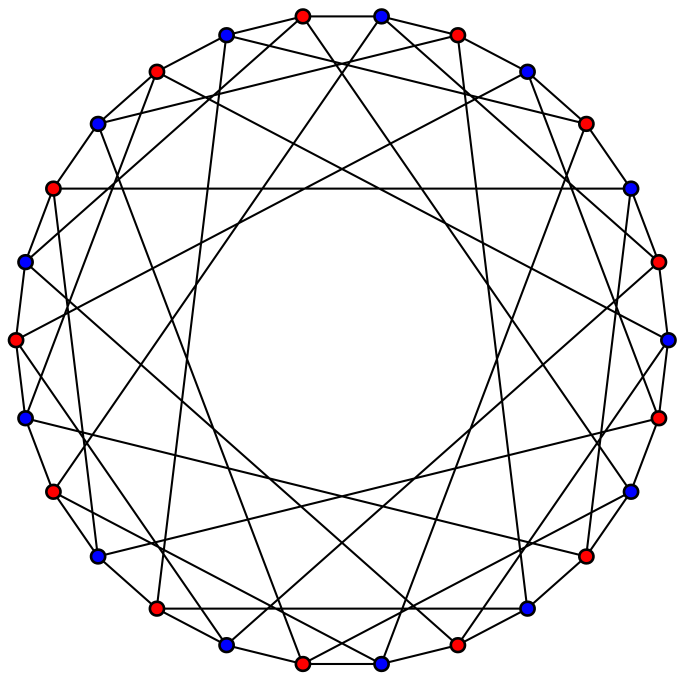
Hyperbolic Coxeter groups and their finite simple cousins



Jon McCammond (U.C. Santa Barbara)

Outline

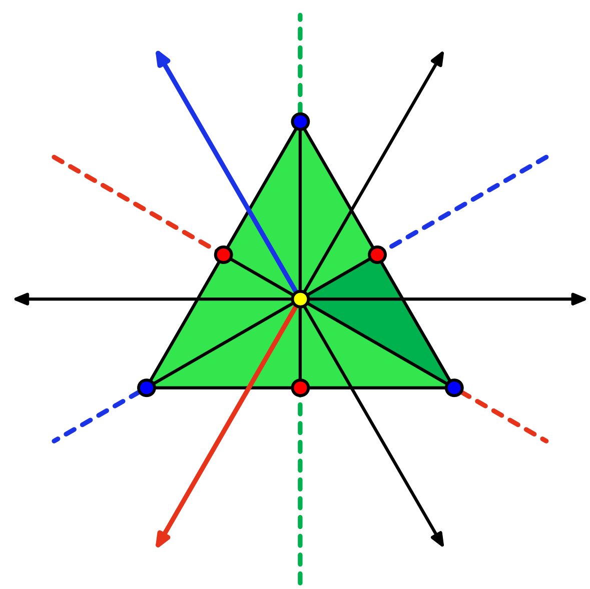
This talk focuses on describing the collection of simply-laced Coxeter groups with a hyperbolic signature. There is one group in this collection that has a surprising relationship with the largest of the sporadic finite simple groups and there are hints of other such connections. I was torn between describing the surprising connection on the one hand and the various examples and partial classification results on the other. I'll start with the former and include the latter as time permits because the former motivates the latter.

I. Hyperbolic Coxeter Groups
II. Finite Simple Groups
III. Theorems and Examples

I. Hyperbolic Coxeter Groups

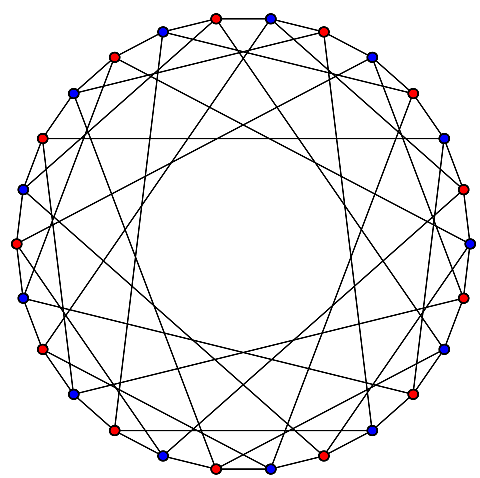
Def: A **Coxeter presentation** is a finite presentation $\langle S \mid R \rangle$ with a relation s^2 for each $s \in S$, and at most one relation $(st)^m$ for each pair of distinct $s, t \in S$. Such a presentation defines a **Coxeter group**. All finite reflection groups, e.g. the isometry groups of regular polytopes, have Coxeter presentations.

