THE NAVIER-STOKES EQUATIONS IN NONENDPOINT BORDERLINE LORENTZ SPACES

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Abstract. It is shown both locally and globally that $L_t^\infty(L_x^{3,q})$ solutions to the three-dimensional Navier-Stokes equations are regular provided $q \neq \infty$. Here $L_x^{3,q}$, $0 < q \leq \infty$, is an increasing scale of Lorentz spaces containing L_x^3. Thus the result provides an improvement of a result by Escauriaza, Seregin and Šverák ((Russian) Uspekhi Mat. Nauk **58** (2003), 3–44; translation in Russian Math. Surveys **58** (2003), 211–250), which treated the case $q = 3$. A new local energy bound and a new ϵ-regularity criterion are combined with the backward uniqueness theory of parabolic equations to obtain the result. A weak-strong uniqueness of Leray-Hopf weak solutions in $L_t^\infty(L_x^{3,q})$, $q \neq \infty$, is also obtained as a consequence.

1. Introduction

This paper addresses certain regularity and uniqueness criteria for the three-dimensional Navier-Stokes equations

$$\partial_t u - \Delta u + \operatorname{div} u \otimes u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1.1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ and $p = p(x,t) \in \mathbb{R}$, with $x \in \mathbb{R}^3$ and $t \geq 0$. The initial condition associated to (1.1) is given by

$$u(x,0) = a(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

Equations (1.1)–(1.2) describes the motion an incompressible fluid in three spatial dimensions with unit viscosity and zero external force. Here u and p are referred to as the fluid velocity and pressure, respectively.

From the classical works of Leray [19] and Hopf [11], it is known that for any divergence-free vector field $a \in L^2(\mathbb{R}^3)$ there exists at least one weak solution to the Cauchy problem (1.1)–(1.2) in $\mathbb{R}^3 \times (0,\infty)$. Such a solution is now called *Leray-Hopf weak solution* whose precise definition will be given next. Let $\dot{C}_0^\infty$ denote the space of all divergence-free infinitely differentiable vector fields with compact support in $\mathbb{R}^3$. Let $\dot{J}$ be the closure of $\dot{C}_0^\infty$ in $L^2(\mathbb{R}^3)$, and $\dot{J}^{1,2}$ be the closure of the same set with respect to the Dirichlet integral.

Definition 1.1. *A Leray-Hopf weak solution of the Cauchy problem* (1.1)–(1.2) *in $Q_\infty := \mathbb{R}^3 \times (0,\infty)$ is a vector field $u : Q_\infty \to \mathbb{R}^3$ such that:*

(i) *$u \in L^\infty(0,\infty; \dot{J}) \cap L^2(0,\infty; \dot{J}^{1,2})$;*

(ii) *The function* $t \to \int_{\mathbb{R}^3} u(x,t)w(x)dx$ *is continuous on* $[0,\infty)$ *for any* $w \in L^2(\mathbb{R}^3)$*;*

(iii) *For any* $w \in \dot{C}_0^\infty(Q_\infty)$ *there holds*

$$\int_{Q_\infty} (-u \cdot \partial_t w - u \otimes u : \nabla w + \nabla u : \nabla w) dxdt = 0;$$

(iv) *The energy inequality*

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxds \leq \int_{\mathbb{R}^3} |a(x)|^2 dx$$

holds for all $t \in [0,\infty)$*, and*

$$\|u(\cdot,t) - a(\cdot)\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{as } t \to 0^+.$$

As of now the problems of uniqueness and regularity of Leray-Hopf weak solutions are still open. Only some partial results are known. The partial uniqueness result of Prodi [24] and Serrin [34], and the partial smoothness result of Ladyzhenskaya [17] can be summarized in the following theorem.

Theorem 1.2. *Suppose that* $a \in \dot{J}$ *and* u*,* u_1 *are two Leray-Hopf weak solutions to the Cauchy problem* (1.1)–(1.2)*. If* $u \in L^s(0,T;L^p(\mathbb{R}^3))$ *for some* $T > 0$*, where*

$$\frac{3}{p} + \frac{2}{s} = 1, \qquad p \in (3,\infty],$$

then $u = u_1$ *in* $Q_T := \mathbb{R}^3 \times (0,T)$ *and, moreover,* u *is smooth on* $\mathbb{R}^3 \times (0,T]$*.*

Here recall that the condition $u \in L^s(0,T;L^p(\mathbb{R}^3))$ means that

$$\|u\|_{L^s(0,T;L^p(\mathbb{R}^3))} := \Big(\int_0^T \|u(\cdot,t)\|^s_{L^p(\mathbb{R}^n)}\, dt\Big)^{\frac{1}{s}} < +\infty \qquad \text{if } s \in [1,\infty),$$

and

$$\|u\|_{L^s(0,T;L^p(\mathbb{R}^3))} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} < +\infty \qquad \text{if } s = \infty.$$

It is obvious that, when $s = p$, $L^p(0,T;L^p(\mathbb{R}^3)) = L^p(Q_T)$. In general, if X is a Banach space with norm $\|\cdot\|_X$, then $L^s(a,b;X)$, $a < b$, means the usual Banach space of strongly measurable X-valued functions $f(t)$ on (a,b) such that the norm

$$\|u\|_{L^s(a,b;X)} := \Big(\int_0^T \|f(t)\|^s_X\, dt\Big)^{\frac{1}{s}} < +\infty \tag{1.3}$$

for $s \in [1,\infty)$, and with the usual modification of (1.3) in the case $s = \infty$.

The endpoint case $p = 3$ and $s = \infty$, which is not covered by Theorem 1.2, was considered harder and has been settled by Escauriaza-Seregin-Šverák in the interesting work [5]:

Theorem 1.3. *Let $a \in \dot{J} \cap L^3(\mathbb{R}^3)$. Suppose that u is a Leray-Hopf weak solution of the Cauchy problem* (1.1)–(1.2)*, and u satisfies the additional condition*

$$u \in L^\infty(0,T; L^3(\mathbb{R}^3)) \tag{1.4}$$

for some $T > 0$. Then $u \in L^5(Q_T)$ and hence it is unique and smooth on $\mathbb{R}^3 \times (0,T]$.

We remark that the condition $a \in L^3(\mathbb{R}^3)$ in the above theorem is superfluous as it can be deduce from condition (1.4). A basic consequence of Theorem 1.3 is that if a Leray-Hopf weak solution u develops a singularity at a first finite time $t_0 > 0$ then there necessarily holds

$$\limsup_{t \uparrow t_0} \|u(\cdot,t)\|_{L^3(\mathbb{R}^3)} = +\infty. \tag{1.5}$$

An improvement of this necessary condition of potential blow up can be found in the recent work [31]. See also the papers [7, 13] for another approach to regularity using certain profile decompositions.

It should be noticed that the uniqueness of u under condition (1.4) had been known earlier (see [15]). Moreover, local versions of the corresponding partial regularity results are also available (see [33], [37], and [5]). In particular, the local regularity result of [5] reads as follows.

Theorem 1.4. *Suppose that the pair of functions (u,p) satisfies the Navier-Stokes equations* (1.1) *in $Q_1(0,0) = B_1(0) \times (-1,0)$ in the sense of distributions and has the following properties:*

$$u \in L^\infty(-1,0; L^2(B_1)) \cap L^2(-1,0; W^{1,2}(B_1)) \tag{1.6}$$

and

$$p \in L^{3/2}(-1,0; L^{3/2}(B_1)).$$

Suppose further that

$$u \in L^\infty(-1,0; L^3(B_1)).$$

Then the velocity function u is Hölder continuous on $\overline{Q}_{1/2}(0,0)$.

The main goal of this paper is to improve Theorems 1.3 and 1.4 by means of Lorentz spaces. Given a measurable set $\Omega \subset \mathbb{R}^3$, recall that the Lorentz space $L^{p,q}(\Omega)$, with $p \in (0,\infty), q \in (0,\infty]$, is the set of measurable functions g on Ω such that the quasinorm $\|g\|_{L^{p,q}(\Omega)}$ is finite. Here we define

$$\|g\|_{L^{p,q}(\Omega)} := \begin{cases} \left(p \displaystyle\int_0^\infty \alpha^q |\{x \in \Omega : |g(x)| > \alpha\}|^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{q}} \text{ if } q < \infty, \\ \displaystyle\sup_{\alpha > 0} \alpha \, |\{x \in \Omega : |g(x)| > \alpha\}|^{\frac{1}{p}} \text{ if } q = \infty. \end{cases}$$

The space $L^{p,\infty}(\Omega)$ is often referred to as the Marcinkiewicz or weak L^p space. It is known that $L^{p,p}(\Omega) = L^p(\Omega)$ and $L^{p,q_1}(\Omega) \subset L^{p,q_2}(\Omega)$ whenever

$q_1 \leq q_2$. On the other hand, if $|\Omega|$ is finite then $L^{p,q}(\Omega) \subset L^r(\Omega)$ for any $0 < q \leq \infty$ and $0 < r < p$. Moreover,

$$\|g\|_{L^r(\Omega)} \leq |\Omega|^{\frac{1}{r}-\frac{1}{p}} \|g\|_{L^{p,q}(\Omega)}.$$

Lorentz spaces can be used to capture logarithmic singularities. For example, in $\mathbb{R}^3$, for any $\beta > 0$ we have

$$|x|^{-1}|\log(|x|/2)|^{-\beta} \in L^{3,q}(B_1(0)) \quad \text{if and only if } q > \frac{1}{\beta}. \tag{1.7}$$

Note that the inequality in (1.7) is strict. Of course, in the case $\beta = 0$, the function $|x|^{-1}$ belongs to the Marcinkiewicz space $L^{3,\infty}(\mathbb{R}^3)$.

