A SUBLINEAR SOBOLEV INEQUALITY FOR p-SUPERHARMONIC FUNCTIONS

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ABSTRACT. We establish a "sublinear" Sobolev inequality of the form

$$\left(\int_{\mathbb{R}^n} u^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \leq C \left(\int_{\mathbb{R}^n} |Du|^q dx\right)^{\frac{1}{q}}$$

for all global p-superharmonic ($1 < p < 2$) functions u in $\mathbb{R}^n$, $n \geq 2$, with $\inf_{\mathbb{R}^n} u = 0$ and $p - 1 < q < 1$. The same result also holds for the class of $\mathcal{A}$-superharmonic functions. More general sublinear trace inequalities, where Lebesgue measure is replaced by a general measure, are also considered.

1. Introduction

A celebrated inequality due to S. L. Sobolev [Sob] states that for any $1 < q < n$ there is a constant $C = C(n, q) > 0$ such that for every function $u \in C_0^\infty(\mathbb{R}^n)$ it holds that

$$\left(\int_{\mathbb{R}^n} u^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \leq C \left(\int_{\mathbb{R}^n} |Du|^q dx\right)^{\frac{1}{q}}. \tag{1.1}$$

Inequality (1.1) also holds for all $u \in C_0^\infty(\mathbb{R}^n)$ in the case $q = 1$ by the Gagliardo–Nirenberg estimate (see [Gag, Nir]). However, it seems to be an open problem whether (1.1) holds for all $u \in C_0^\infty(\mathbb{R}^n)$ and for all or just certain values of q in the "sublinear" range $q \in (0, 1)$.

The main purpose of this note is to show that Sobolev inequality of the form (1.1) remains to hold for certain exponent $q < 1$ when restricted to a class of positive p-superharmonic ($1 < p < 2$) functions u in $\mathbb{R}^n$, $n \geq 2$, with a mild decay at infinity. Precisely, it will be shown that for $1 < p < 2$, inequality (1.1) holds with $q \in (p - 1, 1)$ for all p-superharmonic functions u in $\mathbb{R}^n$, $n \geq 2$, such that $\inf_{\mathbb{R}^n} u = 0$ (see Theorem 1.4 below). More generally, we also consider sublinear trace inequalities of Sobolev type for p-superharmonic functions, where Lebesgue measure on the left-hand side is replaced by a general measure (see Theorem 3.1). We remark that when $1 < p \leq 2 - 1/n$ the 'gradient' Du in (1.1) may not belong to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and thus should be understood by means of an approximation (see (1.4) below).

The same kind of Sobolev inequality also holds for the more general class of positive $\mathcal{A}$-superharmonic functions. We now recall the notion of p-superharmonicity and $\mathcal{A}$-superharmonicity. Let $1 < p < n$, $n \geq 2$, and let $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a vector valued mapping which satisfies the following conditions:

$$\text{the mapping } x \to \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n,$$
$$\text{the mapping } \xi \to \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n,$$

and there are constants $0 < \alpha \le \beta < \infty$ such that for a.e. x in $\mathbb{R}^n$, and for all ξ in $\mathbb{R}^n$,

$$\mathcal{A}(x,\xi)\cdot\xi \ge \alpha\,|\xi|^p\,, \quad |\mathcal{A}(x,\xi)| \le \beta\,|\xi|^{p-1}\,, \tag{1.2}$$
$$[\mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2)]\cdot(\xi_1-\xi_2) > 0, \quad \text{if } \xi_1 \neq \xi_2 \quad .$$

For $u \in W^{1,p}_{\rm loc}(\Omega)$, where Ω is an open set, we define the divergence of $\mathcal{A}(x,\nabla u)$ in the sense of distributions, i.e., if $\varphi \in C_0^\infty(\Omega)$, then

$$\mathrm{div}\mathcal{A}(x,\nabla u)(\varphi) = -\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx.$$

It is well known that every solution $u \in W^{1,p}_{\rm loc}(\Omega)$ to the equation

$$-\mathrm{div}\mathcal{A}(x,\nabla u) = 0 \tag{1.3}$$

has a continuous representative. Such continuous solutions are said to be $\mathcal{A}$-*harmonic* in Ω. If $u \in W^{1,p}_{\rm loc}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \ge 0,$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$, i.e., $-\mathrm{div}\mathcal{A}(x,\nabla u) \ge 0$ in the distributional sense, then u is called a *supersolution* to (1.3) in Ω.