Ex: $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$



Simply-Laced Coxeter Groups

Def: A Coxeter group $W = \langle S \mid R \rangle$ is **simply-laced** (or **small-type**) if for every pair of distinct $s, t \in S$, either $(st)^2$ or $(st)^3$ is a relation in R. Our convention is that s and t are connected by an edge iff st has order 3. Diagrams of small-type Coxeter groups correspond to arbitrary simplicial graphs. For other Coxeter groups labels are added to the edges.



Coxeter Elements

Def: A **Coxeter element** in a Coxeter group is the product of its generators in some order.

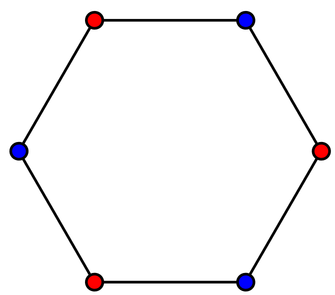
Thm: If the Dynkin diagram has no cycles then all of its Coxeter elements are conjugate. In particular, Coxeter elements in finite Coxeter groups are well-defined up to conjugacy.

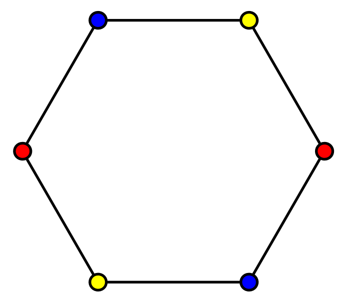
Rem: Coxeter elements are in bijection with acyclic orientations of the Dynkin diagram, and orientations equivalent under the relation generated by “reflection functors” represent conjugate Coxeter elements. (This is closely related to the theory of quivers in representation theory.)

Distinct Coxeter Elements

Ex: Consider the Coxeter group defined by a hexagon.

- there are $6! = 720$ orderings of the generators,
- but only $2^6 - 2 = 62$ different group elements,
- that fall into 5 distinct conjugacy clases and 3 types.

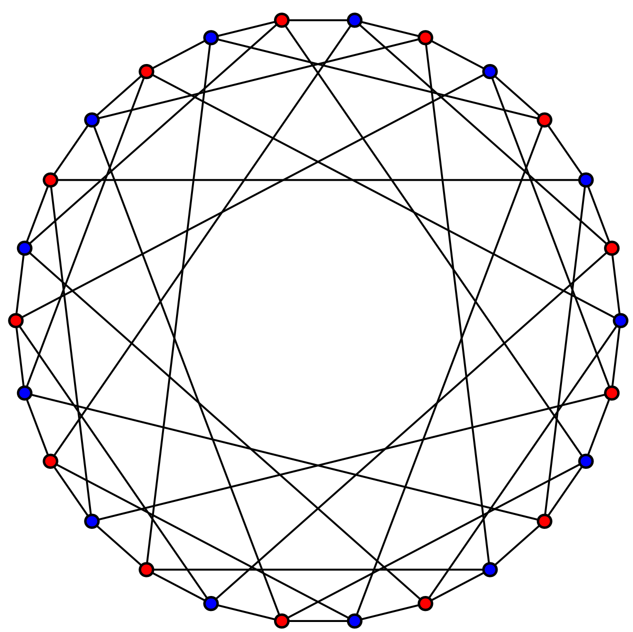




The 3 types are the cyclic version, the bipartite version (on the left), and the antipodal version (on the right).

Buildings

The incidence graphs of finite projective planes are highly symmetric and examples of **buildings**. They have diameter 3, girth 6, distance transitive, and every pair of points lies on an embedded hexagon.



Finite Projective Planes

The construction of $\mathbb{K}P^2$ still works perfectly well when $\mathbb{K}$ is a *finite* field instead of $\mathbb{R}$ or $\mathbb{C}$. Lines through the origin in the vector space $\mathbb{K}^3$ become (projective) points in $\mathbb{K}P^2$, planes through the origin in $\mathbb{K}^3$ become (projective) lines in $\mathbb{K}P^2$.