To the best of our knowledge, a criterion of local regularity for the Navier-Stokes equations in $L^\infty(-1, 0; L^{3,\infty}(B_1))$ is still unknown. See [14, 21] for some partial results, which require a smallness condition. See also [35, 38] for some nonendpoint related results. The first result of this paper provides instead a regularity condition in terms of the borderline space $L^\infty(-1, 0; L^{3,q}(B_1))$ for *any* $q \in (0, \infty)$, and thus excluding only the endpoint case $q = \infty$.

Theorem 1.5. *Suppose that the pair of functions* (u, p) *satisfies the Navier-Stokes equations* (1.1) *in* $Q_1(0, 0) = B_1(0) \times (-1, 0)$ *in the sense of distributions such that* (1.6) *holds and*

$$p \in L^2(-1, 0; L^1(B_1)). \tag{1.8}$$

Suppose further that

$$u \in L^\infty(-1, 0; L^{3,q}(B_1)) \tag{1.9}$$

for some $q \in (3, \infty)$. *Then the velocity function* u *is Hölder continuous on* $\overline{Q}_{1/2}(0, 0)$.

It is worth mentioning that even the regularity at $(0, 0)$ is still unknown for solutions u satisfying the pointwise bound

$$|u(x, t)| \leq C\, |x|^{-1}$$

for a.e. $(x, t) \in Q_1(0, 0)$. A regularity result under this condition is known only for axially symmetric solutions (see [30] and also [3, 4]). On the other hand, in view of (1.7), Theorem 1.5 yields the regularity of u under a logarithmic 'bump' condition

$$|u(x, t)| \leq C\, |x|^{-1}|\log(|x|/2)|^{-\beta}$$

for any $\beta > 0$.

In fact, it is possible to obtain regularity under a weaker pointwise bound condition on the solution. In this case Theorem 1.5 is no longer applicable.

Theorem 1.6. *Suppose that the pair of functions* (u, p) *satisfies the Navier-Stokes equations* (1.1) *in* $Q_1(0, 0)$ *in the sense of distributions such that* (1.6) *holds, and*

$$p \in L^{3/2}(-1, 0; L^1(B_1)).$$

Suppose further that for a.e. $(x,t) \in Q_1(0,0)$, *there holds*

$$|u(x,t)| \le f(t)|x|^{-1}g(x) \tag{1.10}$$

for nonnegative functions $f \in L^\infty((-1,0))$ *and* $g \in L^\infty(B_1(0))$ *such that* $\lim_{x\to 0} g(x) = 0$. *Then* u *is Hölder continuous on* $\overline{Q}_{1/2}(0,0)$.

On the other hand, Theorem 1.5 can be used to deduce the following uniqueness and global regularity results, which give an improvement of Theorem 1.3.

Theorem 1.7. *Let* $a \in \dot{J}$. *Suppose that* u *is a Leray-Hopf weak solution of the Cauchy problem* (1.1)–(1.2), *and it satisfies the additional condition*

$$u \in L^\infty(0,T; L^{3,q}(\mathbb{R}^3)) \tag{1.11}$$

for some $q \in (3,\infty)$ *and* $T > 0$. *Then* u *is smooth on* $\mathbb{R}^3 \times (0,T]$. *Moreover, if in addition* $a \in L^3(\mathbb{R}^3)$ *then* $u \in L^5(Q_T)$ *and hence it is unique in* Q_T *(in the sense of weak-strong uniqueness as in Theorem 1.2).*

Theorem 1.7 implies that the necessary condition of potential blow up (1.5) can now be improved by replacing the L^3 norm with any smaller $L^{3,q}$ quasi-norm provided $q \neq \infty$. We should mention that this kind of potential blow up criterion has recently been extended in [8] to the norms of Besov spaces $\dot{B}^{-1+3/p}_{q,p}(\mathbb{R}^3)$, $3 < p, q < \infty$, using profile decompositions in the framework of "strong" solutions. See also [2] for an earlier related result. In such a setting of strong solutions, the blow up criterion of [8] is more general than ours since for $q > 3$,

$$L^{3,q}(\mathbb{R}^3) \subset \dot{B}^{-1+3/q}_{q,q}(\mathbb{R}^3).$$

However, our blow up criterion here is obtained for Leray-Hopf weak solutions. Moreover, using instead the local regularity criterion, Theorem 1.5, we see that if (x_0,t_0) is a singular point of a Leray-Hopf weak solution u then there holds

$$\limsup_{t\uparrow t_0} \|u(\cdot,t)\|_{L^{3,q}(B_\delta(x_0))} = +\infty$$

for any $\delta > 0$ and $q \neq \infty$. Note that for a Leray-Hopf weak solution u, the associated pressure p can be chosen so that $p \in L^2(0,\infty; L^{3/2}(\mathbb{R}^3))$ since $|u|^2$ belongs to the same space (see, e.g., [29]).

Our approach to Theorems 1.5 and 1.7 is influenced by the above mentioned work of Escauriaza-Seregin-Šverák [5], which reduces the regularity matter to the backward uniqueness problem for parabolic equations with variable lower-order terms. A key ingredient, which makes our results stronger than that of [5], is a new ϵ-regularity criterion for *suitable weak solutions* to the Navier-Stokes equations (see Proposition 3.2). See Definition 2.1 below for the notion of suitable weak solutions. In turn, this kind of ϵ-regularity criterion is a consequence of a new bound for some scaling invariant energy quantities (see Corollary 2.4). Moreover, this new energy bound is also essential in a blow-up procedure needed in the proof of Theorem

1.5. It provides a certain compactness result and thus yields a non-trivial 'ancient solution' (see Proposition 4.2), another important ingredient in the proof of Theorem 1.5.

On the other hand, the proof of Theorem 1.6 is simple. It requires only an ϵ-regularity criterion of Seregin and Šverák in [29, Lemma 3.3].

2. Preliminaries and local energy estimates

Throughout the paper we use the following notations for balls and parabolic cylinders:

$$B_r(x) = \{y \in \mathbb{R}^3 : |x-y| < r\}, \quad x \in \mathbb{R}^3,\ r>0,$$

and

$$Q_r(z) = B_r(x) \times (t-r^2, t) \quad \text{with } z=(x,t).$$

The following scaling invariant quantities will be employed:

$$\begin{aligned}
A(z_0,r) = A(u,z_0,r) &= \sup_{t_0-r^2\leq t\leq t_0} r^{-1}\int_{B_r(x_0)} |u(x,t)|^2 dx,\\
B(z_0,r) = B(u,z_0,r) &= r^{-1}\int_{Q_r(x_0)} |\nabla u(x,t)|^2 dxdt,\\
C(z_0,r) = C(u,z_0,r) &= r^{-3}\int_{t_0-r^2}^{t_0} \|u\|^4_{L^{\frac{12}{5}}(B_r(x_0))}\, dt,\\
C_1(z_0,r) = C_1(u,z_0,r) &= r^{-2}\int_{t_0-r^2}^{t_0} \|u\|^3_{L^3(B_r(x_0))}\, dt,\\
D(z_0,r) = D(p,z_0,r) &= r^{-3}\int_{t_0-r^2}^{t_0} \|p\|^2_{L^{\frac{6}{5}}(B_r(x_0))}\, dt,\\
D_1(z_0,r) = D_1(p,z_0,r) &= r^{-2}\int_{t_0-r^2}^{t_0} \|p\|^{3/2}_{L^{\frac{3}{2}}(B_r(x_0))}\, dt.
\end{aligned}$$

To analyze local properties of solutions, it is often useful to use the notion of suitable weak solutions. Such a notion of weak solutions was introduced in Caffarelli-Kohn-Nirenberg [1] following the work of Scheffer [25]–[28]. Here we use the version introduced by Lin in [20].

