Random weak limits of self-similar Schreier graphs

Tatiana Nagnibeda

University of Geneva
Swiss National Science Foundation

Goa, India, August 12, 2010

A problem about invariant measures

Suppose G is a group. Consider the space Sub(G) of all subgroups of G.

(Tychonoff topology on the space of all subsets: a subset is a configuration in $\{0,1\}^G$; the base is given by cylinders fixing n coordinates).

G acts on Sub(G) : H → $g^{-1}Hg$

Does there exist a non-atomic ergodic measure on Sub(G) invariant under this action? Describe groups for which such measures exist. Classify such measures for a given group.

Example (A.Vershik): The infinite symmetric group S_∞ . Such measures form a one-parameter family.

Recall from Yair's talk: no such measure on Sub(SL(3,Z)).

Schreier graphs

Suppose a group G acts on a set Y, and suppose G is generated by a finite symmetric set S. The Schreier graph Sch(G,S,Y) of this action with respect to the generating set S is a (rooted, labeled) graph:

- the vertex set is Y;
- two vertices y and y' form an edge labeled by s iff there exists a generator s such that s(y) = y'.

If the action is transitive, then Sch(G,S,Y) is the Schreier graph of G with respect to the subgroup $H=\mathrm{Stab}_G(y)$ for any y in Y, with the vertex set H\G, and the root vertex H.

If the action is not transitive, consider orbital Schreier graphs.

Orbital Schreier graphs (labeled, unrooted) are orbits for the action of G by conjugation on Sub(G).

Measures in the space of rooted graphs

The space of all rooted graphs (labeled or not) of bounded degree is equipped with the pointed Gromov-Hausdorff topology:

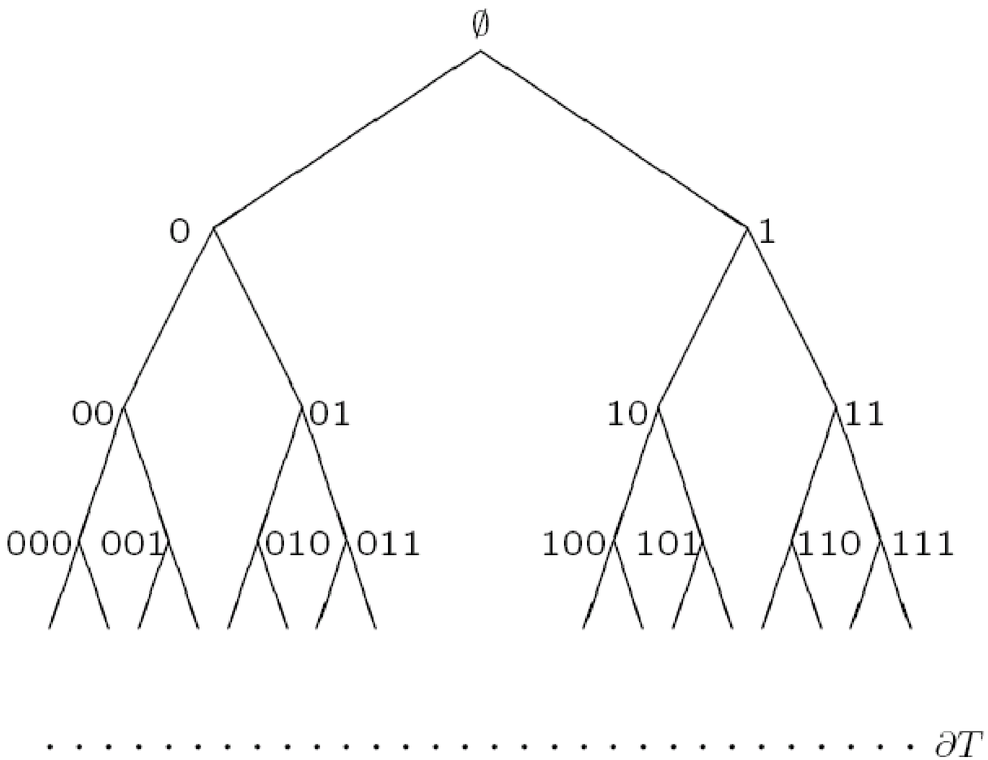
$(\Gamma_n, y_n) \to (\Gamma, y)$ as $n \to \infty$, iff for every k, there exists N such that for all n>N, the ball B(y, k) in Γ is isomorphic to the ball $B(y_n,k)$ in Γ_n.