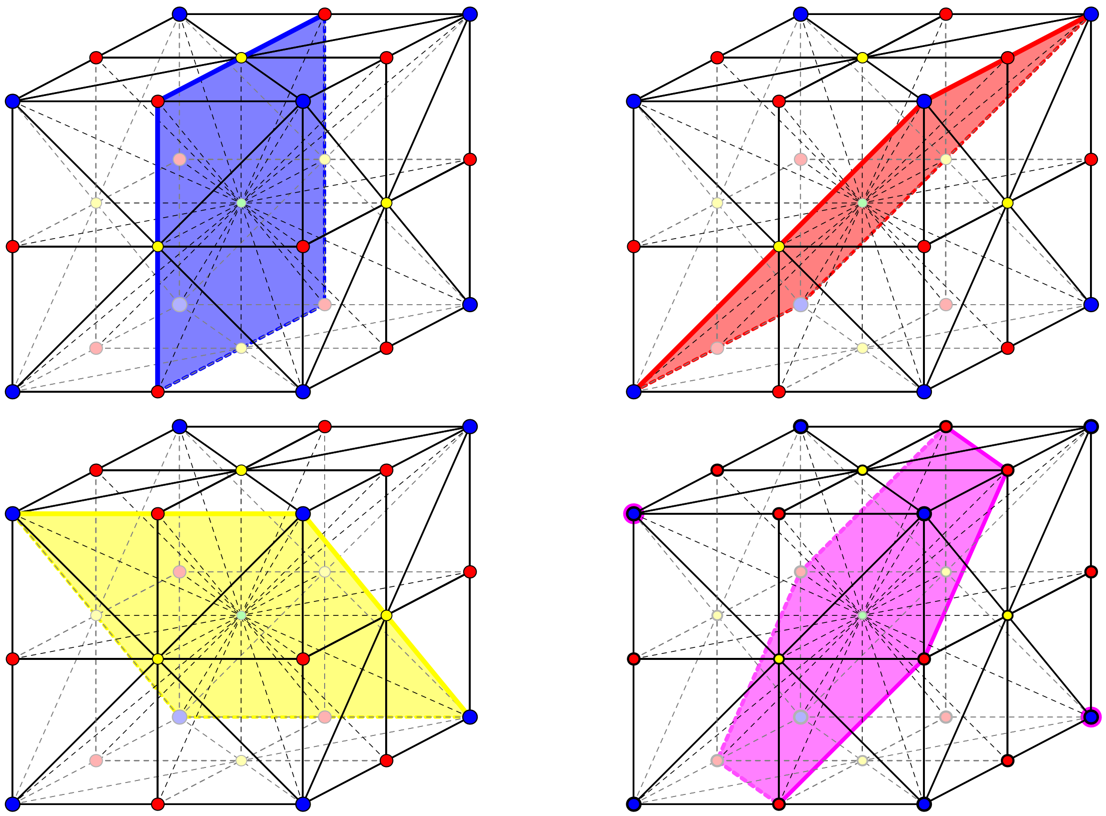
If the field $\mathbb{K}$ has q elements, then $\mathbb{K}P^2$ has
- $q^2 + q + 1$ points and
- $q^2 + q + 1$ lines.

Moreover, there is the usual duality between points and lines.

Rem: Finite projective spaces can be viewed as discrete analogs of symmetric spaces and their automorphism groups as discrete analogs of Lie groups (a.k.a. finite groups of Lie type).

