GLOBAL INTEGRAL GRADIENT BOUNDS FOR QUASILINEAR EQUATIONS BELOW OR NEAR THE NATURAL EXPONENT

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ABSTRACT. We obtain sharp integral potential bounds for gradients of solutions to a wide class of quasilinear elliptic equations with measure data. Our estimates are global over bounded domains that satisfy a mild exterior capacitary density condition. They are obtained in Lorentz spaces whose degrees of integrability lie below or near the natural exponent of the operator involved. As a consequence, nonlinear Calderón-Zygmund type estimates below the natural exponent are also obtained for $\mathcal{A}$-superharmonic functions in the whole space $\mathbb{R}^n$. This answers a question raised in our earlier work [28] and thus greatly improves the result there.

1. INTRODUCTION

The main goal of this paper is to obtain maximal global regularity for gradients of weak solutions to nonhomogeneous quasilinear equations with measure data of the form

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,\mathcal{A}(x,\nabla u) & = & \mu & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

for a given finite measure μ on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

In (1.1) the nonlinearity $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory vector valued function, i.e., $\mathcal{A}(x, \xi)$ is measurable in x for every ξ and continuous in ξ for a.e. x. We assume that $\mathcal{A}$ satisfies the following growth and monotonicity conditions: for some $1 < p \leq n$ there holds

$$|\mathcal{A}(x,\xi)| \leq \beta\, |\xi|^{p-1}, \tag{1.2}$$

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta \rangle \geq \alpha (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 \tag{1.3}$$

for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$. Here α and β are positive constants.

Under a capacitary density condition on Ω, for $2 - \frac{1}{n} < p \leq n$ we show in this paper the following integral gradient bound

$$\int_\Omega |\nabla u|^q dx \leq C \int_{\mathbb{R}^n} [\mathcal{M}_1(\chi_\Omega |\mu|)]^{\frac{q}{p-1}} dx, \tag{1.4}$$

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where q lies below or near the natural exponent p, i.e., $0 < q < p + \epsilon$ for some small $\epsilon > 0$ depending only on n, p, α, β, and Ω. In (1.4), χ_Ω is the characteristic function of Ω and $\mathcal{M}_1$ is the fractional maximal function defined for each nonnegative locally finite measure ν in $\mathbb{R}^n$ by

$$\mathcal{M}_1(\nu)(x) = \sup_{r>0} \frac{r\,\nu(B_r(x))}{|B_r(x)|}, \qquad x \in \mathbb{R}^n.$$

By a capacitary density condition on Ω we mean in this paper the p-capacity uniform thickness condition (with constants $r_0, c_0 > 0$) imposed on $\mathbb{R}^n \setminus \Omega$. That is, there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$ there holds

$$\text{cap}_p(\overline{B_t(x)} \cap (\mathbb{R}^n \setminus \Omega), B_{2t}(x)) \geq c_0\, \text{cap}_p(\overline{B_t(x)}, B_{2t}(x)). \tag{1.5}$$

Here for a compact set $K \subset B_{2t}(x)$ we define its p-capacity by

$$\text{cap}_p(K, B_{2t}(x)) = \inf \left\{ \int_{B_{2t}(x)} |\nabla \varphi|^p dy : \varphi \in C_0^\infty(B_{2t}(x)), \varphi \geq \chi_K \right\}.$$

It is easy to see that domains satisfying (1.5) include those with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$, there is $y \in B_t(x)$ such that $B_{t/c_0}(y) \subset \mathbb{R}^n \setminus \Omega$.

The restriction $q < p + \epsilon$ for a small $\epsilon > 0$ is a natural one in order to obtain (1.4). For one reason by now it is well known that, in general, the structural assumptions (1.2)-(1.3) on the nonlinearity $\mathcal{A}(x, \xi)$ are not enough to ensure higher integrability even locally for gradients of solutions to (1.1) (see, e.g., [21]). For another reason our condition on the domain Ω allows all domains with Lipschitz boundaries, whereas an example given in [15] (see also [20]) makes it clear that global $W^{1,q}$ regularity, $q > 2$, fails in general even for solutions to Laplace equations ($p = 2$) over polygonal domains.

We should mention that, at least in the case $2 \leq p \leq n$, a local version of inequality (1.4) has already been obtained by G. Mingione for the first time in [24] and the possibility of extending such local results to global ones was also mentioned in the same paper. Some of the key ideas in [24] are borrowed in this work in order to obtain (1.4), but technically our presentation is somewhat different from that of [24].

A solution u to the boundary value problem (1.1) is understood in the following sense. For each integer $k > 0$ the truncation

$$T_k(u) := \max\{-k, \min\{k, u\}\}$$

belongs to $W_0^{1,p}(\Omega)$ and satisfies

$$-\text{div}\, \mathcal{A}(x, \nabla T_k(u)) = \mu_k$$

in the sense of distributions in Ω for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \setminus \Omega$ then μ_k^+ and μ_k^- converge respectively to μ^+ and μ^- weakly as measures in $\mathbb{R}^n$. Here for a (signed) measure ν, ν^+ and ν^- stand for its positive and negative parts respectively, i.e., $\nu = \nu^+ - \nu^-$. The existence of such solutions to the measure datum problem (1.1) is now well-known (see, e.g., [7]). Alternatively, one can also adopt the notion of *Solutions Obtained by Limit of Approximations* (SOLA) (see [3, 4, 8]) as having been employed, e.g., in [10, 25].

It is not hard to see that for a nonnegative locally finite measure ν in $\mathbb{R}^n$ we have

$$\mathcal{M}_1(\nu)(x) \leq c(n)\, \mathbf{I}_1(\nu)(x) := c(n) \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-1}}, \qquad x \in \mathbb{R}^n.$$

Thus inequality (1.4) can be viewed as an integral potential bound for gradients of solutions to (1.1). In fact, by a well-known result of Muckenhoupt and Wheeden [26] it is equivalent to use the first order Riesz potential $\mathbf{I}_1$ in place of $\mathcal{M}_1$ on the right-hand side of (1.4).

Inequality (1.4) holds also in the setting of Lorentz spaces. Recall that the Lorentz space $L^{s,\,t}(\Omega)$ with $0 < s < \infty$, $0 < t \leq \infty$, is the set of measurable functions g on Ω such that

$$\|g\|_{L^{s,\,t}(\Omega)} := \left[s \int_0^\infty (\alpha^s |\{x \in \Omega : |g(x)| > \alpha\}|)^{\frac{t}{s}} \frac{d\alpha}{\alpha} \right]^{\frac{1}{t}} < +\infty$$

when $t \neq \infty$; for $t = \infty$ the space $L^{s,\,\infty}(\Omega)$ is set to be the usual weak L^s or Marcinkiewicz space with quasinorm

$$\|g\|_{L^{s,\,\infty}(\Omega)} := \sup_{\alpha > 0} \alpha |\{x \in \Omega : |g(x)| > \alpha\}|^{\frac{1}{s}}.$$

It is easy to see that when $t = s$ the Lorentz space $L^{s,\,s}(\Omega)$ is nothing but the Lebesgue space $L^s(\Omega)$.

We are now ready to state the main result of the paper.

Theorem 1.1. *Let $2 - \frac{1}{n} < p \leq n$ and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition with constants $c_0, r_0 > 0$. Then there exist $\epsilon = \epsilon(n, p, \alpha, \beta, c_0) > 0$ such that for any $0 < q < p + \epsilon$, and $0 < t \leq \infty$ and for any solution u to (1.1) with a finite measure μ there holds*

$$\|\nabla u\|_{L^{q,\,t}(\Omega)} \leq C \left\| \mathcal{M}_1(\chi_\Omega |\mu|)^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)}. \tag{1.6}$$

Here the constants C depends only on n, p, q, t, c_0, and $\mathrm{diam}(\Omega)/r_0$.

Remark 1.2. The space $L^{q,\,t}(\mathbb{R}^n)$ appearing on the right-hand side of (1.6) can be replaced by $L^{q,\,t}(B_0)$ for any ball B_0 of radius, say, $R_0 \leq 2\mathrm{diam}(\Omega)$ that contains Ω. Moreover, it can also be replaced by the space $L^{q,\,t}(\Omega)$ provided the domain Ω satisfies an additional interior density condition: there exist constants $c_1, r_1 > 0$ such that for all $0 < t \leq r_1$ and all $x \in \Omega$ there holds

$$|B_t(x) \cap \Omega| \geq c_1 |B_t(x)|.$$

In particular, (1.6) with $L^{q,\,t}(\Omega)$ in place of $L^{q,\,t}(\mathbb{R}^n)$ holds on any Lipschitz domain Ω.

By the boundedness property of the first order fractional maximal function on Lorentz spaces we obtain the following corollary.

