# QUASILINEAR RICCATI TYPE EQUATIONS WITH DISTRIBUTIONAL DATA IN MORREY SPACE FRAMEWORK

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Abstract. In the framework of Morrey or Lorentz-Morrey spaces, we characterize the existence of solutions to the quasilinear Riccati type equation

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u) &= |\nabla u|^q + \sigma \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with a distributional datum $\sigma$. Here $\mathrm{div}\,\mathcal{A}(x,\nabla u)$ is a quasilinear elliptic operator modelled after the $p$-Laplacian, $p>1$, but with a very general nonlinear structure, and $\Omega$ is a sufficiently flat domain in the sense of Reifenberg. The existence results are obtained in the natural or super-natural range of the gradient growth, i.e., $q \geq p$.

Motivated by the analysis of quasilinear Riccati type equation, a substantial part of the paper is also devoted to the Calderón-Zygmund type gradient regularity for the boundary value problem

$$\begin{cases} \mathrm{div}\,\mathcal{A}(x,\nabla u) &= \mathrm{div}\,|\mathbf{f}|^{p-2}\mathbf{f} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We obtain regularity estimates in some weighted and unweighted function spaces as well as natural Lorentz-Morrey spaces associated to the Riccati type equation above.

## 1. Introduction and Results

One of the main goals of this paper is to address the question of existence for the quasilinear Riccati type equation

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u) &= |\nabla u|^q + \sigma \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the datum $\sigma$ is a signed distribution given on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

In (1.1) the nonlinearity $\mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector valued function, i.e., $\mathcal{A}(x,\xi)$ is measurable in $x$ for every $\xi$ and continuous in $\xi$ for a.e. $x$. Moreover, for a.e. $x$, $\mathcal{A}(x,\xi)$ is continuously differentiable in $\xi$ away from the origin. Our standing assumption is that $\mathcal{A}$ satisfies the following growth and monotonicity conditions: for some $1<p<\infty$ there hold

$$|\mathcal{A}(x,\xi)| \leq \beta\,|\xi|^{p-1}, \quad |\nabla_\xi \mathcal{A}(x,\xi)| \leq \beta\,|\xi|^{p-2} \tag{1.2}$$

and

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta\rangle \geq \alpha(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \tag{1.3}$$

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for any $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus (0,0)$ and a.e. $x \in \mathbb{R}^n$. Here $\alpha$ and $\beta$ are positive constants. The special case $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$ gives rise to the standard $p$-Laplacian $\Delta_p u = \operatorname{div} |\nabla u|^{p-2}\nabla u$.

Equation (1.1) can be viewed as a quasilinear stationary version of a time-dependent viscous Hamilton-Jacobi equation, also known as the Kardar-Parisi-Zhang equation, which appears in the physical theory of surface growth [30, 35].

In this paper we are concerned with (1.1) only in the case of a natural or super-natural growth $q$ in the gradient, i.e., we assume that $q \geq p$. One of the main results we prove in this paper is an existence result for equation (1.1) which allows the datum $\sigma$ to be a distribution of the form $\sigma = \operatorname{div} \boldsymbol{\zeta}$ for a vector field $\boldsymbol{\zeta}$ belonging to a Morrey space. Given $s \geq 1$ and $\theta \in (0, n]$, the Morrey space $\mathcal{L}^{s;\,\theta}(\Omega)$ is the set of functions $f \in L^s(\Omega)$ such that

$$\int_{B_r(z)\cap\Omega} |f(x)|^s dx \leq C\, r^{n-\theta}$$

for all $z \in \Omega$ and $r \in (0, \operatorname{diam}(\Omega)]$ with a constant $C$ independent of $z$ and $r$. In this case the norm of $f$ is given by

$$\|f\|_{\mathcal{L}^{s;\,\theta}(\Omega)} := \sup_{z\in\Omega,\, 0<r\leq \operatorname{diam}(\Omega)} r^{\frac{\theta-n}{s}} \, \|f\|_{L^s(B_r(z)\cap\Omega)}.$$

**Theorem 1.1.** *Let $1 < p \leq q < \infty$, $R_0 > 0$, and assume that $\mathcal{A}$ satisfies (1.2)-(1.3). Then there exist constants $s > 1$, $\delta > 0$ and $c_0 > 0$ such that the following holds. Let $\sigma = \operatorname{div} \boldsymbol{\zeta}$ for a given vector field $\boldsymbol{\zeta}$ on $\Omega$ such that $|\boldsymbol{\zeta}|^{\frac{q}{p-1}} \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon > 0$ with $\frac{q(1+\epsilon)}{q-p+1} \leq n$. If $\Omega$ is $(\delta, R_0)$-Reifenberg flat, $\mathcal{A}$ satisfies the $(\delta, R_0)$-BMO condition with exponent $s$ and*

$$\left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq c_0,$$

*then, there exists a solution $u \in W^{1,\, q(1+\epsilon)}_0(\Omega)$, with $|\nabla u|^q \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$, to the Riccati type equation*

$$\left\{ \begin{array}{rcl} -\operatorname{div} \mathcal{A}(x, \nabla u) &=& |\nabla u|^q + \operatorname{div} \boldsymbol{\zeta} \quad \text{in } \Omega, \\ u &=& 0 \quad \text{on } \partial\Omega. \end{array} \right. \tag{1.4}$$

*The constants depend as follows: $c_0 = c_0\big(n, p, \alpha, \beta, q, \operatorname{diam}(\Omega), \frac{\operatorname{diam}(\Omega)}{R_0}\big)$, $s = s(n, p, \alpha, \beta)$ and $\delta = \delta(n, p, \alpha, \beta, q)$.*

Some remarks are now in order. The notion of $(\delta, R_0)$-Reifenberg flatness and the $(\delta, R_0)$-BMO condition mentioned in the above theorem will be defined precisely shortly. For now it suffices to comment that $(\delta, R_0)$-Reifenberg flat domains are those with boundary that (locally) will lie between two hyperplanes a small distance apart at small scales and include those with $C^1$ boundaries or Lipschitz domains with sufficiently small Lipschitz constants. Moreover, the $(\delta, R_0)$-BMO condition is a small mean-oscillation condition imposed on $\mathcal{A}(\xi, \cdot)$ and is satisfied when $\mathcal{A}$ is continuous or has small jump discontinuities in the $x$-variable.

The condition on $\boldsymbol{\zeta}$ (and therefore on $\sigma$) in Theorem 1.1 is satisfied if $\boldsymbol{\zeta}$ is a vector field such that $|\boldsymbol{\zeta}|^{\frac{q}{p-1}}$ belongs to the weak Lebesgue space $L^{\frac{n(q-p+1)}{q}, \infty}(\Omega)$, $q/(q-p+1) < n$, $p \leq q$, with a small norm. This follows from the continuous embedding

$$L^{\frac{n(q-p+1)}{q},\infty}(\Omega)\hookrightarrow \mathcal{L}^{1+\epsilon;\frac{q(1+\epsilon)}{q-p+1}}(\Omega)$$

that holds for any $\epsilon>0$ provided $\frac{q(1+\epsilon)}{q-p+1}<n$. We note that for $s\geq 1$, the weak Lebesgue space $L^{s,\infty}(\Omega)$ is the set of functions $g$ such that the norm

$$\|g\|_{L^{s,\infty}}=\sup_{\alpha>0}\alpha|\{x\in\Omega:|g(x)|>\alpha\}|^{1/s}<\infty.$$

Another instance where the condition on $\sigma$ is satisfied is when $\sigma$ is a finite signed measure in $\Omega$ such that $\mathfrak{B}_1(|\sigma|)^{\frac{q}{p-1}}$ belongs to $\mathcal{L}^{1+\epsilon;\frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon>0$ with a sufficiently small norm. Here $\mathfrak{B}_1(|\sigma|)$ is the first order Bessel potential of $|\sigma|$:

$$\mathfrak{B}_1(|\sigma|)(x):=\int_{\mathbb{R}^n}g_1(x-y)d|\sigma|(y)$$

with $g_1$ being the Bessel kernel of order 1 defined via its Fourier transform by $\widehat{g_1}(\xi)=(1+|\xi|^2)^{\frac{-1}{2}}$. After extending $\sigma$ by zero outside $\Omega$, we may write $\sigma=\operatorname{div}\boldsymbol{\zeta}$ in $\mathcal{D}'(\Omega)$, with

$$\boldsymbol{\zeta}(x)=-\int_B\nabla_x G(x,y)d\sigma(y), \tag{1.5}$$

where $B$ is a ball of radius diam$(\Omega)$ containing $\Omega$ and $G(x,y)$ is the Green function with zero boundary condition associated to $-\Delta$ on $B$. Then observe that from the pointwise estimate

$$|\nabla_x G(x,y)|\leq C\,|x-y|^{1-n}\leq C(n,\operatorname{diam}(\Omega))\,g_1(x-y) \tag{1.6}$$

that holds for all $x,y\in B$ with $x\neq y$, we have $|\boldsymbol{\zeta}(x)|\leq C\mathfrak{B}_1(|\sigma|)(x)$, from which the assertion follows.

Note that the condition $\mathfrak{B}_1(|\sigma|)^{\frac{q}{p-1}}\in\mathcal{L}^{1+\epsilon;\frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon>0$ is satisfied if $\sigma\in\mathcal{L}^{1+\epsilon;\frac{q(1+\epsilon)}{q-p+1}}(\Omega)$. This is known as the well-known Fefferman-Phong condition [14] which appeared in the analysis of the Schrödinger operator. In particular, these conditions hold if $\sigma$ belongs to the weak Lebesgue space $L^{\frac{n(q-p+1)}{q},\infty}(\Omega)$ with $q/(q-p+1)<n$.