Examples of subspaces: space of marked groups, space of Schreier graphs of a given group Sch(G,S)…

Instead of thinking about invariant non-atomic ergodic measures on Sub(G), we would like to look for ergodic non-atomic measures on Sch(G,S) invariant under the equivalence relation "forgetting the root".

This is possible if G acts on a probability space preserving the measure. Example: groups acting on rooted trees.

Self-similar groups



The rooted binary tree

$T=T_d$ - infinite d-ary tree; $V(T_d) = X^*$, $X=\{0,\dots d-1\}$.

$\mathrm{Aut}(T) = \mathrm{Aut}(T) \wr \mathrm{Sym}_d$; $g= \tau_g(g|_0,\dots g|_{d-1})$

where $\tau_g \in \mathrm{Sym}_d$ and $g|_0,\dots g|_{d-1}$ are restrictions of g on the subtrees rooted in the vertices of the first level.

A finitely generated subgroup $G < \mathrm{Aut}(T)$ is **self-similar** if $g|_v \in G$ for all $g\in G$ and all $v\in V(T)$.

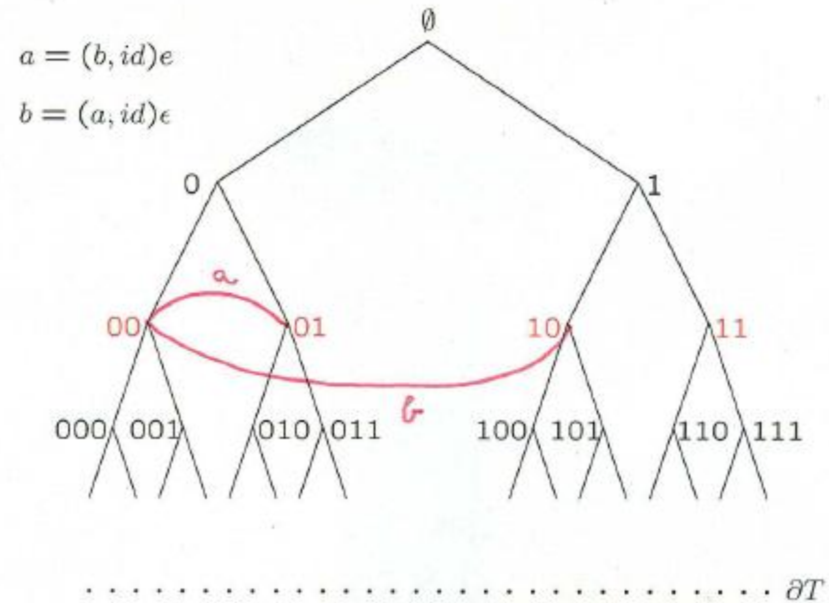
Grigorchuk, Bartholdi, Nekrashevych, Sidki, Sunic, Suschanski…

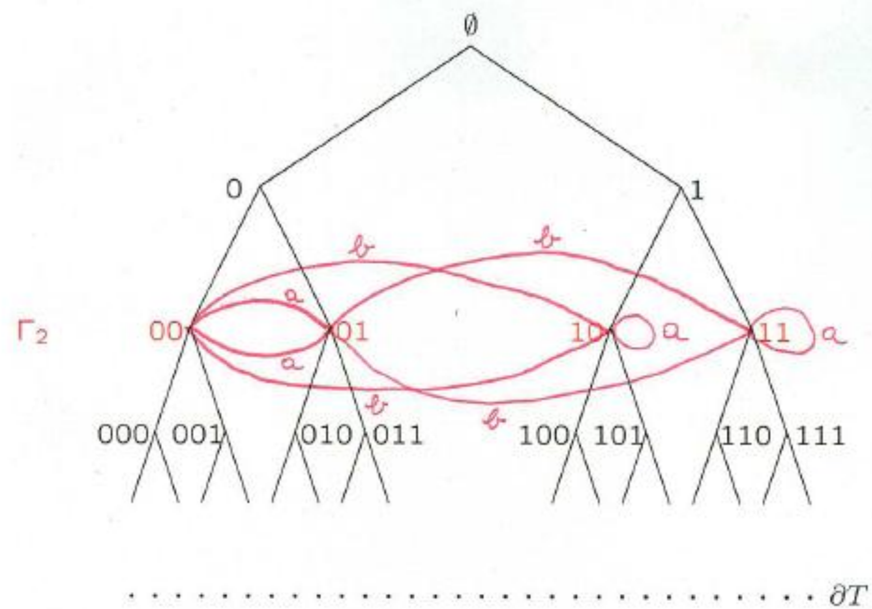
Ex.1. Grigorchuk's group of intermediate growth:

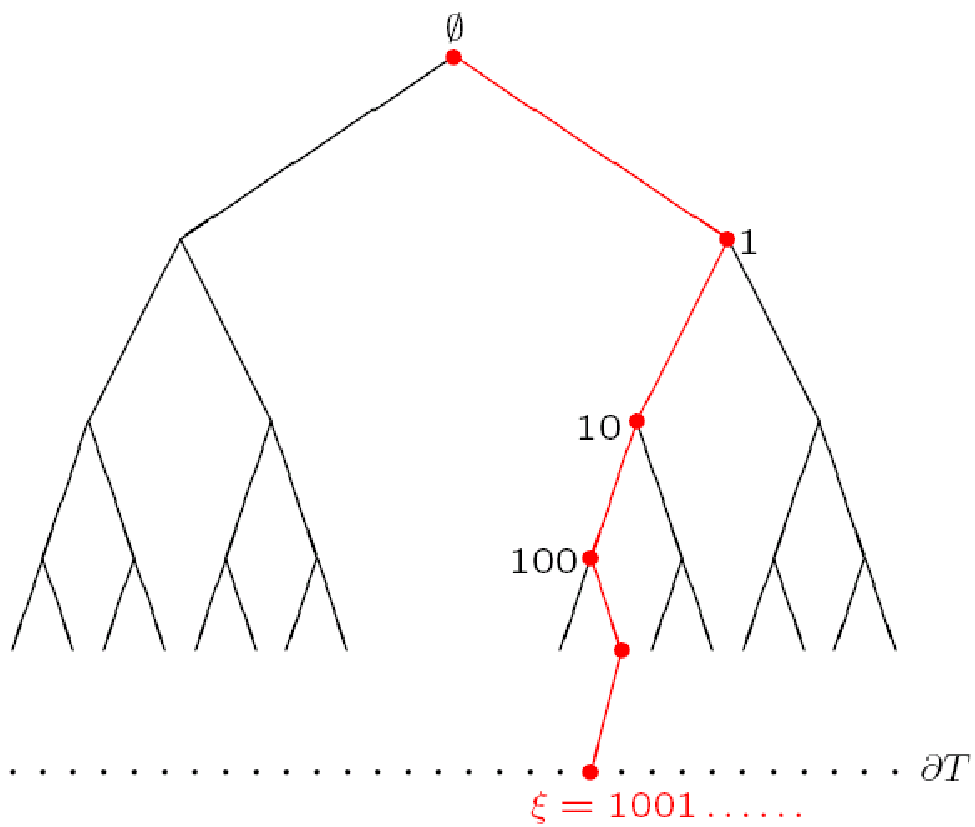
$G= \langle a,b,c,d\rangle$ with $a= \varepsilon(id, id)$; $b=e(a,c)e$; $c=e(a,d)$; $d=e(1,b)$

Ex.2. Basilica group (amenable but cannot be obtained from groups of subexponential growth by direct limits and extensions): $B = \langle a,b\rangle$ with $a=e(b, id)$; $b= \varepsilon(a, id)$

The action on T: $b(0w)=1a(w)$; $b(1w)=0w$; $a(0w)=0b(w)$; $a(1w)=1w$.


$a = (b, id)e$
$b = (a, id)\epsilon$
$\emptyset$
0
1
00
01
10
11
a
b
000
001
010
011
100
101
110
111
∂T


Γ_2
$\emptyset$
0
1
00
01
10
11
a
b
000
001
010
011
100
101
110
111
∂T



∅
1
10
100
∂T
ξ = 1001 . . . . . .