Corollary 1.3. *Let $2-\frac{1}{n}<p\leq n$, $0<t\leq\infty$, and suppose that $\Omega\subset\mathbb{R}^n$ is a bounded domain whose complement satisfies a p-capacity uniform thickness condition with constants $c_0, r_0>0$. Assume that $1<\gamma<\frac{n(p+\epsilon)}{n(p-1)+p+\epsilon}$, where $\epsilon=\epsilon(n,p,\alpha,\beta,c_0)>0$ is as in Theorem 1.1. Then for any solution u to (1.1) with $\mu=f\in L^{\gamma,\,t}(\Omega)$ there holds*

$$\left\| |\nabla u|^{p-1}\right\|_{L^{\frac{n\gamma}{n-\gamma},\,t}(\Omega)}\leq C\,\|f\|_{L^{\gamma,\,t}(\Omega)}\,.$$

Here the constants C depends only on n,p,q,t,c_0 and $\mathrm{diam}(\Omega)/r_0$.

Remark 1.4. For $\mu=f\in L^{\gamma,\,\gamma}(\Omega)=L^{\gamma}(\Omega)$ with $1<\gamma<\frac{np}{n(p-1)+p}$, Boccardo and Gallöuet obtained in [4] the solvability of equation (1.1) with a (unique) solution $u\in W_0^{1,\,\frac{n\gamma(p-1)}{n-\gamma}}(\Omega)$ only under the assumption that Ω is bounded. For $1<\gamma<\frac{np}{n(p-1)+p}$, see also the papers [1, 9, 16]. On the other hand, the Lorentz space borderline case $\gamma=\frac{np}{n(p-1)+p}$, with $p<n$, was first obtained by G. Mingione [24] in the local setting. The possibility of extending such local results to global ones was also mentioned without proof in the same paper. Note that since $\epsilon>0$ we have

$$\frac{np}{n(p-1)+p}<\frac{n(p+\epsilon)}{n(p-1)+p+\epsilon}.$$

We next take this opportunity to discuss a Calderón-Zygmund type estimate below the natural exponent p for $\mathcal{A}$-superharmonic functions in the whole space $\mathbb{R}^n$. For the notion of $\mathcal{A}$-superharmonicity see [13, 17, 18]. Suppose now that u is an $\mathcal{A}$-superharmonic solution to the equation

$$\mathrm{div}\mathcal{A}(x,\nabla u)=\mathrm{div}F \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{1.7}$$

In a recent paper [28] we show that, for $2-\frac{1}{n}<p\leq n$, $\max\{1,p-1\}<q<p$ and under a BMO type smallness condition on the nonlinearity $\mathcal{A}$, there holds

$$\|\nabla u\|_{L^q(\mathbb{R}^n)}\leq C\,\|F\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1}}(\mathbb{R}^n)}, \tag{1.8}$$

provided that $\|\nabla u\|_{L^q(\mathbb{R}^n)}<+\infty$. The following theorem shows that the norm $\|\nabla u\|_{L^q(\mathbb{R}^n)}$ is in fact finite as long as $\nabla u\in L^1(\mathbb{R}^n,\mathbb{R}^n)$. This answers a question raised by ourselves in [28, Remark 3.3].

Theorem 1.5. *Let $2-\frac{1}{n}<p\leq n$, $\max\{1,p-1\}<q<p$, $0<t\leq\infty$, and suppose that F is a vector field in $L^{\frac{q}{p-1},\,\frac{t}{p-1}}(\mathbb{R}^n,\mathbb{R}^n)$. Assume that u is an*

entire $\mathcal{A}$-superharmonic solution of (1.7) *such that* $\nabla u \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. *Then one has the estimate*

$$\|\nabla u\|_{L^{q,\,t}(\mathbb{R}^n)} \leq C\,\|\nabla u\|_{L^1(\mathbb{R}^n)} + C\,\|F\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{t}{p-1}}(\mathbb{R}^n)}, \tag{1.9}$$

where $C = C(n, p, q, t, \alpha, \beta)$.

It is worth mentioning that estimate (1.8), with $\max\{1, p-1\} < q < p$, was conjectured by T. Iwaniec in [14] to hold for all distributional solutions to (1.7). Thus Theorem (1.5) provides a solution to this conjecture when the solution u belongs to the class of $\mathcal{A}$-superharmonic functions. Here the assumption $q > 1$ is essential in our approach to (1.9). As mentioned above the first term in the right-hand side of (1.9) can be dropped if $\mathcal{A}$ satisfies an additional smallness condition of BMO type. In general, we have the following existence result where the exponent q may go below 1.

Theorem 1.6. *Let* $2 - \frac{1}{n} < p < n$, $p - 1 < q \leq p$, *and* $0 < t \leq \infty$. *Suppose that* $F \in L^{\frac{q}{p-1},\frac{t}{p-1}}(\mathbb{R}^n, \mathbb{R}^n)$ *with* $-\mathrm{div} F \geq 0$ *in* $\mathcal{D}'(\mathbb{R}^n)$. *Then there exists an entire nonnegative $\mathcal{A}$-superharmonic solution of* (1.7) *such that*

$$\|u\|_{L^{\frac{nq}{n-q},\,t}(\mathbb{R}^n)} + \|\nabla u\|_{L^{q,\,t}(\mathbb{R}^n)} \leq C\,\|F\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\frac{t}{p-1}}(\mathbb{R}^n)},$$

where $C = C(n, p, q, t, \alpha, \beta)$.

Remark 1.7. If $p - 1 < q \leq \frac{n(p-1)}{n-1}$ then by [27, Theorem 3.1] we have $\mathrm{div} F = 0$. Thus in this case the solution u obtained in Theorem 1.6 is identically zero. This also implies that Theorem 1.6 holds as well in the case $p = n$, with $u \equiv 0$ being a valid nonnegative solution.

The proofs of Theorems 1.1, 1.5 and 1.6 will be presented in Section 4 of the paper.

2. Interior and boundary comparison estimates

Following G. Mingione [24], in order to prove Theorem 1.1 we need to obtain certain local interior and boundary comparison estimates. First let us consider the interior ones. With $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, for each ball $B_{2R} = B_{2R}(x_0) \subset \Omega$ we defined $w \in u + W^{1,p}_0(B_{2R})$ as the unique solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}\ \mathcal{A}(x, \nabla w) = 0 & \text{in } B_{2R}, \\ w = u & \text{on } \partial B_{2R}. \end{cases} \tag{2.1}$$

Then a well-known version of Gehring's lemma applied to the function w defined above yields the following result (see [12, Theorem 6.7] and [12, Remark 6.12]).

Lemma 2.1. *With $u \in W^{1,p}_{\rm loc}(\Omega)$, let w be as in (2.1). Then there exists a constant $\theta_0 = \theta_0(n,p,\alpha,\beta) > 1$ such that for any $t \in (0,p]$ the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0}} \leq C \fint_{B_\rho(z)} |\nabla w|^t dx$$

holds for all balls $B_\rho(z) \subset B_{2R}(x_0)$ for a constant C depending only on n, p, α, β, t.

It is worth mentioning that the approach of using this kind of reverse Hölder's inequalities with arbitrarily small exponents in the context of measure datum problems has been first implemented by G. Mingione in the paper [23].

The following important comparison lemma involving an estimate "below the natural growth exponent" was also established in [23] (see also [10, Lemma 3.3]) for the degenerate case $p \geq 2$. This lemma was later obtained in [11, Lemma 4.2] for the singular case $2 - 1/n < p < 2$.

Lemma 2.2. *With $p > 2 - 1/n$, let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a solution of (1.1) and let w be as in (2.1). Then there is a constant $C = C(n,p,\alpha,\beta)$ such that*

$$\begin{aligned} \fint_{B_{2R}} |\nabla u - \nabla w| dx \;\; &\leq \;\; C \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + C \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} |\nabla u| dx\right)^{2-p}. \end{aligned}$$

Moreover, when $p \geq 2$ the second term on the right-hand side can be dropped.

Next we consider the counterparts of Lemmas 2.1 and 2.2 up to the boundary. As $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick with constants $c_0, r_0 > 0$, there exists $1 < p_0 = p_0(n,p,c_0) < p$ such that $\mathbb{R}^n \setminus \Omega$ is uniformly p_0-thick with constants $c_* = c(n,p,c_0)$ and r_0. This is by now a classical result due to J. Lewis [19] (see also [22]). Moreover, p_0 can be chosen near p so that $p_0 \in (\frac{np}{n+p}, p)$. Thus, since $p_0 < n$, we have

$$\begin{aligned} \text{(2.2)} \qquad {\rm cap}_{p_0}(\overline{B_t(x)} \cap (\mathbb{R}^n \setminus \Omega), B_{2t}(x)) \;\; &\geq \;\; c_*\, {\rm cap}_{p_0}(\overline{B_t(x)}, B_{2t}(x)) \\ &\geq \;\; C(n,p,c_0)\, t^{n-p_0} \end{aligned}$$

for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$.

Now let $x_0 \in \partial\Omega$ be a boundary point and for $0 < 2R \leq r_0$ we set $\Omega_{2R} = \Omega_{2R}(x_0) = B_{2R}(x_0) \cap \Omega$. For $u \in W^{1,p}_0(\Omega)$ we consider the unique solution $w \in u + W^{1,p}_0(\Omega_{2R})$ to the equation

$$\text{(2.3)} \qquad \begin{cases} {\rm div}\, \mathcal{A}(x, \nabla w) = 0 & \text{in } \Omega_{2R}, \\ w = u & \text{on } \partial\Omega_{2R}. \end{cases}$$

In what follows we extend μ and u by zero to $\mathbb{R}^n \setminus \Omega$ and then extend w by u to $\mathbb{R}^n \setminus \Omega_{2R}$.