For $q>p$, an existence result for (1.1) with *measure data* $\sigma$ satisfying the Morrey condition $\mathfrak{B}_1(|\sigma|)^{\frac{q}{p-1}}\in\mathcal{L}^{1+\epsilon;\frac{q(1+\epsilon)}{q-p+1}}(\Omega)$, $\epsilon>0$, was obtained in [45] under stronger regularity conditions on $\mathcal{A}$ and $\partial\Omega$. For $q=p$ and distributional data $\sigma=\operatorname{div}\boldsymbol{\zeta}$, an existence result has been obtained recently in [15] (see also [16]) under the stronger condition $|\boldsymbol{\zeta}|^{\frac{p}{p-1}}\in L^{n/p,\infty}(\Omega)$. For results in the sub-natural case $p-1<q<p$, we refer the readers to [20, 49]. It is known from [23] and [45] that in order for (1.1) to have a $W^{1,q}_{\rm loc}(\Omega)$ solution, $q>p-1$, it is necessary that $\sigma$ be regular and small enough. When $\sigma$ is a nonnegative measure these necessary conditions can be quantified as

$$\int_\Omega\varphi^{\frac{q}{q-p+1}}d\sigma\leq C\int_\Omega|\nabla\varphi|^{\frac{q}{q-p+1}}dx \tag{1.7}$$

for all $\varphi\in C_0^\infty(\Omega)$, $\varphi\geq 0$, with an explicit constant $C=\beta^{\frac{q}{q-p+1}}\left(\frac{q-p+1}{p-1}\right)^{\frac{1-p}{q-p+1}}$ (see [45]). In particular, $\sigma$ must obey the Morrey type condition

$$\sigma(B_r)\leq C\,r^{n-\frac{q}{q-p+1}} \tag{1.8}$$

for every ball $B_r$ with $B_{2r} \subset \Omega$. A consequence of the necessary condition (1.7) is that the existence condition $\mathfrak{B}_1(|\sigma|)^{\frac{q}{p-1}} \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ is sharp in the sense that in general one cannot take $\epsilon = 0$. Indeed, as constructed in [37], for $n = 3$ and $p = q = n-1 = 2$ there exists a nonnegative measure $\sigma$ compactly supported in $\Omega = B_3(0)$ such that $\mathfrak{B}_1(\sigma)^2 \in \mathcal{L}^{1;\,2}(\Omega)$ but (1.7) with $p = q = 2$ fails to hold for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

The requirement in Theorem 1.1 that the distribution $\sigma$ is a divergence of a vector field is sharp. Indeed, as the following theorem shows it is in fact necessary for the existence of a solution $u$ with $|\nabla u|^q \in \mathcal{L}^{(1+\epsilon);\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ to equation (1.1).

**Theorem 1.2.** *Let $p > 1$, $q > p-1$, and let $\mathcal{A}$ satisfy the first inequality in (1.2). Suppose that $\sigma$ is a distribution in a bounded domain $\Omega$ such that the Riccati type equation*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = |\nabla u|^q + \sigma \quad \text{in } \mathcal{D}'(\Omega) \tag{1.9}$$

*admits a solution $u \in W^{1,\, q(1+\epsilon)}(\Omega)$ with $|\nabla u|^q \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon > 0$ with $\frac{q(1+\epsilon)}{q-p+1} \leq n$. Then there exists a vector field $\boldsymbol{\zeta}$ on $\Omega$ such that $\sigma = \operatorname{div} \boldsymbol{\zeta}$ and $|\boldsymbol{\zeta}|^{\frac{q}{p-1}} \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ with*

$$\left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq C \left( \| |\nabla u|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} + \| |\nabla u|^q \|^{\frac{q}{p-1}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \right).$$

**Remark 1.3.** Theorems 1.1 and 1.2 also hold in the case $\frac{q(1+\epsilon)}{q-p+1} > n$ as long as we replace the Morrey space $\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ by the Lebesgue space $L^{1+\epsilon}(\Omega)$ after noting that $1 + \epsilon > n(q-p+1)/q$. Moreover, these theorems can be stated in the framework of Lorentz-Morrey spaces $\mathcal{L}^{1+\epsilon,\, t;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon > 0$ and for any $0 < t \leq \infty$. This follows directly from the proofs of these theorems and estimates in Lorentz-Morrey spaces obtained in Theorem 1.11 below. The precise definition of Lorentz-Morrey space shall be given shortly in this section.

A final comment on the existence result: Theorem 1.1 is an extension of a result proved in [45] in that the existence result can be applied when $\sigma$ not only is a measure but also a distribution in divergence form. For sign-changing measures $\sigma$, the divergence structure allows usage of the self-cancelation property of $\sigma$ in an important way. To further illustrate this, we present the following example.

**Example 1.4.** With $1 < p < n$ and $q \geq p$, let $s = \frac{q}{q-p+1}$. Then $0 < s < n$. Fix $\epsilon > 0$ such that $\epsilon + s < n$ and define

$$f(x) = |x|^{-\epsilon - s} \sin(|x|^{-\epsilon}) \tag{1.10}$$

for $x \in B_1(0) \setminus \{0\}$. By Proposition 5.1 in the Appendix below, the nonnegative function $|f|$ violates the necessary condition (1.8). Thus the equation

$$\begin{cases} -\Delta_p u &= |\nabla u|^q + \lambda\, |f| \quad \text{in } B_1(0), \\ \quad\ u &= 0 \quad \text{on } \partial B_1(0) \end{cases} \tag{1.11}$$

has no solution for any real number $\lambda \neq 0$.

On the other hand, one can write

$$f(x) = \operatorname{div} \mathbf{g}(x) + g(x) \quad \text{in } \mathcal{D}'(B_1(0)),$$

where $\mathbf{g}(x) = \frac{1}{\epsilon}\, x\, |x|^{-s}\cos(|x|^{-\epsilon})$ and $g(x) = \frac{s-n}{\epsilon}\, |x|^{-s}\cos(|x|^{-\epsilon})$. Moreover, if $G(x,y)$ is the Green function with zero boundary condition associated to $-\Delta$ on $B_1(0)$ then $g$ can be written as $g(x) = \operatorname{div}\mathbf{h}$ with

$$\mathbf{h}(x) = -\int_{B_1(0)} \nabla_x G(x,y) g(y) dy.$$

Thus it is easy to see that $f = \operatorname{div}(\mathbf{g}+\mathbf{h})$, where $|\mathbf{g}+\mathbf{h}|^{\frac{q}{p-1}}$ belongs to the weak Lebesgue space $L^{\frac{n(q-p+1)}{q},\infty}(B_1(0))$. By Theorem 1.1, there exists a solution to (1.11) with datum $f$ in place of $|f|$ provided $|\lambda|$ is sufficiently small. This shows that the self-cancellation property of $f$ plays an important role in this existence result. We remark that such a strongly oscillating datum $f$ has been considered in [38] in the analysis of Schrödinger operator on the energy space.

The study of (1.1) naturally leads us to the question of whether weak solutions to the nonhomogeneous nonlinear boundary value problems of the form

$$\begin{cases} \operatorname{div}\mathcal{A}(x,\nabla u) = \operatorname{div}\, |\mathbf{f}|^{p-2}\mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.12}$$

enjoy a Calderón-Zygmund type property. Thus a substantial part of this paper is also devoted to finding minimal conditions on the nonlinearity $\mathcal{A}$ and on the boundary of the domain so that the gradient, $\nabla u$, of a solution to (1.12) is as regular as the data $\mathbf{f}$. In particular, we will give various function spaces $\mathcal{S}$ such that $\mathbf{f}\in\mathcal{S}$ implies $\nabla u\in\mathcal{S}$. The regularity estimate that correspond to the special case $\mathcal{S} = \mathcal{L}^{q(1+\epsilon),\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega;\mathbb{R}^n)$ will be used in the proof of Theorem 1.1.

By now it is well known that, in general, the structural assumptions (1.2)-(1.3) on the nonlinearity $\mathcal{A}(x,\xi)$ are not enough to ensure that a solution to (1.12) enjoys a Calderón-Zygmund type regularity property. Even in the standard $L^q$-gradient theory for linear equations, where $\mathcal{A}(x,\xi) = A(x)\xi$ for an $n\times n$ bounded and uniformly elliptic coefficient matrix $A(x) = (A_{i,j}(x))$, over smooth domains, there are counter examples (see, e.g., [41]) that justify restricting coefficients to satisfy additional conditions, say small mean oscillations in the $x$-variable. On the other hand, global estimates require some regularity of the boundary. The example given in [28] make it clear that one should not expect global $L^q$-integrability of gradient of solutions to linear equations over certain polygonal domains.

Our additional regularity assumption on the nonlinearity $\mathcal{A}$ is the following $(\delta, R_0)$-BMO condition. To formulate it, for each ball $B$ we let

$$\overline{\mathcal{A}}_B(\xi) = \fint_B \mathcal{A}(x,\xi)dx = \frac{1}{|B|}\int_B \mathcal{A}(x,\xi)dx,$$

and define the following function that measures the oscillation of $\mathcal{A}(\cdot,\xi)$ over $B$:

$$\Upsilon(\mathcal{A},B)(x) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\mathcal{A}(x,\xi) - \overline{\mathcal{A}}_B(\xi)|}{|\xi|^{p-1}}.$$

**Definition 1.5.** *Given two positive numbers $\delta$ and $R_0$, we say that $\mathcal{A}(x,\xi)$ satisfies a $(\delta, R_0)$-BMO with exponent $s > 0$, if*

$$[\mathcal{A}]_s^{R_0} := \sup_{y\in\mathbb{R}^n,\, 0<r\leq R_0} \left( \fint_{B_r(y)} \Upsilon(\mathcal{A}, B_r(y))(x)^s dx \right)^{\frac{1}{s}} \leq \delta.$$

In the linear case, where $\mathcal{A}(x,\xi) = A(x)\xi$ for an elliptic matrix $A$, we see that

$$\Upsilon(\mathcal{A}, B)(x) \leq |A(x) - \overline{A}_B|$$

for almost every $x \in \mathbb{R}^n$ and thus one may think of Definition 1.5 as a natural extension of the standard small BMO condition to the nonlinear setting. For general nonlinearities $\mathcal{A}(x,\xi)$ of at most linear growth, i.e., $p = 2$, the above $(\delta, R_0)$-BMO condition was introduced in [9], whereas such a condition for general $p \in (1,\infty)$ appears first in [47]. We remark that the $(\delta, R_0)$-BMO condition allows the nonlinearity $\mathcal{A}(x,\xi)$ to have certain discontinuity in $x$, and it can be used as an appropriate substitute for the Sarason VMO (vanishing mean oscillation [52]) condition (see, e.g., [6, 9, 21, 25, 43, 53, 56]).

The domain over which we solve our equations may be nonsmooth but should satisfy some flatness condition. Essentially, at each boundary point and every scale, we require the boundary of the domain to be between two hyperplanes separated by a distance proportional to the scale. Absence of such flatness may result in a limited regularity of solutions, as demonstrated in the counterexample given [28] (see also [39] for details). The following defines the relevant geometry precisely.