Definition 2.1. *Let ω be an open set in $\mathbb{R}^3$ and let $-\infty < a < b < \infty$. We say that a pair (u,p) is a suitable weak solution to the Navier-Stokes equations in $Q=\omega\times(a,b)$ if the following conditions hold:*

(i) $u \in L^\infty(a,b;L^2(\omega)) \cap L^2(a,b;W^{1,2}(\omega))$ and $p \in L^{3/2}(\omega\times(a,b))$;

(ii) (u,p) *satisfies the Navier-Stokes equations in the sense of distributions. That is,*

$$\int_a^b\int_\omega \{-u\,\psi_t + \nabla u : \nabla\psi - (u\otimes u) : \nabla\psi - p\,\mathrm{div}\,\psi\}\, dxdt = 0$$

for all vector fields $\psi \in C_0^\infty(\omega \times (a,b); \mathbb{R}^3)$, *and*

$$\int_{\omega\times\{t\}} u(x,t)\cdot\nabla\phi(x)\,dx = 0$$

for a.e. $t \in (a,b)$ *and all real valued functions* $\phi \in C_0^\infty(\omega)$*;*

(iii) (u,p) *satisfies the local generalized energy inequality*

$$\begin{aligned}\int_\omega |u(x,t)|^2\phi(x,t)dx + 2\int_a^t\int_\omega |\nabla u|^2\phi(x,s)dxds \\ \leq \int_a^t\int_\omega |u|^2(\phi_t+\Delta\phi)dxds + \int_a^t\int_\omega (|u|^2+2p)u\cdot\nabla\phi dxds.\end{aligned}$$

for a.e. $t\in(a,b)$ *and any nonnegative function* $\phi\in C_0^\infty(\mathbb{R}^3\times\mathbb{R})$ *vanishing in a neighborhood of the parabolic boundary* $\partial' Q = \omega\times\{t=a\}\cup\partial\omega\times[a,b]$.

A proof of the following lemma can be found in [10, Lemma 6.1].

Lemma 2.2. *Let* $I(s)$ *be a bounded nonnegative function in the interval* $[R_1,R_2]$*. Assume that for every* $s,\rho\in[R_1,R_2]$ *and* $s<\rho$ *we have*

$$I(s)\leq [A(\rho-s)^{-\alpha}+B(\rho-s)^{-\beta}+C]+\theta I(\rho)$$

with $A,B,C\geq 0$, $\alpha>\beta>0$ *and* $\theta\in[0,1)$*. Then there holds*

$$I(R_1)\leq c(\alpha,\theta)[A(R_2-R_1)^{-\alpha}+B(R_2-R_1)^{-\beta}+C].$$

In the next lemma $L^{-1,2}(B_r(x_0))$ stands for the dual of the Sobolev space $W_0^{1,2}(B_r(x_0))$. The latter is defined as the completion of $C_0^\infty(B_r(x_0))$ under the Dirichlet norm

$$\|\varphi\|_{W_0^{1,2}(B_r(x_0))} = \Big(\int_{B_r(x_0)}|\nabla\varphi|^2dx\Big)^{1/2}.$$

Lemma 2.3. *Suppose that* (u,p) *is a suitable weak solution to the Navier-Stokes equations in* $Q=\omega\times(a,b)$*. Let* $z_0=(x_0,t_0)$ *and* $r>0$ *be such that* $Q_r(z_0)\subset Q$*. Then there holds*

$$\begin{aligned}A(z_0,r/2)+B(z_0,r/2) \quad &\leq \quad C\left[r^{-3}\int_{t_0-r^2}^{t_0}\big\||u|^2\big\|^2_{L^{-1,2}(B_r(x_0))}dt\right]^{1/2} \\ &\quad + Cr^{-3}\int_{t_0-r^2}^{t_0}\big\||u|^2+2p\big\|^2_{L^{-1,2}(B_r(x_0))}dt.\end{aligned}$$

Proof. For $z_0=(x_0,t_0)$ and $r>0$ such that $Q_r(z_0)\subset Q$, we consider the cylinders

$$Q_s(z_0)=B_s(x_0)\times(t_0-s^2,t_0)\subset Q_\rho(z_0)=B_\rho(x_0)\times(t_0-\rho^2,t_0),$$

where $r/2\leq s<\rho\leq r$.

Let $\phi(x,t)=\eta_1(x)\eta_2(t)$ where $\eta_1\in C_0^\infty(B_\rho(x_0))$, $0\leq\eta_1\leq 1$ in $\mathbb{R}^n$, $\eta_1\equiv 1$ on $B_s(x_0)$, and

$$|\nabla^\alpha\eta_1|\leq\frac{c}{(\rho-s)^{|\alpha|}}$$

for all multi-indices α with $|\alpha| \leq 3$. The function $\eta_2(t)$ is chosen so that $\eta_2 \in C_0^\infty(t_0-\rho^2, t_0+\rho^2)$, $0 \leq \eta_2 \leq 1$ in $\mathbb{R}$, $\eta_2(t) \equiv 1$ for $t \in [t_0-s^2, t_0+s^2]$, and

$$|\eta_2'(t)| \leq \frac{c}{\rho^2 - s^2} \leq \frac{c}{r(\rho-s)}.$$

Then

$$|\nabla \phi_t| \leq \frac{c}{r(\rho-s)^2} \leq \frac{c}{(\rho-s)^3}, \quad |\nabla \Delta \phi| \leq \frac{c}{(\rho-s)^3},$$

$$|\nabla^2 \phi| \leq \frac{c}{(\rho-s)^2}, \quad |\nabla \phi| \leq \frac{c}{\rho-s}.$$

We next define

$$I(s) = I_1(s) + I_2(s),$$

where

$$I_1(s) = \sup_{t_0-s^2 \leq t \leq t_0} \int_{B_s(x_0)} |u(x,t)|^2 dx = s\, A(z_0, s)$$

and

$$I_2(s) = \int_{t_0-s^2}^{t_0} \int_{B_s(x_0)} |\nabla u(x,t)|^2 dx dt = s\, B(z_0, s).$$

Using ϕ as a test function in the generalized energy inequality we find

$$\begin{aligned} I(s) \qquad\qquad\qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.1)\\ \leq\ & \int_{t_0-\rho^2}^{t_0} \left\| |u|^2 \right\|_{L^{-1,2}(B_\rho(x_0))} \|\nabla \phi_t + \nabla \Delta \phi\|_{L^2(B_\rho(x_0))}\, dt \\ & + \int_{t_0-\rho^2}^{t_0} \Big\{ \left\| |u|^2 + 2p \right\|_{L^{-1,2}(B_\rho(x_0))} \times \\ & \qquad\qquad \times \left\| \nabla u \cdot \nabla \phi + u \cdot \nabla^2 \phi \right\|_{L^2(B_\rho(x_0))} \Big\} dt \\ =:\ & J_1 + J_2. \end{aligned}$$

By the choice of test function we have

$$\begin{aligned} J_1 \ &\leq\ C \frac{\rho^{3/2}}{(\rho-s)^3} \int_{t_0-\rho^2}^{t_0} \left\| |u|^2 \right\|_{L^{-1,2}(B_\rho(x_0))} dt \\ &\leq\ C \frac{\rho^{5/2}}{(\rho-s)^3} \left[\int_{t_0-\rho^2}^{t_0} \left\| |u|^2 \right\|^2_{L^{-1,2}(B_\rho(x_0))} dt \right]^{1/2}. \end{aligned} \tag{2.2}$$

Also,

$$\begin{aligned} J_2 \ \leq\ & C \int_{t_0-\rho^2}^{t_0} \Big\{ \left\| |u|^2 + 2p \right\|_{L^{-1,2}(B_\rho(x_0))} \times \\ & \qquad \times \Big[\frac{\|\nabla u\|_{L^2(B_\rho(x_0))}}{\rho-s} + \frac{\|u\|_{L^2(B_\rho(x_0))}}{(\rho-s)^2} \Big] \Big\} dt, \end{aligned}$$

and thus by Hölder's inequality we get

$$(2.3)\ J_2 \le \frac{C}{\rho - s}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} I_2(\rho)^{1/2} + \frac{C\rho}{(\rho - s)^2}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} I_1(\rho)^{1/2}.$$

Combining inequalities (2.1)–(2.3) and using $\rho \le r$ we arrive at

$$\begin{aligned} I(s) &\le \frac{Cr^{5/2}}{(\rho - s)^3}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} + \\ &+ \frac{C}{\rho - s}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} I_2(\rho)^{1/2} + \\ &+ \frac{Cr}{(\rho - s)^2}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} I_1(\rho)^{1/2}. \end{aligned}$$

By Young's inequality this yields

$$\begin{aligned} I(s) &\le \frac{Cr^{5/2}}{(\rho - s)^3}\left[\int_{t_0-\rho^2}^{t_0} \left\||u|^2\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt\right]^{1/2} \\ &+ \left\{\frac{C}{(\rho - s)^2} + \frac{Cr^2}{(\rho - s)^4}\right\}\int_{t_0-\rho^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_\rho(x_0))} dt \\ &+ \frac{1}{2} I(\rho), \end{aligned}$$

which implies in particular that

$$\begin{aligned} I(s) &\le \frac{Cr^{5/2}}{(\rho - s)^3}\left[\int_{t_0-r^2}^{t_0} \left\||u|^2\right\|^2_{L^{-1,2}(B_r(x_0))} dt\right]^{1/2} \\ &+ \frac{Cr^2}{(\rho - s)^4}\int_{t_0-r^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_r(x_0))} dt + \frac{1}{2} I(\rho). \end{aligned}$$