Lemma 2.3. *With $u \in W_0^{1,p}(\Omega)$, let w be as in (2.3). Then there exists a constant $\theta_0 = \theta_0(n,p,\alpha,\beta,c_0) > 1$ such that the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0}} \leq C \fint_{B_{11\rho/4}(z)} |\nabla w|^p dx$$

holds for all balls $B_{11\rho/4}(z) \subset B_{2R}(x_0)$ for a constant C depending only on n, p, α, β, c_0.

Proof. By Gehring's lemma it is enough to show that there exists $\epsilon \in (0,1)$ such that

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^p dx\right)^{1/p} \leq C \left(\fint_{B_{11\rho/4}(z)} |\nabla w|^{\epsilon p} dx\right)^{1/\epsilon p} \tag{2.4}$$

for all balls $B_{11\rho/4}(z) \subset B_{2R}(x_0)$.

Inequality (2.4) obviously holds when $B_\rho(z) \subset \mathbb{R}^n \setminus \Omega$. Next suppose that $B_\rho(z) \subset \Omega$. Let $\varphi \in C_0^\infty(B_\rho(z))$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_{\rho/2}(z)$ and $|\nabla \varphi| \leq c/\rho$. Then using

$$\phi = (w - \overline{w}_{B_\rho(z)})\varphi^p, \qquad \text{with} \quad \overline{w}_{B_\rho(z)} = \fint_{B_\rho(z)} w dy,$$

as a test function for (2.3) we find

$$\int_{B_\rho(z)} |\nabla w|^p \varphi^p dx \leq C \int_{B_\rho(z)} |\nabla w|^{p-1} |\nabla \varphi| \varphi^{p-1} |w - \overline{w}_{B_\rho(z)}| dx.$$

Thus by Hölder's inequality we get

$$\int_{B_{\rho/2}(z)} |\nabla w|^p dx \leq \frac{C}{\rho^p} \int_{B_\rho(z)} |w - \overline{w}_{B_\rho(z)}|^p dx.$$

This yields

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^p dx\right)^{1/p} \leq C \left(\fint_{B_\rho(z)} |\nabla w|^{mp} dx\right)^{1/mp}$$

by Poincaré-Sobolev inequality, where $m = n/(n+p)$ if $np/(n+p) \geq 1$ and $m = 1/p$ if $np/(n+p) < 1$. Hence, we obtain (2.4) with $\epsilon = m$.

Finally, we consider the case $B_\rho(z) \cap \partial\Omega \neq \emptyset$. In this case we choose $z_0 \in \partial\Omega$ such that $|z - z_0| = \operatorname{dist}(z, \partial\Omega)$. Then $|z - z_0| < \rho$ and thus

$$B_{\rho/2}(z) \subset B_{3\rho/2}(z_0) \subset B_{7\rho/4}(z_0) \subset B_{11\rho/4}(z) \subset B_{2R}(x_0).$$

Let $\varphi \in C_0^\infty(B_{7\rho/4}(z_0))$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_{3\rho/2}(z_0)$ and $|\nabla \varphi| \leq c/\rho$. Using $\phi = w\varphi^p$ as a test function for (2.3) we find

$$\int_{B_{3\rho/2}(z_0)} |\nabla w|^p dx \leq \frac{C}{\rho^p} \int_{B_{7\rho/4}(z_0)} |w|^p dx.$$

Recall now that $\mathbb{R}^n \setminus \Omega$ is uniformly p_0-thick for some $p_0 \in (\frac{np}{n+p}, p)$. Thus $p < \frac{p_0 n}{n-p_0}$ and by Hölder's inequality we get

$$\begin{aligned}\left(\fint_{B_{\rho/2}(z)} |\nabla w|^p dx\right)^{1/p} &\leq \frac{C}{\rho}\left(\fint_{B_{7\rho/4}(z_0)} |w|^p dx\right)^{1/p}\\ &\leq \frac{C}{\rho}\left(\fint_{B_{7\rho/4}(z_0)} |w|^{\frac{np_0}{n-p_0}} dx\right)^{\frac{n-p_0}{np_0}}.\end{aligned}$$

On the other hand, with $K = \{w = 0\} \cap \overline{B}_{7\rho/4}(z_0)$, by a Sobolev type inequality (see Lemma 8.11 and Remark 8.14 in [22])

$$\begin{aligned}&\left(\fint_{B_{7\rho/4}(z_0)} |w|^{\frac{np_0}{n-p_0}} dx\right)^{\frac{n-p_0}{np_0}}\\ &\leq C\left(\frac{1}{\mathrm{cap}_{p_0}(K, B_{7\rho/2}(z_0))}\int_{B_{7\rho/4}(z_0)} |\nabla w|^{p_0} dx\right)^{1/p_0}\\ &\leq C\left(\rho^{p_0}\fint_{B_{7\rho/4}(z_0)} |\nabla w|^{p_0} dx\right)^{1/p_0},\end{aligned}$$

where we used (2.2) in the last inequality which is valid since $7\rho/4 < 11\rho/4 \leq 2R \leq r_0$. These yield

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^p dx\right)^{1/p} \leq C\left(\fint_{B_{11\rho/4}(z)} |\nabla w|^{p_0} dx\right)^{1/p_0},$$

and thus we get (2.4) with $\epsilon = p_0/p \in (0, 1)$. □

On the other hand, arguing as in [12, Remark 6.12] (see also [11, Lemma 3.2]) we have

Lemma 2.4. *Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}$ be an integrable function such that*

$$\left(\fint_{B_\rho} |f|^{\theta_0} dx\right)^{\frac{1}{\theta_0}} \leq C \fint_{B_{11\rho/2}} |f| dx$$

for all concentric balls $B_\rho \subset B_{11\rho/2} \subset A$, where $\theta_0 > 1$ and $C \geq 0$. Then for every $t \in (0, 1]$ and $\theta \in (0, \theta_0]$ there exists a constant $C_0 = C_0(n, C, t)$ such that

$$\left(\fint_{B_\rho} |f|^{\theta} dx\right)^{\frac{1}{\theta}} \leq C_0\left(\fint_{B_{11\rho/2}} |f|^t dx\right)^{\frac{1}{t}}$$

for all concentric balls $B_\rho \subset B_{11\rho/2} \Subset A$.

Thus combining the last two lemmas we obtain the following reverse Hölder type inequality, a version of Lemma 2.1 up to the boundary.

Lemma 2.5. *With $u \in W_0^{1,p}(\Omega)$, let w be as in (2.3). Then there exists a constant $\theta_0 = \theta_0(n,p,\alpha,\beta,c_0) > 1$ such that for every $t \in (0,p]$ the reverse Hölder type inequality*

$$\left(\fint_{B_{\rho/2}(z)} |\nabla w|^{\theta_0 p} dx\right)^{\frac{1}{\theta_0 p}} \leq C \left(\fint_{B_{3\rho}(z)} |\nabla w|^t dx\right)^{\frac{1}{t}}$$

holds for all balls $B_{3\rho}(z) \subset B_{2R}(x_0)$ for a constant $C = C(n,p,t,\alpha,\beta,c_0)$.

We also have a counterpart of Lemma 2.2 up to the boundary.

Lemma 2.6. *With $p > 2-1/n$, let $u \in W_0^{1,p}(\Omega)$ be a solution of (1.1) and let w be as in (2.3). Then there is a constant $C = C(n,p,\alpha,\beta)$ such that*

$$\begin{aligned}\fint_{B_{2R}} |\nabla u - \nabla w| dx \quad &\leq \quad C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} |\nabla u| dx\right)^{2-p}.\end{aligned}$$

Moreover, when $p \geq 2$ the second term on the right-hand side can be dropped.