**Definition 1.6.** *Given $\delta \in (0,1)$ and $R_0 > 0$, we say that $\Omega$ is a $(\delta, R_0)$-Reifenberg flat domain if for every $x_0 \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{y_1, y_2, \ldots, y_n\}$, which may depend on $r$ and $x_0$, so that in this coordinate system $x_0 = 0$ and that*

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

For more on Reifenberg flat domains and their many applications, we refer to the papers [22, 29, 31, 32, 51, 55]. For our purpose it suffices to know that Reifenberg flat domains can be very rough. They include Lipschitz domains with sufficiently small Lipschitz constants (see [55]) and even some domains with fractal boundaries.

In this paper, most of the Calderón-Zygmund type estimates for equation (1.12) follow from the following global weighted Lorentz space estimates that involve an $A_\infty$ weight. Recall that for a nonnegative locally integrable function $w$, called *a weight function*, the weighted Lorentz space $L^{q,\,t}_w(\Omega)$ with $0 < q < \infty$, $0 < t \leq \infty$, is the set of measurable functions $g$ on $\Omega$ such that

$$\|g\|_{L^{q,\,t}_w(\Omega)} := \begin{cases} \left[ q \displaystyle\int_0^\infty (\alpha^q w(\{x \in \Omega : |g(x)| > \alpha\}))^{\frac{t}{q}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}} < \infty \text{ if } t < \infty, \\ \displaystyle\sup_{\alpha>0} \alpha\, w(\{x \in \Omega : |g(x)| > \alpha\})^{\frac{1}{q}} < \infty \text{ if } t = \infty. \end{cases}$$

In the above for any measurable set $E$, $w(E) = \int_E w(x)\, dx$. We remark that $L^{q,\,\infty}_w(\Omega)$ is the usual (weighted) Marcinkiewicz space (weak Lebesgue space). As usual, when $w \equiv 1$ we write

$L^{q,t}_w(\Omega)$ as $L^{q,t}(\Omega)$. It is easy to see that when $t = q$ the weighted Lorentz space $L^{q,q}_w(\Omega)$ is nothing but the weighted Lebesgue space $L^q_w(\Omega)$, which is equivalently defined as

$$g \in L^q_w(\Omega) \Longleftrightarrow \int_\Omega |g(x)|^q w(x)dx < \infty.$$

The class of weights considered in this paper is the class of $A_\infty$ weights. Several equivalent definitions of this class of weights can be given. For our purpose we choose the following one.

**Definition 1.7.** *We say that a weight $w$ is an $A_\infty$ weight if there are two positive constants $\Theta$ and $\nu$ such that*

$$w(E) \leq \Theta \left(\frac{|E|}{|B|}\right)^\nu w(B).$$

*for every ball $B \subset \mathbb{R}^n$ and every measurable subsets $E$ of $B$. The pair $(\Theta, \nu)$ is called the $A_\infty$ constants of $w$ and is denoted by $[w]_\infty$.*

**Theorem 1.8.** *Let $1 < p < \infty$, $0 < q < \infty$, $0 < t \leq \infty$, $R_0 > 0$, and let $w \in A_\infty$. Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Then there exist positive constants $s = s(n, p, \alpha, \beta) > 1$ and $\delta = \delta(n, p, \alpha, \beta, q, t, [w]_\infty) \in (0, 1)$ such that the following holds. Given $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n) \cap L^{q,t}_w(\Omega, \mathbb{R}^n)$, the boundary value problem (1.12) in a $(\delta, R_0)$-Reifenberg flat domain $\Omega$, with $[\mathcal{A}]^{R_0}_s \leq \delta$, has a unique weak solution $u \in W^{1,p}_0(\Omega)$ satisfying $\nabla u \in L^{q,t}_w(\Omega, \mathbb{R}^n)$ with the estimate*

$$\|\nabla u\|_{L^{q,t}_w(\Omega)} \leq C \left\| \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} \right\|_{L^{q,t}_w(\Omega)}. \tag{1.13}$$

*Here the constant $C$ depends only on $n, p, \alpha, \beta, q, t, [w]_\infty$, and* $\mathrm{diam}(\Omega)/R_0$.

In (1.13), $\mathfrak{M}$ denotes the Hardy-Littlewood maximal function defined for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$\mathfrak{M}f(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|dy, \qquad x \in \mathbb{R}^n.$$

It is well known that the $A_\infty$ class is the union of $A_s$ weights for all $s \in (1, \infty)$. Here a weight $w$ is an $A_s$ weight, $1 < s < \infty$, if the quantity

$$[w]_s := \sup_{\text{balls } B \subset \mathbb{R}^n} \left(\fint_B w(x)dx\right) \left(\fint_B w(x)^{\frac{-1}{s-1}} dx\right)^{s-1} < +\infty.$$

The quantity $[w]_s$ will be referred to as the $A_s$ constant of $w$.

It is also known that $A_s$ weights with $s \in (1, \infty)$ can be characterized by the boundedness of $\mathfrak{M}$ on $L^{s,t}_w(\mathbb{R}^n)$ with $t \in (0, \infty]$ (see [40, 44]).

**Lemma 1.9.** *Let $0 < t \leq \infty$ and $1 < s < \infty$. Then for any $w \in A_s$ there exists a constant $C = C(n, s, t, [w]_s)$ such that*

$$\|\mathfrak{M}f\|_{L^{s,t}_w(\mathbb{R}^n)} \leq C\|f\|_{L^{s,t}_w(\mathbb{R}^n)} \tag{1.14}$$

*for all $f \in L^{s,t}_w(\mathbb{R}^n)$. Conversely, if (1.14) holds for all $f \in L^{s,t}_w(\mathbb{R}^n)$, then $w \in A_s$.*

Theorem 1.8 and Lemma 1.9 yield the following estimate on weighted Lorentz spaces that involves no maximal function. However, the weights are now restricted to a smaller Muckenhoupt class.

**Theorem 1.10.** *Let $1 < p < q < \infty$, $0 < t \leq \infty$, $R_0 > 0$, and let $w \in A_{q/p}$. Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Then there exist positive constants $s = s(n, p, \alpha, \beta) > 1$ and $\delta = \delta(n, p, \alpha, \beta, q, t, [w]_{q/p}) \in (0, 1)$ such that the following holds. Given $\mathbf{f} \in L^{q,\,t}_w(\Omega, \mathbb{R}^n)$, the boundary value problem (1.12) in a $(\delta, R_0)$-Reifenberg flat domain $\Omega$, with $[\mathcal{A}]_s^{R_0} \leq \delta$, has a unique weak solution $u \in W_0^{1,\,p}(\Omega)$ satisfying $\nabla u \in L^{q,\,t}_w(\Omega, \mathbb{R}^n)$ with the estimate*

$$\|\nabla u\|_{L^{q,\,t}_w(\Omega)} \leq C \, \|\mathbf{f}\|_{L^{q,\,t}_w(\Omega)} .$$

*Here the constant $C$ depends only on $n, p, \alpha, \beta, q, t, [w]_{q/p}$, and $\mathrm{diam}(\Omega)/R_0$.*

We observe that the global gradient estimates for solutions of (1.12) obtained in Theorem 1.10 extend results in [2, 4, 6, 9, 13, 39] to arbitrary $p$, and those in [7, 8, 10, 11, 24, 33, 34, 40, 46] to more general nonlinear structures or domains. Moreover, Theorem 1.10 also generalizes the result of [5] to the setting of weighted Lorentz spaces. In these respects, our results are actually new even in the unweighted setting.

An immediate but important application of the above theorem is the following gradient estimate on the Lorentz-Morrey space. The Lorentz-Morrey function space is defined as follows. A function $g \in L^{q,\,t}(\Omega)$, $0 < q < \infty$, $0 < t \leq \infty$ is said to belong to the Morrey-Lorentz function space $\mathcal{L}^{q,\,t;\,\theta}(\Omega)$, $0 < \theta \leq n$, if

$$\|g\|_{\mathcal{L}^{q,\,t;\,\theta}(\Omega)} := \sup_{z\in\Omega,\, 0<r\leq \mathrm{diam}(\Omega)} r^{\frac{\theta-n}{q}} \, \|g\|_{L^{q,\,t}(B_r(z)\cap\Omega)} < +\infty.$$

When $s = t$ the space $\mathcal{L}^{s,\,t;\,\theta}(\Omega)$ becomes the usual Morrey space $\mathcal{L}^{s;\,\theta}(\Omega)$ introduced earlier.

**Theorem 1.11.** *Let $1 < p < q < \infty$, $0 < t \leq \infty$, $0 < \theta \leq n$, and let $R_0 > 0$. Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Then there exist positive constants $s = s(n, p, \alpha, \beta) > 1$ and $\delta = \delta(n, p, \alpha, \beta, q, t, \theta) \in (0, 1)$ such that the following holds. Given $\mathbf{f} \in \mathcal{L}^{q,\,t;\,\theta}(\Omega, \mathbb{R}^n)$, the boundary value problem (1.12) in a $(\delta, R_0)$-Reifenberg flat domain $\Omega$, with $[\mathcal{A}]_s^{R_0} \leq \delta$, has a unique weak solution $u \in W_0^{1,\,p}(\Omega)$ satisfying $\nabla u \in \mathcal{L}^{q,\,t;\,\theta}(\Omega, \mathbb{R}^n)$ with the estimate*

$$\|\nabla u\|_{\mathcal{L}^{q,\,t;\,\theta}(\Omega)} \leq C \, \|\mathbf{f}\|_{\mathcal{L}^{q,\,t;\,\theta}(\Omega)} .$$

*Here the constant $C$ depends only on $n, p, \alpha, \beta, q, t, \theta$, and $\mathrm{diam}(\Omega)/R_0$.*

We note that the Lorentz-Morrey bound in Theorem 1.11 is obtained by applying Theorem 1.10 with an appropriate choice of weight functions exactly as in proof of Theorem 2.3 in [40]. On the other hand, by Morrey-Sobolev Embedding Theorem (see [17, Theorem 7.19]), Theorem 1.11 in its turn yields the following global Hölder regularity of solutions. This extends the results of [39, 40] to more general nonlinear structures.