Schreier graphs of self-similar groups

$G<Aut(T)$, transitive on levels X_n. $G = <S>$, S finite

Γ_n = Sch (G, X_n, S) = Sch (G,$Stab_G(v)$,S) for any $v \in X_n$

Vertices = X_n ; $|Vert(\Gamma_n)| = d^n$. Edges = $\{(v,s(v)) \mid s \text{ in } S\}$

$$Stab_G(\xi) = \bigcap_{n\in\mathbb{N}} Stab_G(\xi_n)$$

and thus $\{\Gamma_n\}$ is a family of graph coverings.

G acts on $\partial T=\{\ \xi=x_0x_1x_2\ldots..\ \}$ by homeomorphisms. For $\xi \in \partial T$,

Γ_ξ = Sch (G, $Stab_G(\xi)$, S) the infinite (orbital) Schreier graph.

Then **$(\Gamma_n, x_0\ldots x_n) \rightarrow (\Gamma_\xi, \xi)$** as $n \rightarrow \infty$

in the rooted Gromov-Hausdorff convergence.

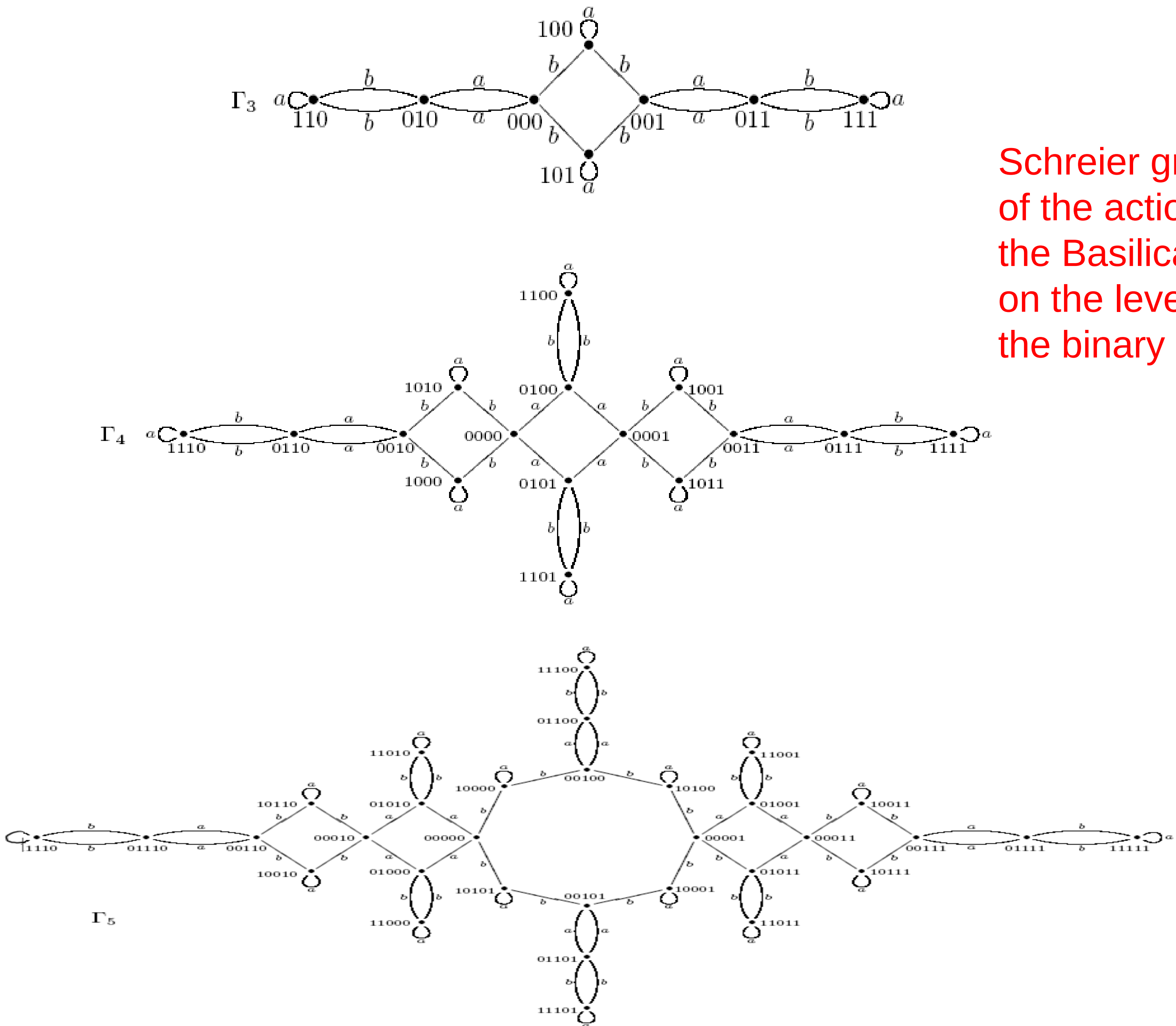
The union $\bigcup_{\xi\in\partial T} \Gamma_\xi$ is the inverse limit of the projective sequence $\{\Gamma_n\}_n$.