Proof. A proof of this lemma can be obtained by the method of [23, 10, 11, 25] that was implemented for the interior situation, i.e., Lemma 2.2. Here, to avoid a scaling argument, we choose to present a slightly different approach based on a technique in [2]. Note that u, w, and μ are all zero outside Ω. Since both u and w are solutions we find

$$\int_{\Omega_{2R}} \langle \mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla w), \nabla\varphi\rangle dx = \int_{\Omega_{2R}} \varphi d\mu \tag{2.5}$$

for every $\varphi \in W_0^{1,p}(\Omega_{2R})$. Thus choosing $\varphi = T_k(u-w)$, $k>0$, in (2.5) we have

$$\int_{\{|u-w|<k\}\cap\Omega_{2R}} g(u,w)(x) dx \leq ck|\mu|(\Omega_{2R}), \tag{2.6}$$

where we set

$$g(u,w)(x) = (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla(u-w)|^2.$$

For $k, \lambda \geq 0$ we now put

$$\Phi(k,\lambda) = |\{|u-w| > k, g(u,w) > \lambda\} \cap \Omega_{2R}|.$$

As the map $\lambda \mapsto \Phi(k,\lambda)$ is nonincreasing we find that

$$\Phi(0,\lambda) \leq \frac{1}{\lambda}\int_0^\lambda \Phi(0,s) ds \leq \Phi(k,0) + \frac{1}{\lambda}\int_0^\lambda [\Phi(0,s) - \Phi(k,s)] ds.$$

Thus

$$\begin{aligned}\Phi(0,\lambda) &\leq |\{|u-w|>k\}\cap\Omega_{2R}| \\ &\quad + \frac{1}{\lambda}\int_0^\lambda |\{|u-w|<k, g(u,w)>s\}\cap\Omega_{2R}|ds \\ &\leq |\{|u-w|>k\}\cap\Omega_{2R}| + \frac{1}{\lambda}\int_{\{|u-w|<k\}\cap\Omega_{2R}} g(u,w)dx \\ &\leq |\{|u-w|>k\}\cap\Omega_{2R}| + \frac{1}{\lambda} c\, k|\mu|(\Omega_{2R}),\end{aligned}$$

where (2.6) was used in the last inequality. Using Sobolev inequality this gives

$$\Phi(0,\lambda) \leq c\, k^{-\frac{n}{n-1}} \|\nabla u - \nabla w\|_{L^1(\Omega_{2R})}^{\frac{n}{n-1}} + \frac{1}{\lambda} c\, k|\mu|(\Omega_{2R})$$

which holds for all $k>0$. Choosing

$$k = \left[\lambda \|\nabla u - \nabla w\|_{L^1(\Omega_{2R})}^{\frac{n}{n-1}} / |\mu|(\Omega_{2R})\right]^{\frac{n-1}{2n-1}}$$

in the above inequality we arrive at

$$\lambda^{\frac{n}{2n-1}}\Phi(0,\lambda) \leq c\,|\mu|(\Omega_{2R})^{\frac{n}{2n-1}} \|\nabla u - \nabla w\|_{L^1(\Omega_{2R})}^{\frac{n}{2n-1}}.$$

Letting $\lambda = s^p$ this yields

$$\left\|g(u,w)^{\frac{1}{p}}\right\|_{L^{\frac{np}{2n-1},\infty}(\Omega_{2R})} \leq c\,|\mu|(\Omega_{2R})^{\frac{1}{p}} \|\nabla u - \nabla w\|_{L^1(\Omega_{2R})}^{\frac{1}{p}},$$

and by Hölder's inequality

$$\left\|g(u,w)^{\frac{1}{p}}\right\|_{L^1(\Omega_{2R})} \leq c\,|\mu|(\Omega_{2R})^{\frac{1}{p}} |\Omega_{2R}|^{1-\frac{2n-1}{np}} \|\nabla u - \nabla w\|_{L^1(\Omega_{2R})}^{\frac{1}{p}}, \tag{2.7}$$

where we used the fact that $p > 2 - 1/n$.

We next consider separately the case $p \geq 2$ and the case $2 - 1/n < p < 2$. For $p \geq 2$ using the pointwise bound

$$|\nabla u - \nabla w| \leq g(u,w)^{\frac{1}{p}}$$

coupled with inequality (2.7) we easily obtain the desired result. For $2 - 1/n < p < 2$ we write

$$\begin{aligned}|\nabla u - \nabla w| &= g(u,w)^{\frac{1}{2}}(|\nabla u|^2 + |\nabla w|^2)^{\frac{2-p}{4}} \\ &\leq c\, g(u,w)^{\frac{1}{2}}(|\nabla u - \nabla w|^{\frac{2-p}{2}} + |\nabla u|^{\frac{2-p}{2}}) \\ &\leq c\, g(u,w)^{\frac{1}{p}} + (1/2)|\nabla u - \nabla w| + c\, g(u,w)^{\frac{1}{2}}|\nabla u|^{\frac{2-p}{2}},\end{aligned}$$

and thus

$$|\nabla u - \nabla w| \leq c\, g(u,w)^{\frac{1}{p}} + c\, g(u,w)^{\frac{1}{2}}|\nabla u|^{\frac{2-p}{2}}.$$

Using this and Hölder's inequality we get

$$\begin{aligned}\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})} &\le c\left\|g(u,w)^{\frac{1}{p}}\right\|_{L^1(\Omega_{2R})}\\ &\quad + c\left\|g(u,w)^{\frac{1}{p}}\right\|^{\frac{p}{2}}_{L^1(\Omega_{2R})}\|\nabla u\|^{\frac{2-p}{2}}_{L^1(\Omega_{2R})}.\end{aligned}$$

By (2.7) this yields

$$\begin{aligned}&\|\nabla u-\nabla w\|_{L^1(\Omega_{2R})}\le c\,|\mu|(\Omega_{2R})^{\frac{1}{p}}|\Omega_{2R}|^{1-\frac{2n-1}{np}}\|\nabla u-\nabla w\|^{\frac{1}{p}}_{L^1(\Omega_{2R})}\\ &+c\,|\mu|(\Omega_{2R})^{\frac{1}{2}}|\Omega_{2R}|^{\left(1-\frac{2n-1}{np}\right)\frac{p}{2}}\|\nabla u-\nabla w\|^{\frac{1}{2}}_{L^1(\Omega_{2R})}\|\nabla u\|^{\frac{2-p}{2}}_{L^1(\Omega_{2R})},\end{aligned}$$

or

$$\begin{aligned}\|\nabla u-\nabla w\|^{\frac{1}{2}}_{L^1(\Omega_{2R})} &\le c\,|\mu|(\Omega_{2R})^{\frac{1}{p}}|\Omega_{2R}|^{1-\frac{2n-1}{np}}\|\nabla u-\nabla w\|^{\frac{1}{p}-\frac{1}{2}}_{L^1(\Omega_{2R})}\\ &\quad +c\,|\mu|(\Omega_{2R})^{\frac{1}{2}}|\Omega_{2R}|^{\left(1-\frac{2n-1}{np}\right)\frac{p}{2}}\|\nabla u\|^{\frac{2-p}{2}}_{L^1(\Omega_{2R})}.\end{aligned}$$

Thus using Young's inequality for the first term on the right-hand side we get

$$\begin{aligned}\|\nabla u-\nabla w\|^{\frac{1}{2}}_{L^1(\Omega_{2R})} &\le c\,|\mu|(\Omega_{2R})^{\frac{1}{2(p-1)}}|\Omega_{2R}|^{\left(1-\frac{2n-1}{np}\right)\frac{p}{2(p-1)}}\\ &\quad +c\,|\mu|(\Omega_{2R})^{\frac{1}{2}}|\Omega_{2R}|^{\left(1-\frac{2n-1}{np}\right)\frac{p}{2}}\|\nabla u\|^{\frac{2-p}{2}}_{L^1(\Omega_{2R})}.\end{aligned}$$

The desired result is easily seen to follow from the last inequality. □

3. Applications of comparison estimates

Our approach to Theorem 1.1 is based on following technical lemma which allows ones to work with balls instead of cubes. A version of this lemma appeared for the first time in [31]. It can be viewed as a version of the Calderón-Zygmund-Krylov-Safonov decomposition that has been used in [6, 24]. A proof of this lemma, which uses Lebesgue Differentiation Theorem and the standard Vitali covering lemma, can be found in [5] with obvious modifications to fit the setting here.

Lemma 3.1. *Assume that $A\subset\mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1>0$ such that*

$$|B_t(x)\cap A|\ge c_1\,|B_t(x)| \tag{3.1}$$

holds for all $x\in A$ and $0<t\le r_1$. Fix $0<r\le r_1$ and let $C\subset D\subset A$ be measurable sets for which there exists $0<\epsilon<1$ such that

1. *$|C|<\epsilon\, r^n|B_1|$ and*
2. *for all $x\in A$ and $\rho\in(0,r]$, if $|C\cap B_\rho(x)|\ge\epsilon\,|B_\rho(x)|$, then $B_\rho(x)\cap A\subset D$.*

Then we have the estimate

$$|C|\le (c_1)^{-1}\epsilon\,|D|.$$

We now recall that for a function $f \in L^1_{\rm loc}(\mathbb{R}^n)$ the Hardy-Littlewood maximal function of f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B_r(x)} |f(y)|dy.$$

In order to apply Lemma 3.1 we need the following proposition, whose proof relies essentially on the comparison estimates obtained in the previous section.

