Homework on $Out(F_n)$

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Problems labeled with an asterisk are more difficult/technical and constitute the take-home final.

1 Folding and applications

H is a finitely generated subgroup of F_n.

1. Find a basis of the subgroup

$$H = \langle b\overline{a}baaba, ab\overline{a}baba, ab\overline{a}b\overline{a}\overline{b}a\overline{b} \rangle < \langle a, b \rangle$$

2. Prove that type 1 folds are π_1-isomorphisms, type 2 folds are π_1-epimorphisms (and induce the obvious map $F_k \to F_{k-1}$ in suitable bases), and immersions are π_1-injective. Hint: For the first two claims choose suitable standard markings. For the last, show that immersions can be completed to covering maps by adding edges.

3. Given $w \in F_n$ give an algorithm to decide whether $w \in H$. E.g. show $a \notin H$ of #1.

4. Given $w \in F_n$ give an algorithm to decide whether w is conjugate into H.

5. Can you tell if H is normal in F_n?

6. Can you tell if H has finite index in F_n?

7. Suppose H is a finitely generated normal subgroup of F_n. Show that either H has finite index in F_n or $H = \{1\}$.

8. Given a homomorphism $h : F_n \to F_m$, can you tell if h is injective, surjective, bijective? Answer: Injective iff there are no folds of the second kind. Surjective iff the last map is a homeomorphism. In particular, show that F_n is *hopfian*, i.e. every epimorphism $F_n \to F_n$ is an automorphism.

9. Let $h : \langle a, b\rangle \to \langle a, b\rangle$ be given by $h(a) = abbab$, $h(b) = bababbab$. Show that h is an automorphism and compute h^{-1}. (You can do this by messing about. But try to do it algorithmically, that is, decompose h into a product of Nielsen generators and then compose the inverses in opposite order. The point is that this can be programmed on a computer.)

10. Show that for every homomorphism $h : F_n \to F_m$ there is a free factorization $F_n = A * B$ such that h kills A and is injective on B.

11. Show that for every finitely generated $H \subset F_n$ there is a subgroup $H' \subset F_n$ such that $H \subset H'$, H is a free factor in H', and H' has finite index in F_n. This is called Marshall Hall's theorem. You can find H' algorithmically. Do it for H in the example from #1. Hint: Add some edges to G to turn an immersion $G \to Y$ into a covering map.

12. Can you always compute the normalizer

$$N(H) = \{\gamma \in F_n \mid \gamma H \gamma^{-1} = H\}?$$

What can you say about the index $[N(H) : H]$? (Answer: it is always finite and bounded by the number of vertices in the graph representing H. Recall that $N(H)/H$ is the deck group.) E.g. show that $N(H) = H$ for H as in #1.

13. If T and T' are two maximal trees, show that there is a sequence $T = T_0, T_1, \cdots, T_k = T'$ of maximal trees such that any two consecutive trees differ in only one edge, as in the lecture.

14.* This is a bit more ambitious. Consider the simplicial complex whose vertices are non-closed edges of G, and a collection of edges spans a simplex if their union is a forest. Draw some examples. Can you make a conjecture about the homotopy type of the complex?

15.* Read the wonderful paper *Topology of finite graphs* by John Stallings (Inventiones 71 (1983) 551-565.)

2 Outer space

16. Consider a graph with two vertices and four edges, all joining the two vertices. Once a marking is provided, this graph defines a simplex with missing faces in Outer space $\mathcal{X}_3$. How many faces are missing? How many simplices-with-missing-faces in $\mathcal{X}_3$ contain this simplex?

17. Construct a simplicial complex $\hat{\mathcal{X}}_n$ by "putting the missing faces back in". Thus $\hat{\mathcal{X}}_n \setminus \mathcal{X}_n$ is a subcomplex of $\hat{\mathcal{X}}_n$. Show that simplices in $\hat{\mathcal{X}}_n \setminus \mathcal{X}_n$ have infinitely many "superfaces" (i.e. simplices containing them) while simplices not contained in $\hat{\mathcal{X}}_n \setminus \mathcal{X}_n$ have only finitely many superfaces. Draw the picture in rank 2. In particular, $\mathcal{X}_n$ is locally compact.

