$

to denote the spatial average of a function  $f$  over the ball  $B_\rho(x_0)$ .

Note that for any  $\varphi \in C_0^\infty(B_\rho(x_0))$ , we have

$$\begin{aligned}
-\int_{B_\rho(x_0)} \tilde{p}_{x_0, \rho} \Delta \varphi dx &= \int_{B_\rho(x_0)} (u_i - [u_i]_{x_0, \rho})(u_j - [u_j]_{x_0, \rho}) D_{ij} \varphi dx \\
&= \int_B u_i u_j D_{ij} \varphi dx,
\end{aligned}$$

which follows from the properties  $-R_i R_j(\Delta \varphi) = D_{ij} \varphi$  and  $\operatorname{div} u = 0$ . Thus, as  $p$  also solves

$$-\Delta p = \operatorname{div} \operatorname{div}(u \otimes u)$$

in the distributional sense, we see that  $h_{x_0, \rho}$  is harmonic in  $B_\rho(x_0)$  for a.e.  $t$ . Then for  $r \in (0, \rho/2]$  it holds that

$$\left( \int_{B_r(x_0)} |h_{x_0, \rho}|^2 dx \right)^{\frac{1}{2}} \leq C \int_{B_\rho(x_0)} |h_{x_0, \rho}| dx$$

and

$$\left( \int_{B_r(x_0)} |h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,r}|^2 dx \right)^{\frac{1}{2}} \leq C \frac{r}{\rho} \int_{B_\rho(x_0)} |h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,\rho}| dx.$$

Then using  $p = \tilde{p}_{x_0,\rho} + h_{x_0,\rho}$ , they give

$$(4.9) \quad \begin{aligned} & \int_{B_r(x_0)} |p(x,t)|^2 dx \\ & \leq 2 \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^2 dx + C \frac{r^3}{\rho^6} \|h_{x_0,\rho}\|_{L^1(B_\rho(x_0))}^2, \end{aligned}$$

and

$$(4.10) \quad \begin{aligned} & \int_{B_r(x_0)} |p(x,t) - [p(\cdot,t)]_{x_0,r}|^2 dx \\ & \leq 2 \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^2 dx + C \frac{r^5}{\rho^8} \|h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,\rho}\|_{L^1(B_\rho(x_0))}^2. \end{aligned}$$

Now by (4.9) and  $h_{x_0,\rho} = p - \tilde{p}_{x_0,\rho}$  we obtain

$$(4.11) \quad \begin{aligned} \int_{B_r(x_0)} |p(x,t)|^2 dx & \leq 2 \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^2 dx + \\ & \quad + C \frac{r^3}{\rho^6} \left( \|\tilde{p}_{x_0,\rho}\|_{L^1(B_\rho(x_0))}^2 + \|p\|_{L^1(B_\rho(x_0))}^2 \right) \\ & \leq C \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^2 dx + C \frac{r^3}{\rho^6} \|p\|_{L^1(B_\rho(x_0))}^2, \end{aligned}$$

where we used Hölder's inequality and the fact that  $r/\rho \leq 1/2$ .

On the other hand, by the Calderón-Zygmund estimate and Lemma 2.5 we find

$$(4.12) \quad \begin{aligned} \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^2 dx & \leq C \int_{B_\rho(x_0)} |u - [u]_{x_0,\rho}|^4 dx \\ & \leq C \left( \int_{B_\rho(x_0)} |\nabla u|^2 dx \right)^{3/2} \left( \int_{B_\rho(x_0)} |u|^2 dx \right)^{1/2}. \end{aligned}$$

Combining (4.11) and (4.12) we have

$$\begin{aligned} \|p\|_{L^2(B_r(x_0))} & \leq C \frac{r^{3/2}}{\rho^3} \|p\|_{L^1(B_\rho(x_0))} + \\ & \quad + C \left( \int_{B_\rho(x_0)} |\nabla u|^2 dx \right)^{3/4} \left( \int_{B_\rho(x_0)} |u|^2 dx \right)^{1/4}. \end{aligned}$$

Integrating the last bound with respect to  $r^{-3/2} dt$  over the interval  $(t_0 - r^2, t_0)$  and using Hölder's inequality we obtain inequality (4.6).