A lower semicontinuous function $u : \Omega \to (-\infty,\infty]$ is called $\mathcal{A}$-*superharmonic* if u is not identically infinite in each component of Ω, and if for all open sets D such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in D, it follows that $h \le u$ on ∂D implies $h \le u$ in D.

In the special case $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$, $\mathcal{A}$-superharmonicity is often referred to as p-superharmonicity. It is worth mentioning that the latter can also be defined equivalently using the language of viscosity solutions (see [JLM]).

We recall here the fundamental connection between supersolutions of (1.3) and $\mathcal{A}$-superharmonic functions [HKM].

Proposition 1.1 ([HKM])**.** (i) *If v is $\mathcal{A}$-superharmonic on Ω then*

$$v(x) = \mathrm{ess}\liminf_{y\to x}\, v(y), \qquad x \in \Omega.$$

Moreover, if $v \in W^{1,p}_{\rm loc}(\Omega)$ then

$$-\mathrm{div}\mathcal{A}(x,\nabla v) \ge 0.$$

(ii) *If $u \in W^{1,p}_{\rm loc}(\Omega)$ is such that*

$$-\mathrm{div}\mathcal{A}(x,\nabla u) \ge 0,$$

then there is an $\mathcal{A}$-superharmonic function v such that $u = v$ a.e.

(iii) *If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\rm loc}(\Omega)$ and*

$$-\mathrm{div}\mathcal{A}(x,\nabla v) \ge 0.$$

Note that an $\mathcal{A}$-superharmonic function u does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncation $\min\{u,k\}$ does for every integer k due to Proposition 1.1(iii). Using this we let Du stand for the a.e. defined function

$$Du = \lim_{k\to\infty} \nabla\,(\min\{u,k\}). \tag{1.4}$$

If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then Du coincides with the regular distributional gradient ∇u of u. In general, we have the following gradient estimates [KM1] (see also [HKM]).

Proposition 1.2 ([KM1])**.** *Suppose u is $\mathcal{A}$-superharmonic in Ω, $0 < r < \frac{n(p-1)}{n-1}$, and $0 < s < \frac{n}{n-1}$. Then $Du \in L^r_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}(x, Du) \in L^s_{\mathrm{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then Du coincides with the distributional gradient ∇u of u.*

We can now extend the definition of the divergence of $\mathcal{A}(x,\nabla u)$ to those u which are merely $\mathcal{A}$-superharmonic in Ω. For such u we set

$$-\mathrm{div}\mathcal{A}(x,\nabla u)(\varphi) = -\mathrm{div}\mathcal{A}(x,Du)(\varphi) = \int_\Omega \mathcal{A}(x,Du)\cdot\nabla\varphi\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Note that by Proposition 1.2 and the dominated convergence theorem,

$$-\mathrm{div}\mathcal{A}(x,\nabla u)(\varphi) = \lim_{k\to\infty}\int_\Omega \mathcal{A}(x,\nabla\min\{u,k\})\cdot\nabla\varphi\,dx \ge 0$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $\varphi \ge 0$.

Since $-\mathrm{div}\mathcal{A}(x,\nabla u)$ is a nonnegative distribution in Ω for an $\mathcal{A}$-superharmonic u, it follows that there is a nonnegative (not necessarily finite) Radon measure denoted by $\mu[u]$ such that

$$-\mathrm{div}\mathcal{A}(x,\nabla u) = \mu[u] \quad \text{in} \quad \Omega.$$

An important contribution to the theory of $\mathcal{A}$-superharmonic functions is the following pointwise estimates by Wolff's potentials obtained by T. Kilpeläinen and J. Malý.