The class of self-similar groups contains:

- Examples of infinite torsion groups. E.g. the group G and many other examples (Aleshin, Suschanski, Grigorchuk, Gupta-Sidki…)
- Examples of groups of intermediate growth. G, other examples by Erschler, Nekrashevych.
- Examples of amenable groups that do not belong to the closure of {Finite}∪{Abelian} under taking direct products, subgroups, quotients, extenstions and direct limits (counterexample to Day's question). Groups of intermedite growth.
- Example of an amenable group that does not belong to the closure of {Subexponential} under the same operations. Basilica group B: Grigorchuk-Żuk '02, Bartholdi-Virág '05.
- Examples of exponential but not uniformly exponential groups. Recent examples by Nekrashevych.

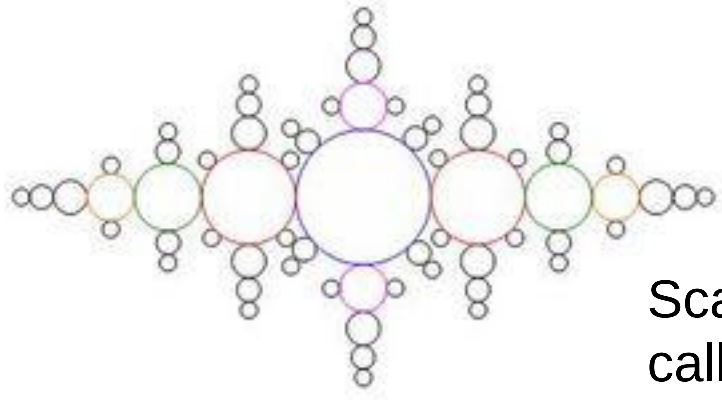
Schreier graphs of self-similar groups provide:

- First examples of regular graphs with the Laplacian spectrum Cantor set. (Kadison-Kaplansky Conjecture $\Longrightarrow$ the Laplacian spectrum of a Cayley graph of a torsion-free group is an interval).
- Examples of regular planar graphs of polynomial growth of degree log2/logα with α irrational. (I. Bondarenko).
- Examples of amenable actions of nonamenable groups. (free groups F_n also provide such examples).
- Approximating sequences for fractals via Nekrashevych's notion of Limit Space of G, defined if the action of G on T is contracting – see below. (With applications to spectra on fractals: Rogers-Teplyaev for the Basilica Julia set).



Schreier graphs of the action of the Basilica group on the levels of the binary tree

Γ_6



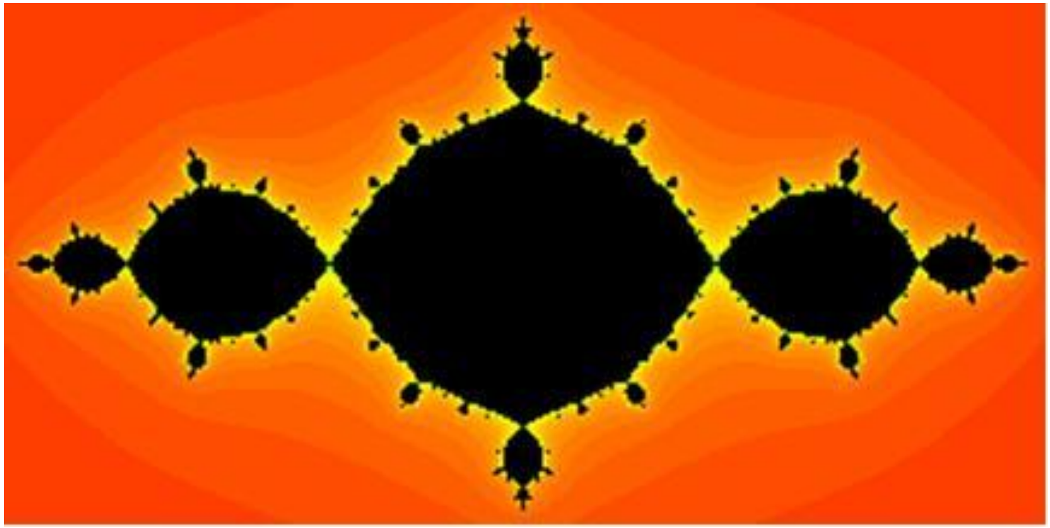
Scaling limit
called Limit space of G
(Nekrashevych)