**Corollary 1.12.** *Let $1 < p < q < \infty$, $0 < \theta < \min\{n, q\}$, and let $R_0 > 0$. Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Then there exist positive constants $s = s(n, p, \alpha, \beta) > 1$ and $\delta = \delta(n, p, \alpha, \beta, q, \theta) \in (0, 1)$ such that the following holds. Given $\mathbf{f} \in \mathcal{L}^{q;\,\theta}(\Omega, \mathbb{R}^n)$, the unique* *solution* $u \in W_0^{1,p}(\Omega)$ *to* (1.12) *in a* $(\delta, R_0)$*-Reifenberg flat domain* $\Omega$*, with* $[\mathcal{A}]_s^{R_0} \leq \delta$*, is* $C^{0,1-\theta/q}(\overline{\Omega})$ *Hölder continuous and for any ball* $B_r$

$$\mathrm{osc}_{B_r \cap \overline{\Omega}}\, u \leq C \, \|\mathbf{f}\|_{\mathcal{L}^{q;\,\theta}(\Omega)} \, r^{1-\theta/q}.$$

*Here the constant* $C$ *depends only on* $n, p, \alpha, \beta, q, \theta$*, and* $\mathrm{diam}(\Omega)/R_0$*.*

We point out that the proof of Theorem 1.8 is different from that of the weighted $L^q$ gradient estimates given in [46]. We follow the ideas pioneered in [11] and implemented in [6, 9, 57] rather than relying on a local version of Fefferman-Stein sharp maximal functions and $C^{1,\alpha}$ regularity of homogeneous equations, as is done in [46]. Specifically, we make use of weak compactness, Lipschitz regularity of reference homogeneous equations, and a weighted variant of the Vitali covering lemma. Some of the ideas in the recent paper [47] are also employed to handle difficulties arising from the general nonlinearity $\mathcal{A}(x, \xi)$ considered in this paper. Moreover, our approach is somewhat direct and thus avoids an approximation procedure on the data and the domains as was carried out in [5].

The paper is organized as follows. In Section 2, certain weighted and unweighted local interior and boundary estimates will be given. Global gradient estimates in Lorentz spaces, i.e., Theorem 1.10 will be proved in Section 3. Then the Riccati type equation (1.1) is studied in Section 4 where Theorems 1.1 and 1.2 are proved. Finally, in Section 5 we verify that the function $|f|$, where $f$ is as in (1.10), fails to satisfy the condition (1.8).

## 2. Local interior and boundary estimates

In this section we obtain certain unweighted and weighted local interior and boundary estimates for weak solutions $u$ of (1.12). They will be essential for our global estimates later. We first recall basic existence and uniqueness results together with accompanying local and global energy estimates in the following proposition.

**Proposition 2.1.** *If* $\mathcal{A}$ *satisfies* (1.3) *and the first bound in* (1.2)*, then corresponding to a given* $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$ *there exists a unique weak solution* $u \in W_0^{1,p}(\Omega)$ *to* (1.12) *such that*

$$\int_{B_r(y)\cap\Omega} |\phi|^p |\nabla u|^p dx \leq C \left( \int_{B_r(y)\cap\Omega} |\phi|^p |\mathbf{f}|^p dx + \int_{B_r(y)\cap\Omega} |\nabla\phi|^p |u|^p dx \right) \tag{2.1}$$

*holds for all* $\phi \in C_0^\infty(B_r(y))$ *and any ball* $B_r(y)$ *intersecting* $\Omega$*. In particular, the following global energy estimate holds:*

$$\int_\Omega |\nabla u|^p dx \leq C \int_\Omega |\mathbf{f}|^p dx.$$

*Here the constant* $C$ *depends only on* $p, \alpha$ *and* $\beta$*.*

*Proof.* Since $\mathbf{f} \in L^p(\Omega, \mathbb{R}^n)$ we see that the distribution $\mathrm{div}|\mathbf{f}|^{p-2}\mathbf{f}$ belongs to the dual of $W_0^{1,p}(\Omega)$. Thus the existence and uniqueness follow from the theory of monotone operators (see, e.g., [36, 42]). To obtain (2.1) one uses $|\phi|^p u$ as a test function in (1.12) and argue, e.g., as in the proof of Lemma 4.3 in [8]. In doing so one employs (1.3), the first inequality in (1.2), along with Hölder and Young's inequalities. □

2.1. **Unweighted local estimates.** In this subsection we obtain various unweighted local interior and boundary estimates. First let us consider the interior ones. For $u \in W^{1,p}_{\rm loc}(\Omega)$ being a weak solution of (1.12) and for each ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$, we defined $w \in u + W^{1,p}_0(B_{2R})$ as the unique solution to the Dirichlet problem

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{2R}, \\ w &= u \quad \text{on } \partial B_{2R}. \end{cases} \tag{2.2}$$

Then a well-known variant of Gehring's lemma applied to the function $w$ defined above yields the following result (see [18, Theorem 6.7]).

**Lemma 2.2.** *With $u \in W^{1,p}_{\rm loc}(\Omega)$, let $w$ be as in (2.2). Then there exists a constant $\theta_0 = \theta_0(n, p, \alpha, \beta) > 1$ such that the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0}} \leq C \fint_{B_\rho(z)} |\nabla w|^p dx$$

*holds for all balls $B_\rho(z) \subset B_{2R}$ for a constant $C$ depending only on $n, p, \alpha, \beta$.*

The next lemma gives an estimate for the difference $\nabla u - \nabla w$ in terms of the data $\mathbf{f}$.

**Lemma 2.3.** *Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a weak solution of (1.12) and let $w$ be as in (2.2). Then there is a constant $C = C(n, p, \alpha, \beta)$ such that*

$$\fint_{B_{2R}} |\nabla u - \nabla w|^p dx \leq C \fint_{B_{2R}} |\mathbf{f}|^p dx$$

*for $p \geq 2$, and*

$$\fint_{B_{2R}} |\nabla u - \nabla w|^p dx \ \leq \ C \left(\fint_{B_{2R}} |\mathbf{f}|^p dx\right)^{p-1} \left(\fint_{B_{2R}} |\nabla u|^p dx\right)^{2-p}$$

*for $1 < p < 2$.*

*Proof.* Using $u - w$ as a test function in (1.12) and (2.2) we have

$$\int_{B_{2R}} \langle \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla w), \nabla u - \nabla w \rangle dx = \int_{B_{2R}} \langle |\mathbf{f}|^{p-2}\mathbf{f}, \nabla(u - w)\rangle dx.$$

By (1.3) and Hölder's inequality this gives

$$\begin{aligned} \int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla(u-w)|^2 dx &\leq C \int_{B_{2R}} |\mathbf{f}|^{p-1} |\nabla(u-w)| dx \\ &\leq C \left(\int_{B_{2R}} |\mathbf{f}|^p dx\right)^{\frac{p-1}{p}} \left(\int_{B_{2R}} |\nabla u - \nabla w|^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.3}$$

Thus for $p \geq 2$ we get

$$\int_{B_{2R}} |\nabla(u-w)|^p dx \leq C \left(\int_{B_{2R}} |\mathbf{f}|^p dx\right)^{\frac{p-1}{p}} \left(\int_{B_{2R}} |\nabla u - \nabla w|^p dx\right)^{\frac{1}{p}},$$

which gives the desired estimate.

For $1 < p < 2$ we write

$$|\nabla u - \nabla w|^p = (|\nabla u|^2 + |\nabla w|^2)^{\frac{(p-2)p}{4}} |\nabla w - \nabla u|^p (|\nabla u|^2 + |\nabla w|^2)^{\frac{(2-p)p}{4}}$$

and apply Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p} > 1$ to get

$$\begin{aligned}
\int_{B_{2R}} |\nabla u - \nabla w|^p dx &\leq \left(\int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2 dx\right)^{\frac{p}{2}} \times \\
&\quad \times \left(\int_{B_{2R}} (|\nabla u|^2 + |\nabla w|^2)^{\frac{p}{2}} dx\right)^{\frac{2-p}{2}} \\
&\leq C \left(\int_{B_{2R}} |\mathbf{f}|^p dx\right)^{\frac{p-1}{2}} \left(\int_{B_{2R}} |\nabla u - \nabla w|^p dx\right)^{\frac{1}{2}} \left(\int_{B_{2R}} |\nabla u|^p dx\right)^{\frac{2-p}{2}}.
\end{aligned}$$

Here we used (2.3) and the estimate $\int_{B_{2R}} |\nabla w|^p dx \leq C \int_{B_{2R}} |\nabla u|^p dx$ in the last inequality. Thus we arrive at the inequality

$$\left(\int_{B_{2R}} |\nabla u - \nabla w|^p dx\right)^{\frac{1}{2}} \leq C \left(\int_{B_{2R}} |\mathbf{f}|^p dx\right)^{\frac{p-1}{2}} \left(\int_{B_{2R}} |\nabla u|^p dx\right)^{\frac{2-p}{2}},$$

which gives the desired estimate in the case $1 < p < 2$. □

Next with $u$ and $w$ being as in (2.2) where $B_{2R} = B_{2R}(x_0)$ we further define another function $v \in w + W_0^{1,p}(B_R)$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_R}(\nabla v) &= 0 \quad \text{in } B_R, \\ v &= w \quad \text{on } \partial B_R, \end{cases} \tag{2.4}$$

where $B_R = B_R(x_0)$.