Proposition 3.2. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, c_0, so that the following holds for any $T > 1$ and any $\lambda > 0$. Suppose that $u \in W^{1,p}_0(\Omega)$ is a weak solution of (1.1) with $\mathcal{A}$ satisfying (1.2)-(1.3). Assume that for some ball $B_\rho(y)$ with $16\rho \le r_0$ we have*

$$B_\rho(y) \cap \{\mathcal{M}(\chi_\Omega|\nabla u|) \le \lambda\} \cap \{[\mathcal{M}_1(\chi_\Omega|\mu|)]^{\frac{1}{p-1}} \le \epsilon(T)\lambda\} \neq \emptyset,$$

where $\epsilon(T)$ is defined by

$$\epsilon(T) = \begin{cases} T^{-p\theta_0+1} & if\ p \ge 2, \\ T^{(-p\theta_0+1)/(p-1)} & if\ 2 - \frac{1}{n} < p < 2. \end{cases} \tag{3.2}$$

Then there holds

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega|\nabla u|) > AT\lambda\} \cap B_\rho(y)| < T^{-p\theta_0}|B_\rho(y)|. \tag{3.3}$$

Proof. By hypothesis, there exists $x_0 \in B_\rho(y)$ such that for any $r > 0$

$$\fint_{B_r(x_0)} \chi_\Omega|\nabla u|dz \le \lambda \quad \text{and } r\fint_{B_r(x_0)} \chi_\Omega d|\mu| \le [\epsilon(T)\lambda]^{p-1}. \tag{3.4}$$

We first claim that for $x \in B_\rho(y)$ there holds

$$\mathcal{M}(\chi_\Omega|\nabla u|)(x) \le \max\left\{\mathcal{M}(\chi_{B_{2\rho}(y)\cap\Omega}|\nabla u|)(x), 3^n\lambda\right\}. \tag{3.5}$$

Indeed, for $r \le \rho$ we have $B_r(x) \cap \Omega \subset B_{2\rho}(y) \cap \Omega$ and thus

$$\fint_{B_r(x)} \chi_\Omega|\nabla u|dz = \fint_{B_r(x)} \chi_{B_{2\rho}(y)\cap\Omega}|\nabla u|dz,$$

whereas for $r > \rho$ we have $B_r(x) \subset B_{3r}(x_0)$ from which by (3.4) yields

$$\fint_{B_r(x)} \chi_\Omega|\nabla u|dz \le 3^n \fint_{B_{3r}(x_0)} \chi_\Omega|\nabla u|dz \le 3^n\lambda.$$

In view of (3.5) we see that (3.3) trivially holds provided $A \ge 3^n$ and $B_{4\rho}(y) \subset \mathbb{R}^n \setminus \Omega$. Thus it is enough to consider the case $B_{4\rho}(y) \subset \Omega$ and the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$.

First we consider the case that $B_{4\rho}(y) \subset \Omega$. Let $w \in u + W^{1,p}_0(B_{4\rho}(y))$ be the unique solution to the Dirichlet problem

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } B_{4\rho}(y), \\ w &= u \quad \text{on } \partial B_{4\rho}(y). \end{cases} \tag{3.6}$$

By weak type $(1,1)$ estimates for the maximal function we have

$$\begin{aligned}
&\left|\{x\in\mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\}\cap B_\rho(y)\right|\\
&\quad\le \left|\{x\in\mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla w|) > AT\lambda/2\}\cap B_\rho(y)\right|\\
&\qquad + \left|\{x\in\mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u-\nabla w|) > AT\lambda/2\}\cap B_\rho(y)\right|\\
&\quad\le C(AT\lambda)^{-p\theta_0}\int_{B_{2\rho}(y)}|\nabla w|^{p\theta_0}dx + C(AT\lambda)^{-1}\int_{B_{2\rho}(y)}|\nabla u-\nabla w|dx.
\end{aligned}$$

Note that by Lemma 2.1 we have

$$\begin{aligned}
\left(\fint_{B_{2\rho}(y)}|\nabla w|^{p\theta_0}dx\right)^{\frac{1}{p\theta_0}} &\le C\fint_{B_{4\rho}(y)}|\nabla w|dx\\
&\le C\fint_{B_{4\rho}(y)}|\nabla u|dx + C\fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx
\end{aligned}$$

and thus

$$\begin{aligned}
(3.7)\qquad &\left|\{x\in\mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\}\cap B_\rho(y)\right|\\
&\le C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\fint_{B_{4\rho}(y)}|\nabla u|dx\right)^{p\theta_0}\\
&\quad + C(AT\lambda)^{-p\theta_0}|B_\rho(y)|\left(\fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx\right)^{p\theta_0}\\
&\quad + C(AT\lambda)^{-1}|B_\rho(y)|\fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx.
\end{aligned}$$

On the other hand, by Lemma 2.2 we have

$$\begin{aligned}
(3.8)\qquad \fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx &\le C\left[\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]^{\frac{1}{p-1}}\\
&\quad + C\left[\frac{|\mu|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]\left(\fint_{B_{5\rho}(x_0)}|\nabla u|dx\right)^{2-p},
\end{aligned}$$

where the last term should be dropped when $p\ge 2$. Thus by (3.4) and the definition of $\epsilon(T)$ we get

$$\fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx \le CT^{-p\theta_0+1}\lambda$$

if $p\ge 2$ and

$$\fint_{B_{4\rho}(y)}|\nabla u-\nabla w|dx \le CT^{(-p\theta_0+1)/(p-1)}\lambda + CT^{-p\theta_0+1}\lambda$$

if $2-\frac{1}{n}<p<2$. In any case, since $T>1$, we have

$$\fint_{B_{4\rho}(y)} |\nabla u - \nabla w| dx \leq C T^{-p\theta_0+1}\lambda. \tag{3.9}$$

At this point combining (3.7),(3.9) and using $T > 1$ we find

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq \quad (CA^{-p\theta_0} + CA^{-1})T^{-p\theta_0}|B_\rho(y)|. \end{aligned}$$

We now choose A so that $A \geq 3^n$ and $2CA^{-1} \leq 1/2$, i.e., $A \geq \max\{3^n, 4C\}$. Then we have

$$\left|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{B_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \leq (1/2)T^{-p\theta_0}|B_\rho(y)|,$$

which in view of (3.5) yields (3.3).

Finally, we consider the case that $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$. Let $y_0 \in \partial\Omega$ be a boundary point such that $|y - y_0| = \text{dist}(y, \partial\Omega)$. Define $w \in u + W_0^{1,p}(\Omega_{16\rho}(y_0))$ as the unique solution to the Dirichlet problem

$$\begin{cases} \text{div } \mathcal{A}(x, \nabla w) &= 0 \quad \text{in } \Omega_{16\rho}(y_0), \\ w &= u \quad \text{on } \partial\Omega_{16\rho}(y_0). \end{cases}$$

Here we also extend u by zero to $\mathbb{R}^n \setminus \Omega$ and then extend w by u to $\mathbb{R}^n \setminus \Omega_{16\rho}(y_0)$. As in (3.7) in this case we have

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq \quad C(AT\lambda)^{-p\theta_0}|B_\rho(y)| \left(\fint_{B_{12\rho}(y)} |\nabla u| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-p\theta_0}|B_\rho(y)| \left(\fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx\right)^{p\theta_0} \\ &\quad + C(AT\lambda)^{-1}|B_\rho(y)| \fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx, \end{aligned} \tag{3.10}$$

where Lemma 2.5 is used in stead of Lemma 2.1. Since

$$B_{12\rho}(y) \subset B_{16\rho}(y_0) \subset B_{20\rho}(y) \subset B_{21\rho}(x_0)$$

by Lemma 2.6, as in (3.9), we find

$$\fint_{B_{12\rho}(y)} |\nabla u - \nabla w| dx \leq C T^{-p\theta_0+1}\lambda. \tag{3.11}$$

Inequalities (3.10)-(3.11) and the fact that $T > 1$ now yield

$$\begin{aligned} &\left|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \\ &\leq \quad (CA^{-p\theta_0} + CA^{-1})T^{-p\theta_0}|B_\rho(y)|, \end{aligned}$$

and thus we arrive at

$$\left|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_{\Omega_{2\rho}(y)}|\nabla u|) > AT\lambda\} \cap B_\rho(y)\right| \leq (1/2)T^{-p\theta_0}|B_\rho(y)|.$$

provided $A \geq \max\{3^n, 4C\}$. The last bound and (3.5) yield (3.3) as desired. □

Remark 3.3. By approximation Proposition 3.2 continues to hold without assuming that $u \in W_0^{1,p}(\Omega)$. To this end, let $u_k = T_k(u)$ for each integer $k > 0$. Then by our notion of solutions $u_k \in W_0^{1,p}(\Omega)$ solves

$$-\operatorname{div} \mathcal{A}(x, \nabla u_k) = \mu_k \tag{3.12}$$

for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \setminus \Omega$ then μ_k^+ and μ_k^- converge respectively to μ^+ and μ^- weakly as measures in $\mathbb{R}^n$. This implies in particular that

$$\limsup_{k\to\infty} |\mu_k|(B_r(z)) \leq |\mu|(\overline{B_r(z)}) \tag{3.13}$$

for any ball $B_r(z) \subset \mathbb{R}^n$. To show (3.3) it is enough to consider the case $B_{4\rho}(y) \subset \Omega$ as the case $B_{4\rho}(y) \cap \partial\Omega \neq \emptyset$ is just similar. By working with (3.12) then instead of (3.8) we have

$$\begin{aligned} \fint_{B_{4\rho}(y)} |\nabla u_k - \nabla w_k| dx \quad &\leq \quad C \left[\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\right]^{\frac{1}{p-1}} \\ &\quad + C \left[\frac{|\mu_k|(B_{5\rho}(x_0))}{\rho^{n-1}}\right] \left(\fint_{B_{5\rho}(x_0)} |\nabla u_k| dx\right)^{2-p} \end{aligned}$$

and the last term should be dropped when $p \geq 2$. Here w_k is the solution of (3.6) with u_k in place of u. Thus using (3.4) and (3.13) we have the following analogue of (3.9)

$$\limsup_{k\to\infty} \fint_{B_{4\rho}(y)} |\nabla u_k - \nabla w_k| dx \leq C T^{-p\theta_0+1} \lambda,$$

from which we obtain, for large enough A,

$$\begin{aligned} &\limsup_{k\to\infty} |\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ &\qquad \leq (1/2) T^{-p\theta_0} |B_\rho(y)|, \end{aligned} \tag{3.14}$$