Since this holds for all $r/2 \le s < \rho \le r$ by Lemma 2.2 we find

$$\begin{aligned} I(r/2) &\le Cr^{-1/2}\left[\int_{t_0-r^2}^{t_0} \left\||u|^2\right\|^2_{L^{-1,2}(B_r(x_0))} dt\right]^{1/2} \\ &+ Cr^{-2}\int_{t_0-r^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_r(x_0))} dt. \end{aligned}$$

Thus

$$\begin{aligned} A(z_0, r/2) + B(z_0, r/2) &\le C\left[r^{-3}\int_{t_0-r^2}^{t_0} \left\||u|^2\right\|^2_{L^{-1,2}(B_r(x_0))} dt\right]^{1/2} \\ &+ Cr^{-3}\int_{t_0-r^2}^{t_0} \left\||u|^2 + 2p\right\|^2_{L^{-1,2}(B_r(x_0))} dt \end{aligned}$$

as desired. $\square$

Note that for $f \in L^{6/5}(B_r(x_0))$ and for $\varphi \in C_0^\infty(B_r(x_0))$ we have

$$\begin{aligned}
\left| \int_{B_r(x_0)} \varphi(x) f(x) dx \right| &\leq C \int_{B_r(x_0)} \left[\int_{B_r(x_0)} \frac{|\nabla \varphi(y)|}{|x-y|^2} dy \right] |f(x)| dx \\
&= C \int_{B_r(x_0)} |\nabla \varphi(y)| \left[\int_{B_r(x_0)} \frac{|f(x)| dx}{|x-y|^2} \right] dy \\
&\leq C \left\| \nabla \varphi \right\|_{L^2(B_r(x_0))} \left\| \mathbf{I}_1(\chi_{B_r(x_0)} |f|) \right\|_{L^2(B_r(x_0))}.
\end{aligned}$$

Here $\mathbf{I}_1$ is the first order Riesz's potential defined by

$$\mathbf{I}_1(\mu)(x) = c \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x-y|^2}, \qquad x \in \mathbb{R}^3,$$

for a nonnegative locally finite measure μ in $\mathbb{R}^3$. Thus we find

$$\|f\|_{L^{-1,2}(B_r(x_0))} \leq C \left\| \mathbf{I}_1(\chi_{B_r(x_0)} |f|) \right\|_{L^2(B_r(x_0))} \leq C \|f\|_{L^{\frac{6}{5}}(B_r(x_0))} \tag{2.4}$$

by the embedding property of Riesz's potentials.

Using (2.4) we obtain the following important consequence of Lemma 2.3.

Corollary 2.4. *Suppose that (u, p) is a suitable weak solution to the Navier-Stokes equations in $Q = \omega \times (a, b)$. Let $z_0 = (x_0, t_0)$ and $r > 0$ be such that $Q_r(z_0) \subset Q$. Then there holds*

$$A(z_0, r/2) + B(z_0, r/2) \leq C[C(z_0, r)^{1/2} + C(z_0, r) + D(z_0, r)].$$

3. ϵ-REGULARITY CRITERIA

As demonstrated in [5], the proof of Theorem 1.4 above relies heavily on the following ϵ-regularity criterion for suitable weak solutions to the Navier-Stokes equations (see [5, Lemma 2.2], see also [1, 18, 23]).

Proposition 3.1. *There exist positive constants ϵ_0 and C_k, $k = 0, 1, 2, \ldots$, such that the following holds. Suppose that the pair (u, p) is a suitable solution to the Navier-Stokes equations in $Q_1(z_0)$ and satisfies the smallness condition*

$$C_1(u, z_0, 1) + D_1(p, z_0, 1) \leq \epsilon_0.$$

Then $\nabla^k u$ is Hölder continuous on $\overline{Q}_{1/2}(z_0)$ for any integer $k \geq 0$, and

$$\max_{z \in \overline{Q}_{1/2}(z_0)} |\nabla^k u(z)| \leq C_k.$$

To prove Theorem 1.5 we use instead a new different version of ϵ-regularity criterion.

Proposition 3.2. *There exist positive constants ϵ_1 and C_k, $k = 0, 1, 2, \ldots$, such that the following holds. Suppose that the pair (u, p) is a suitable solution to the Navier-Stokes equations in $Q_8(z_0)$ and satisfies the smallness condition*

$$C(u, z_0, 8) + D(p, z_0, 8) \leq \epsilon_1. \tag{3.1}$$

Then $\nabla^k u$ is Hölder continuous on $\overline{Q}_{1/2}(z_0)$ for any integer $k \geq 0$, and

$$\max_{z \in \overline{Q}_{1/2}(z_0)} |\nabla^k u(z)| \leq C_k.$$

The proof of Proposition 3.2 will be given at the end of this section. It requires the following two preliminary results. The first one is by now a well-known lemma that can be found in [20, Lemma 2.1].

Lemma 3.3. *Suppose that (u, p) is a suitable weak solution to the Navier-Stokes equations in $Q = \omega \times (a, b)$. Let $z_0 = (x_0, t_0)$ and let $\rho > 0$ be such that $Q_\rho(z_0) \subset Q = \omega \times (a, b)$. For any $r \in (0, \rho]$ we have*

$$C_1(z_0, r) \leq C\Big(\frac{\rho}{r}\Big)^3 A(z_0, \rho)^{3/4} B(z_0, \rho)^{3/4} + C\Big(\frac{r}{\rho}\Big)^3 A(z_0, \rho)^{3/2}.$$

In what follows we shall use the following notation to denote the spatial average of a function f over a ball $B_r(x_0)$:

$$[f]_{x_0, r} := \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} f(x)\, dx.$$

Lemma 3.4. *Suppose that (u, p) is a suitable weak solution to the Navier-Stokes equations in $Q = \omega \times (a, b)$. Let $z_0 = (x_0, t_0)$ and let $\rho > 0$ be such that $Q_\rho(z_0) \subset Q = \omega \times (a, b)$. For any $r \in (0, \rho/4]$ we have*

$$D_1(z_0, r) \leq C\Big(\frac{\rho}{r}\Big)^{3/2} A(z_0, \rho)^{3/4} B(z_0, \rho)^{3/4} + C\Big(\frac{r}{\rho}\Big)^{3/2} D(z_0, \rho)^{3/4}.$$

Proof. Let $h_{x_0,\rho} = h_{x_0,\rho}(\cdot, t)$ be a function on $B_\rho(x_0)$ for a.e. t such that

$$h_{x_0,\rho} = p - \tilde{p}_{x_0,\rho} \quad \text{in } B_\rho(x_0),$$

where $\tilde{p}_{x_0,\rho}$ is defined by

$$\tilde{p}_{x_0,\rho} = R_i R_j[(u_i - [u_i]_{x_0,\rho})(u_j - [u_j]_{x_0,\rho})\chi_{B_\rho(x_0)}].$$

Here $R_i = D_i(-\Delta)^{-\frac{1}{2}}$, $i = 1, 2, 3$, is the i-th Riesz transform. Note that for any $\varphi \in C_0^\infty(B_\rho(x_0))$, we have

$$\begin{aligned} -\int_{B_\rho(x_0)} \tilde{p}_{x_0,\rho} \Delta\varphi dx &= \int_{B_\rho(x_0)} (u_i - [u_i]_{x_0,\rho})(u_j - [u_j]_{x_0,\rho}) D_{ij}\varphi\, dx \\ &= \int_B u_i u_j D_{ij}\varphi\, dx, \end{aligned}$$

which follows from the properties $-R_i R_j(\Delta\varphi) = D_{ij}\varphi$ and $\operatorname{div} u = 0$. Thus, as p also solves

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes u)$$

in the distributional sense, we see that $h_{x_0,\rho}$ is harmonic in $B_\rho(x_0)$ for a.e. t.