Likewise, using (4.10) instead of (4.9) and arguing similarly we obtain inequality (4.7). We remark that in this case we also need to use the elementary fact that

$$\|h_{x_0,\rho} - [h_{x_0,\rho}]_{x_0,\rho}\|_{L^1(B_\rho(x_0))} \leq 2 \|h_{x_0,\rho} - [p(\cdot, t)_{x_0,\rho}]_{x_0,\rho}\|_{L^1(B_\rho(x_0))}.$$

As for (4.8), we first bound

$$\begin{aligned} \int_{B_r(x_0)} |p(x, t)| dx &\leq \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}| dx + \int_{B_r(x_0)} |h_{x_0,\rho}| dx \\ &\leq \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}| dx + Cr^3 \left( \int_{B_{\rho/2}(x_0)} |h_{x_0,\rho}|^2 dx \right)^{1/2} \\ &\leq \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}| dx + C \frac{r^3}{\rho^{\frac{3}{2}+\sigma}} \|h_{x_0,\rho}\|_{L^{-\sigma,2}(B_\rho(x_0))} \\ &\leq \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}| dx + \\ &\quad + C \frac{r^3}{\rho^{\frac{3}{2}+\sigma}} \left( \|\tilde{p}_{x_0,\rho}\|_{L^{-\sigma,2}(B_\rho(x_0))} + \|p\|_{L^{-\sigma,2}(B_\rho(x_0))} \right). \end{aligned}$$

Here we used Lemma 4.1 in the third inequality.

Now using Hölder's inequality, Lemma 2.3 with  $\sigma \in [0, 3/2)$ , and  $r/\rho \leq 1/2$ , we have

$$\begin{aligned} \int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}| dx &+ C \frac{r^3}{\rho^{\frac{3}{2}+\sigma}} \|\tilde{p}_{x_0,\rho}\|_{L^{-\sigma,2}(B_\rho(x_0))} \\ &\leq C \rho^{3/2-\sigma} \|\tilde{p}_{x_0,\rho}\|_{L^{\frac{6}{3+2\sigma}}(B_\rho(x_0))}. \end{aligned}$$

Thus,

$$(4.13) \quad \begin{aligned} \int_{B_r(x_0)} |p(x, t)| dx &\leq C \rho^{3/2-\sigma} \|\tilde{p}_{x_0,\rho}\|_{L^{\frac{6}{3+2\sigma}}(B_\rho(x_0))} \\ &\quad + C \frac{r^3}{\rho^{\frac{3}{2}+\sigma}} \|p\|_{L^{-\sigma,2}(B_\rho(x_0))}. \end{aligned}$$

As before, the  $L^{\frac{6}{3+2\sigma}}$  norm of  $\tilde{p}_{x_0,\rho}$  is treated using Calderón-Zygmund estimate and Lemma 2.5 which give

$$(4.14) \quad \begin{aligned} &\|\tilde{p}_{x_0,\rho}\|_{L^{\frac{6}{3+2\sigma}}(B_\rho(x_0))} \\ &\leq C \left( \int_{B_\rho(x_0)} |\nabla u|^2 dx \right)^{3/4-\sigma/2} \left( \int_{B_\rho(x_0)} |u|^2 dx \right)^{1/4+\sigma/2}. \end{aligned}$$

Combining (4.13), (4.14) we have

$$\begin{aligned} \int_{B_r(x_0)} |p(x, t)| dx &\leq C \frac{r^3}{\rho^{3/2+\sigma}} \|p\|_{L^{-\sigma, 2}(B_\rho(x_0))} + \\ &+ C \rho^{3/2-\sigma} \left( \int_{B_\rho(x_0)} |\nabla u|^2 dx \right)^{3/4-\sigma/2} \left( \int_{B_\rho(x_0)} |u|^2 dx \right)^{1/4+\sigma/2}, \end{aligned}$$

from which integrating in  $t$  we obtain (4.8).  $\square$

We now recall the following  $\epsilon$ -regularity criterion for suitable weak solutions to the Navier-Stokes equations (see [16, Lemma 3.3]).

