Cohomology computations for Coxeter groups and their relatives

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Suppose G acts on a CW complex $\tilde{X}$ (written $G \curvearrowright \tilde{X}$). Put $X = \tilde{X}/G$ and let $p : \tilde{X} \to X$ be the projection. Let

- $C_k(\tilde{X}) =$ the free abelian group on the k-cells of $\tilde{X}$.
- It is a G-module.
- Given a an arbitrary G-module M, put

$$C^*_G(\tilde{X}; M) := \mathrm{Hom}_G(C_*(\tilde{X}), M.$$

- We can regard M as defining a (not locally constant) coefficient system on the orbit space X. On a cell σ of X, it is defined by

$$\sigma \mapsto \mathrm{Hom}_G(\mathbf{Z}(p^{-1}(\sigma)), M)$$

$X = \tilde{X}/G$.

- If $G \curvearrowright \tilde{X}$ freely, then this system on X is locally constant. Write
$$C^*(X; M) := C^*_G(\tilde{X}; M).$$
- BG denotes a CW complex with fundamental group G and with universal cover, EG, contractible. (BG is also called a $K(G, 1)$.)

Definition (of group cohomology)

$$H^*(G; M) := H^*(BG; M).$$

Freeing up the action

If $G \curvearrowright \tilde{X}$ is not free, then there is a free action on the homotopy equivalent space, $EG \times \tilde{X}$. The orbit space is denoted

$$EG \times_G \tilde{X}$$

and is called the *Borel construction* on X. The G-map $EG \times \tilde{X} \to \tilde{X}$ induces a homo, $H^*_G(\tilde{X}; M) \to H^*(EG \times_G \tilde{X}; M)$, which is sometimes an iso.

We want to compute $H^*(G; M)$ or possibly $H^*_G(\tilde{X}; M)$ for M a G-module and $\tilde{X}$ a G-space for

- $M = \mathbf{Z}G$, or
- $\ell^2 G$, the square summable functions on G, or
- $\mathcal{N}(G)$, an associated von Neumann algebra, or
- a “Hecke - von Neumann algebra” used for “weighted ℓ^2-cohomology”.

Topological interpretation of $H^*(X; \mathbf{Z}G)$

Suppose X is compact (i.e., a finite complex). Then

$$H^*(X; \mathbf{Z}G) = H^*_c(\tilde{X}),$$

the point being that the G-equivariant functions $p^{-1}(\sigma) \to \mathbf{Z}G$ can be identified with the finitely supported functions $p^{-1}(\sigma) \to \mathbf{Z}$. Even if the G-action on $\tilde{X}$ is only assumed to be proper, $H^*_G(\tilde{X}; \mathbf{Z}G) = H^*_c(\tilde{X})$. (Proper means that the cell stabilizers are finite subgroups. Similarly, $H^*_G(\tilde{X}; \ell^2 G)$ just means that we are using square summable cochains on $\tilde{X}$

Why are we interested in $\mathbf{Z}G$ coeffiicents?

- The rank of $H^1(G; \mathbf{Z}G)$ tells us the number of ends of G.
- Suppose $H^*(G; \mathbf{Z}G)$ is concentrated in a single degree, say n. Then G is a PD group $\iff H^n(G; \mathbf{Z}G) = \mathbf{Z}$ and G is a *duality group* $\iff H^n(G; \mathbf{Z}G)$ is torsion-free.

Example

$H^*(\mathbf{Z}^n; \mathbf{Z}\mathbf{Z}^n) = H^*_c(\mathbf{R}^n)$, which is concentrated in degree $* = n$, where it is $\cong \mathbf{Z}$.

Why are we interested in $\ell^2 G$ coefficients?

Because Hilbert G modules have a "dimension" with respect to the von Neumann algebra $\mathcal{N}(G)$. Hence we can define ℓ^2-Betti numbers:

$$\ell^2 b^i(Y, G) := \dim_{\mathcal{N}(G)} H^i_G(Y; \ell^2 G).$$

Example

If G is a (higher genus) surface gp, then $H^*(G; \ell^2 G) = H^*(\mathbf{H}^2; \ell^2 G)$ which is concentrated in degree 1 and $\ell^2 b^1(G) = -\chi(G)$. ($\mathbf{H}^2$ means the hyperbolic plane.)