**Lemma 2.4.** *Let $w$ and $v$ be as in (2.2) and (2.4), respectively. There exist constants $s = s(n, p, \alpha, \beta) > 1$ and $C = C(n, p, \alpha, \beta) > 0$ such that*

$$\fint_{B_R} |\nabla v - \nabla w|^p dx \leq C \left(\fint_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^s dx\right)^{\min\{p, \frac{p}{p-1}\}/s} \fint_{B_{2R}} |\nabla w|^p dx.$$

*Proof.* The proof of this lemma is just similar to that of Lemma 3.9 (for $p \geq 2$) and Lemma 3.10 (for $1 < p < 2$) in [47]. Thus we omit its proof. □

**Corollary 2.5.** *Let $s = s(n, p, \alpha, \beta) > 1$ be as in Lemma 2.4. For any $\epsilon > 0$ there exists a small $\delta = \delta(n, p, \alpha, \beta, \epsilon) > 0$ such that if $u \in W^{1,p}_{\rm loc}(\Omega)$ is a weak solution of (1.12) with*

$$\fint_{B_{2R}} |\nabla u|^p \leq 1, \qquad \fint_{B_{2R}} |\mathbf{f}|^p dx \leq \delta^p, \qquad \left(\fint_{B_R} \Upsilon(\mathcal{A}, B_R)(x)^s dx\right)^{\frac{1}{s}} \leq \delta,$$

*where $B_{2R} \Subset \Omega$, then there is a function $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$ such that*

$$\|\nabla v\|_{L^\infty(B_{R/2})} \leq C_0 = C_0(n, p, \alpha, \beta), \qquad \text{and} \quad \fint_{B_R} |\nabla u - \nabla v|^p dx \leq \epsilon^p.$$

*Proof.* We first observe that the monotonicity condition (1.3) implies the following ellipticity condition:

$$\langle \nabla_\xi \mathcal{A}(x,\xi)\lambda, \lambda\rangle \geq 2^{\frac{p-2}{2}}\alpha|\xi|^{p-2}|\lambda|^2 \tag{2.5}$$

for every $(\lambda,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and for a.e. $x \in \mathbb{R}^n$. Let $w$ and $v$ be as in (2.2) and (2.4), respectively. By standard regularity theory (see [12, 54]), using (2.5) and the second bound in (1.2), we have

$$\begin{aligned}\|\nabla v\|^p_{L^\infty(B_{R/2})} &\leq C_1 \fint_{B_R} |\nabla v|^p dx \leq C_2 \fint_{B_R} |\nabla w|^p dx\\ &\leq C_3 \fint_{B_{2R}} |\nabla w|^p dx \leq C_4 \fint_{B_{2R}} |\nabla u|^p dx \leq C_4.\end{aligned}$$

Thus the lemma follows from Lemma 2.3 and Lemma 2.4. □

Next, we consider the corresponding unweighted local boundary estimates. Suppose that the domain $\Omega$ is $(\delta, R_0)$-Reifenberg flat with $\delta < 1/2$. Let $x_0 \in \partial\Omega$, $R \in (0, R_0/10)$, and let $u \in W^{1,p}_0(\Omega)$ be a weak solution of (1.12). On $\Omega_{10R}(x_0)$ we define $w \in u + W^{1,p}_0(\Omega_{10R}(x_0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{10R}(x_0),\\ w = u & \text{on } \partial\Omega_{10R}(x_0).\end{cases} \tag{2.6}$$

We now extend $u$ by zero to $\mathbb{R}^n \setminus \Omega$ and then extend $w$ by $u$ to $\mathbb{R}^n \setminus \Omega_{10R}(x_0)$. Analogous to Lemma 2.2 we have the following boundary counterpart.

**Lemma 2.6.** *With $u \in W^{1,p}_0(\Omega)$, let $w$ be as in (2.6). Then there exists a constant $\theta_0 = \theta_0(n,p,\alpha,\beta) > 1$ such that the reverse Hölder type inequality*

$$\left(\fint_{B_{t/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0}} \leq C \fint_{B_{3t}(z)} |\nabla w|^p dx$$

*holds for all balls $B_{3t}(z) \subset B_{10R}(x_0)$ for a constant $C$ depending only on $n, p, \alpha, \beta$.*

Lemma 2.6 follows from [48, Lemma 2.3]. We remark that, for a $(\delta, R_0)$-Reifenberg flat domains with $\delta < 1/2$, the exterior density condition

$$|B_r(x) \cap (\mathbb{R}^n \setminus \Omega)| \geq c|B_r(x)|$$

holds with $c = [(1-\delta)/2]^n \geq 4^{-n}$ for all $x \in \partial\Omega$ and $0 < r < R_0$.

The next lemma can be proved similarly to the proof of Lemma 2.3.

**Lemma 2.7.** *Let $u$ and $w$ be as in (2.6). Then there is a constant $C = C(n,p,\alpha,\beta)$ such that*

$$\fint_{B_{10R}(x_0)} |\nabla u - \nabla w|^p dx \leq C \fint_{B_{10R}(x_0)} |\mathbf{f}|^p \chi_\Omega dx$$

*for $p \geq 2$, and*

$$\fint_{B_{10R}(x_0)} |\nabla u - \nabla w|^p dx \leq C \left(\fint_{B_{10R}(x_0)} |\mathbf{f}|^p \chi_\Omega dx\right)^{p-1} \left(\fint_{B_{10R}(x_0)} |\nabla u|^p dx\right)^{2-p}$$

*for* $1 < p < 2$.

With $x_0 \in \partial\Omega$ and $0 < R < R_0/10$ as above, we now set $\rho = R(1-\delta)$. ¿From the definition of Reifenberg flat domains we deduce that there exists a coordinate system $\{z_1, z_2, \cdots, z_n\}$ with the origin $0 \in \Omega$ such that in this coordinate system $x_0 = (0, \ldots, 0, -\rho\delta/(1-\delta)) \in \partial\Omega$ and

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{z = (z_1, z_2, \ldots, z_n) : z_n > -2\rho\delta/(1-\delta)\}.$$

Thus if $\delta < 1/2$ we have

$$B_\rho^+(0) \subset \Omega \cap B_\rho(0) \subset B_\rho(0) \cap \{z = (z_1, z_2, \ldots, z_n) : z_n > -4\delta\rho\}. \tag{2.7}$$

Here $B_\rho^+(0) := B_\rho(0) \cap \{z = (z_1, z_2, \ldots, z_n) : z_n > 0\}$ denotes an upper half ball in the corresponding coordinate system.

With this $\rho$, we define another function $v \in w + W_0^{1,p}(\Omega_\rho(0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_\rho}(\nabla v) &= 0 \quad \text{in } \Omega_\rho(0), \\ v &= w \quad \text{on } \partial\Omega_\rho(0). \end{cases} \tag{2.8}$$

We then set $v$ to be equal to $w$ in $\mathbb{R}^n \setminus \Omega_\rho(0)$. Similar to Lemma 2.4, we have the following boundary comparison estimate.

**Lemma 2.8.** *Let $w$ and $v$ be as in (2.6) and (2.8), respectively. There are constants $s = s(n, p, \alpha, \beta) > 1$ and $C = C(n, p, \alpha, \beta) > 0$ such that*

$$\fint_{B_\rho(0)} |\nabla v - \nabla w|^p dx \leq C \left( \fint_{B_\rho(0)} \Upsilon(\mathcal{A}, B_\rho(0))(x)^s dx \right)^{\min\{p, \frac{p}{p-1}\}/s} \fint_{B_{6\rho}(0)} |\nabla w|^p dx.$$

As the boundary of $\Omega$ can be very irregular, the $L^\infty$-norm of $\nabla v$ up to it could be unbounded. Therefore, we consider another equation:

$$\begin{cases} \operatorname{div} \overline{\mathcal{A}}_{B_\rho(0)}(\nabla V) &= 0 \quad \text{in } B_\rho^+(0), \\ V &= 0 \quad \text{on } T_\rho, \end{cases} \tag{2.9}$$

where $T_\rho$ is the flat portion of $\partial B_\rho^+(0)$. A function $V \in W^{1,p}(B_\rho^+(0))$ is a weak solution of (2.9) if its zero extension to $B_\rho(0)$ belongs to $W^{1,p}(B_\rho(0)$ and if

$$\int_{B_\rho^+(0)} \overline{\mathcal{A}}_{B_\rho(0)}(\nabla V) \cdot \nabla \varphi dx = 0$$

for all $\varphi \in W_0^{1,p}(B_\rho^+(0))$.

We shall need the following key perturbation result obtained earlier in [50, Theorem 2.12].

**Theorem 2.9.** *Suppose that $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)-(1.3). For any $\epsilon > 0$ there exists a small $\delta = \delta(n, p, \alpha, \beta, \epsilon) > 0$ such that if $v \in W^{1,p}(\Omega_\rho(0))$ is a solution of (2.8) under the geometric setting (2.7), then there exists a weak solution $V \in W^{1,p}(B_\rho^+(0))$ of (2.9) whose zero extension to $B_\rho(0)$ satisfies*

$$\|\nabla V\|^p_{L^\infty(B_{\rho/4}(0))} \leq C \fint_{B_\rho(0)} |\nabla v|^p dx,$$

*and*

$$\fint_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p dx \leq \epsilon^p \fint_{B_\rho(0)} |\nabla v|^p dx.$$

*Here* $C = C(n, p, \alpha, \beta)$.

**Corollary 2.10.** *Let* $s = s(n, p, \alpha, \beta) > 1$ *be as in Lemma 2.8. For any* $\epsilon > 0$ *there exists a small* $\delta = \delta(n, p, \alpha, \beta, \epsilon) > 0$ *such that the following holds. If* $\Omega$ *is* $(\delta, R_0)$*-Reifenberg flat and if* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of* (1.12) *with*

$$\fint_{B_{10R}(x_0)} |\nabla u|^p \leq 1, \qquad \fint_{B_{10R}(x_0)} |\mathbf{f}|^p \chi_\Omega dx \leq \delta^p, \qquad [\mathcal{A}]_s^{R_0} \leq \delta,$$

*where* $x_0 \in \partial\Omega$ *and* $R \in (0, R_0/10)$, *then there is a function* $V \in W^{1,\infty}(B_{R/10}(x_0))$ *such that*

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))} \leq C_0 = C_0(n, p, \alpha, \beta),$$

*and*

$$\fint_{B_{R/10}(x_0)} |\nabla u - \nabla V|^p dx \leq \epsilon^p. \tag{2.10}$$

*Proof.* First recall that $u$ is extended by zero to $\mathbb{R}^n \setminus \Omega$. With $x_0 \in \partial\Omega$ and $R \in (0, R_0/10)$, we set $\rho = R(1-\delta)$. By the remark make after Lemma 2.7, via a translation and a rotation we may assume that $0 \in \Omega$, $x_0 = (0, \dots, 0, -\rho\delta/(1-\delta))$ and the geometric setting

$$B_\rho^+(0) \subset \Omega_\rho(0) \subset B_\rho(0) \cap \{x_n > -4\delta\rho\}. \tag{2.11}$$

Moreover, for $\delta < 1/45$ we have

$$B_\rho(0) \subset B_{6\rho}(0) \subset B_{10R}(x_0), \quad \text{and } B_{R/10}(x_0) \subset B_{\rho/8}(0).$$

We now choose $w$ and $v$ as in (2.6) and (2.8) corresponding to these $R$ and $\rho$. Then there holds

$$\fint_{B_\rho(0)} |\nabla v|^p dx \leq C \fint_{B_{6\rho}(0)} |\nabla w|^p dx \leq C \fint_{B_{10R}(x_0)} |\nabla u|^p dx \leq C.$$