In equality (3.3) (with $2A$ in place of A) follows from (3.14) by writing

$$\begin{aligned} &|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u|) > 2AT\lambda\} \cap B_\rho(y)| \\ &\quad \leq \quad |\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \\ &\qquad + |\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega |\nabla u - \nabla u_k|) > AT\lambda\} \cap B_\rho(y)| \end{aligned}$$

and using the weak type $(1, 1)$ bound of the maximal function.

We remark that the above argument works equally well for *Solutions Obtained by Limit of Approximations* (SOLA) as property (3.13) holds also for the approximating measures in that case (see, e.g., [10, Section 5]).

Proposition 3.2 can be restated as follows.

Proposition 3.4. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, c_0, so that the following holds for any $T > 1$ and any $\lambda > 0$. Let u be a solution of (1.1) with $\mathcal{A}$ satisfying (1.2)-(1.3). Suppose that for some ball $B_\rho(y)$ with $16\rho \leq r_0$ we have*

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega|\nabla u|) > AT\lambda\} \cap B_\rho(y)| \geq T^{-p\theta_0}\,|B_\rho(y)|.$$

Then there holds

$$B_\rho(y) \subset \{\mathcal{M}(\chi_\Omega|\nabla u|) > \lambda\} \cup \{[\mathcal{M}_1(\chi_\Omega|\mu|)]^{\frac{1}{p-1}} > \epsilon(T)\lambda\},$$

where $\epsilon(T)$ is as defined in (3.2).

We can now apply Lemma 3.1 and the last proposition to get the following result.

Lemma 3.5. *There exist constants $A, \theta_0 > 1$, depending only on n, p, α, β, c_0, so that the following holds for any $T > 1$. Let u be a solution of (1.1) with $\mathcal{A}$ satisfying (1.2)-(1.3). Let B_0 be a ball of radius R_0. Fix a real number $0 < r \leq \min\{r_0, 2R_0\}/16$ and suppose that there exists $N > 0$ such that*

$$|\{x \in \mathbb{R}^n : \mathcal{M}(\chi_\Omega|\nabla u|) > N\}| < T^{-p\theta_0}\, r^n |B_1|. \tag{3.15}$$

Then for any integer $k \geq 0$ there holds

$$\begin{aligned} &|\{x \in B_0 : \mathcal{M}(\chi_\Omega|\nabla u|) > N(AT)^{k+1}\}| \\ &\quad\leq c(n)T^{-p\theta_0}|\{x \in B_0 : \mathcal{M}(\chi_\Omega|\nabla u|) > N(AT)^k\}| \\ &\qquad + c(n)|\{x \in B_0 : [\mathcal{M}_1(\chi_\Omega|\mu|)]^{\frac{1}{p-1}} > \epsilon(T)N(AT)^k\}|, \end{aligned}$$

where $\epsilon(T)$ is as defined in (3.2).

Proof. Let A and $\theta_0 > 1$ be as in Proposition 3.4 and set

$$C = \{x \in B_0 : \mathcal{M}(\chi_\Omega|\nabla u|) > N(AT)^{k+1}\}$$

and

$$D = \Big[\{\mathcal{M}(\chi_\Omega|\nabla u|) > N(AT)^k\} \cup \{[\mathcal{M}_1(\chi_\Omega|\mu|)]^{\frac{1}{p-1}} > \epsilon(T)N(AT)^k\}\Big] \cap B_0.$$

Since $AT > 1$ the assumption (3.15) implies that $|C| < T^{-p\theta_0}\, r^n |B_1|$. Moreover, if $x \in B_0$ and $\rho \in (0, r]$ such that $|C \cap B_\rho(x)| \geq T^{-p\theta_0}\,|B_\rho(x)|$, then $16\rho \leq r_0$ and thus by using Proposition 3.4 with $\lambda = N(AT)^k$ we have

$$B_\rho(x) \cap B_0 \subset D.$$

Thus the hypotheses of Lemma 3.1 are satisfied with $A = B_0$ and $\epsilon = T^{-p\theta_0}$ (note that condition (3.1) holds for all $0 < t \leq 2R_0$). Since $T > 1$, this yields

$$\begin{aligned} |C| &\leq c(n)\, T^{-p\theta_0}\,|D| \\ &\leq c(n)\, T^{-p\theta_0}\,|\{x \in B_0 : \mathcal{M}(\chi_\Omega|\nabla u|) > N(AT)^k\}| + \\ &\quad + c(n)\,|\{x \in B_0 : [\mathcal{M}_1(\chi_\Omega|\mu|)]^{\frac{1}{p-1}} > \epsilon(T)N(AT)^k\}|. \end{aligned}$$

□

Remark 3.6. From its proof we see that Lemma 3.5 also holds if B_0 is replaced Ω provided that $A = \Omega$ satisfies (3.1) with some constants c_1, $r_1 > 0$. Of course, in this case r should be chosen so that $0 < r \leq \min\{r_0, r_1\}/16$.

4. Global Lorentz estimates

We are now ready to prove the main theorem of the paper.

Proof of the Theorem 1.1. Let B_0 be a ball of radius $R_0 \leq 2\mathrm{diam}(\Omega)$ that contains Ω. Note then that $\mathrm{diam}(\Omega) \leq 2R_0$. As usual we set u and μ to be zero in $\mathbb{R}^n \setminus \Omega$. We are planning to show that

$$\|\nabla u\|_{L^{q,\,t}(\Omega))} \leq C \left\| [\mathcal{M}_1(|\mu|)]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(B_0)}, \tag{4.1}$$

where $0 < q < p+\epsilon$ and $0 < t \leq \infty$. Here $\epsilon > 0$ is a small number depending only on n, p, α, β, and c_0. In what follows we consider only the case $t \neq \infty$ as for $t = \infty$ the proof is similar. Moreover, to prove (4.1) we may assume that

$$\|\nabla u\|_{L^1(\Omega)} \neq 0.$$

Let $r = \min\{r_0, \mathrm{diam}(\Omega)\}/16$. For $T > 1$ to be determined, we claim that there exists $N > 0$ such that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}| < T^{-p\theta_0}\, r^n |B_1|.$$

To see this, we first use the weak type $(1, 1)$ estimate for the maximal function to get

$$|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}| < \frac{C(n)}{N} \int_\Omega |\nabla u| dx.$$

Then we choose $N > 0$ so that

$$\frac{C(n)}{N} \int_\Omega |\nabla u| dx = T^{-p\theta_0}\, r^n |B_1|. \tag{4.2}$$

Let $A, \theta_0 > 1$ be as in Lemma 3.5 and let $\epsilon(T)$ be as in (3.2). For $0 < t < \infty$ we now consider the sum

$$S = \sum_{k=1}^{\infty} \left[(AT)^{qk} |\{x \in B_0 : \mathcal{M}(|\nabla u|)(x) > N(AT)^k\}| \right]^{\frac{t}{q}}.$$

Note that we have

$$C^{-1}\, S \leq \|\mathcal{M}(|\nabla u/N|)\|^t_{L^{q,\,t}(B_0)} \leq C\, (|B_0|^{\frac{t}{q}} + S). \tag{4.3}$$

By Lemma 3.5 we find

$$\begin{aligned} S &\leq C\sum_{k=1}^{\infty}\Big[(AT)^{qk}T^{-p\theta_0}|\{x\in B_0:\mathcal{M}(|\nabla u|)(x)>N(AT)^{k-1}\}|\Big]^{\frac{t}{q}}\\ &\quad + C\sum_{k=1}^{\infty}\Big[(AT)^{qk}|\{x\in B_0:[\mathcal{M}_1(|\mu|)]^{\frac{1}{p-1}}>\epsilon(T)N(AT)^{k-1}\}|\Big]^{\frac{t}{q}}\\ &\leq C[(AT)^{q}T^{-p\theta_0}]^{\frac{t}{q}}(S+|B_0|^{\frac{t}{q}})+C_2\left\|[\mathcal{M}_1(|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|^{t}_{L^{q,\,t}(B_0)}. \end{aligned}$$

