Twisted doubles and nonpositive curvature

C S Aravinda, TIFR Centre for Applicable Mathematics

10th August, 2010

In this talk, we consider closed manifolds M^n and discuss the question of whether they admit riemannian metrics of nonpositive or negative sectional curvature, whenever they have a chance to do so.

When the dimension is 2 and M is orientable, Gauss-Bonnet theorem indicates that all closed surfaces other than the 2-sphere have a chance to admit nonpositively curved metrics and classical uniformization theorem for surfaces confirms this.

That is, if M is either the torus or a surface of higher genus then $\mathcal{X}(M) \leq 0$. Hence, by Gauss-Bonnet, M has a chance to admit metrics of nonpositive curvature. The uniformization theorem then implies that one can, indeed, prescribe metrics of constant curvature of appropriate signature.

In dimensions bigger than or equal to 3, we have the following two general consequences of nonpositive curvature condition that help in our study.

Fact 1: The universal cover $\widetilde{M}$ of M is diffeomorphic to $\mathbb{R}^n$.

Fact 2: Each element of $\pi_1(M)$ has infinite order.

In particular, if $M = S^n$ or if $\pi_1(M)$ has elements of finite order then M cannot admit metrics of nonpositive curvature.

We shall now try to see:

1. How much more control does $\pi_1(M)$ *have on the geometry or the topology of* M*?* and

2. Whenever M *has a chance to admit metrics of nonpositive curvature does it really do so?*

Answer(s) to Question 1:
If M has nonpositive curvature, *Fact 1* implies that $\pi_n(M) = 0$ for all $n \geq 2$.

Therefore, if M^n and N^n are two closed manifolds and $\pi_1(M) = \pi_1(N)$, then $\pi_n(M) = \pi_n(N)$ for all $n \geq 1$.

Moreover, standard arguments from topology show that M and N are homotopically equivalent.

Going further, one has the following two celebrated theorems:

Mostow's strong rigidity theorem: *If $n \geq 3$ and if M and N are locally symmetric spaces (of non-compact type) with $\pi_1(M) = \pi_1(N)$, then M and N are isometric to each other, up to scaling by a constant.*

Farrell-Jones' topological rigidity theorem: *If $n \geq 5$ and if M and N are nonpositively curved with $\pi_1(M) = \pi_1(N)$, then M and N are homeomorphic.*

Farrell and Jones also showed that 'smooth rigidity' (i.e., that M and N are diffeomorphic) fails in general.

In dimension 3, one has the following answer:

If $f : M \to N$ is a homotopy equivalence between closed 3-manifolds such that N supports a Riemannian metric of nonpositive curvature, then f is homotopic to a homeomorphism.

When N is nonpositively curved, the geometrization conjecture implies that either N supports a hyperbolic metric or N is sufficiently large in the sense of Waldhausen.

D. Gabai proved the above result when N is hyperbolic, provided the Poincaré Conjecture is true.

And, Waldhausen proved this for sufficiently large aspherical 3-manifolds provided the Poincaré Conjecture is true.

Now that the solutions to both the Poincaré conjecture as well as the geometrization conjecture are widely accepted to be correct, the dimension 3 case of topological rigidity follows.

The case of dimension 4 is still wide open!

Let us now consider the following more precise form of Question 2:

If M has nonpositive curvature and if N is homemorphic to M, then does N admit metrics of nonpositive curvature?

We have the following two cases:

Case i: M has negative curvature, and

Case ii: M has nonpositive curvature.

While the more subtle *Case i* is still open, the answer in *Case ii* is NO, in general.

The negative answer follows from the following three different constructions:

1. B. Okun's construction of exotic differential structures on certain higher rank locally symmetric spaces.

2. Construction of exotic smooth structures on certain 'doubles' of finite volume real hyperbolic manifolds (A-Farrell).

3. Construction of exotic *PL*-structures on certain 'doubles' of finite volume real hyperbolic manifolds by P. Ontaneda.