With this decomposition of the pressure p, we have

$$\begin{aligned}\int_{B_r(x_0)}|p(x,t)|^{3/2}dx &= C\int_{B_r(x_0)}|\tilde{p}_{x_0,\rho}+h_{x_0,\rho}|^{3/2}dx\\ &\leq C\int_{B_\rho(x_0)}|\tilde{p}_{x_0,\rho}|^{3/2}dx+C\int_{B_r(x_0)}|h_{x_0,\rho}|^{3/2}dx.\end{aligned}$$

Next, as $h_{x_0,\rho}$ is harmonic in $B_\rho(x_0)$, for $r\in(0,\rho/4]$ there holds

$$\begin{aligned}\Big(\fint_{B_r(x_0)}|h_{x_0,\rho}|^{3/2}dx\Big)^{2/3} &\leq \Big(\fint_{B_r(x_0)}|h_{x_0,\rho}|^{2}dx\Big)^{1/2}\\ &\leq C\Big(\fint_{B_{\rho/4}(x_0)}|h_{x_0,\rho}|^{2}dx\Big)^{1/2}\\ &\leq C\Big(\fint_{B_{\rho/2}(x_0)}|h_{x_0,\rho}|^{6/5}dx\Big)^{5/6}.\end{aligned}$$

This gives

$$\begin{aligned}\int_{B_r(x_0)}|p(x,t)|^{3/2}dx &\leq C\int_{B_\rho(x_0)}|\tilde{p}_{x_0,\rho}|^{3/2}dx\\ &\quad+C\frac{r^3}{\rho^{15/4}}\Big(\int_{B_{\rho/2}(x_0)}|h_{x_0,\rho}|^{6/5}dx\Big)^{5/4}.\end{aligned}$$

Thus using $h_{x_0,\rho}=p-\tilde{p}_{x_0,\rho}$ again we find

$$\begin{aligned}\int_{B_r(x_0)}|p(x,t)|^{3/2}dx &\leq C\int_{B_\rho(x_0)}|\tilde{p}_{x_0,\rho}|^{3/2}dx\\ &\quad+C\frac{r^3}{\rho^{15/4}}\Big(\int_{B_{\rho}(x_0)}|\tilde{p}_{x_0,\rho}|^{6/5}dx\Big)^{5/4}\\ &\quad+C\frac{r^3}{\rho^{15/4}}\Big(\int_{B_{\rho}(x_0)}|p|^{6/5}dx\Big)^{5/4}.\end{aligned}$$

By Hölder's inequality this yields

$$\begin{aligned}\int_{B_r(x_0)}|p(x,t)|^{3/2}dx &\leq C\Big[1+\Big(\frac{r}{\rho}\Big)^3\Big]\int_{B_\rho(x_0)}|\tilde{p}_{x_0,\rho}|^{3/2}dx\\ &\quad+C\frac{r^3}{\rho^{15/4}}\Big(\int_{B_{\rho}(x_0)}|p|^{6/5}dx\Big)^{5/4}.\end{aligned}\tag{3.2}$$

On the other hand, by the Calderón-Zygmund estimate and a Sobolev interpolation inequality (see, e.g., (1.1) of [18]) we find

$$
\begin{aligned}
\int_{B_\rho(x_0)} |\tilde{p}_{x_0,\rho}|^{3/2}dx \le C \int_{B_\rho(x_0)} |u-[u]_{x_0,\rho}|^3 dx \\
\le \; C\Big(\int_{B_\rho(x_0)} |\nabla u|^2 dx\Big)^{3/4}\Big(\int_{B_\rho(x_0)} |u-[u]_{x_0,\rho}|^2 dx\Big)^{3/4} \\
\le \; C\Big(\int_{B_\rho(x_0)} |\nabla u|^2 dx\Big)^{3/4}\Big(\int_{B_\rho(x_0)} |u|^2 dx\Big)^{3/4},
\end{aligned} \tag{3.3}
$$

where we used the bound

$$
\int_{B_r(x_0)} |u-[u]_{x_0,r}|^2 dx \le \int_{B_r(x_0)} |u|^2 dx
$$

in the last inequality.

Combining (3.2), (3.3) and using $r/\rho \le 1/4$ we have

$$
\begin{aligned}
\int_{B_r(x_0)} |p(x,t)|^{3/2}dx \;\; &\le \;\; C\Big(\int_{B_\rho(x_0)} |\nabla u|^2 dx\Big)^{3/4}\Big(\int_{B_\rho(x_0)} |u|^2 dx\Big)^{3/4} \\
&\quad + C\frac{r^3}{\rho^{15/4}}\Big(\int_{B_\rho(x_0)} |p|^{6/5} dx\Big)^{5/4}.
\end{aligned}
$$

Integrating the last bound with respect to dt/r^2 over the interval $(t_0 - r^2, t_0)$ and using Hölder's inequality we obtain

$$
D_1(z_0,r) \;\; \le \;\; C\Big(\frac{\rho}{r}\Big)^{3/2} A(z_0,\rho)^{3/4}B(z_0,\rho)^{3/4} + C\Big(\frac{r}{\rho}\Big)^{3/2} D(z_0,\rho)^{3/4}
$$

as desired. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 3.3 and Corollary 2.4 we have

$$
\begin{aligned}
C_1(z_0,1) \;\; &\le \;\; CA(z_0,1)^{3/4}B(z_0,1)^{3/4} + CA(z_0,1)^{3/2} \\
&\le \;\; C[A(z_0,1)+B(z_0,1)]^{3/2} \\
&\le \;\; C[C(z_0,2)^{1/2} + C(z_0,2) + D(z_0,2)]^{3/2}.
\end{aligned}
$$

Thus using (3.1) we find

$$
C_1(z_0,1) \;\; \le \;\; C(\epsilon_1^{1/2} + \epsilon_1)^{3/2}. \tag{3.4}
$$

On the other hand, using Lemma 3.4 with $r=1$ and $\rho=4$ there holds

$$
D_1(z_0,1) \le C[A(z_0,4)+B(z_0,4)]^{3/2} + CD(z_0,4)^{3/4},
$$

which by Corollary 2.4 and (3.1) yields

$$
\begin{aligned}
D_1(z_0,1) \;\; &\le \;\; C[C(z_0,8)^{1/2} + C(z_0,8) + D(z_0,8)]^{3/2} + CD(z_0,8)^{3/4} \\
&\le \;\; C\epsilon_1^{3/2} + C\epsilon_1^{3/4}.
\end{aligned}
$$

Now choosing ϵ_1 sufficiently small in (3.4) and the last bound, we can make

$$C_1(z_0, 1) + D_1(z_0, 1) \leq \epsilon_0,$$

and thus Lemma 3.1 implies the desired regularity result. $\square$

4. Proof of Theorems 1.5 and 1.6

This section is devoted to the proof of Theorems 1.5 and 1.6. We shall need the following lemma.

Lemma 4.1. *Suppose that the pair of functions (u, p) satisfies the Navier-Stokes equations in $Q_1(0,0) = B_1(0) \times (-1, 0)$ in the sense of distributions and has the properties (1.6), (1.8), and (1.9) for some $q \in (3, \infty]$. Then (u, p) forms a suitable solution to the Navier-Stokes equations in $Q_{5/6}$ with a generalized energy equality, $u \in L^4(Q)$, and $p \in L^2(Q_{5/6})$. Moreover, the inequality*

$$\|u(\cdot, t)\|_{L^{3,q}(B_{3/4})} \leq \|u\|_{L^\infty(-(3/4)^2, 0; L^{3,q}(B_{3/4}))} \tag{4.1}$$

holds for all $t \in [-(3/4)^2, 0]$, *and the function*

$$t \to \int_{B_{3/4}} u(x,t) w(x) dx$$

is continuous on $[-(3/4)^2, 0]$ for any $w \in L^{3/2, q/(q-1)}(B_{3/4})$. Here it is understood as usual that $q/(q-1) = 1$ in the case $q = \infty$.

Proof. By Sobolev inequality we have $u \in L^2(-1, 0; L^6(B_1))$, which using (1.9) and the interpolative inequality

$$\|u(\cdot, t)\|_{L^4(B_1)} \leq C \|u(\cdot, t)\|^{\frac{1}{2}}_{L^{3,q}(B_1)} \|u(\cdot, t)\|^{\frac{1}{2}}_{L^6(B_1)}$$

yields

$$u \in L^4(Q). \tag{4.2}$$

Thus by Hölder's inequality the nonlinear term

$$u \cdot \nabla u \in L^{4/3}(Q). \tag{4.3}$$

As above, we have a decomposition

$$p = \tilde{p} + h,$$

where $\tilde{p} = R_i R_j [(u_i u_j) \chi_{B_1}]$, and h is harmonic in B_1. By Calderón-Zygmund estimate we have

$$\|\tilde{p}\|_{L^2(-1,0;L^2(B_1))} \leq C \|u\|^2_{L^4(-1,0;L^4(B_1))} = C \|u\|^2_{L^4(Q)}, \tag{4.4}$$

and by harmonicity and assumption (1.8) there holds

$$\begin{aligned} \|h\|_{L^2(-1,0;L^\infty(B_{5/6}))} &\leq C \|h\|_{L^2(-1,0;L^1(B_1))} \\ &= C \|p - \tilde{p}\|_{L^2(-1,0;L^1(B_1))} \\ &\leq C \Big[\|p\|_{L^2(-1,0;L^1(B_1))} + \|u\|^2_{L^4(Q)} \Big]. \end{aligned} \tag{4.5}$$

Estimates (4.4)–(4.5) imply in particular that the pressure

$$p \in L^2(Q_{5/6}). \tag{4.6}$$

Using the inclusions (1.6), (4.2), (4.3), (4.6), and the local interior regularity of non-stationary Stokes systems we eventually find

$$\int_{Q_{3/4}} (|u|^4 + |\partial_t u|^{4/3} + |\nabla^2 u|^{4/3} + |\nabla p|^{4/3}) dxdt < +\infty.$$

It then follows that

$$u \in C(-[(3/4)^2, 0]; L^{4/3}(B_{3/4}))$$

and thus the function

$$g_\varphi(t) := \int_{B_{3/4}} u(x,t)\varphi(x)dx$$

is continuous on $[-(3/4)^2, 0]$ for any $\varphi \in C_0^\infty(B_{3/4})$. This yields

$$\Big| \int_{B_{3/4}} u(x,t)\varphi(x)dx \Big| \leq C \, \|\varphi\|_{L^{3/2,q/(q-1)}(B_{3/4})} \, \|u\|_{L^\infty(-(3/4)^2,0;L^{3,q}(B_{3/4}))}$$

for *any* $t \in [-(3/4)^2, 0]$ and any $\varphi \in C_0^\infty(B_{3/4})$. Thus by the density of $C_0^\infty(B_{3/4})$ in $L^{3/2,q/(q-1)}(B_{3/4})$ we see that

$$\|u(\cdot,t)\|_{L^{3,q}(B_{3/4})} \leq C \, \|u\|_{L^\infty(-(3/4)^2,0;L^{3,q}(B_{3/4}))}$$

for any $t \in [-(3/4)^2, 0]$. Then it can be seen, again by density, that the function $g_\varphi(t)$ above is actually continuous on $[-(3/4)^2, 0]$ for any $\varphi \in L^{3/2,q/(q-1)}(B_{3/4})$.