Two methods of proof

First Method

Find a direct sum decomposition of G-module as $M = \bigoplus_T \, M^T$ and a corresponding decomposition of cochain complexes as so that each summand gives constant coefficients except that they are 0 on a certain subcomplex $X(T)$, giving

$$C^*(X;M) = \bigoplus_T C^*(X, X(T)) \otimes M^T)$$

This gives corresponding decomposition in cohomology. (This method was used for Coxeter groups and locally finite buildings.)

Second Method

We compute $H^*(EG \times_G \tilde{X}; M)$ by using a spectral sequence which decomposes at E_2 as a direct sum:

$$E_2^{pq} = \bigoplus_T H^p(X_T, \partial X_T; H^q(BG_T; M))$$

where X_T is certain subcomplex of X. Furthermore, the spectral sequence degenerates at E_2. Ignoring torsion, the terms on the RHS can be rewritten as $H^p(X_T, \partial X_T) \otimes H^q(BG_T; M)$. In both methods the space X is the same: the fundamental chamber for standard complex with a Coxeter group action.)

Which groups G are we interested in?

- Coxeter groups
- Artin groups
- Bestvina-Brady groups
- graph product of groups.

Coxeter groups

$M = (m_{st})$ a symmetric $S \times S$ matrix with 1's on the diagonal and off-diagonal entries integers ≥ 2 or ∞. (M is called a *Coxeter matrix*.)

$$W := \langle S \mid (st)^{m_{st}} \rangle_{(s,t) \in S \times S} \rangle$$

(W, S) is called a *Coxeter system*. W is *right-angled* (a RACG) if each off-diagonal $m_{st} = 2$ or ∞.

Notation

$$\begin{aligned}\mathcal{S} &:= \{T \subset S \mid |W_T| < \infty\}\\ &= \text{ the poset of } \textit{spherical subsets}\end{aligned}$$

$L = L(W, S)$ is the *nerve* of (W, S), ie, the simplicial complex with vertex set S and simplices the nonempty elements of $\mathcal{S}$.
$K =$ geometric realization of $\mathcal{S} \cong$ the cone on L.
$K_s =$ the geometric realization of $\mathcal{S}_{\geq\{s\}} \cong \mathrm{Cone}(\mathrm{Lk}(s))$, where $\mathrm{Lk}(s)$ denotes the link of s in L.

$$K^{S-T} := \bigcup_{s \in S-T} K_s, \quad \partial K := K^S, \quad K_T := \bigcap_{s \in T} K_s$$

Artin groups

As before, (m_{st}) is a Coxeter matrix. Introduce generators $\{g_s\}_{s \in S}$ and for each $s \neq t$ with $m_{st} < \infty$, relations

$$g_s g_t \cdots = g_t g_s \cdots$$

setting equal the alternating words of length m_{st}. (NB each generator g_s has infinite order.) The result is the *Artin group* A. Let W be associated Coxeter gp. There is a a certain cell cx X' on which W acts freely. $X := X'/W$ is the *Salvetti cx*. $\pi_1(X) = A$.

The $K(\pi, 1)$-Conjecture

$X = BA \quad$ (ie X is a $K(A, 1)$).

Definition

If each $m_{st} = 2$ or ∞, then A is *right-angled* (a RAAG).

Example

If A is a RAAG, then X is a certain union of subtori of T^S and the $K(\pi, 1)$-Conjecture is true.

The setup

Γ a graph with $\mathrm{Vert}(\Gamma) = S$; L the flag cx determined by the graph and (W, S) the *RACS* with nerve L. Let $\{X_s\}_{s\in S}$ be a family of pointed spaces. Their *polyhedral product* is defined by

$$\mathcal{T}_L X_s := \bigcup_{T\in\mathcal{S}} X_T$$

where $X_T = \prod_{s\in T} X_s \subset \prod_{s\in S} X_s$.