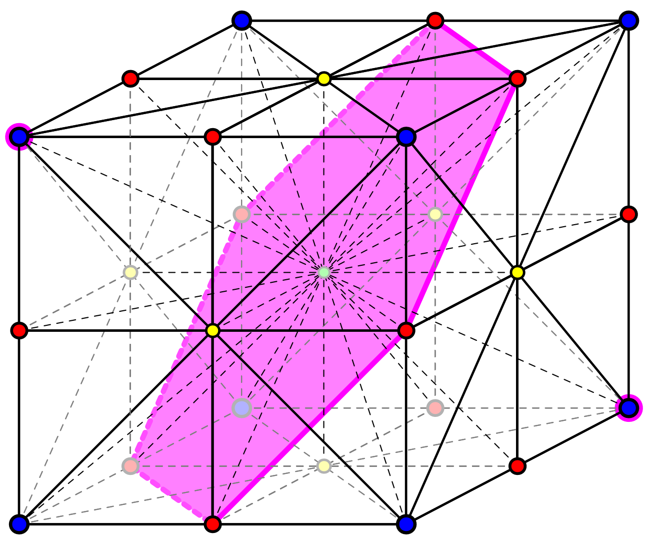
A Sample Finite Projective Plane

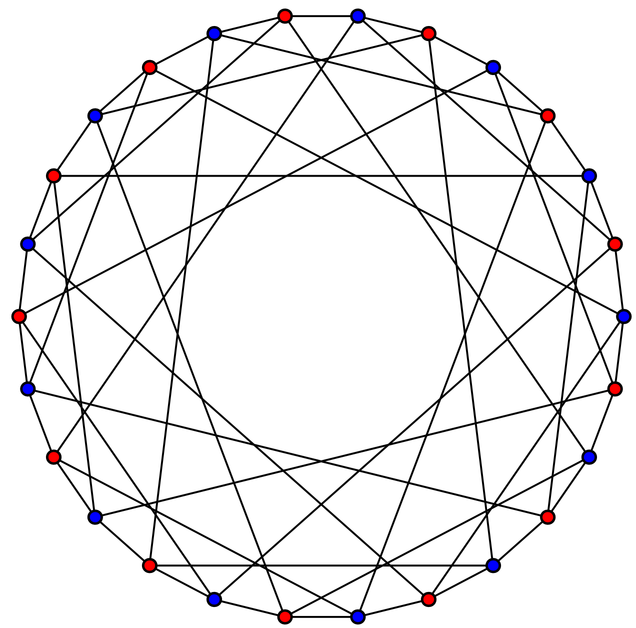
The finite projective plane over $\mathbb{F}_3$ with its $9+3+1=13$ points and 13 lines can be visualized using a cube.



Incidence Graphs

Let $\mathrm{Inc}(\mathbb{K}P^2)$ denote the incidence graph of $\mathbb{K}P^2$: draw a red dot for every projective point in $\mathbb{K}P^2$, a blue dot for every projective line and connect a red dot to a blue one iff the point lies on the line. $\mathrm{Inc}(\mathbb{F}_3P^2)$ is shown.





Bilinear Forms

The normal vectors $\vec{u}_i$ arising from the basic reflections of a finite reflection group determine a matrix $M = [\vec{u}_i \cdot \vec{u}_j]_{(i,j)}$ where the dot products are 1 along the diagonal and $\cos(\pi - \pi/n)$ otherwise (where n is label on the edge connecting v_i and v_j or $n = 3$ when there is no label, or $n = 2$ when there is no edge).

The formula can be followed blindly for any Dynkin diagram. Define a bilinear form on $V = \mathbb{R}^n$ by setting $B(\vec{x}, \vec{y}) = \vec{x}\ M\ \vec{y}^{\ T}$. B is the form associated with the corresponding Coxeter group and this turns $\mathbb{R}^n$ into a metric vector space.

After rescaling, simply-laced diagrams produce a matrix with 2's on the diagonal and 0 and -1 off diagonal.

Metric vector spaces

A *metric vector space* is a vector space with a form defining lengths and angles. Let $\mathbb{R}^{p,q,r}$ be the metric vector space on which we impose the bilinear form defined by the matrix

$$\begin{bmatrix} I_p & & \\ & -I_q & \\ & & O_r \end{bmatrix}$$

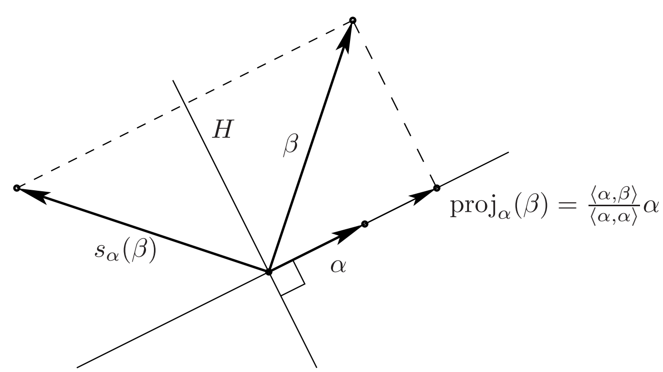
Every real metric vector space is isometric to $\mathbb{R}^{p,q,r}$ for canonically defined p, q, and r. The triple (p, q, r) is the *signature* of the form. Drop r when $r = 0$, and drop q and r when both are 0. The space $\mathbb{R}^{p,1}$ contains the hyperboloid model of $\mathbf{H}^p$.