Theorem 1.3 ([KM1, KM2])**.** *Let u be an $\mathcal{A}$-superharmonic function in $\mathbb{R}^n$ with $\inf_{\mathbb{R}^n} u = 0$. If $-\mathrm{div}\,\mathcal{A}(x,\nabla u) = \mu$, then for all $x \in \mathbb{R}^n$*

$$C_1\mathbf{W}_{1,p}\mu(x) \le u(x) \le C_2\mathbf{W}_{1,p}\mu(x), \tag{1.5}$$

where the constants C_1, C_2 depend only on n, p and the structural constants α, β. Here $\mathbf{W}_{1,p}\mu$ is the Wolff's potential of μ defined by

$$\mathbf{W}_{1,p}\mu(x) = \int_0^\infty \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n.$$

We are now ready to state the main result of the paper.

Theorem 1.4. *Let $1 < p < 2$ and $p-1 < q < 1$. For any $\mathcal{A}$-superharmonic function u in $\mathbb{R}^n$ such that $\inf_{\mathbb{R}^n} u = 0$ we have*

$$\left(\int_{\mathbb{R}^n} u^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \leq C \left(\int_{\mathbb{R}^n} |Du|^q dx\right)^{\frac{1}{q}}$$

with a constant $C = C(n, p, q, \alpha, \beta)$.

The proof if Theorem 1.4 is presented in the next section. It relies on the upper bound in (1.5), which serves as a 'representation' for $\mathcal{A}$-superharmonic functions. We mention here that Wolff potential estimates for k-convex functions have recently been used as an important tool in [Ver] to give a new and purely elliptic proof of the so-called Hessian Sobolev inequality of N. S. Trudinger and X.-J. Wang.

2. Proof of Theorem 1.4

Proof. Let $\mu = \mu[u]$ be the measure associated to the $\mathcal{A}$-superharmonic function u. We first observe that in the case $1 < p < 2$ we have

$$\mathbf{W}_{1,p}\mu(x) = \int_0^\infty \left[\frac{\mu(B_t(x))}{t^{n-p}}\right]^{\frac{1}{p-1}} \frac{dt}{t} \leq C \left\{\int_0^\infty \frac{\mu(B_t(x))}{t^{n-p}} \frac{dt}{t}\right\}^{\frac{1}{p-1}},$$

and thus by Fubini's theorem we find

$$\mathbf{W}_{1,p}\mu(x) \leq C \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-p}} = C\ \{\mathbf{I}_p\mu(x)\}^{\frac{1}{p-1}}. \tag{2.1}$$

Here for $0 < \alpha < n$, $\mathbf{I}_\alpha\mu$, stands for the Riesz's potential of order α of the measure μ defined by

$$\mathbf{I}_\alpha\mu(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \qquad x \in \mathbb{R}^n,$$

where

$$\gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$

Using the convolution identity (see [Stei, p. 118])

$$|\cdot|^{-n+(p-1)} * |\cdot|^{-n+1}(z) = \frac{\gamma_n(p-1)\gamma_n(1)}{\gamma_n(p)} |z|^{-n+p}, \qquad |z| \neq 0,$$

and Fubini's theorem (using that $\mu \geq 0$) we can write

$$\mathbf{I}_p\mu = \mathbf{I}_{p-1}(\mathbf{I}_1\mu),$$

and thus in view of (2.1) we get

$$\mathbf{W}_{1,p}\mu(x) \leq C \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-p}} = C\ \{\mathbf{I}_{p-1}(\mathbf{I}_1\mu)(x)\}^{\frac{1}{p-1}}.$$

Now by Theorem 1.3 and the last inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^n} u^{\frac{nq}{n-q}} dx &\leq C \int_{\mathbb{R}^n} (\mathbf{W}_{1,p}\mu(x))^{\frac{nq}{n-q}}\, dx \\
&\leq C \int_{\mathbb{R}^n} (\mathbf{I}_{p-1}(\mathbf{I}_1\mu)(x))^{\frac{nq}{(n-q)(p-1)}}\, dx \\
&\leq C \left\{ \int_{\mathbb{R}^n} (\mathbf{I}_1\mu(x))^{\frac{q}{p-1}}\, dx \right\}^{\frac{n}{n-q}},
\end{aligned} \tag{2.2}$$

where we used the standard Sobolev embedding in the last inequality. This is possible since $\frac{q}{p-1} > 1$ by our hypothesis.