Let $\{G_s\}_{s\in S}$ be a family of groups. Their *graph product* G is defined by

$$G = \prod_{\Gamma} G_s := \pi_1(\mathcal{T}_L BG_s)$$

Example

- If each $G_s = \mathbf{Z}/2$, then $G = \prod_{\Gamma} G_s$ is a RACG.
- If each $G_s = \mathbf{Z}$, then G is a RAAG.

Bestvina-Brady groups

Let A_L be the RAAG associated to a flag cx L. Let $\varphi : A_L \to \mathbf{Z}$ send each standard generator to 1. The *Bestvina-Brady group* is $BB_L := \operatorname{Ker} \varphi$.

Theorem (Bestvina-Brady)

If L is acyclic, then BB_L is type FP (or FL), but not finitely presented if $\pi_1(L) \neq 1$.

General form of the results

In every case, there is a Coxeter system (W, S) in the background. $\mathcal{S}$ is the poset of spherical subsets of S and K is the geometric realization of $\mathcal{S}$. There are explicit computations in almost all cases and they all have the same general form:

$$H^*(G; M) = \bigoplus_{\substack{T \in \mathcal{S} \\ p \leq *}} H^p(?, ?) \otimes M^{T,p},$$

where $(?, ?)$ is a pair of subcomplexes of K and $M^{T,p}$ is an abelian gp or G-module.

It turns out that there are two distinct possibilities for $(?, ?)$. In the first case (the locally finite case),

$(?, ?) = (K, K^{S-T})$, and there is no shifting of degrees in cohomology. (Remember $K^{S-T} = \bigcup_{s \in S-T} K_s$.) In the second case (the locally infinite case),

$(?, ?) = (K_T, \partial K_T)$,

and cohomology is shifted in degrees. (Remember $K_T = \bigcap_{s \in T} K_s$.)

Here

- ∂K_T is the (barycentric subdivision of) the link of the simplex T in L and $K_T = \mathrm{Cone}(\partial K_T)$
- K^{S-T} (the union of mirrors indexed by $S - T$) is homotopy equivalent to the complement of the simplex T in L, and K is the cone on ∂K.

As an example of the first case:

Theorem (D)

$H^*(W;\mathbf{Z}W) = \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes M^T$, *for a certain free abelian gp* M^T.

Remarks

- (DDJMO) A similar formula holds for any locally finite bldg of type (W, S).
- In particular since a graph product of finite groups is a locally finite *RAB*, a similar formula holds for such graph products.

The next two results are examples of the second case:

Theorem (D - Leary)

A the Artin gp associated to (W,S) and X its Salvetti cx. Then

$$H^*(X;\ell^2 A) \cong H^*(K,\partial K)\otimes \ell^2(A)$$

In particular, $\ell^2 b^i(X;A) = b^i(K,\partial K)$. If $K(\pi,1)$-Conjecture holds for A, then we can replace the left hand side by $H^*(A;\ell^2 A)$.

I should be saying "reduced" ℓ^2-cohomology and writing $\mathcal{H}^*(X)$.

Theorem (Jensen-Meier)

If A is a RAAG, then

$$H^*(A;\mathbf{Z}A) = \bigoplus_{T\in\mathcal{S}} H^{*-|T|}(K_T,\partial K_T)\otimes \textit{free abelian gp}$$

This theorem was originally proved by using the first theorem and result of DJ that any *RAAG* is commensurable with a *RACG*.

Theorem

Suppose $G = \prod_\Gamma G_s$ is a graph product, where each G_s is infinite. Then

$$H^n(G;\mathbf{Z}G) = \bigoplus_{T\in\mathcal{S}} \bigoplus_{p+q=n} H^p(K_T, \partial K_T; H^q(G_T;\mathbf{Z}G))$$

Similarly,

Theorem

Still supposing each G_s is infinite,

$$\ell^2 b^n(G) = \sum_{T\in\mathcal{S}} \sum_{p+q=n} b^p(K_T, \partial K_T) \cdot \ell^2 b^q(G_T)$$

- Here G_T denotes the direct product $\prod_{s \in T} G_s$. So, ignoring torsion
$$H^*(G_T; \mathbf{Z}G_T) = \bigotimes_{\sum i_s = *} H^{i_s}(G_s; \mathbf{Z}G_T)$$
- I should be putting a Gr in front of the LHS for "associated graded group".