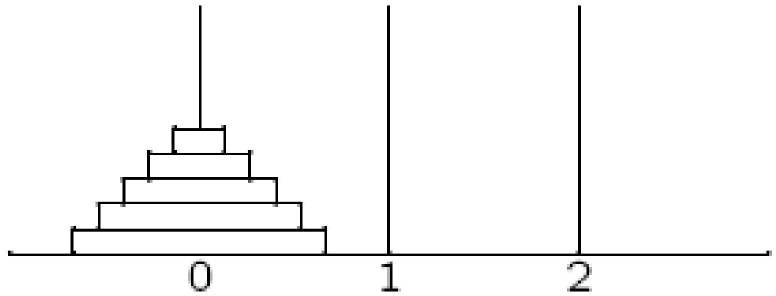
$n \to \infty$

. . .

B = IMG (z^2-1)

Julia set of z^2-1

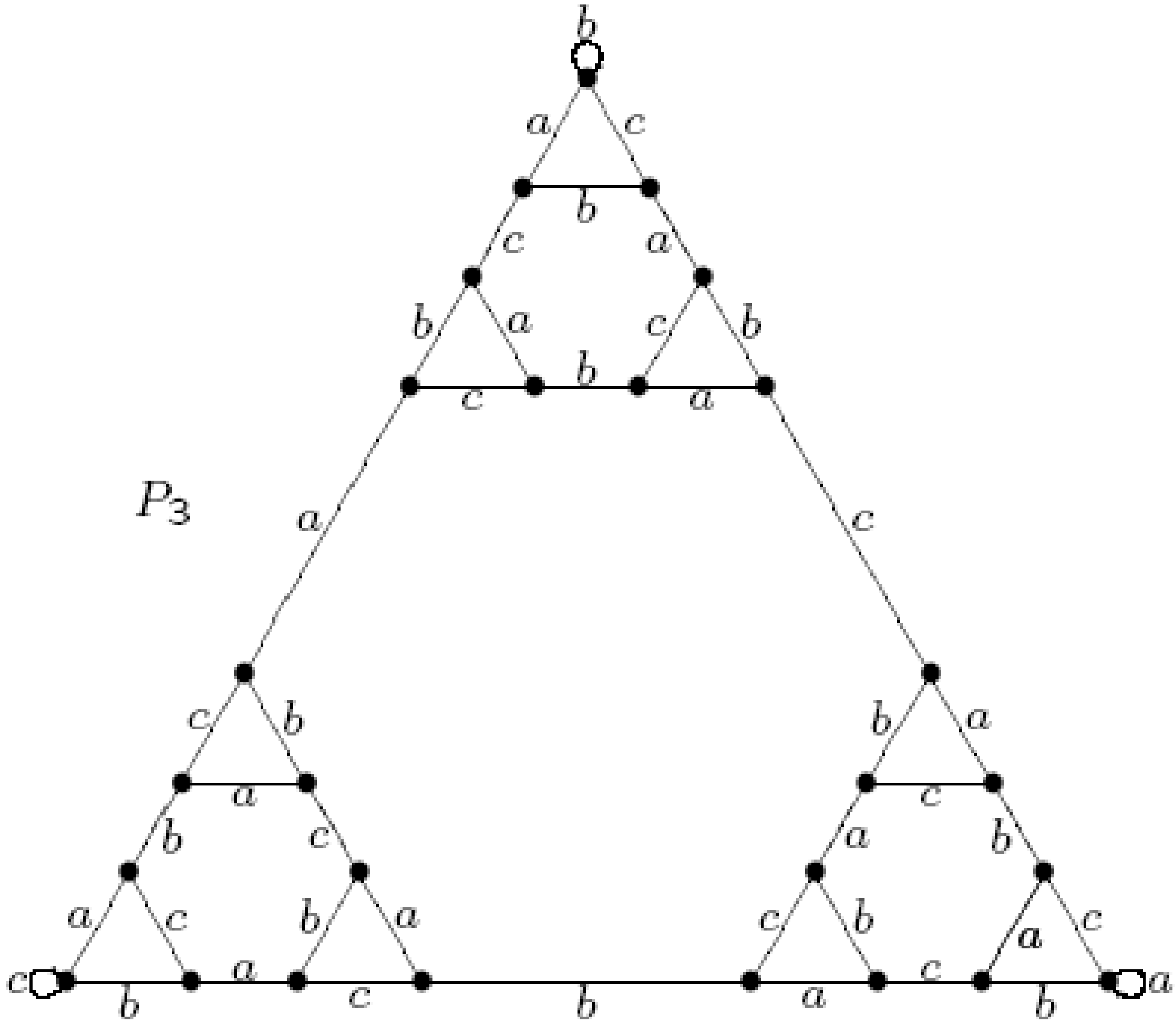




The Hanoi Towers game on three pegs.

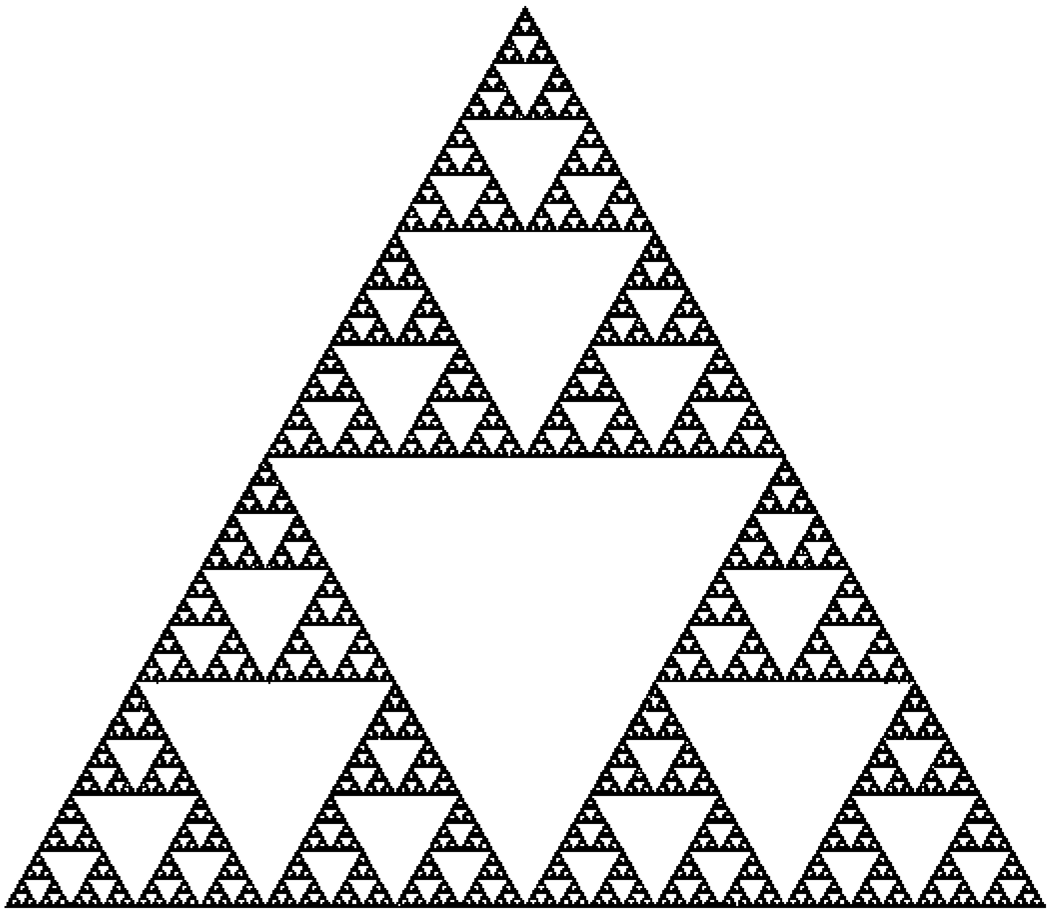
Given n disks of different sizes, the game consists in taking all n disks from a peg to an other by moving each disk so that at each step we have an allowed configuration. A configuration is allowed if no disk is placed on top of a smaller disk. Words of length n in the alphabet $\{0,1,2\}$ encode the configurations of n disks on three pegs.