Finally, using (4.2) and a standard mollification in $\mathbb{R}^{3+1}$ combined with a truncation in time of test functions, we obtain the local generalized energy equality in $Q_{5/6}$. □

We now proceed with the proof of Theorem 1.5.

Proof of Theorem 1.5. Henceforth, let the hypothesis of Theorem 1.5 be enforced. Notice that by Lemma 4.1 (u, p) forms a suitable weak solution to the Navier-Stokes equations in $Q_{5/6}(0, 0)$. As in [5], the proof of Theorem 1.5 goes by a contradiction. Suppose that $z_0 = (x_0, t_0) \in \overline{Q}_{1/2}(0, 0)$ is a singular point. By definition, this means that there exists no neighborhood $\mathcal{N}$ of z_0 such that u has a Hölder continuous representative on $\mathcal{N} \cap B_1(0) \times (-1, 0])$. By Lemma 3.3 of [29], there exist $c_0 > 0$ and a sequence of numbers $\epsilon_k \in (0, 1)$ such that $\epsilon_k \to 0$ as $k \to +\infty$ and

$$A(z_0, \epsilon_k) = \sup_{t_0 - \epsilon_k^2 \leq s \leq t_0} \frac{1}{\epsilon_k} \int_{B(x_0,\epsilon_k)} |u(x,s)|^2 dx \geq c_0 \tag{4.7}$$

for any $k \in \mathbb{N}$. Moreover, by Lemma 4.1 we have in particular

$$u(\cdot, t_0) \in L^{3,q}(B_{3/4}(0)). \tag{4.8}$$

Recall that we can decompose

$$p = \tilde{p} + h,$$

where h is harmonic in B_1, and $\tilde{p} = R_i R_j[(u_i u_j)\chi_{B_1}]$.

For each $Q = \omega \times (a, b)$, where $\omega \Subset \mathbb{R}^3$ and $-\infty < a < b \leq 0$, we choose a large $k_0 = k_0(Q) \geq 1$ so that for any $k \geq k_0$ there hold the implications

$$x \in \omega \Longrightarrow x_0 + \epsilon_k x \in B_{2/3},$$

and

$$t \in (a, b) \Longrightarrow t_0 + \epsilon_k^2 t \in (-(2/3)^2, 0),$$

where the sequence $\{\epsilon_k\}$ is as in (4.7).

Given such a $Q = \omega \times (a, b)$, let us set

$$u_k(x, t) = \epsilon_k u(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad p_k(x, t) = \epsilon_k^2 p(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t),$$

and

$$\tilde{p}_k(x, t) = \epsilon_k^2\, \tilde{p}(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t), \quad h_k(x, t) = \epsilon_k^2\, h(x_0 + \epsilon_k x, t_0 + \epsilon_k^2 t)$$

for any $(x, t) \in Q$ and $k \geq k_0(Q)$.

The following proposition provides a non-trivial *ancient solution* (see [32] for this notion) that is essential in the proof of Theorem 1.5.

Proposition 4.2. (i) *There exist subsequence of* (u_k, p_k)*, still denoted by* (u_k, p_k)*, and a pair of functions*

$$(u_\infty, p_\infty) \in L^\infty(-\infty, 0; L^{3,q}(\mathbb{R}^3) \times L^\infty(-\infty, 0; L^{3/2,q/2}(\mathbb{R}^3), \tag{4.9}$$

with $\operatorname{div} u_\infty = 0$ *in* $\mathbb{R}^3 \times (-\infty, 0)$*, such that*

$$u_k \to u_\infty \quad \text{in} \quad C([a, b]; L^s(\omega)), \tag{4.10}$$

$$\tilde{p}_k \to p_\infty \quad \text{weakly}^* \text{ in} \quad L^\infty(a, b; L^{3/2,q/2}(\omega),$$

for any $s \in (1, 3)$*, and any* $\omega \Subset \mathbb{R}^3$*,* $-\infty < a < b \leq 0$*.*

(ii) *Moreover, for any* $Q = \omega \times (a, b)$ *with* $\omega \Subset \mathbb{R}^3$*,* $-\infty < a < b \leq 0$*,*

$$|u_\infty|^2, \nabla u_\infty \in L^2(Q), \quad \partial_t u_\infty, \nabla^2 u_\infty, \nabla p_\infty \in L^{4/3}(Q),$$

and (u_∞, p_∞) *forms a suitable weak solution of the Navier-Stokes equations in any such* Q*.*

(iii) *Additionally,* u_∞ *satisfies the lower bound*

$$\sup_{t \in [-1, 0]} \int_{B_1(0)} |u_\infty(x, t)|^2 dx \geq c_0, \tag{4.11}$$

where $c_0 > 0$ *is the constant in* (4.7)*.*

Proof. For each $Q = \omega \times (a, b)$, where $\omega \Subset \mathbb{R}^3$, $-\infty < a < b \leq 0$, and for every $t \in [a, b]$ we have

$$\|u_k(\cdot, t)\|_{L^{3,q}(\omega)} \leq \left\|u(\cdot, t_0 + \epsilon_k^2 t)\right\|_{L^{3,q}(B_{3/4})} \leq \|u\|_{L^\infty(-1,0;L^{3,q}(B_1))}. \tag{4.12}$$

By Calderón-Zygmund estimate, for a.e. $t \in (a,b)$ there holds

$$\begin{aligned}
(4.13)\qquad \|\tilde{p}_k(\cdot,t)\|_{L^{3/2,q/2}(\omega)} &\leq \left\|\tilde{p}(\cdot,t_0+\epsilon_k^2 t)\right\|_{L^{3/2,q/2}(B_{3/4})} \\
&\leq \operatorname*{ess\,sup}_{t'\in(-(3/4)^2,0)} \left\|\tilde{p}(\cdot,t')\right\|_{L^{3/2,q/2}(B_{3/4})} \\
&\leq C\,\|u\|^2_{L^\infty(-1,0;L^{3,q}(B_1))}.
\end{aligned}$$

On the other hand, by harmonicity we have

$$\begin{aligned}
\int_a^b \sup_{x\in\omega}|h_k(x,t)|^2 dt &\leq \epsilon_k^2 \int_{-(3/4)^2}^0 \sup_{x\in\omega}|h(x_0+\epsilon_k x,s)|^2 ds \\
&\leq \epsilon_k^2\,\|h\|^2_{L^2(-1,0;L^\infty(B_{3/4}))} \\
&\leq C\,\epsilon_k^2\,\|h\|^2_{L^2(-1,0;L^1(B_{5/6}))}
\end{aligned}$$

provided $k \geq k_0(Q)$. Thus again by Calderón-Zygmund estimate we find

$$\begin{aligned}
(4.14)\qquad &\int_a^b \sup_{x\in\omega}|h_k(x,t)|^2 dt \\
&\leq C\,\epsilon_k^2\,\|p-\tilde{p}\|^2_{L^2(-1,0;L^1(B_{5/6}))} \\
&\leq C\,\epsilon_k^2\Big[\,\|p\|^2_{L^2(-1,0;L^1(B_1))} + \|u\|^4_{L^\infty(-1,0;L^{3,q}(B_1))}\Big].
\end{aligned}$$

Using the last estimates for $\tilde{p}_k$ and h_k and Hölder's inequality we have the following uniform bound for p_k:

$$\begin{aligned}
(4.15)\qquad \|p_k\|_{L^2(a,b;L^{6/5}(\omega))} &\leq \|\tilde{p}_k\|_{L^2(a,b;L^{6/5}(\omega))} + \|h_k\|_{L^2(a,b;L^{6/5}(\omega))} \\
&\leq C(Q)\Big[\,\|p\|_{L^2(-1,0;L^1(B_1))} + \|u\|^2_{L^\infty(-1,0;L^{3,q}(B_1))}\Big]
\end{aligned}$$

for any $k \geq k_0(Q)$. Here the constant $C(Q)$ is independent of such k.
With regard to u_k, with $k \geq k_0(Q)$, we have

$$\begin{aligned}
(4.16)\qquad \|u_k\|_{L^4(a,b;L^{12/5}(\omega))} &\leq C(Q)\,\|u_k\|_{L^\infty(a,b;L^{3,q}(\omega))} \\
&\leq C(Q)\,\|u\|_{L^\infty(-1,0;L^{3,q}(B_1))}.
\end{aligned}$$