Linear Representations

Let $V = \mathbb{R}^n$, let Γ be a Dynkin diagram and let W be the Coxeter group it defines. Using the bilinear form B we can define a linear representation of W. For each generator s_i define a reflection $\rho_i : V \to V$ by setting

$$\rho_i(\vec{v}) = \vec{v} - 2\frac{B(\vec{e_i},\vec{v})}{B(\vec{e_i},\vec{e_i})}\vec{e_i}$$

This mimics the usual formula for a reflection.



Generalized Orthogonal Groups

For any bilinear form B, let $O(V, B)$ denote the set of invertible linear transformations T of the n-dimensional vector space V that preserve this bilinear form: $B(T\vec{x}, T\vec{y}) = B(\vec{x}, \vec{y})$.

Rem: $O(V, B)$ is a subset of $GL(V)$ and it inherits a Lie group structure.

Thm (Tits) The homomorphism $W \to O(V, B)$ is an embedding.

By the earlier remark, (V, B) is a metric vector space isomorphic to some $\mathbb{R}^{p,q,r}$, and $O(V, B) = O(\mathbb{R}^{p,q,r})$. We can use its signature to coarsely classify Coxeter groups into types.

Types of Coxeter Groups

Let W be a Coxeter group whose matrix M has p positive, q negative and r zero eigenvalues. We say W is **spherical** when $q = 0$, W is **hyperbolic** when $q = 1$ and W is **higher rank** when $q > 1$. When $r > 0$ we add the adjective **weakly**.

Ex: The Coxeter group defined by:

- a hexagon is weakly spherical (a.k.a. affine),
  Spectrum = $[4^1\ 3^2\ 1^2\ 0^1]$
- $\mathrm{Inc}(\mathbb{F}_3 P^2)$ is hyperbolic,
  Spectrum = $[6^1\ (2+\sqrt{3})^{12}\ (2-\sqrt{3})^{12}\ (-2)^1]$
- the 1-skeleton of the 4-cube is weakly hyperbolic.
  Spectrum = $[6^1\ 4^4\ 2^6\ 0^4\ (-2)^1]$
- the 1-skeleton of the 5-cube is higer rank.
  Spectrum = $[7^1\ 5^5\ 3^{10}\ 1^{10}\ (-1)^5\ (-3)^1]$

Graph Spectra

When W is a simply-laced Coxeter group its matrix M is $2I - A$ where A is the adjacency matrix of the graph Γ defining M. Because $A + M = 2I$ the spectrum of one determines the other. More precisely, if λ is an eigenvalue of one of these matrices then $2 - \lambda$ is an eigenvalue of the other with the exact same eigenvectors.

If we label the eigenvectors of A as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, then Γ defines a hyperbolic Coxeter group iff $\lambda_1 > 2 \geq \lambda_2$.

Many patterns are easier to see in the graph spectrum and there is an extensive literature on the topic. For example: (1) a graph is bipartite iff its spectrum is symmetric about 0, (2) $k \geq \lambda_1$ where k is the maximum vertex degree with equality iff Γ is regular, and (3) $\lambda_1 > \lambda_2$ when Γ is connected (Perron-Frobenius).

Hyperbolic vs. Hyperbolic

There are other notions of a hyperbolic Coxeter group. There are Coxeter groups that are Gromov-hyperbolic (classified by Moussong and unrelated to the ones defined here). There are the Coxeter groups studied by Vinberg and his students. These act on hyperbolic space with either a compact or finite volume fundamental domain that is either simplicial or polytopal. Dimension bounds are known in each case.

	Simplex	**Polytope**
Compact	5	≤ 30
Finite Volume	10	≤ 996

Coxeter groups that are hyperbolic in the sense defined here satisfy no dimension bounds and include the Vinberg examples.

II. Classification of Finite Simple Groups

Recall the classification theorem for finite simple groups.

Thm: Every finite simple group is either
1. Cyclic ($\mathbb{Z}_p$, p prime),
2. Alternating (Alt_n, $n \geq 5$),
3. A finite group of Lie type,
4. One of 26 sporadic exceptions.