On the other hand, by Fubini's theorem we have

$$\mathbf{I}_1\mu(x) = \frac{n-1}{\gamma_n(1)} \int_0^\infty \frac{\mu(B_t(x))}{t^n} dt = \frac{n-1}{\gamma_n(1)} \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \frac{\mu(B_t(x))}{t^n} dt. \tag{2.3}$$

Recall that $\mu = -\mathrm{div}\, \mathcal{A}(x, Du)$ and $\mathcal{A}(x, Du) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ (by Proposition 1.2). Therefore, the Gauss-Green formula for L^1_{loc} vector fields with divergence measure (see [DMM, Theorem 5.4]) yields that for any $x \in \mathbb{R}^n$ and for *almost every* $t > 0$,

$$\begin{aligned}
\mu(B_t(x)) &= -\int_{B_t(x)} \mathrm{div}\, \mathcal{A}(y, Du) dy \\
&= \int_{\partial B_t(x)} \mathcal{A}(y, Du) \cdot \frac{(x-y)}{|x-y|} d\mathcal{H}^{n-1}(y).
\end{aligned}$$

Plugging this into (2.3) gives

$$\begin{aligned}
\mathbf{I}_1\mu(x) &= \frac{n-1}{\gamma_n(1)} \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \int_{\partial B_t(x)} \mathcal{A}(y, Du) \cdot \frac{(x-y)}{|x-y|^{n+1}} d\mathcal{H}^{n-1}(y) dt \\
&= \frac{n-1}{\gamma_n(1)} \lim_{\epsilon\to 0^+} \int_{|x-y|>\epsilon} \mathcal{A}(y, Du) \cdot \frac{(x-y)}{|x-y|^{n+1}} dy.
\end{aligned}$$

When $Du \in L^q(\mathbb{R}^n)$ we have $\mathcal{A}(y, Du) \in L^{\frac{q}{p-1}}(\mathbb{R}^n)$ by (1.2), and thus the limit above exists for almost every $x \in \mathbb{R}^n$ and equals $\sum_{j=1}^n \mathbf{R}_j \mathcal{A}_j(\cdot, Du)(x)$. Here $\mathcal{A}_j$ is the j-th component of $\mathcal{A}$ and $\mathbf{R}_j$ is the j-th Riesz transform. That is,

$$\mathbf{I}_1\mu(x) = \frac{n-1}{\gamma_n(1)} \sum_{j=1}^n \mathbf{R}_j \mathcal{A}_j(\cdot, Du)(x) \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Using this identity in (2.2) and the boundedness of Riesz transform in $L^{\frac{q}{p-1}}(\mathbb{R}^n)$, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^n} u^{\frac{nq}{n-q}} dx &\leq C \left\{ \int_{\mathbb{R}^n} |\mathcal{A}(y, Du)|^{\frac{q}{p-1}} dy \right\}^{\frac{n}{n-q}} \\
&\leq C \left\{ \int_{\mathbb{R}^n} |Du|^q dy \right\}^{\frac{n}{n-q}},
\end{aligned}$$

by (1.2). This completes the proof of the theorem. □

3. Further generalization, open question, and motivation

Let σ be a nonnegative locally finite measure in $\mathbb{R}^n$. Adams [Ad] (see also [AH, Theorem 7.2.2]) showed that the trace inequality

$$\left(\int_{\mathbb{R}^n}|u|^{q_1}d\sigma\right)^{\frac{1}{q_1}} \leq C\left(\int_{\mathbb{R}^n}|\nabla u|^q dx\right)^{\frac{1}{q}}, \tag{3.1}$$

where $u \in C_0^\infty(\mathbb{R}^n)$, and $q_1 > q > 1$, holds if and only if σ satisfies the condition

$$\sigma(B_r(x)) \leq C r^{(n-q)q_1/q} \tag{3.2}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Here the constant C is independent of x and r.