For each $\varphi \in C_0^\infty(\mathbb{R}^3\times\mathbb{R})$ that vanishes in a neighborhood of the parabolic boundary $\partial' Q = \omega\times\{t=a\}\cup\partial\omega\times[a,b]$, we define

$$\varphi_k(x,t) = \epsilon_k^{-1}\varphi(\epsilon_k^{-1}(x-x_0),\epsilon_k^{-2}(t-t_0)).$$

Then with $k \geq k_0(Q)$ we see that φ_k vanishes in a neighborhood of the parabolic boundary of $Q_{3/4}(0,0)$. Using φ_k as a test function in the generalized energy equality for (u,p) at $t = t_0 + \epsilon_k^2\tau$ with a.e. $\tau \in (a,b)$ we find

$$\begin{aligned}\int_{B_{3/4}} |u(x,t)|^2\varphi_k(x,t)dx + 2\int_{-(3/4)^2}^{t}\int_{B_{3/4}} |\nabla u|^2\varphi_k(x,s)dxds\\ = \int_{-(3/4)^2}^{t}\int_{B_{3/4}} |u|^2(\partial_t\phi_k + \Delta\phi_k)dxds\\ + \int_{-(3/4)^2}^{t}\int_{B_{3/4}} (|u|^2+2p)u\cdot\nabla\varphi_k dxds.\end{aligned}$$

Hence by making a change of variables we obtain

$$\begin{aligned}\int_\omega |u_k(y,\tau)|^2\varphi(y,\tau)dy + 2\int_a^\tau\int_\omega |\nabla u_k|^2\varphi(y,s')dyds'\\ = \int_a^\tau\int_\omega |u_k|^2(\phi_t+\Delta\phi)dyds' + \int_a^\tau\int_\omega (|u_k|^2+2p_k)u_k\cdot\nabla\varphi dyds'\end{aligned}$$

for a.e. $\tau \in (a,b)$.

Thus each u_k is a suitable solution in Q for any $Q = \omega\times(a,b)$, with $\omega \Subset \mathbb{R}^3$ and $-\infty < a < b \le 0$, and any $k \ge k_0(Q)$. Then, given such a Q, it follows from Corollary 2.4 and inequalities (4.15)–(4.16) (applied to an appropriate enlargement of Q) that

$$\|u_k\|_{L^\infty(a,b;L^2(\omega))} + \|\nabla u_k\|_{L^2(a,b;L^2(\omega))} \le C(Q) \tag{4.17}$$

for all sufficiently large k depending only on Q.

Using (4.17) and Sobolev inequality we have

$$\|u_k\|_{L^2(a,b;L^6(\omega))} \le C(Q),$$

which by (4.12), interpolation, and Hölder's inequality gives

$$\|u_k\|_{L^4(\omega\times(a,b))} + \|u_k\cdot\nabla u_k\|_{L^{4/3}(\omega\times(a,b))} \le C. \tag{4.18}$$

From the bounds (4.13) and (4.14) for $\tilde{p}_k$ and h_k we also have

$$\|p_k\|_{L^s(\omega\times(a,b))} \le C(Q,s)\,\|p_k\|_{L^2(a,b;L^{3/2,q/2}(\omega))} \le C \tag{4.19}$$

for any $s \in (0,3/2)$.

Using (4.17)–(4.19), it follows from the local interior regularity of solutions to non-stationary Stokes equations we find

$$\|\partial_t u_k\|_{L^{4/3}(\omega\times(a,b))} + \left\|\nabla^2 u_k\right\|_{L^{4/3}(\omega\times(a,b))} + \|\nabla p_k\|_{L^{4/3}(\omega\times(a,b))} \le C \tag{4.20}$$

for all sufficiently large k depending only on Q.

At this point, using (4.12)–(4.13) and a diagonal process we may assume that

$$u_k \to u_\infty \quad \text{weakly}^* \text{ in} \quad L^\infty(a,b;L^{3,q}(\omega)$$
$$\tilde{p}_k \to p_\infty \quad \text{weakly}^* \text{ in} \quad L^\infty(a,b;L^{3/2,q/2}(\omega)),$$

for a pair of functions (u_∞, p_∞) satisfying (4.9), with $\operatorname{div} u_\infty = 0$ in $\mathbb{R}^3 \times (-\infty, 0)$.

Estimates (4.17) and (4.20) now yield

$$u_k \to u_\infty \quad \text{in} \quad C([a,b];L^{4/3}(\omega)). \tag{4.21}$$

For any $s \in (1,3)$, the uniform bound (4.12), and the interpolation inequality

$$\begin{aligned}&\left\|u_k(\cdot,t)-u_k(\cdot,t')\right\|_{L^s(\omega)}\\ &\leq \; C(s)\left\|u_k(\cdot,t)-u_k(\cdot,t')\right\|_{L^{4/3}(\omega)}^{\frac{12}{5}\left(\frac{1}{s}-\frac{1}{3}\right)}\left\|u_k(\cdot,t)-u_k(\cdot,t')\right\|_{L^{3,q}(\omega)}^{\frac{12}{5}\left(\frac{3}{4}-\frac{1}{s}\right)}\end{aligned}$$

imply that each $u_k \in C([a,b];L^s(\omega)$. Thus by using (4.21) and interpolating we obtain (4.10) for any $s \in (1,3)$. This completes the proof of (i).

On the other hand, by (4.14) we have

$$h_k \to 0 \quad \text{strongly in} \quad L^2(a,b;L^\infty(\omega)),$$

for any $-\infty < a < b \leq 0$ and $\omega \Subset \mathbb{R}^3$, and thus in the limit (u_∞, p_∞) satisfies the Navier-Stokes equations in the sense of distributions in $\omega \times (a,b)$. Now (ii) follows from (i), (4.17) and (4.20) via an argument as in the proof of Lemma 4.1.

Finally, note that by (4.7) and a change of variables we have

$$\sup_{-1\leq t\leq 0}\int_{B(0,1)}|u_k(x,t)|^2dx = \sup_{t_0-\epsilon_k^2\leq s\leq t_0}\frac{1}{\epsilon_k}\int_{B(x_0,\epsilon_k)}|u(y,s)|^2dy \geq c_0.$$

Thus using (4.10) with $s=2$ we obtain (4.11), which proves (iii). □

We now continue with the proof of Theorem 1.5. By (i) of Proposition 4.2, we have

$$\int_{-M}^0 (\|u_\infty(\cdot,t)\|^4_{L^{3,q}(\mathbb{R}^3)} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(\mathbb{R}^3)})dt < +\infty$$

for any real number $M>0$. Note that for a.e. t,

$$\|u_\infty(\cdot,t)\|^4_{L^{3,q}(\mathbb{R}^3\setminus \overline{B}_R(0))} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(\mathbb{R}^3\setminus \overline{B}_R(0))} \to 0$$

as $R \to +\infty$. We thus have

$$\int_{-M}^0 (\|u_\infty(\cdot,t)\|^4_{L^{3,q}(\mathbb{R}^3\setminus \overline{B}_R(0))} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(\mathbb{R}^3\setminus \overline{B}_R(0))})dt \to 0$$

as $R\to+\infty$. This yields that for any $M>200$ there exists $N = N(\epsilon_1, M) > 10$ such that

$$\int_{-M}^0 (\|u_\infty(\cdot,t)\|^4_{L^{3,q}(\mathbb{R}^3\setminus \overline{B}_N(0))} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(\mathbb{R}^3\setminus \overline{B}_N(0))})dt \leq \epsilon_2,$$

where

$$\epsilon_2 = 8^2|B_1(0)|^{-1/3}\epsilon_1 \tag{4.22}$$

with ϵ_1 being as found in Proposition 3.2.