By Theorem 2.9 for any $\eta > 0$ we can find a $\delta = \delta(n, p, \alpha, \beta, \eta) \in (0, 1/45)$ such that, under (2.11), there is a function $V \in W^{1,p}(B_\rho(0)) \cap W^{1,\infty}(B_{\rho/4}(0))$ such that

$$\|\nabla V\|^p_{L^\infty(B_{R/10}(x_0))} \leq C \, \|\nabla V\|^p_{L^\infty(B_{\rho/4}(0))} \leq C \fint_{B_\rho(0)} |\nabla v|^p dx \leq C,$$

and

$$\fint_{B_{\rho/8}(0)} |\nabla v - \nabla V|^p dx \leq \eta^p \fint_{B_\rho(0)} |\nabla v|^p dx \leq C\eta^p.$$

By Lemma 2.7 we find

$$\fint_{B_{\rho/8}(0)} |\nabla u - \nabla w|^p dx \leq C \fint_{B_{10R}(x_0)} |\nabla u - \nabla w|^p dx \leq C \begin{cases} \delta^p & \text{if } p \geq 2, \\ \delta^{p(p-1)} & \text{if } 1 < p < 2. \end{cases}$$

Also, by Lemma 2.8 we have

$$\fint_{B_{\rho/8}(0)} |\nabla w - \nabla v|^p dx \le C\,[\mathcal{A}]_s^{R_0} \fint_{B_{6\rho}(0)} |\nabla w|^p dx \le C\,\delta^{\min\{p, \frac{p}{p-1}\}}.$$

Therefore, using the last three bounds and the inequality

$$\fint_{B_{R/10}(x_0)} |\nabla u - \nabla V|^p dx = \fint_{B_{\rho/8}(0)} |\nabla(u-w) + \nabla(w-v) + \nabla(v-V)|^p dx,$$

we obtain inequality (2.10) as desired. □

2.2. **Weighted estimates for upper-level sets.** The goal of this section is to prove Theorem 2.14 below. To that end, we first recall the following lemma.

**Lemma 2.11.** *Let $\Omega$ be a $(\delta, R_0)$-Reifenberg flat domain with $\delta < 1/8$, and let $w$ be an $A_\infty$ weight. Suppose that $\{B_r(y_i)\}_{i=1}^L$ is a finite collection of balls with centers $y_i \in \overline{\Omega}$ and radius $r < R_0/4$ that covers $\Omega$. Let $C \subset D \subset \Omega$ be measurable sets for which there exists $0 < \epsilon < 1$ such that*

1. *$w(C) < \epsilon\, w(B_r(y_i))$ for all $i = 1, \dots, L$, and*
2. *for all $y \in \Omega$ and $\rho \in (0, 2r]$, if $w(C \cap B_\rho(y)) \ge \epsilon\, w(B_\rho(y))$, then $B_\rho(y) \cap \Omega \subset D$.*

*Then we have the estimate*

$$w(C) \le A\,\epsilon\, w(D)$$

*for a constant $A$ depending only on $n$ and the $A_\infty$ constants of $w$.*

The proof of the above lemma follows Lebesgue Differentiation Theorem and the standard Vitali covering lemma (see [39]). In the unweighted case various versions of this lemma have been obtained (see, e.g., [57, 6]). A very similar lemma was also obtained in [11] based on the Calderón-Zygmund decomposition using cubes.

We next use Corollaries 2.5 and 2.10 to obtain the following technical result.

**Proposition 2.12.** *With $\mathcal{A}$ satisfying (1.2)-(1.3), there exist constants $\Lambda = \Lambda(n, p, \alpha, \beta) > 1$ and $s = s(n, p, \alpha, \beta) > 1$ such that the following holds. For any $\epsilon > 0$ there is a small $\delta = \delta(n, p, \alpha, \beta, \epsilon) > 0$ such that if $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.12) where $\Omega$ is $(\delta, R_0)$-Reifenberg flat, $[\mathcal{A}]_s^{R_0} \le \delta$, and*

$$B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} \le 1\} \cap \{x \in \mathbb{R}^n : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} \le \delta\} \ne \emptyset \tag{2.12}$$

*for some ball $B_\rho(y)$ with $\rho < R_0/600$, then there holds*

$$|\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)| < \epsilon\, |B_\rho(y)|. \tag{2.13}$$

*Proof.* By (2.12) there exists $x_0 \in B_\rho(y)$ such that for any $r > 0$

$$\fint_{B_r(x_0)} |\nabla u|^p dz \le 1 \quad \text{and} \quad \fint_{B_r(x_0)} |\mathbf{f}|^p \chi_\Omega dx \le \delta^p. \tag{2.14}$$

Here recall that $u$ is extended by zero outside $\Omega$. By the first inequality in (2.14) we see that for $x \in B_\rho(y)$ there holds

$$\mathfrak{M}(|\nabla u|^p)(x) \le \max\left\{\mathfrak{M}(\chi_{B_{2\rho}(y)}|\nabla u|^p)(x), 3^n\right\}. \tag{2.15}$$

In order to prove (2.13) we consider separately the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. First we consider the latter. Let $y_0 \in B_{4\rho}(y) \cap \partial\Omega$. We have

$$B_{2\rho}(y) \subset B_{6\rho}(y_0) \subset B_{600\rho}(y_0) \subset B_{605\rho}(x_0).$$

Since $6\rho < R_0/100$ by Corollary 2.10 there exists $s = s(n, p, \alpha, \beta) > 1$ such that the following holds. For any $\eta \in (0, 1)$ there is a small $\delta = \delta(n, p, \alpha, \beta, \eta) > 0$ such that if $\Omega$ is $(\delta, R_0)$-Reifenberg flat and $[\mathcal{A}]_s^{R_0} \le \delta$, then one can find a function $V \in W^{1,\infty}(B_{6\rho}(y_0))$ with

$$\|\nabla V\|_{L^\infty(B_{2\rho}(y))} \le \|\nabla V\|_{L^\infty(B_{6\rho}(y_0))} \le C_0 = C_0(n, p, \alpha, \beta), \tag{2.16}$$

and

$$\fint_{B_{2\rho}(y)} |\nabla u - \nabla V|^p dx \le C \fint_{B_{6\rho}(y_0)} |\nabla u - \nabla V|^p dx \le \eta^p. \tag{2.17}$$

In view of (2.15), (2.16) and the triangle inequality we see that for $\Lambda = \max\{3^n, 2C_0\}$ there hold the inclusions

$$\begin{aligned} &\left\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\right\} \cap B_\rho(y) \\ &\subset \left\{x \in \mathbb{R}^n : \mathfrak{M}(\chi_{B_{2\rho}(y)}|\nabla u|^p)^{\frac{1}{p}} > \Lambda\right\} \cap B_\rho(y) \\ &\subset \left\{x \in \mathbb{R}^n : \mathfrak{M}(\chi_{B_{2\rho}(y)}|\nabla u - \nabla V|^p)^{\frac{1}{p}} > \Lambda/2\right\} \cap B_\rho(y). \end{aligned}$$

Thus by weak-type $(1,1)$ bound for the Hardy-Littlewood maximal function and inequality (2.17) we find

$$|\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)| \le \frac{C}{\Lambda^p} \int_{B_{2\rho}(y)} |\nabla u - \nabla V|^p dx \le \frac{C}{C_0^p} |B_{2\rho}(y)|\, \eta^p.$$

This gives the estimate (2.13) in the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$, provided $\eta$ is appropriately chosen. The case $B_{4\rho}(y) \subset \Omega$ can be done in a similar way using Corollary 2.5 instead of Corollary 2.10. □

Proposition 2.12 is now used to obtain the following result that involves $A_\infty$ weights.

**Proposition 2.13.** *Let $w$ be an $A_\infty$ weight in $\mathbb{R}^n$ and let $\mathcal{A}$ satisfy (1.2)-(1.3). There exist constants $\Lambda = \Lambda(n, p, \alpha, \beta) > 1$ and $s = s(n, p, \alpha, \beta) > 1$ such that the following holds. For any $\epsilon > 0$ there is a small $\delta = \delta(n, p, \alpha, \beta, \epsilon, [w]_\infty) > 0$ such that if $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.12) where $\Omega$ is $(\delta, R_0)$-Reifenberg flat, $[\mathcal{A}]_s^{R_0} \le \delta$, and*

$$w(\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)) \ge \epsilon\, w(B_\rho(y))$$

*for some ball $B_\rho(y)$ with $\rho < R_0/600$, then there holds*

$$B_\rho(y) \subset \{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > 1\} \cup \{x \in \mathbb{R}^n : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} > \delta\}.$$

*Proof.* Suppose that $(\Theta, \nu)$ is a pair of $A_\infty$ constants of $w$. Let $\Lambda, s > 1$ be as in Proposition 2.12. Given $\epsilon > 0$, we choose $\delta = \delta(\epsilon, \Theta, \nu)$ as in Proposition 2.12 with $[\epsilon/(2\Theta)]^{1/\nu}$ replacing $\epsilon$. By contradiction, suppose that the last inclusion fails for this $\delta$, then we must have that

$$B_\rho(y) \cap \{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} \leq 1\} \cap \{x \in \mathbb{R}^n : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} \leq \delta\} \neq \emptyset.$$

Thus by Proposition 2.12 if $\Omega$ is $(\delta, R_0)$-Reifenberg flat and $[\mathcal{A}]_s^{R_0} \leq \delta$, there holds

$$|\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)| \leq \left(\frac{\epsilon}{2\Theta}\right)^{1/\nu} |B_\rho(y)|. \tag{2.18}$$

Now using the $A_\infty$ characterization of $w$, we get from (2.18) that

$$\begin{aligned} & w(\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)) \\ \leq \; & \Theta \left[\frac{|\{x \in \mathbb{R}^n : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\} \cap B_\rho(y)|}{|B_\rho(y)|}\right]^\nu w(B_\rho(y)) \\ \leq \; & \frac{\epsilon}{2} w(B_\rho(y)) < \epsilon\, w(B_\rho(y)). \end{aligned}$$

This yields a contradiction and thus the proof is complete. □

The last proposition is designed so that Lemma 2.11 can be applied. Indeed, they yield the following result.