The proof of Theorem 1.4 reveals that the condition (3.2) remains to be sufficient for the validity of (3.1) even in the sublinear case $p-1 < q < 1$, $q_1 > q$, provided u is as in Theorem 1.4. Indeed, by the result of Adams (see [AH, Theorem 7.2.2] or [Ad]), condition (3.2) implies that σ satisfies the inequality

$$\left(\int_{\mathbb{R}^n}|\mathbf{I}_{p-1}f|^{\frac{q_1}{p-1}}d\sigma\right)^{\frac{p-1}{q_1}} \leq C\left(\int_{\mathbb{R}^n}|f|^{\frac{q}{p-1}}dx\right)^{\frac{p-1}{q}} \tag{3.3}$$

for all $f \in L^{\frac{q}{p-1}}(\mathbb{R}^n)$ since $q_1 > q > p-1$. This allows us to obtain the following analogue of (2.2):

$$\begin{aligned}\int_{\mathbb{R}^n}u^{q_1}d\sigma &\leq C\int_{\mathbb{R}^n}(\mathbf{W}_{1,p}\mu(x))^{q_1}\,d\sigma\\ &\leq C\int_{\mathbb{R}^n}(\mathbf{I}_{p-1}(\mathbf{I}_1\mu)(x))^{\frac{q_1}{p-1}}\,d\sigma\\ &\leq C\left\{\int_{\mathbb{R}^n}(\mathbf{I}_1\mu(x))^{\frac{q}{p-1}}\,dx\right\}^{\frac{q_1}{q}}.\end{aligned}$$

Then the rest of the proof can be done as before. In summary, we have

Theorem 3.1. *Let $1 < p < 2$, $p-1 < q < 1$, and $q_1 > q$. Suppose that σ is a nonnegative locally finite measure in $\mathbb{R}^n$ satisfying condition (3.2). Then for any $\mathcal{A}$-superharmonic function u in $\mathbb{R}^n$ with $\inf_{\mathbb{R}^n} u = 0$, the trace inequality (3.1) holds with a constant $C = C(n,p,q,q_1,\alpha,\beta)$.*

When σ is the Lebesgue measure of $\mathbb{R}^n$, i.e., $d\sigma = dx$, then obviously Theorem 3.1 implies Theorem 1.4. We remark here that condition (3.2) is independent of p.

It is also possible to give a version of Theorem 3.1 in the case $0 < q_1 \leq q$ (and $p-1 < q < 1$). Indeed, suppose that σ satisfies the inequality (3.3) for all $f \in L^{\frac{q}{p-1}}(\mathbb{R}^n)$, where $0 < q_1 \leq q$, then the conclusion of Theorem 3.1 also holds by a similar argument. Now characterizations of inequality (3.3) in terms of σ in the case $0 < q_1 \le q$ and $q > p-1$ are also available. For example, in the case $q_1 = q$, then inequality (3.3) holds if and only if

$$\sigma(K) \le C\, \mathrm{cap}_{p-1,\frac{q}{p-1}}(K) \tag{3.4}$$

for every compact set $K \subset \mathbb{R}^n$ (see, e.g., [AH, Theorem 7.2.1]). Here $\mathrm{cap}_{p-1,\frac{q}{p-1}}(K)$ is a Riesz capacity of the compact set K defined by

$$\mathrm{cap}_{p-1,\frac{q}{p-1}}(K) = \inf\left\{\int_{\mathbb{R}^n} f^{\frac{q}{p-1}}\, dx : f \ge 0 \text{ and } \mathbf{I}_{p-1} f \ge 1 \text{ on } K\right\}.$$

On the other hand, in the case $0 < q_1 < q$, inequality (3.3) holds if and only if

$$\int_{\mathbb{R}^n}\left\{\int_0^\infty \left(\frac{\sigma(B_t(x))}{t^{n-q}}\right)^{\frac{p-1}{q-p+1}} \frac{dt}{t}\right\}^{\frac{q_1(q-p+1)}{(q-q_1)(p-1)}} d\sigma(x) < +\infty, \tag{3.5}$$

(see [COV1, COV2]). However, we observe that unlike (3.2), conditions (3.4) and (3.5) generally depend on p.