**Lemma 4.3.** *There exists a positive number  $\epsilon_\star$  such that the following property holds. If  $(u, p)$  be a suitable solution to the Navier-Stokes equations in  $Q_{R_\star}(z_0)$  for some  $R_\star > 0$  such that*

$$\sup_{0 < r < R_\star} A(z_0, r) \leq \epsilon_\star,$$

then  $z_0$  is a regular point of  $u$ .

We are now ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* Our assumption is that

$$A((0, 0), 1) + B((0, 0), 1) + \int_{-1}^0 \|p\|_{L^1(B_1(0))} dt \leq \epsilon,$$

where  $\epsilon \in (0, 1)$  is to be determined. By Lemma 4.3, it is enough to show that

$$\sup_{0 < r < 1/4} [A(z, r) + B(z, r)] \leq C \epsilon^{1/2} \leq \epsilon_\star.$$

for every  $z \in Q_{1/2}(0, 0)$ . Here  $C$  is independent of  $r$  and  $z$ . By translation invariance, it suffices to consider the case  $z = 0$ . Moreover, it suffices to show a discrete version, i.e., we just need to show that

$$(4.15) \quad A((0, 0), \theta^n) + B((0, 0), \theta^n) \leq \epsilon^{1/2}$$

for a fixed  $\theta \in (0, 1/4]$  and for all  $n = 1, 2, \dots$ . The discretization enables us to use an inductive argument in the spirit of [1, Section 4] and [18].

Let  $\theta \in (0, 1/4]$  be determined later and define

$$r_n = \theta^n, \quad n \in \mathbb{N}.$$

By our hypothesis, inequality (4.15) holds in the case  $n = 1$  provided  $\epsilon_0$  is sufficiently small (depending on  $\theta$ ). Suppose now that it holds for  $n = 1, \dots, m-1$  with an  $m \geq 2$ . Let  $\phi_m = \chi \psi_m$ , where  $0 \leq \chi \leq 1$  is a smooth cutoff function which equals 1 on  $Q_{\theta^2}(0, 0)$  and vanishes in  $\mathbb{R}^3 \times (-\infty, 0) \setminus Q_{2\theta^3}(0, 0)$ , and  $\psi_m$  is given by

$$\psi_m(x, t) = (r_m^2 - t)^{-3/2} e^{-\frac{|x|^2}{4(r_m^2 - t)}}, \quad t < r_m^2.$$

Then it can be seen that  $\phi_m \geq 0$ ,  $(\partial_t + \Delta)\phi_m = 0$  in  $Q_{\theta^2}(0, 0)$ , and

$$|(\partial_t + \Delta)\phi_m| \leq C \quad \text{on } Q_{2\theta/3}(0, 0),$$

$$2^{-3/2} r_m^{-3} \leq \phi_m \leq r_m^{-3}, \quad |\nabla\phi_m| \leq Cr_m^{-4} \quad \text{on } Q_{r_m}(0, 0), \quad m \geq 2,$$

$$\phi_m \leq Cr_k^{-3}, \quad |\nabla\phi_m| \leq Cr_k^{-4} \quad \text{on } Q_{r_{k-1}}(0, 0) \setminus Q_{r_k}(0, 0), \quad 1 < k \leq m.$$

Here the constant  $C = C(\theta)$  is independent of  $m$ .

