Flows, Fixed Points and Rigidity for Kleinian Groups

Kingshook Biswas,
Department of Mathematics,
RKM Vivekananda University.

Theorem

(Mostow Rigidity) Any isomorphism $f : \pi_1(M) \to \pi_1(N)$ between fundamental groups of closed hyperbolic manifolds M, N of dimension $n \geq 3$ is induced by an isometry $\tilde{f} : M \to N$.

Sketch of proof:

Step 1. Fixing a basepoint p in the universal cover $\mathbb{H}^n$, f induces a map F between orbits $\pi_1(M) \cdot p$ and $\pi_1(N) \cdot p$ conjugating actions.

Step 2. Orbits are dense in $\partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$, and F is a quasi-isometry, extends to a quasi-conformal map $F : \partial \mathbb{H}^n \to \partial \mathbb{H}^n$ conjugating actions.

Step 3. If F is not conformal then "zoom-in" near point of differentiability where DF not conformal to get linear non-conformal map A conjugating actions.

Step 4. Constant ellipse field induced by A invariant under $\pi_1(N)$ cocompact Kleinian group, not possible, contradiction.

Mostow rigidity: f quasi-isometric, **equivariant** pairing of **orbits** of two cocompact Kleinian groups $\Rightarrow$ f at bounded distance from an isometry performing the same pairing.

Question : Rigidity for non-equivariant "pairings"?

Theorem

(Schwartz '97) Let $M_1 = \mathbb{H}^n / G_1, M_2 = \mathbb{H}^n / G_2$ be closed hyperbolic manifolds of dimension $n \geq 3$, and $\mathcal{J}_1, \mathcal{J}_2$ collections of lifts of finitely many geodesics in M_1, M_2 respectively. If f is a quasi-isometry such that $f(\mathcal{J}_1) = \mathcal{J}_2$ then f is at bounded distance from an isometry ϕ such that $\phi(\mathcal{J}_1) = \mathcal{J}_2$. Moreover G_1 and $\phi^{-1} G_2 \phi$ are commensurable.

Generalizations : Replace cyclic subgroups $\gamma \subset G$ (geodesics) with infinite index quasiconvex subgroups $H \subset G$. Consider collection $\mathcal{J}$ of limit sets on boundary.

Definition. Let G be a cocompact Kleinian group. A G-symmetric pattern is a G-invariant collection $\mathcal{J}$ of closed subsets of $\mathbb{H}^n$, none of which are singletons, and whose only accumulations (in Hausdorff topology) are singletons.

Example : Collection of translates of limit sets of any infinite index quasiconvex subgroup $H \subset G$.

Theorem

(B., Mj '08) Let G_1, G_2 be cocompact Kleinian groups in dimension $n \geq 3$, and $H_i \subset G_i$ infinite index quasiconvex subgroups satisfying one of the two following conditions: (1) H_i is a codimension duality group. (2) H_i is an odd-dimensional Poincare Duality Group. Then any quasi-conformal pairing f between the corresponding patterns of limit sets $\mathcal{J}_1, \mathcal{J}_2$ is conformal and $G_1, f^{-1}G_2 f$ are commensurable.

Theorem

(Mj. '09) Let G_1, G_2 be word-hyperbolic groups and $H_i \subset G_i$ codimension one filling subgroups. Suppose G_1, G_2 are Poincare Duality groups and Hausdorff dimension of ∂G_i is strictly larger than topological dimension of G_i plus two. If there is a quasi-conformal pairing between the patterns of limit sets given by H_1, H_2 then G_1, G_2 are commensurable.

Theorem

Let G_1, G_2 be cocompact Kleinian groups in dimension $n \geq 3$ and $\mathcal{J}_i$ be G_i-symmetric patterns, $i = 1, 2$. Then any quasi-conformal pairing f between $\mathcal{J}_1$ and $\mathcal{J}_2$ is conformal, and $G_1, f^{-1}G_2 f$ are commensurable.

Observation: The subgroup of Homeo$(\partial\mathbb{H}^n)$ preserving a symmetric pattern $\mathcal{J}$ is closed and totally disconnected.