Thus for $q<p\theta_0$, i.e., $q<p+\epsilon$ with $\epsilon=p(\theta_0-1)$, and T sufficiently large we have

$$S\leq C\left(|B_0|^{\frac{t}{q}}+\left\|[\mathcal{M}_1(|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|^{t}_{L^{q,t}(B_0)}\right).$$

By (4.3) this yields

$$\|\nabla u/N\|_{L^{q,\,t}(\Omega)}\leq C\left(|B_0|^{\frac{1}{q}}+\left\|[\mathcal{M}_1(|\mu|/N^{p-1})]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}\right),$$

and thus

$$\|\nabla u\|_{L^{q,\,t}(\Omega))}\leq C\left(|B_0|^{\frac{1}{q}}N+\left\|[\mathcal{M}_1(|\mu|)]^{\frac{1}{p-1}}\right\|_{L^{q,\,t}(B_0)}\right). \tag{4.4}$$

On the other hand, by (4.2) and the condition $p>2-\frac{1}{n}$ we get

$$\begin{aligned} N &\leq C\,r^{-n}\,\|\nabla u\|_{L^1(\Omega)}\\ &\leq C\,\min\{r_0,\mathrm{diam}(\Omega)\}^{-n}\,|\Omega|^{1-\frac{n-1}{n(p-1)}}\,\|\nabla u\|_{L^{\frac{n(p-1)}{n-1},\infty}(\Omega)}\\ &\leq C\,\min\{r_0,\mathrm{diam}(\Omega)\}^{-n}\,|\Omega|^{1-\frac{n-1}{n(p-1)}}\,|\mu|(\Omega)^{\frac{1}{p-1}}\\ &\leq C\,\min\{r_0,\mathrm{diam}(\Omega)\}^{-n}\,\mathrm{diam}(\Omega)^{n}\left[\frac{|\mu|(\Omega)}{\mathrm{diam}(\Omega)^{n-1}}\right]^{\frac{1}{p-1}}, \end{aligned}$$

where the third inequality follows from standard estimates for equations with measure data (see, e.g., [2, 7]). Thus for any $x\in B_0$ we have

$$N\leq C(n,p,\mathrm{diam}(\Omega)/r_0)\,[\mathcal{M}_1(|\mu|)(x)]^{\frac{1}{p-1}},$$

which holds since $R_0\leq 2\mathrm{diam}(\Omega)$. Combining the last inequality with (4.4) we obtain (4.1) as desired. □

Next, we present the proof of Theorem 1.5.

Proof of the Theorem 1.5. Since u is $\mathcal{A}$-superharmonic there is a nonnegative measure $\mu[u]$ such that

$$-\,\mathrm{div}\mathcal{A}(x,\nabla u)=-\mathrm{div}F=\mu[u] \tag{4.5}$$

in the sense of distributions in $\mathbb{R}^n$. Moreover, for each integer $k>0$ the function $u_k=T_k(u)\in W^{1,p}_{\rm loc}(\mathbb{R}^n)$ is also $\mathcal{A}$-superharmonic and satisfies

$\mu[u_k] \to \mu[u]$ weakly as measures in $\mathbb{R}^n$. Here $\mu[u_k]$ is the nonnegative measure generated by the $\mathcal{A}$-superharmonic function u_k.

Thus it is easily seen that Lemma 3.5 holds also for solutions of (4.5) with $\Omega = B_0 = \mathbb{R}^n$ and, say, with $r = 1$. More precisely, there exist constants $A, \theta_0 > 1$, depending only on n, p, α, and β, such that the following holds for any $T > 1$. Suppose that u is an $\mathcal{A}$-superharmonic solution of (4.5) such that

$$|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|) > N\}| < T^{-p\theta_0}\,|B_1| \tag{4.6}$$

for some $N > 0$. Then for any integer $k \geq 0$, and with $\epsilon(T)$ as in (3.2), there holds

$$\begin{aligned}|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|) > N(AT)^{k+1}\}| \\ \leq\ & c(n)T^{-p\theta_0}|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|) > N(AT)^{k}\}| \\ & + c(n)|\{x \in \mathbb{R}^n : [\mathcal{M}_1(\mu[u])]^{\frac{1}{p-1}} > \epsilon(T)N(AT)^{k}\}|.\end{aligned} \tag{4.7}$$

To continue, for $T > 1$ to be chosen later, we now take

$$N = \frac{C(n)}{T^{-p\theta_0}|B_1|}\,\|\nabla u\|_{L^1(\mathbb{R}^n)} > 0 \tag{4.8}$$

with $C(n)$ large enough so that condition (4.6) holds true.

For $0 < t < \infty$ (the case $t = \infty$ is similar) we next consider the sums

$$S^+ = \sum_{k=1}^{\infty} \Big[(AT)^{qk}|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N(AT)^{k}\}|\Big]^{\frac{t}{q}}$$

and

$$S^- = \sum_{k=-\infty}^{0} \Big[(AT)^{qk}|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N(AT)^{k}\}|\Big]^{\frac{t}{q}}.$$

By (4.7) we find

$$\begin{aligned}S^+ \ \leq\ & C\sum_{k=1}^{\infty} \Big[(AT)^{qk}T^{-p\theta_0}|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N(AT)^{k-1}\}|\Big]^{\frac{t}{q}} \\ & + C\sum_{k=1}^{\infty} \Big[(AT)^{qk}|\{x \in \mathbb{R}^n : [\mathcal{M}_1(\mu[u])]^{\frac{1}{p-1}} > \epsilon(T)N(AT)^{k-1}\}|\Big]^{\frac{t}{q}} \\ \leq\ & C[(AT)^{q}T^{-p\theta_0}]^{\frac{t}{q}}\Big(S^+ + |\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}|^{\frac{t}{q}}\Big) \\ & + C_1\Big\|[\mathcal{M}_1(\mu[u]/N^{p-1})]^{\frac{1}{p-1}}\Big\|^{t}_{L^{q,\,t}(\mathbb{R}^n)}.\end{aligned}$$

Thus for $q < p\theta_0$, i.e., $q < p + \epsilon$ with $\epsilon = p(\theta_0 - 1)$, and T sufficiently large we have

$$
\begin{aligned}
S^+ &\leq C\,|\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla u|)(x) > N\}|^{\frac{t}{q}} \\
&\quad + C \left\| [\mathcal{M}_1(\mu[u]/N^{p-1})]^{\frac{1}{p-1}} \right\|^t_{L^{q,t}(\mathbb{R}^n)} \\
&\leq C \left(S^- + \left\| [\mathcal{M}_1(\mu[u]/N^{p-1})]^{\frac{1}{p-1}} \right\|^t_{L^{q,t}(\mathbb{R}^n)} \right).
\end{aligned} \tag{4.9}
$$

On the other hand, by the weak type $(1,1)$ bound for the maximal function and (4.8) there holds

$$
\begin{aligned}
S^- &\leq \sum_{k=-\infty}^{0} \left[(AT)^{qk} \frac{C(n)}{N(AT)^k} \int_\Omega |\nabla u| dx \right]^{\frac{t}{q}} \\
&= \sum_{k=-\infty}^{0} \left[(AT)^{k(q-1)} T^{-p\theta_0} |B_1| \right]^{\frac{t}{q}} \\
&\leq C(q, t, p, \theta_0, A, T),
\end{aligned} \tag{4.10}
$$

where the last inequality follows since $q > 1$. Note that we have

$$C^{-1}\,(S^+ + S^-) \leq \|\mathcal{M}(|\nabla u/N|)\|^t_{L^{q,\,t}(\mathbb{R}^n)} \leq C\,(S^+ + S^-),$$

and thus by (4.9) and (4.10) this yields

$$\|\nabla u/N\|_{L^{q,\,t}(\mathbb{R}^n)} \leq C \left(1 + \left\| [\mathcal{M}_1(\mu[u]/N^{p-1})]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} \right).$$

We therefore have

$$
\begin{aligned}
\|\nabla u\|_{L^{q,\,t}(\Omega))} &\leq C \left(N + \left\| [\mathcal{M}_1(\mu[u])]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} \right) \\
&\leq C \left(\|\nabla u\|_{L^1(\mathbb{R}^n)} + \left\| [\mathcal{M}_1(\mu[u])]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} \right) \\
&\leq C \left(\|\nabla u\|_{L^1(\mathbb{R}^n)} + \left\| [\mathbf{I}_1(\mu[u])]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} \right).
\end{aligned}
$$

Notice that, by the second equality in (4.5), the equality

$$\mathbf{I}_1(\mu[u]) = c(n) \sum_{j=1}^{n} R_j f_j \tag{4.11}$$

holds a.e. in $\mathbb{R}^n$, where $F = (f_1, f_2, \ldots, f_n)$ and $R_j f_j$ denotes the j-th Riesz transform of the function f_j (see [27], page 1580). Since $q > p-1$ this yields

$$
\begin{aligned}
\left\| [\mathbf{I}_1(\mu[u])]^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} &= \|\mathbf{I}_1(\mu[u])\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\,\frac{t}{p-1}}(\mathbb{R}^n)} \\
&\leq C\,\|F\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\,\frac{t}{p-1}}(\mathbb{R}^n)},
\end{aligned}
$$

and the desired estimate follows. □

Finally, we prove Theorem 1.6.