We now fix such numbers M and N and consider any z_1 that

$$z_1 = (x_1,t_1) \in (\mathbb{R}^3 \setminus \overline{B}_{2N}(0)) \times (-M/2, 0].$$

Then there holds

$$Q_8(z_1) = B_8(x_1) \times (t_1 - 8^2, t_1) \subset (\mathbb{R}^3 \setminus \overline{B}_N(0)) \times (-M, 0],$$

and hence

$$\int_{t_1-8^2}^{t_1} (\|u_\infty(\cdot,t)\|^4_{L^{3,q}(B_8(x_1))} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(B_8(x_1))})dt \leq \epsilon_2. \tag{4.23}$$

Since

$$\begin{aligned} &\|u_\infty(\cdot,t)\|^4_{L^{12/5}(B_8(x_1))} + \|p_\infty(\cdot,t)\|^2_{L^{6/5}(B_8(x_1))} \\ &\leq \ |B_8(x_1)|^{1/3} \left\{ \|u_\infty(\cdot,t)\|^4_{L^{3,q}(B_8(x_1))} + \|p_\infty(\cdot,t)\|^2_{L^{3/2,q/2}(B_8(x_1))} \right\} \end{aligned}$$

we see from (4.22)–(4.23) that

$$C(u_\infty, z_1, 8) + D(p_\infty, z_1, 8) \leq \epsilon_1. \tag{4.24}$$

The smallness property (4.24) and Proposition 3.2 now yield that $\nabla^k u_\infty$, $k = 0, 1, 2, \ldots$, is Hölder continuous on $(\mathbb{R}^3 \setminus \overline{B}_{2N}(0)) \times (-M/2, 0]$, and

$$\max_{z \in \overline{Q}_{1/2}(z_1)} |\nabla^k u_\infty(z)| \leq C_k. \tag{4.25}$$

Let $\omega_\infty = \operatorname{curl} u_\infty$ be the vorticity of u_∞. Then ω_∞ satisfies the equation

$$\partial_t \omega_\infty - \Delta\omega_\infty + (u_\infty \cdot \nabla)\omega_\infty - (\omega_\infty \cdot \nabla)u_\infty = 0$$

on the set $(\mathbb{R}^3 \setminus \overline{B}_{4N}(0)) \times (-M/4, 0]$, which by (4.25) gives

$$|\partial_t \omega_\infty - \Delta\omega_\infty| \leq C(|\omega_\infty| + |\nabla\omega_\infty|) \tag{4.26}$$

with

$$|\omega_\infty| \leq C < +\infty \tag{4.27}$$

on the set $(\mathbb{R}^3 \setminus \overline{B}_{4N}(0)) \times (-M/4, 0]$, for a universal constant $C > 0$.

We now claim that

$$\omega_\infty = 0 \quad \text{on } (\mathbb{R}^3 \setminus \overline{B}_{4N}(0)) \times (-M/4, 0]. \tag{4.28}$$

To see this, by applying the backward uniqueness theorem (see [5, Theorem 5.1] and [6]) and the bounds (4.26)–(4.27), it is enough to show that

$$\omega_\infty(y, 0) = 0 \quad \text{for all } y \in \mathbb{R}^3 \setminus \overline{B}_{4N}(0)). \tag{4.29}$$

Note that for any $y \in \mathbb{R}^3$ we have

$$\begin{aligned} &\int_{B_1(y)} |u_\infty(x,0)|dx \\ &\leq \ \int_{B_1(y)} |u_\infty(x,0) - u_k(x,0)|dx + \int_{B_1(y)} |u_k(x,0)|dx \\ &\leq \ \int_{B_1(y)} |u_\infty(x,0) - u_k(x,0)|dx + |B_1(0)|^{\frac{2}{3}} \|u_k(\cdot,0)\|_{L^{3,q}(B_1(y))} \\ &\leq \ \|u_\infty - u_k\|_{C([-M/4,0];L^1(B_1(y)))} + |B_1(0)|^{\frac{2}{3}} \|u(\cdot,t_0)\|_{L^{3,q}(B_{\epsilon_k}(x_0+\epsilon_k y))}. \end{aligned}$$

Thus sending $k \to +\infty$ we see that

$$\int_{B_1(y)} |u_\infty(x,0)| dx = 0$$

for all $y \in \mathbb{R}^3$, which yields (4.29) as desired. Here we have used (i) of Proposition 4.2 and (4.8).

At this point using (4.28) combined with the argument on pages 227–229 of [5], which ultimately employs the theory of unique continuation for parabolic inequalities, we see that in fact

$$\omega_\infty(\cdot, t) = 0 \quad \text{in the whole } \mathbb{R}^3$$

for a.e. $t \in (-M/4, 0)$. Thus $u_\infty(\cdot, t)$ is globally harmonic and by a Liouville theorem it follows that $u_\infty(\cdot, t) = 0$ for a.e. $t \in (-M/4, 0)$. This leads to a contradiction to the lower bound (4.11) and hence completes the proof of Theorem 1.5.

We next prove Theorem 1.6.

Proof of Theorem 1.6. Arguing as in the proof of Lemma 4.1 we see that (u, p) forms a suitable solution to the Navier-Stokes equations in $Q_{5/6}$.

Suppose that $(x_0, t_0) \in \overline{Q}_{1/2}(0,0)$ is a singular point. Then we must have that $x_0 = 0$. By Lemma 3.3 of [29], there exist $c_0 > 0$ and a sequence of numbers $\epsilon_k \in (0, 1/8)$ such that $\epsilon_k \to 0$ as $k \to +\infty$ and

$$\sup_{t_0 - \epsilon_k^2 \le s \le t_0} \frac{1}{\epsilon_k} \int_{B(0,\epsilon_k)} |u(x,s)|^2 dx \ge c_0.$$

Using a change of variables and the condition (1.10), we then have

$$\begin{aligned} 0 < c_0 &\le \sup_{-1 \le t \le 0} \int_{B(0,1)} |\epsilon_k u(\epsilon_k y, t_0 + \epsilon_k^2 t)|^2 dy \\ &\le \|f\|^2_{L^\infty((-1,0))} \int_{B_1(0)} |y|^{-2} g(\epsilon_k y)^2 dy. \end{aligned}$$

By the property of g, this is impossible to hold for all $k \in \mathbb{N}$, and thus the proof of Theorem 1.6 is complete. $\square$

5. Proof of Theorems 1.7

We shall prove Theorems 1.7 in this section. First observe that under the hypothesis of Theorem 1.7, we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^4(\mathbb{R}^3)} &\le C \|u(\cdot,t)\|^{\frac{1}{2}}_{L^{3,q}(\mathbb{R}^3)} \|u(\cdot,t)\|^{\frac{1}{2}}_{L^6(\mathbb{R}^3)} \\ &\le C \|u(\cdot,t)\|^{\frac{1}{2}}_{L^{3,q}(\mathbb{R}^3)} \|\nabla u(\cdot,t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Thus,

$$u \in L^4(Q_T) \quad \text{and} \quad u \cdot \nabla u \in L^{4/3}(Q_T), \tag{5.1}$$

where the latter follows from Hölder's inequality. Using these inclusions, the coercive estimates (see [9, 22, 36]) and the uniqueness theorem (see [16]) for the Stokes problem, we can introduce the *associated pressure* p such that

$$\partial_t u, \nabla^2 u, \nabla p \in L^{4/3}(\mathbb{R}^3 \times (\delta, T))$$

for any $\delta \in (0, T)$. Moreover, it follows from the pressure equation and the global condition (1.11) that

$$p \in L^\infty(0, T; L^{3/2, q/2}(\mathbb{R}^3)). \tag{5.2}$$

Arguing as in the proof of Lemma 4.1 we see that (u, p) forms a suitable weak solution in any bounded cylinder of Q_T. By (1.11) and (5.2), we have

$$\int_0^T (\|u(\cdot, t)\|^4_{L^{3,q}(\mathbb{R}^3)} + \|p(\cdot, t)\|^2_{L^{3/2,q/2}(\mathbb{R}^3)})dt \leq C(T) < +\infty. \tag{5.3}$$

We next fix a $\delta \in (0, T)$ and set $r_0 = \sqrt{\delta/128}$. Then using (5.3) we find a large number $R = R(T, \delta) > 0$ such that

$$C(u, z_0, 8r_0) + D(p, z_0, 8r_0) \leq \epsilon_1$$

for any $z_0 = (x_0, t_0) \in \mathbb{R}^3 \setminus B_R(0) \times [\delta, T]$. Thus by scaling and Proposition 3.2, there holds

$$\sup_{\mathbb{R}^3 \setminus B_R(0) \times [\delta, T]} |u| \leq C(\delta).$$

On the other hand, for any $z_0 = (x_0, t_0) \in \overline{B}_R(0) \times [\delta, T]$ and with r_0 as above, (u, p) is a suitable solution in $Q_{r_0}(z_0)$. Thus by scaling, Theorem 1.5, and the compactness of $\overline{B}_R(0) \times [\delta, T]$, we have

$$\sup_{\overline{B}_R(0) \times [\delta, T]} |u| \leq C(\delta).$$

Combining the last two bounds we obtain

$$\sup_{\mathbb{R}^3 \times [\delta, T]} |u| \leq C(\delta) < +\infty$$

which holds for any $\delta \in (0, T)$. Thus u is smooth on $\mathbb{R}^3 \times (0, T]$, and using $u \in L^4(Q_T)$ (see (5.1)) and interpolation we see that $u \in L^5(\mathbb{R}^3 \times (\delta, T))$ for any $\delta \in (0, T)$.

On the other hand, if $a \in \dot{J} \cap L^3$ then by local strong solvability and weak-strong uniqueness $u \in L^5(\mathbb{R}^3 \times (0, \delta_0))$ for some $\delta_0 > 0$ (see, e.g., [5, Theorem 7.4] and [12]). Thus we conclude that $u \in L^5(\mathbb{R}^3 \times (0, T))$ and hence by Theorem 1.2 it is unique in Q_T.

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