Using  $\phi_m$  as a test function in the generalized energy inequality, we find that

$$(4.16) \quad A((0, 0), r_m) + B((0, 0), r_m) \leq C(I + II + III),$$

where

$$\begin{aligned} I &= r_m^2 \int_{Q_{\theta}(0, 0)} |u|^2 dxdt, \\ II &= r_m^2 \int_{Q_{\theta}(0, 0)} |u|^3 |\nabla\phi_m| dxdt, \\ III &= r_m^2 \left| \int_{Q_{\theta}(0, 0)} p(u \cdot \nabla\phi_m) dxdt \right|. \end{aligned}$$

By the hypothesis, we have

$$I \leq r_m^2 \epsilon \leq \epsilon^{\frac{3}{4}}.$$

By the above properties of  $\phi_m$ , we have

$$\begin{aligned} II &= r_m^2 \sum_{k=1}^{m-1} \int_{Q_{r_k} \setminus Q_{r_{k+1}}} |u|^3 |\nabla\phi_m| dxdt + r_m^2 \int_{Q_{r_m}} |u|^3 |\nabla\phi_m| dxdt \\ &\leq Cr_m^2 \sum_{k=1}^{m-1} r_k^{-4} \int_{Q_{r_k}} |u|^3 dxdt. \end{aligned}$$

Thus by Lemma 2.6 and inductive hypothesis, it follows that

$$II \leq Cr_m^2 \sum_{k=1}^{m-1} r_k^{-2} \epsilon^{3/4} \leq C\epsilon^{3/4}.$$

As for the term  $III$ , we write

$$\phi_m = \chi_1 \phi_m = \sum_{k=1}^{m-1} (\chi_k - \chi_{k+1}) \phi_m + \chi_m \phi_m,$$

where  $\chi_k$ ,  $k = 1, 2, \dots, m$ , is a smooth cutoff function such that  $0 \leq \chi_k \leq 1$ ,  $\chi_k = 1$  in  $Q_{7r_k/8}(0, 0)$ ,  $\chi_k = 0$  in  $\mathbb{R}^3 \times (-\infty, 0) \setminus Q_{r_k}(0, 0)$ , and  $|\nabla\chi_k| \leq C/r_k$ .

Then

$$\begin{aligned}
III &\leq r_m^2 \left| \sum_{k=1}^{m-1} \int_{Q_{r_k}} pu \cdot \nabla[(\chi_k - \chi_{k+1})\phi_m] dxdt \right| \\
&\quad + r_m^2 \left| \int_{Q_{r_m}} pu \cdot \nabla(\chi_m \phi_m) dxdt \right| \\
&= r_m^2 \left| \sum_{k=1}^{m-1} \int_{Q_{r_k}} (p - [p]_{0,r_k})u \cdot \nabla[(\chi_k - \chi_{k+1})\phi_m] dxdt \right| \\
&\quad + r_m^2 \left| \int_{Q_{r_m}} (p - [p]_{0,r_m})u \cdot \nabla(\chi_m \phi_m) dxdt \right|,
\end{aligned}$$

where we used the fact that  $u$  is divergence-free. Then by Hölder's inequality and the properties of  $\phi_m$ , we see that

$$\begin{aligned}
III &\leq Cr_m^2 \sum_{k=2}^m r_k^{-4} \int_{Q_{r_k}} |(p - [p]_{0,r_k})u| dxdt \\
&\quad + C\theta^{-2} \int_{Q_\theta} |(p - [p]_{0,\theta})u| dxdt \\
&\leq Cr_m^2 \sum_{k=2}^m r_k^{-4} \int_{-r_k^2}^0 \|p - [p]_{0,r_k}\|_{L^2(B_{r_k})} \|u\|_{L^2(B_{r_k})} dxdt \\
&\quad + C\theta^{-2} \int_{-\theta^2}^0 \|p - [p]_{0,\theta}\|_{L^2(B_\theta)} \|u\|_{L^2(B_\theta)} dxdt.
\end{aligned}$$

By inductive hypothesis, this gives

$$\begin{aligned}
(4.17) \quad III &\leq Cr_m^2 \sum_{k=2}^m r_k^{-2} \epsilon^{\frac{1}{4}} r_k^{-3/2} \int_{-r_k^2}^0 \|p - [p]_{0,r_k}\|_{L^2(B_{r_k})} dxdt \\
&\quad + C\epsilon^{\frac{1}{2}} \theta^{-3/2} \int_{-\theta^2}^0 \|p - [p]_{0,\theta}\|_{L^2(B_\theta)} dxdt.
\end{aligned}$$

Here the constant  $C$  could depend on  $\theta$ .