Artin groups

Suppose

- $A = A_L$ is the Artin group associated to (W, S), and X_L is the associated Salvetti complex.
- For each $T \subset S$, A_T is the subgp generated by T. When T is spherical $H^*(A_T; \mathbf{Z}A_T)$ is free abelian and concentrated in degree $|T|$ (ie A_T is a duality gp)

Theorem

$$H^n(X_L; \mathbf{Z}A_L) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes H^{|T|}(A_T; \mathbf{Z}A_L)$$

Bestvina-Brady groups

- Let A_L be the *RAAG* associated to the *RACS* (W, S), where $L =$ nerve (W, S) (ie A_L is a graph product of $\mathbf{Z}$'s).
- $BB_L = \mathrm{Ker}(A_L \to \mathbf{Z})$, the map which sends each generator to 1.
- If L is acyclic, then BB_L is called a *Bestvina-Brady group*.

Theorem

Suppose BB_L is Bestvina-Brady. Then the cohomology of BB_L with group ring coefficients is isomorphic to that of A_L shifted up in degree by 1*:*

$$H^n(BB_L; \mathbf{Z}BB_L) = \bigoplus_{T \in \mathcal{S}_{>\emptyset}} H^{n-|T|+1}(K_T, \partial K_T) \otimes \mathbf{Z}(BB_L / BB_L \cap A_T).$$

L^2-cohomology of BB_L

Let $L^2b^k(BB_L)$ be the k^{th} L^2-Betti number of BB_L.

Theorem

Suppose BB_L is Betvina-Brady. Then

$$L^2b^k(BB_L) = \sum_{s \in S} b^k(K_s, \partial K_s)$$

where $b^k(K_s, \partial K_s)$ $(= \overline{b}^{k-1}(\mathrm{Lk}(s)))$ is the ordinary Betti number.

Idea of proofs

- Suppose $\mathcal{P}$ is a poset, $\{X_a\}_{a\in\mathcal{P}}$ is a poset of spaces and $X = \bigcup_{a\in\mathcal{P}} X_a$
- There is a spectral sequence with

$$E_1^{p,q} = C^p(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^q(\mathcal{V}))$$

converging to $H^*(X)$, where the (nonconstant) coefficient system $\mathcal{H}^q(\mathcal{V})$ associates to a simplex $\sigma \in \mathrm{Flag}(\mathcal{P})$ the abelian gp $H^q(X_{\min a})$

- Want conditions to insure a decomposition:

$$E_2^{p,q} = E_\infty^{p,q} = \bigoplus_{a\in\mathcal{P}} H^p(\mathrm{Flag}(\mathcal{P}_{\leq a}), \mathrm{Flag}(\mathcal{P}_{<a}); H^q(X_a))$$

Put $X_{<a} := \bigcup_{b<a} X_b$.

Main Lemma

The condition we need for this decomposition to hold is that $H^*(X_a) \to H^*(X_{<a})$ is the 0-map, $\forall a \in \mathcal{P}$

In all situations in which we will apply this lemma, $\mathcal{P} = \mathcal{S}$ so that $\text{Flag}(\mathcal{P}) = K$ and $\forall T \in \mathcal{S}$,

$$(\text{Flag}(\mathcal{P}_{\leq T}), \text{Flag}(\mathcal{P}_{<T})) = (K_T, \partial K_T).$$

The key point

for applying this to graph products is that when each G_s is infinite, $H^0(G_s; \mathbf{Z}G_s) = 0$, so by Künneth Formula, $H^*(G_T; \mathbf{Z}G_T) \to H^*(G_U; \mathbf{Z}G_T)$ is the 0-map whenever $U < T$.

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