A natural question to ask is a localized version of (1.1), i.e., sublinear Sobolev inequality on bounded domains $\Omega \subset \mathbb{R}^n$ for $\mathcal{A}$-superharmonic functions in Ω with zero boundary values. For example, for $u \in W_0^{1,p}(\Omega)$, $1 < p < 2$, with $-\mathrm{div}\,\mathcal{A}(x, \nabla u) \ge 0$, and $p - 1 < q < 1$, is it true that the Sobolev inequality

$$\left(\int_\Omega u^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \le C \left(\int_\Omega |\nabla u|^q dx\right)^{\frac{1}{q}} \tag{3.6}$$

holds with a constant C independent of u? To the best of our knowledge, this question is open even when Ω is a ball of $\mathbb{R}^n$. In fact, it is not known even if one replaces the $L^{nq/(n-q)}(\Omega)$ norm on the left-hand side by $L^r(\Omega)$ norm for any $0 < r < nq/(n-q)$ (allowing the constant C to depend on the diameter of Ω).

Our interest in inequality (3.6) mostly lies in the case $1 < p \le 2 - 1/n$. This is motivated from the work [DM2] (see also [DM1, KM]) in which interesting pointwise estimates by means of Riesz or Wolff's potentials for gradients of solutions to p-Laplace type equations with (signed) measure data were obtained in the case $p > 2 - 1/n$. However, similar pointwise gradient bounds for the case $1 < p \le 2 - 1/n$ still remain unknown even for *nonnegative* measure data. The work [DM2] suggests that one of the main obstacles in handling this case is the lack of a sublinear Sobolev inequality of the form (3.6). This in a sense justifies our restriction on the range of p and q in the paper.

Finally, we mention that inequality (3.6) is closely related to the following Sobolev–Poincaré inequality

$$\inf_{a\in\mathbb{R}}\left(\int_\Omega |u-a|^{\frac{nq}{n-q}} dx\right)^{\frac{n-q}{nq}} \le C \left(\int_\Omega |\nabla u|^q dx\right)^{\frac{1}{q}}, \tag{3.7}$$

where u is no longer required to have any zero boundary condition. Even when Ω is a ball in $\mathbb{R}^n$, it is known by means of a counter example in [BK] that inequality (3.7) generally fails in the case $q \in (0,1)$ if we assume only that $u \in W^{1,1}(\Omega)$.

On the other hand, the main result of [BK] states that, for Ω being a Lipschitz or John domain, if $u \in W^{1,1}_{\rm loc}(\Omega)$ is such that $|\nabla u| \in WRH_1^{\Omega}$ (weak reverse Hölder) then inequality (3.7) holds with a constant C depending only on n, q and the WRH_1^{Ω} constants of $|\nabla u|$. Here by definition $|\nabla u| \in WRH_1^{\Omega}$ if there exist constants $A > 0$ and $1 < \sigma \leq \sigma'$ such that

$$\frac{1}{|Q|}\int_Q |\nabla u| dx \leq A\left(\frac{1}{|\sigma Q|}\int_{\sigma Q} |\nabla u|^{1/2} dx\right)^2 \tag{3.8}$$

for all cubes Q such that $\sigma' Q \subset \Omega$. (A cube Q is always assumed to have faces perpendicular to the coordinate directions and rQ is the concentric dilate of Q by a factor $r > 0$.)

We observe that in the case $1 < p \leq 2 - 1/n$, by simply looking at the fundamental solution $U(x) = c\,|x|^{\frac{p-n}{p-1}}$, it is easy to see that the condition $|\nabla u| \in WRH_1^{\Omega}$ in general fails for p-superharmonic functions u.

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