**Theorem 2.14.** *Suppose that $\mathcal{A}$ satisfies (1.2)-(1.3). Let $w$ be an $A_\infty$ weight and let $\Lambda = \Lambda(n, p, \alpha, \beta) > 1$ and $s = s(n, p, \alpha, \beta) > 1$ be as in Proposition 2.13. Then for any $\epsilon > 0$ there exists $\delta = \delta(n, p, \alpha, \beta, \epsilon, [w]_\infty) > 0$ such that the following holds. Suppose that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.12) in a $(\delta, R_0)$-Reifenberg flat domain $\Omega$, with $[\mathcal{A}]_s^{R_0} \leq \delta$. Suppose also that $\{B_r(y_i)\}_{i=1}^L$ is a sequence of balls with centers $y_i \in \overline{\Omega}$ and a common radius $0 < r < R_0/1200$ that covers $\Omega$. If for all $i = 1, \ldots, L$*

$$w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\}) < \epsilon\, w(B_r(y_i)), \tag{2.19}$$

*then for any $t > 0$ and any integer $k \geq 1$ there holds*

$$\begin{aligned} w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda^k\})^t \;\leq\; & \sum_{i=1}^k (B\, \epsilon^t)^i\, w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} > \delta \Lambda^{k-i}\})^t \\ & + (B\, \epsilon^t)^k\, w(\{x \in \Omega : \mathfrak{M}|\nabla u|^p)^{\frac{1}{p}} > 1\})^t, \end{aligned}$$

*where the constant $B = B(n, t, [w]_\infty)$.*

*Proof.* The theorem will be proved by induction in $k$. Given $\epsilon > 0$, we take $\delta = \delta(\epsilon, [w]_\infty)$ as in Proposition 2.13. The case $k = 1$ follows from Proposition 2.13 and Lemma 2.11. Indeed, let

$$C = \{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\}$$

and

$$D = \{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > 1\} \cup \{x \in \Omega : [\mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)]^{\frac{1}{p}} > \delta\}.$$

Then from assumption (2.19) it follows that $w(C) < \epsilon\, w(B_r(y_i))$ for all $i = 1, \dots, L$. Moreover, if $y \in \Omega$ and $\rho \in (0, 2r)$ such that $w(C \cap B_\rho(y)) \geq \epsilon\, w(B_\rho(y))$, then $0 < \rho < R_0/600$ and $B_\rho(y) \cap \Omega \subset D$ by Proposition 2.13. Thus the hypotheses of Lemma 2.11 are satisfied which yield

$$\begin{aligned} w(C)^t &\leq B(n,t,[w]_\infty)\, \epsilon^t\, w(D)^t \\ &\leq B(n,t,[w]_\infty)\, \epsilon^t\, w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} > \delta\})^t + \\ &\quad + B(n,t,[w]_\infty)\, \epsilon^t\, w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > 1\})^t. \end{aligned}$$

This proves the case $k = 1$. Suppose now that the conclusion of the lemma holds for some $k > 1$. Normalizing $u$ to $u_\Lambda = u/\Lambda$ and $\mathbf{f}_\Lambda = \mathbf{f}/\Lambda$, we see that for every $i = 1, \dots, L$ there holds

$$\begin{aligned} w(\{x \in \Omega : \mathfrak{M}(|\nabla u_\Lambda|^p)^{\frac{1}{p}} > \Lambda\}) &= w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda^2\}) \\ &\leq w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda\}) \\ &< \epsilon\, w(B_r(y_i)). \end{aligned}$$

Here we used $\Lambda > 1$ in the first inequality. By inductive hypothesis it follows that

$$\begin{aligned} w(\{x \in \Omega : \mathfrak{M}(|\nabla u_\Lambda|^p)^{\frac{1}{p}} > \Lambda^k\})^t &\leq \sum_{i=1}^{k} (B\,\epsilon^t)^i\, w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}_\Lambda|^p \chi_\Omega)^{\frac{1}{p}} > \delta\Lambda^{k-i}\})^t \\ &\quad + (B\,\epsilon^t)^k\, w(\{x \in \Omega : \mathfrak{M}(|\nabla u_\Lambda|^p)^{\frac{1}{p}} > 1\})^t. \end{aligned}$$

Now applying the case $k = 1$ to the last term we conclude that

$$\begin{aligned} w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > \Lambda^{k+1}\})^t &\leq \sum_{i=1}^{k+1} (B\,\epsilon^t)^i\, w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}} > \delta\Lambda^{k+1-i}\})^t \\ &\quad + (B\,\epsilon^t)^{k+1}\, w(\{x \in \Omega : \mathfrak{M}(|\nabla u|^p)^{\frac{1}{p}} > 1\})^t, \end{aligned}$$

which completes the proof of the theorem. □

## 3. Global weighted Lorentz space estimates

We devote this section to the proof of Theorem 1.8.

**Proof of Theorem 1.8.** We shall consider only the case $t \in (0, \infty)$ as for $t = \infty$ the proof is just similar. Choose a finite number of points $\{y_i\}_{i=1}^L \subset \Omega$ and a ball $B_0$ of radius $2\text{diam}(\Omega)$ such that

$$\Omega \subset \bigcup_{i=1}^{L} B_r(y_i) \subset B_0,$$

where $r = \min\{R_0/1200, \text{diam}(\Omega)\}$. We claim that we can choose $N$ large such that for $u_N = u/N$ and for all $i = 1, \dots, L$

$$w(\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > \Lambda\}) < \epsilon\, w(B_r(y_i)). \tag{3.1}$$

Here $\Lambda = \Lambda(n, p, \alpha, \beta)$ is as in Proposition 2.13. Indeed, the weak-type $(1, 1)$ estimate for the maximal function there exists a constant $C(n) > 0$ such that

$$|\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > \Lambda\}| < \frac{C(n)}{(\Lambda N)^p} \int_{\Omega} |\nabla u|^p dx.$$

By Lemma 1.7 this yields

$$w(\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > \Lambda\}) < \Theta \left[ \frac{C(n)}{(\Lambda N)^p |B_0|} \int_{\Omega} |\nabla u|^p dx \right]^{\nu} w(B_0),$$

where $(\Theta, \nu)$ is a pair of $A_\infty$ constants of $w$. Also, as $w \in A_\infty$, there exist $C_1 > 1$ and $p_1 > 1$ depending only on $n, \Theta, \nu$ such that

$$w(B_0) \le C_1 \left[ \frac{|B_0|}{|B_r(y_i)|} \right]^{p_1} w(B_r(y_i))$$

for all $i = 1, \dots, L$ (see, e.g., [19]). Thus one obtains (3.1) provided we choose $N > 0$ so that

$$\begin{aligned} \frac{C(n)}{(\Lambda N)^p |B_0|} \int_{\Omega} |\nabla u|^p dx &= \left( \frac{\epsilon}{C_1 \Theta} \right)^{1/\nu} \left[ \frac{|B_r(y_i)|}{|B_0|} \right]^{p_1/\nu} \\ &= \left( \frac{\epsilon}{C_1 \Theta} \right)^{1/\nu} [r/(2\text{diam}(\Omega))]^{np_1/\nu}. \end{aligned} \tag{3.2}$$

With this $N$ we now consider the sum:

$$S = \sum_{k=1}^{\infty} \Lambda^{tk} w(\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > \Lambda^k\})^{\frac{t}{q}}.$$

It is easy to see that for $0 < t < \infty$ we have

$$C^{-1} S \le \|\mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}\|^t_{L^{q,\,t}_w(\Omega)} \le C(w(\Omega)^{\frac{t}{q}} + S). \tag{3.3}$$

By (3.1) and Theorem 2.14 we find

$$\begin{aligned} S &\leq \sum_{k=1}^{\infty} \Lambda^{tk} \left[ \sum_{j=1}^{k} (B\,\epsilon^{\frac{t}{q}})^j w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}/N|^p \chi_\Omega)^{\frac{1}{p}} > \delta \Lambda^{k-j}\})^{\frac{t}{q}} \right] \\ &\quad + \sum_{k=1}^{\infty} \Lambda^{tk} (B\,\epsilon^{\frac{t}{q}})^k w(\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > 1\})^{\frac{t}{q}}. \end{aligned}$$

Here the constants $B = B(n, t/q, [w]_\infty)$, $\epsilon = \Lambda^{-q} B^{-q/t} 2^{-q/t}$, and $\delta = \delta(n, p, \alpha, \beta, \epsilon, [w]_\infty)$ is determined by Theorem 2.14 which ultimately depends only on $n, p, \alpha, \beta, t, q$, and $[w]_\infty$.

Thus we have

$$\begin{aligned} S &\leq \sum_{j=1}^{\infty} (\Lambda^t B\,\epsilon^{\frac{t}{q}})^j \left[ \sum_{k=j}^{\infty} \Lambda^{t(k-j)} w(\{x \in \Omega : \mathfrak{M}(|\mathbf{f}/N|^p \chi_\Omega)^{\frac{1}{p}} > \delta \Lambda^{k-j}\})^{\frac{t}{q}} \right] \\ &\quad + \sum_{k=1}^{\infty} (\Lambda^t B\,\epsilon^{\frac{t}{q}})^k w(\{x \in \Omega : \mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}(x) > 1\})^{\frac{t}{q}} \\ &\leq C \left[ \|\mathfrak{M}(|\mathbf{f}/N|^p \chi_\Omega)^{\frac{1}{p}}\|^t_{L^{q,\,t}_w(\Omega)} + w(\Omega)^{\frac{t}{q}} \right] \sum_{k=1}^{\infty} (\Lambda^t B\,\epsilon^{\frac{t}{q}})^k \\ &\leq C \left[ \|\mathfrak{M}(|\mathbf{f}/N|^p \chi_\Omega)^{\frac{1}{p}}\|^t_{L^{q,\,t}_w(\Omega)} + w(\Omega)^{\frac{t}{q}} \right], \end{aligned}$$

by our choice of $\epsilon$. At this point, $C$ depends only on $n, p, \alpha, \beta, q, t$, and $[w]_\infty$.

Thus by (3.3) we obtain

$$\|\mathfrak{M}(|\nabla u_N|^p)^{\frac{1}{p}}\|^t_{L^{q,\,t}_w(\Omega)} \leq C \left[ w(\Omega)^{\frac{t}{q}} + \|\mathfrak{M}(|\mathbf{f}/N|^p \chi_\Omega)^{\frac{1}{p}}\|^t_{L^{q,\,t}_w(\Omega)} \right].$$

This gives

$$\|\nabla u\|_{L^{q,\,t}_w(\Omega)} \leq C \left[ N w(\Omega)^{\frac{1}{q}} + \|\mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}}\|_{L^{q,\,t}_w(\Omega)} \right]. \tag{3.4}$$

The first term on the right-hand side of (3.4) can be controlled as follows. Using (3.2) and Proposition 2.1 we obtain that

$$\begin{aligned} N w(\Omega)^{\frac{1}{q}} &\leq C\,|B_0|^{-1} w(\Omega)^{\frac{1}{q}} \|\nabla u\|_{L^p(\Omega)} \leq C\,w(\Omega)^{\frac{1}{q}} |B_0|^{-1} \|\mathbf{f}\chi_\Omega\|_{L^p(B_0)} \\ &\leq C\,w(\Omega)^{\frac{1}{q}} \mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}}(z) \end{aligned}$$

for every $z \in \Omega$. Here $C$ depends only on $n, p, \alpha, \beta, q, t, [w]_\infty$, and $\mathrm{diam}(\Omega)/R_0$. This yields

$$N w(\Omega)^{\frac{1}{q}} \leq C\,\|\mathfrak{M}(|\mathbf{f}|^p \chi_\Omega)^{\frac{1}{p}}\|_{L^{q,\,t}_w(\Omega)},$$

which in view of (3.4) completes the proof of Theorem 1.8.