We now let  $A(k) = A((0,0), r_k)$ ,  $B(k) = B((0,0), r_k)$ , and

$$U(k) = r_k^{-3/2} \int_{-r_k^2}^0 \|p - [p]_{0,r_k}\|_{L^2(B_{r_k})} dxdt.$$

By Lemma 4.2 and Hölder's inequality, for  $2 \leq k \leq m$  we have

$$U(k) \leq (C\theta)U(k-1) + C\theta^{-3/2}A(k-1)^{1/4}B(k-1)^{3/4},$$

where  $C \geq 1$  is independent of  $k$  and  $\theta$ . Choosing  $\theta = \frac{1}{4C}$  and iterating this inequality we obtain

$$U(k) = (1/4)^{k-1}U(1) + C\theta^{-3/2} \sum_{\ell=1}^{k-1} (1/4)^{\ell-1} A(k-\ell)^{1/4} B(k-\ell)^{3/4}.$$

Then by inductive hypothesis we find

$$\begin{aligned} U(k) &\leq U(1) + C \sum_{\ell=1}^{k-1} (1/4)^{\ell-1} \epsilon^{\frac{1}{2}} \\ &\leq \theta^{-3/2} \int_{-\theta^2}^0 \|p - [p]_{0,\theta}\|_{L^2(B_\theta)} dxdt + C\epsilon^{\frac{1}{2}}. \end{aligned}$$

Combining this with (4.17) we arrive at

$$III \leq C\epsilon^{\frac{1}{4}} \theta^{-3/2} \int_{-\theta^2}^0 \|p - [p]_{0,\theta}\|_{L^2(B_\theta)} dxdt + C\epsilon^{\frac{3}{4}},$$

which by Lemma 4.2 gives

$$\begin{aligned} III &\leq C\epsilon^{\frac{1}{4}} [D((0,0), 2\theta) + A((0,0), 2\theta)^{1/4} B((0,0), 2\theta)^{3/4}] + C\epsilon^{\frac{3}{4}} \\ &\leq C(\epsilon^{\frac{5}{4}} + \epsilon^{\frac{3}{4}}) \leq 2C\epsilon^{\frac{3}{4}}. \end{aligned}$$

Combining 4.16 and the estimates for  $I$ ,  $II$  and  $III$  we obtain

$$A((0,0), r_m) + B((0,0), r_m) \leq C\epsilon^{\frac{3}{4}} \leq \epsilon^{\frac{1}{2}}$$

provided  $\epsilon$  is small enough. This proves (4.15) and the proof is complete.  $\square$

Using Lemma 4.2 and a covering argument we obtain the following consequence of Theorem 1.5.

**Corollary 4.4.** *Let  $\sigma \in [0, 1]$ . There exists a number  $\epsilon \in (0, 1)$  with the following property. If  $(u, p)$  be a suitable solution to the Navier-Stokes equations in  $Q_1$  such that*

$$A((0,0), 1) + B((0,0), 1) + \int_{-1}^0 \|p\|_{L^{-\sigma,2}(B_1(0))} dt \leq \epsilon,$$

*then  $u$  is regular in  $Q_{1/2}$ .*

Finally, we prove Theorem 1.7.

*Proof of Theorem 1.7.* By Hölder's inequality it follows that

$$\int_{-1}^0 \|p\|_{L^{-\sigma,2}(B_1(0))} dt \leq D_\sigma((0,0), 1)^{\frac{2-\sigma}{2}}.$$

Thus by Corollary 4.4, Proposition 3.1, and a covering argument we obtain Theorem 1.7.  $\square$

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