Proof of the Theorem 1.6. Since $-\mathrm{div}F \geq 0$ in $\mathcal{D}'(\mathbb{R}^n)$ there is a nonnegative measure μ in $\mathbb{R}^n$ such that

$$-\mathrm{div}F = \mu.$$

For each integer $m > 0$ let B_m denote the ball of radius m and centered at the origin of $\mathbb{R}^n$. Also, let μ_{B_m} be the restriction of μ to the ball B_m. Then there exists a nonnegative $\mathcal{A}$-superharmonic function u_m in B_m such that

$$\begin{cases} -\mathrm{div}\,\mathcal{A}(x,\nabla u_m) &= \mu_{B_m} \quad \text{in } B_m \\ u_m &= 0 \quad \text{on } \partial B_m. \end{cases} \tag{4.12}$$

By Theorem 1.1 we have

$$\|\nabla u_m\|_{L^{q,\,t}(B_m)} \leq C \left\| \mathcal{M}_1(\mu)^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)} \leq C \left\| \mathbf{I}_1(\mu)^{\frac{1}{p-1}} \right\|_{L^{q,\,t}(\mathbb{R}^n)}, \tag{4.13}$$

where C is independent of m. Thus for $q > 1$ the Sobolev inequality on Lorentz spaces (see [32, Theorem 2.10.2]) yields

$$\|u_m\|_{L^{\frac{nq}{n-q},\,t}(B_m)} \leq C \, \|\mathbf{I}_1(\mu)\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\,\frac{t}{p-1}}(\mathbb{R}^n)}, \tag{4.14}$$

Inequality (4.14) holds also for $p-1 < q \leq 1$. To see this first note that by [29, Theorem 2.1] (see also [18, 22, 30]) we have a pointwise bound

$$u_m(x) \leq C\,\mathbf{W}_{1,\,p}(\mu)(x), \qquad x \in \mathbb{R}^n, \tag{4.15}$$

where $C = C(n,p,\alpha,\beta)$ and

$$\mathbf{W}_{1,\,p}(\mu)(x) := \int_0^\infty \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}$$

is the Wolff's potential of μ. Since $1/(p-1) > 1$ we find

$$\begin{aligned} \mathbf{W}_{1,\,p}(\mu)(x) &\leq C \left[\int_0^\infty \frac{\mu(B_t(x))}{t^{n-p}} \frac{dt}{t}\right]^{\frac{1}{p-1}} \\ &= C\,[\mathbf{I}_p(\mu)(x)]^{\frac{1}{p-1}} \\ &= C\,\{\mathbf{I}_{p-1}[\mathbf{I}_1(\mu)](x)\}^{\frac{1}{p-1}}. \end{aligned}$$

Here for $0 < \alpha < n$ and a nonnegative measure ν, $\mathbf{I}_\alpha(\nu)$ is the (unnormalized) Riesz potential of ν defined by

$$\mathbf{I}_\alpha(\nu)(x) := \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n.$$

Thus by (4.15) and a Sobolev inequality [32, Theorem 2.10.2] we have

$$\begin{aligned} \|u_m\|_{L^{\frac{nq}{n-q},\,t}(B_m)} &\leq C\,\|\mathbf{I}_{p-1}[\mathbf{I}_1(\mu)]\|^{\frac{1}{p-1}}_{L^{\frac{nq}{(n-q)(p-1)},\,\frac{t}{p-1}}(\mathbb{R}^n)} \\ &\leq C\,\|\mathbf{I}_1(\mu)\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\,\frac{t}{p-1}}(\mathbb{R}^n)} \end{aligned}$$

as claimed.

At this point we use [17, Theorem 1.17] to find a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ and an $\mathcal{A}$-superharmonic function u in $\mathbb{R}^n$ such that

$$u(x) = \lim_{j\to\infty} u_{m_j}(x)$$

a.e. in $\mathbb{R}^n$ and $\nabla u_{m_j} \to \nabla u$ a.e. in the set $\{u < +\infty\}$. Note that by (4.14) and Fatou lemma u is finite a.e. and

$$\|u\|_{L^{\frac{nq}{n-q},\, t}(\mathbb{R}^n)} \leq C\, \|\mathbf{I}_1(\mu)\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\, \frac{t}{p-1}}(\mathbb{R}^n)}.$$

Likewise, it follows from (4.11), (4.13) and Fatou lemma that

$$\|\nabla u\|_{L^{q,\, t}(\mathbb{R}^n)} \leq C\, \|\mathbf{I}_1(\mu)\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\, \frac{t}{p-1}}(\mathbb{R}^n)} \leq C\, \|F\|^{\frac{1}{p-1}}_{L^{\frac{q}{p-1},\, \frac{t}{p-1}}(\mathbb{R}^n)}.$$

Finally, (4.12) and the weak continuity result of [30] imply that u is a solution of (1.7). □

References

[1] A. Alvino, V. Ferone, and G. Trombetti, *Estimates for the gradient of solutions of nonlinear elliptic equations with L^1-data*, Ann. Mat. Pura Appl. **178** (2000), 129–142.

[2] P. Bénilan, L. Boccardo, T. Gallöuet, R. Gariepy, M. Pierre , and J. L. Vazquez, *An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa (IV) **22** (1995), 241–273.

[3] L. Boccardo and T. Gallöuet, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149–169.

[4] L. Boccardo and T. Gallöuet, *Nonlinear elliptic equations with right-hand side measures*, Comm. Partial Differential Equations **17** (1992), 641–655.

[5] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. **57** (2004), 1283–1310.

[6] L. Caffarelli and I. Peral, *On $W^{1,\,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998) 1–21.

[7] G. Dal Maso, F. Murat, A. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Super. Pisa (IV) **28** (1999), 741–808.

[8] A. Dall'Aglio, *Approximated solutions of equations with L^1 data. Applications to the H-convergence of quasilinear paprabolic equations*, Ann. Mat. Pura Appl. **170** (1996), 207–240.

[9] T. Del Vecchio, *Nonlinear elliptic equations with measure data*, Potential Anal. **4** (1995) 185–203.

[10] F. Duzaar and G. Mingione, *Gradient estimates via non-linear potentials*, Amer. J. Math. **133**, (2011), 1093–1149.

[11] F. Duzaar and G. Mingione, *Gradient estimates via linear nonlinear potentials*, J. Funt. Anal. **259** (2010), 2961–2998.

[12] E. Giusti, *Direct methods in the calculus of variations*, World Scienticfic Publishing Co., Inc., River Edge, NJ, 2003.

[13] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Oxford Univ. Press, Oxford, 1993.

[14] T. Iwaniec, *Projections onto gradient fields and L^p-estimates for degenerated elliptic operators*, Studia Math. **75** (1983), 293–312.

[15] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), 161–219.

[16] T. Kilpeläinen and G. Li, *Estimates for p-Poisson equations*, Differ. Int. Equ. **13** (2000), 791–800.

[17] T. Kilpeläinen and J. Malý, *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Sup. Pisa (IV) **19** (1992), 591–613.

[18] T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.

[19] J. L. Lewis, *Uniformly fat sets*, Trans. Amer. Math. Soc. **308** (1988), 177–196.

[20] T. Mengesha and N. C. Phuc, *Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains*, J. Diff. Equ. **250** (2011), 1485–2507.

[21] N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (III) **17** (1963), 189–206.

[22] P. Mikkonen, *On the Wolff potential and quasilinear elliptic equations involving measures*, Ann. Acad. Sci. Fenn., Ser AI, Math. Dissert. **104** 1996, 1–71.

[23] G. Mingione, *The Calderón-Zygmund theory for elliptic problems with measure data*, Ann. Scuola Norm. Super. Pisa Cl. Sci. (V) **6** (2007), 195–261.

[24] G. Mingione, *Gradient estimates below the duality exponent,* Math. Ann. **346** (2010), 571–627.

[25] G. Mingione, *Nonlinear measure data problems*, Milan J. Math. **79** (2011), 429–496.

[26] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

[27] N. C. Phuc and M. Torres, *Characterizations of the existence and removable singularities of divergence measure vector fields*, Indiana U. Math. J. **57** (2008), 15731597.

[28] N. C. Phuc, *On Calderón-Zygmund theory for p- and $\mathcal{A}$-superharmonic functions*, Calc. Var. Part. Diff. Equ. To appear.

[29] N. C. Phuc and I. E. Verbitsky, *Quasilimear and Hessian equations of Lane-Emden type*, Ann. Math. **168** (2008), 859–914.

[30] N. S. Trudinger and X.-J. Wang, *On the weak continuity of elliptic operators and applications to potential theory*, Amer. J. Math. **124** (2002), 369–410.

[31] L. Wang, *A geometric approach to the Calderón-Zygmund estimates*, Acta Math. Sin. (Engl. Ser.) **19** (2003), 381–396.

[32] W. P. Ziemer, *Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation*, Grad. Texts in Math., vol. **120**, Springer-Verlag, New York, 1989.

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