## 4. Quasilinear Riccati type equations

In this section we prove Theorems 1.1 and 1.2. Let us begin with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** The proof of this theorem follows an idea in the important papers [26, 27] that treated the case $q = p$ in a slightly different framework. Let $B$ and $G(x, y)$ be as in (1.5). Then we can write

$$|\nabla u(x)|^q = -\mathrm{div} \int_B \nabla_x G(x, y)|\nabla u(y)|^q \chi_\Omega(y) dy \quad \text{in } \mathcal{D}'(\Omega).$$

Thus from equation (1.9) it follows that $\sigma = \mathrm{div}\, \boldsymbol{\zeta}$ in $\mathcal{D}'(\Omega)$ with

$$\boldsymbol{\zeta} = -\mathcal{A}(x, \nabla u) + \int_B \nabla_x G(x, y)|\nabla u(y)|^q \chi_\Omega(y) dy.$$

Note that by the first inequality in (1.2) we have

$$\left\| |\mathcal{A}(x, \nabla u)|^{\frac{q}{p-1}} \right\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq \beta^{\frac{q}{p-1}} \, \| |\nabla u|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)}.$$

On the other hand, using the pointwise bound (1.6) and estimates for Riesz potentials in Morrey spaces [1] we find

$$\left\| \left| \int_B \nabla_x G(\cdot, y)|\nabla u(y)|^q \chi_\Omega(y) dy \right|^{\frac{q}{p-1}} \right\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq C \, \| |\nabla u|^q \|^{\frac{q}{p-1}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)}.$$

The above two estimates show that $|\boldsymbol{\zeta}|^{\frac{q}{p-1}} \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ with the desired estimate. $\square$

We next apply the Morrey space estimate obtained Theorem 1.11 to prove Theorem 1.1.

**Proof of Theorem 1.1.** We apply the Schauder Fixed Point Theorem to prove the theorem. We prove it in several short steps.

**Step 1.** Suppose that $\mu \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ and that $\mathbf{g}$ is a vector field on $\Omega$ such that $|\mathbf{g}|^{\frac{q}{p-1}} \in \mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)$ for some $\epsilon > 0$ and $\frac{q(1+\epsilon)}{q-p+1} \leq n$. Then, as above using the Green function we can write $\mu = \mathrm{div}\, \mathbf{h}$ in $\mathcal{D}'(\Omega)$, where $\mathbf{h}$ is a gradient vector field such that

$$\left\| |\mathbf{h}|^{\frac{q}{p-1}} \right\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq C \, \|\mu\|^{\frac{q}{p-1}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)}.$$

Thus by Theorem 1.11 (applied with $\mathbf{f} = |\mathbf{g} + \mathbf{h}|^{\frac{2-p}{p-1}}(\mathbf{g} + \mathbf{h})$) the equation

$$\begin{cases} \mathrm{div}\, \mathcal{A}(x, \nabla u) &= \mu + \mathrm{div}\, \mathbf{g} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in W_0^{1,\, q(1+\epsilon)}(\Omega)$ such that

$$\| |\nabla u|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq C \left\| |\mathbf{g} + \mathbf{h}|^{\frac{q}{p-1}} \right\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)}.$$

This implies that

$$\| |\nabla u|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq C_0 \left[ \|\mu\|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} + \left\| |\mathbf{g}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \right]^{\frac{q}{p-1}} \tag{4.1}$$

for a constant $C_0$ depends only on $n, p, \alpha, \beta, q, \operatorname{diam}(\Omega)$, and $\operatorname{diam}(\Omega)/R_0$.

**Step 2.** We now let $\boldsymbol{\zeta}$ be as in the theorem and suppose that

$$\left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq c_0 := \frac{q-p+1}{q} \left( \frac{p-1}{C_0 q} \right)^{\frac{p-1}{q-p+1}}, \tag{4.2}$$

where $C_0$ is the constant in the bound (4.1). For each $t \in [0, \infty)$ we define

$$g(t) = C_0 \left[ t + \left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \right]^{\frac{q}{p-1}} - t.$$

Then $g'(t) = 0$ if and only if

$$t = t_0 = \left( \frac{p-1}{C_0 q} \right)^{\frac{p-1}{q-p+1}} - \left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)},$$

and it is easy to see from (4.2) that $g$ has exactly one root $T$ in the interval $(0, t_0]$.

**Step 3.** We now set

$$E = \left\{ v \in W_0^{1,1}(\Omega) : v \in W_0^{1,\, q(1+\epsilon)}(\Omega) \text{ and } \| |\nabla v|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \leq T \right\}.$$

Then E is convex and closed under the strong topology of $W_0^{1,1}(\Omega)$.

Given $v \in E$, let $u \in W_0^{1,\, q(1+\epsilon)}(\Omega)$ be the unique solution to the equation

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) &= |\nabla v|^q + \operatorname{div} \boldsymbol{\zeta} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

By (4.1) in **Step 1** and the fact that $T$ is a root of $g$ in **Step 2** we have

$$\begin{aligned} \| |\nabla u|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} &\leq C_0 \left[ \| |\nabla v|^q \|_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} + \left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \right]^{\frac{q}{p-1}} \\ &\leq C_0 \left[ T + \left\| |\boldsymbol{\zeta}|^{\frac{q}{p-1}} \right\|^{\frac{p-1}{q}}_{\mathcal{L}^{1+\epsilon;\, \frac{q(1+\epsilon)}{q-p+1}}(\Omega)} \right]^{\frac{q}{p-1}} \\ &= T. \end{aligned} \tag{4.4}$$

Therefore, we can define a map $S : E \to E$ by letting $S(v) = u$ for each $v \in E$, where $u$ is the unique solution of (4.3). Using (4.4) and the convergence result of [3] it is easy to see that $S(E)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$.

Moreover, if $\{v_k\} \subset E$ is a sequence that strongly converges in $W_0^{1,1}(\Omega)$ to $v$, then by Vitali Convergence Theorem we find that $\nabla v_k \to \nabla v$ in $L^q(\Omega, \mathbb{R}^n)$. Thus by Remark 2.1 in [3] it follows that a subsequence of $\{S(v_k)\}$ converges to $S(v)$ in $W_0^{1,1}(\Omega)$. As the limit is independent of the subsequence we see that $\{S(v_k)\}$ converges to $S(v)$ in $W_0^{1,1}(\Omega)$. This shows that the map $S$ is continuous on $E$. Now we can apply Schauder Fixed Point Theorem to conclude that $S$ has a fixed point $u$ in $E$. The fixed point $u$ is the solution to the problem (1.4) as desired. □

## 5. Appendix

This section is devoted to the following elementary result that was used in Example 1.10.

**Proposition 5.1.** *Let* $f(x) = |x|^{-\epsilon-s}\sin(|x|^{-\epsilon})$ *with* $\epsilon > 0$*,* $s > 0$*, and* $\epsilon + s < n$*. Then there exists a sequence of positive numbers* $\{r_k\}$ *decreasing to zero such that*

$$\int_{B_{r_k}(0)} f^+(x)dx \geq c\, r_k^{n-s-\epsilon}$$

*for a constant* $c > 0$ *independent of* $k$*. Here* $f^+$ *is the positive part of* $f$*.*

*Proof.* Let $r_k = (\pi/6 + 2k\pi)^{\frac{-1}{\epsilon}}$ for $k = 1, 2, 3, \ldots$ Then we have

$$\begin{aligned}\int_{B_{r_k}(0)} f^+(x)dx &= s(n)\int_0^{r_k} t^{n-\epsilon-s}\sin^+(t^{-\epsilon})\frac{dt}{t}\\ &= \frac{s(n)}{\epsilon}\int_{r_k^{-\epsilon}}^{\infty} x^{\frac{n-\epsilon-s}{-\epsilon}}\sin^+(x)\frac{dx}{x}\\ &\geq \frac{s(n)}{2\epsilon}\sum_{i=0}^{\infty}\int_{\pi/6+2k\pi+2i\pi}^{5\pi/6+2k\pi+2i\pi} x^{\frac{n-s}{-\epsilon}}dx,\end{aligned}$$

where $s(n)$ is the area of the unit sphere in $\mathbb{R}^n$. Thus using the elementary observation

$$2\int_{\pi/6+2k\pi+2i\pi}^{5\pi/6+2k\pi+2i\pi} x^{\frac{n-s}{-\epsilon}}dx \geq \int_{5\pi/6+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{n-s}{-\epsilon}}dx,$$

we find that

$$\begin{aligned}\int_{B_{r_k}(0)} f^+(x)dx &\geq \frac{s(n)}{6\epsilon}\sum_{i=0}^{\infty} 3\int_{\pi/6+2k\pi+2i\pi}^{5\pi/6+2k\pi+2i\pi} x^{\frac{n-s}{-\epsilon}}dx\\ &\geq \frac{s(n)}{6\epsilon}\sum_{i=0}^{\infty}\left(\int_{\pi/6+2k\pi+2i\pi}^{5\pi/6+2k\pi+2i\pi} x^{\frac{n-s}{-\epsilon}}dx + \int_{5\pi/6+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{n-s}{-\epsilon}}dx\right)\\ &\geq \frac{s(n)}{6\epsilon}\sum_{i=0}^{\infty}\int_{\pi/6+2k\pi+2i\pi}^{\pi/6+2k\pi+2(i+1)\pi} x^{\frac{n-s}{-\epsilon}}dx\\ &= \frac{s(n)}{6\epsilon}\int_{\pi/6+2k\pi}^{\infty} x^{\frac{n-s}{-\epsilon}}dx = \frac{s(n)}{6(n-s-\epsilon)} r_k^{n-s-\epsilon}.\end{aligned}$$

This completes the proof of the proposition. □

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