18. Show that $\mathcal{X}_n$ has finitely many $Out(F_n)$-orbits of simplices with missing faces. In particular, the action on the spine K_n is cocompact.

19. Prove that the stabilizer in $Out(F_n)$ of a graph Γ with marking g is isomorphic to the isometry group of Γ.

20. In $\mathcal{X}_2$, sketch an orbit of the automorphism $a \mapsto a$, $b \mapsto ab$, and also of $a \mapsto b$, $b \mapsto ab$.

21. Here are some basic facts from simplicial topology. Let $f : X \to Y$ be a simplicial map between simplicial complexes.

    (a) If $y, y' \in Y$ belong to the same open simplex $\mathring{\sigma}$, then $f^{-1}(y)$ is homeomorphic to $f^{-1}(y')$. In fact, $f^{-1}(\mathring{\sigma})$ is homeomorphic to $\mathring{\sigma} \times f^{-1}(y)$.

    (b) Let U be a standard neighborhood of a closed simplex $\sigma \subset Y$ (i.e. U intersects a simplex τ if and only if $\tau \supset \sigma$ and the intersection $U \cap \tau$ is convex in that case). Show that $f^{-1}(U)$ deformation retracts to $f^{-1}(\sigma)$. Hint: Induction on skeleta. It's easier if X, Y are finite dimensional, and that's all that's needed for our applications.

    (c) Suppose each $f^{-1}(y)$ is contractible ($y \in Y$). Show that $f^{-1}(\sigma)$ is contractible for every closed simplex $\sigma \subset Y$. Then show that f is a homotopy equivalence. Hint: For the first statement we may as well assume $Y = \sigma$. Using (b) find an open cover of $f^{-1}(\sigma)$ consisting of contractible sets with all nonempty intersections contractible and with the nerve contractible. For the second statement induct on the number of simplices when Y is finite. Alternatively, construct directy a homotopy inverse by induction on skeleta.

(d) Suppose each $f^{-1}(y)$ is acyclic. Show that f is a homology isomorphism.

(e) Now suppose we have simplicial maps $X \xrightarrow{f} Y \xrightarrow{\pi} Z$. Also suppose that for every $z \in Z$ the map

$$f|_{(\pi f)^{-1}(z)} : (\pi f)^{-1}(z) \to \pi^{-1}(z)$$

is a homology isomorphism. Then f is a homology isomorphism. Moreover, this also holds "in a range" i.e. if $f|_{(\pi f)^{-1}(z)}$ are isomorphisms in H_i for $i \leq n$ then so is f.

(f) All of the above continues to hold in the category of simplicial-with-missing faces complexes.

22. This is a more detailed outline of contractibility of $\mathcal{X}_n$, essentially following ideas of Steiner.

(a) Let $\pi : \mathcal{X}'_n \to \mathcal{X}_n$ be the forget-the-basepoint map from Autre espace to Outer space. Show that π is simplicial and that the preimage $\pi^{-1}(\Gamma)$ of any point Γ is a tree homeomorphic to the universal cover of Γ; in particular it is contractible. Deduce that π is a homotopy equivalence. Thus it suffices to prove that $\mathcal{X}'_n$ is contractible.

(b) Show that for every $\Gamma \in \mathcal{X}'_n$ there is a unique metric R_Γ on the rose with identity marking and a unique (basepoint preserving) map $f_\Gamma : R_\Gamma \to \Gamma$ such that

- f_Γ is (the inverse of) the marking of Γ,
- f_Γ is a local isometry when restricted to each open edge of R_Γ.

(c) Show that $\Gamma \mapsto [R_\Gamma]$ defines a retraction of $\mathcal{X}'_n$ onto a simplex with missing faces. Here $[R_\Gamma]$ denotes R_Γ with metric rescaled so the volume is 1. (In particular, prove continuity.)