Theorem

Let G_1, G_2 be cocompact Kleinian groups in dimension $n \geq 3$. If f is a quasi-conformal map which is not conformal then the closure of the subgroup of Homeo$(\partial\mathbb{H}^n)$ generated by G_1 and $f^{-1}G_2 f$ contains a non-trivial one parameter subgroup $(f_t)_{t\in\mathbb{R}}$.

Corollary

(1) Mostow Rigidity.

(2) If G is a cocompact Kleinian group in dimension $n \geq 3$ and f is a quasi-conformal map which is not conformal then $\overline{< G, f >}$ contains a non-trivial one-parameter subgroup.

Sketch of proof of Main Theorem:

Observation: For G cocompact, any sequence of isometries can be approximated by elements of G upto bounded error.

Step 1. Given G_1, G_2, f, upgrade f to a non-conformal linear map: zoom-in at point of differentiability of f, approximate zoom-in, zoom-out by elements of G_1, G_2, replace G_1, G_2 by conformal conjugates and f by linear map A.

Step 2. Linear map A must pair poles of G_1, G_2:

Theorem

Let G_1, G_2 be cocompact Kleinian groups in dimension $n \geq 3$ and f a C^2 diffeomorphism of $\partial\mathbb{H}^n$. If $\overline{< G_1, f^{-1}G_2 f >}$ does not contain a flow then f preserves poles; if in addition f is linear then f pairs poles.

Theorem

(Hyperbolic flows) Let G be a cocompact Kleinian group in dimension $n \geq 3$. If x_0 is **not** *a fixed point of G, and is a fixed point of a C^2 diffeomorphism f of $\partial\mathbb{H}^n$ such that $Df(x_0)$ is conjugate to a conformal linear map λO with $\lambda \neq 1$, O orthogonal, then $\overline{< G, f >}$ contains a one-parameter subgroup conjugate to a flow of affine linear maps.*

Theorem

(Parabolic flows) Let G be a cocompact Kleinian group in dimension $n \geq 3$. If x_0 is a fixed point of a C^2 diffeomorphism f of $\partial \mathbb{H}^n$ such that that $Df(x_0) = Id, D^2 f(x_0) \neq 0$, then $\overline{< G, f >}$ contains a one-parameter subgroup conjugate to a flow of translations.

Step 3. Linear map A non-conformal implies group $\hat{G} := \overline{< G_1, A^{-1} G_2 A >}$ indiscrete:

Take g_1 in G_1 with poles not in $\{0, \infty\}$, then A pairs poles of g_1 with some g_2 in G_2.

Conjugate G_1, G_2 to send poles of g_1, g_2 to $0, \infty$ and get new pole-pairing map $\mu =$ linear map A post and pre composed with conformal maps ("eccentric map"), $\mu(0) = 0, \mu(\infty) = \infty$.

A **non-conformal** implies μ **non-linear**.

Zoom-in on fixed point 0 of μ using g_1, zoom-out using G_2, to get sequence of **non-linear** pole-pairing maps $\mu_n = g_2^{-q_n} \circ \mu \circ g_1^{p_n}$ converging to a **linear** map.

For any g in G_2 with poles a, b, conjugates $\mu_n^{-1} g \mu_n$ are in $\hat{G}$, with fixed points $a_n = \mu_n^{-1}(a), b_n = \mu_n^{-1}(b)$.

Use "Scattering lemma" (Schwartz) to see poles of some g in G_2 have **infinitely** many distinct images a_n, b_n under μ_n^{-1}'s.

Points a_n, b_n are fixed points of maps $\mu_n g \mu_n^{-1}$, and also poles of some g_n in G_1.

Zoom-in, zoom-out on these maps using g_1 to get maps F_n in $\hat{G}$ with fixed points a_n, b_n, which are conformal conjugates of linear maps.

Wlog $a_n \to 0, b_n \to \infty$, conjugate by dilation to move b_n much closer to ∞ than a_n is to 0. Then F_n's look like affine maps, converging to a linear map F.

Compositions $F^{-1} F_n$ look like identity plus infinitesimal affine maps, apply Euler's formula to get a flow.