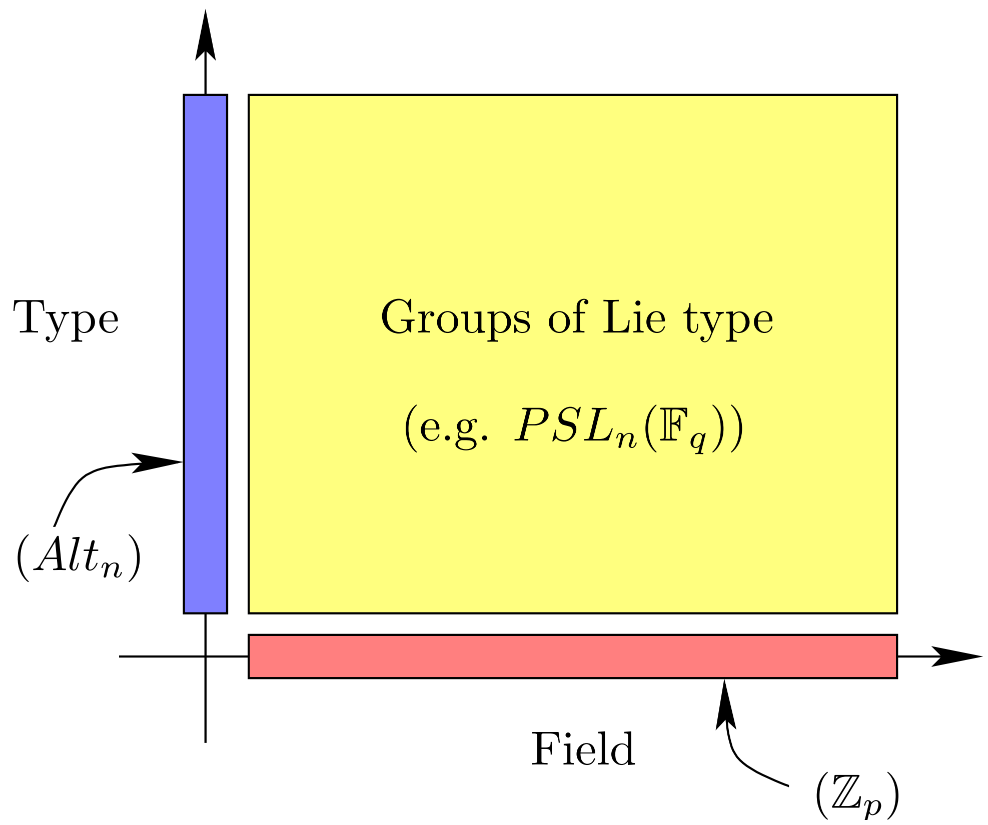
Finite Groups of Lie Type

The finite groups of Lie type are 16 infinite families of finite groups all of the form $X_n(q)$ where X_n is a Cartan-Killing type and q is a power of a prime. The X_n indicates the bilinear form and dictates the construction, and q is order of the finite field over which the construction is carried out.

$A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, $E_8(q)$, $E_7(q)$, $E_6(q)$, $F_4(q)$, $G_2(q)$.

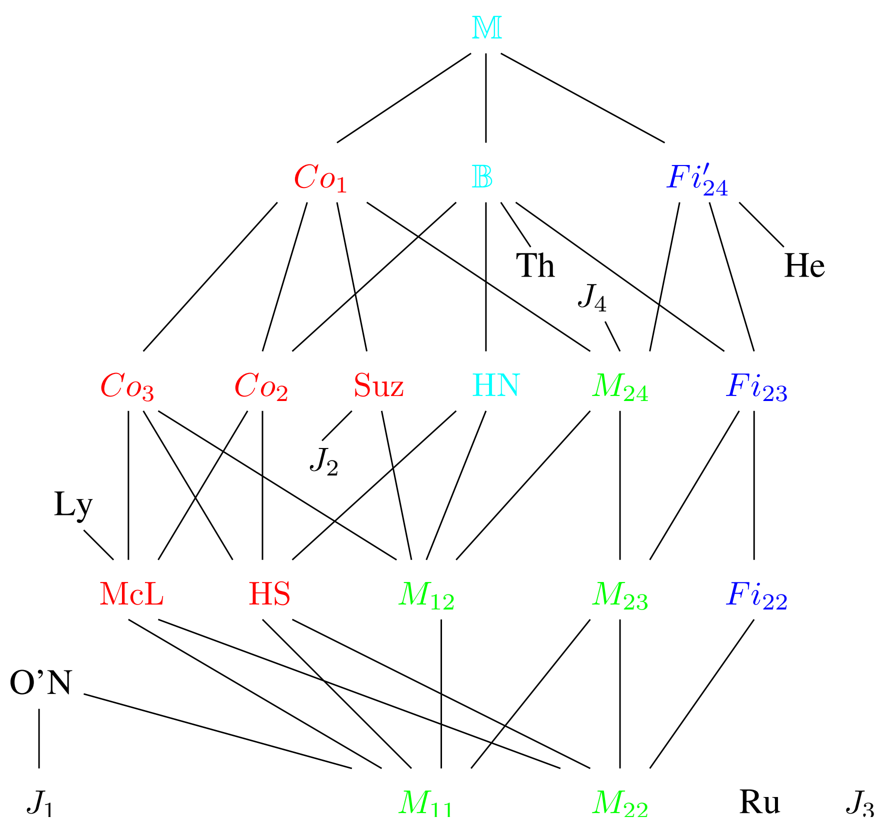
Type $A_n(q)$ comes from the automorphism groups of finite projective spaces over $\mathbb{F}_q$. In addition to these 9, diagram symmetries lead to twisted versions ($^2A_n(q)$, $^2D_n(q)$, $^3D_4(q)$, $^2E_6(q)$) including some ($^2B_2(q)$, $^2G_2(q)$, $^2F_4(q)$) whose construction is characteristic dependent.