(d)* For every $t \geq 0$ define Γ_t as follows (this clean description is due to Skora). View R_Γ as embedded in the product $R_\Gamma \times \Gamma$ as the graph of f_Γ. Let N_t be the *horizontal t-neighborhood* i.e. the set of points (x, y) such that $d(f_\Gamma(x), y) \leq t$. This space is naturally decomposed into components of its intersection with horizontal slices $y = const$. Let Γ_t be the decomposition space obtained by crushing these subsets to points. Show that Γ_t is naturally a metric graph and after rescaling determines a point $[\Gamma_t]$ in $\mathcal{X}'_n$. (You may want to ponder this construction for the function $|\cdot| : [-1, 1] \to [0, 1]$, a prototypical fold.)

(e)* Show that $t \mapsto [\Gamma_t]$ is a path in $\mathcal{X}'_n$ from R_Γ to Γ (for large t it is constant). This is the *folding path* associated with Γ.

(f)* The folding path varies continuously with Γ.

(g) Reparametrizing the folding paths in reverse on $[0, 1]$ gives a strong deformation retraction from $\mathcal{X}'_n$ to a simplex with missing faces. Conclude that $\mathcal{X}'_n$ is contractible.

23. Consider a graph Γ with two vertices and three edges, all joining the two vertices (a Θ-graph). Say all three lengths are $\frac{1}{3}$, the basepoint is in the middle of one edge, the marking to the rose $\langle a, b\rangle$ sends this edge to the vertex and the other two edges to ab and bab (parallel orientations). Draw the folding path to the simplex of roses associated with this graph (by this I mean draw all simplices this graph passes through and indicate the intersections with each simplex).

24. Prove that every folding path can be subdivided into finitely many subpaths each of which is a straight line segment in a simplex with missing faces (but the parametrization may not be linear).

25. Prove that $\mathbb{Z}^m$ cannot act freely and properly discontinuously on a contractible complex of dimension $< m$. Deduce that no subgroup of $Out(F_n)$ is isomorphic to $\mathbb{Z}^{2n-2}$ ($n \geq 2$) (recall that the dimension of the spine is $2n - 3$). Find a subgroup isomorphic to $\mathbb{Z}^{2n-3}$.

26.* This exercise can be used to deduce integral homological stability of $Aut(F_n)$ from [Hatcher-Vogtmann]. Let G be a group and $H < G$ a subgroup. Suppose G acts simplicially on a complex X_G. Also suppose that

- X_H is an H-invariant subcomplex of X_G
- X_G and X_H are n-connected,
- if an element of G leaves a simplex invariant, it fixes it pointwise. In particular, the orbit spaces X_G/G and X_H/H are decomposed into "simplices", i.e. images of simplices in X_G and X_H.
- inclusion induced map $X_H/H \to X_G/G$ is a homeomorphism.
- for every point $x \in X_H$ inclusion

  $$Stab_H(x) \hookrightarrow Stab_G(x)$$

  induces an isomorphism in homology H_i for $i \leq n$.

Then $H \hookrightarrow G$ induces an isomorphism in H_i for $i \leq n$.

Hint: Apply #18(e) to $X_H \times EH/H \to X_G \times EG/G \to X_G/G$.

3 Lipschitz metric and train tracks

27. By considering the simplex of roses from #22, for $\epsilon > 0$ find examples of graphs Γ, Γ' such that $d(\Gamma, \Gamma') < \epsilon$ and $d(\Gamma', \Gamma) > 1/\epsilon$.

28. Let $R \in \mathcal{X}_2$ be the rose with identity marking and edges of length 1/2. Let f be given by $a \mapsto a$, $b \mapsto ab$. Show that $d(R, f^k(R)) \sim \log k$.

29. For the same R and f given by $a \mapsto b$, $b \mapsto ab$, show that $d(R, f^k(R))$ is bounded above and below by a linear function of k. In fact, if R is replaced by a suitable graph in the same simplex, $k \mapsto d(R, f^k(R))$ is a linear function on the nose. Hint: use the train track metric. By connecting consecutive graphs in the orbit with a folding path one gets an *axis* of f.

30. For f in #29, show that axes for f and for f^{-1} are distinct lines.

31. For f in #28 find a sequence Γ_k with $d(\Gamma_k, f(\Gamma_k)) \to 0$. Prove that for any such sequence for large k there is going to be a proper invariant subgraph (up to homotopy).