A Mnemonic for the Classification



(plus 26 sporadic exceptions)

The Sporadic Finite Simple Groups



A line means that one is an image of a subgroup in the other.

The Monster Finite Simple Group

The **Monster** finite simple group $\mathbb{M}$ is the largest of the sporadic finite simple groups with order

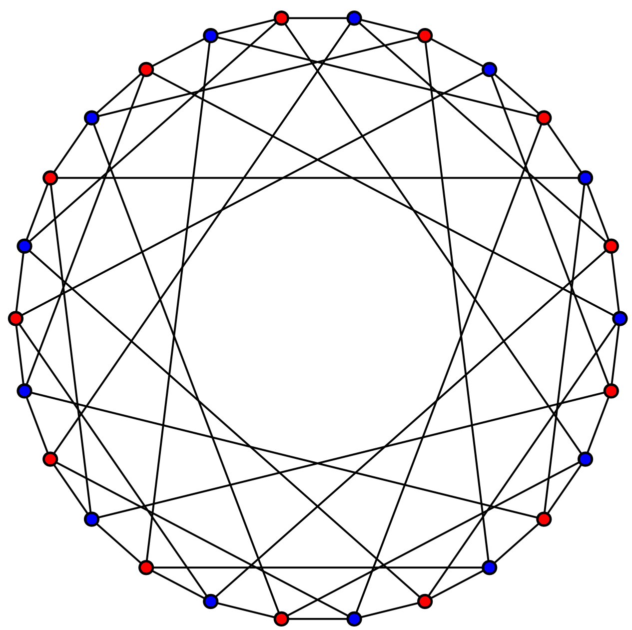
$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$$

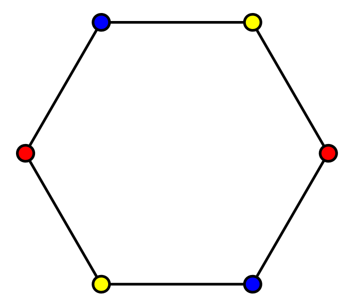
(which is 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000, or $\sim 10^{54}$)

The **Bimonster**, $\mathbb{M} \wr \mathbb{Z}_2$, is a related group of size $\sim 10^{108}$.

A One-relator Coxeter Presentation of the Bimonster

Thm (Conway,Ivanov-Norton) If W is the simply-laced Coxeter group defined by $\mathrm{Inc}(\mathbb{F}_3P^2)$, and $u \in W$ is the fourth power of the antipodal Coxeter element of a hexagonal subgraph in $\mathrm{Inc}(\mathbb{F}_3P^2)$, then $W/\langle\langle u\rangle\rangle \cong \mathbb{M} \wr \mathbb{Z}_2$.





III. Theorems and Examples

The other sporadics are related to the Monster images of subgroups of Cox(Inc($\mathbb{F}_3 P^2$)) and thus more connections between simply-laced hyperbolic Coxeter groups and sporadic finite simple groups should be expected. But first some partial classification results and large classes of examples.

Thm: Let Γ be a graph which (1) is a line graph, (2) has every vertex of degree at most 3, (3) is the graph of a Steiner triple system, or (4) is the graph of a latin square. Then the complement defines a hyperbolic Coxeter group.

There are also several sporadic highly regular examples with strong ties to the other sporadics.

[Switch to whiteboard for the remainder of the time]