Set
a:= moving a disk between peg 0 and 1,
b:= moving a disk between peg 0 and 2,
c:= moving a disk between peg 1 and 2.



The orbit graph of $H^{(3)}$ at level 3

The limit space of the Hanoi towers group $H^{(3)}$ is the Sierpinski gasket



Limit space (Nekrashevych)

Finite Schreier graphs of a contracting action of a self-similar group $G<Aut(T)$ provide an approximating sequence for a compact space called Limit Space of G.

Self-similarity graph of G: connect finite Schreier graphs with vertical edges of the type (v,0v) and (v,1v), where v is a vertex of the n-th level, and 0v,1v are vertices of the (n+1)-st level. The resulting infinite graph is Gromov-hyperbolic (first such example: Sierpinski graph, Kaimanovich '03).

The hyperbolic boundary of this graph = limit space of G (independent of the generating set, up to homeomorphism).

G is contracting iff $\exists$ a finite $N \subset G$ s.t. $\forall g \in G$, $g|_v \in N$ for all v long enough.

Random weak limits

Definition. (Benjamini-Schramm). Let $\{\Gamma_n\}_n$ be a sequence of finite graphs of bounded degree. Consider them as rooted graphs by choosing a root in each Γ_n uniformly at random. This defines a sequence of prob. measures on the space of rooted graphs, and one can consider its weak limit, “the random weak limit of $\{\Gamma_n\}_n$“ . The random weak limit is a probability distribution on the limits of the sequence of graphs $\{\Gamma_n\}$ in rooted G-H convergence, for all possible choices of roots in Γ_n, up to isomorphism of rooted graphs.

Aldous-Lyons, “Processes on unimodular random networks”: what measures on the space of rooted graphs (bounded degree) can arise in this way? In particular, can the δ-measure on a Cayley graph of an arbitrary f.g. group be approximated in this sense by finite graphs?

If yes, would imply that all f.g. groups are sofic.

Invariant measures on the space of Schreier graphs

1. The uniform measure λ on the inverse limit $\{\Gamma_\xi \mid \xi \in \partial T\}$
2. The random weak limit M of the sequence of Schreier graphs is an ergodic probability measure on the space of (isomorphism classes of) orbital Schreier graphs for the action of G on ∂T.
3. Consider f: $\lim_{\leftarrow} \Gamma_n \to$ space of rooted graphs, identifying isomorphic copies in $\{\Gamma_\xi \mid \xi \in \partial T\}$. Then $M = f(\lambda)$.

The problem arises of understanding isomorphisms of orbital Schreier graphs. For labeled graphs there are natural algebraic conditions that guarantee that the random weak limit is non-atomic, e.g. "weakly branched" (implying an easy answer to the question of existence of non-atomic measures of Sub(G)).

Non-labeled Schreier graphs

The question is more interesting for random weak limits of non-labeled (combinatorial) Schreier graphs.

For example, in the Grigorchuk's group the non-labeled RWL is atomic (though the labeled one is just the Lebesgue measure).

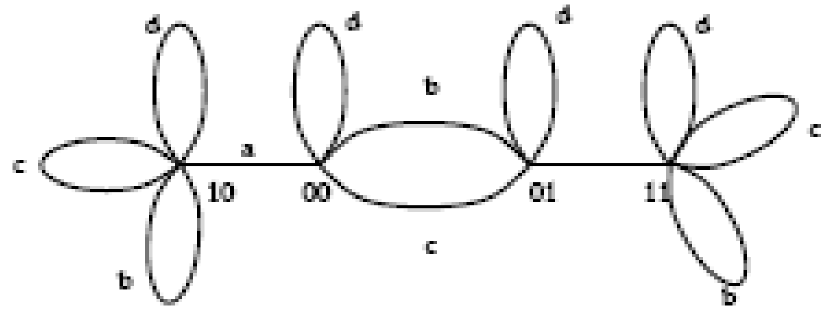
It can be atomic, as for example for the Grigorchuk group. Or nonatomic:

Theorem. The random weak limit of Schreier graphs of the Basilica group and of the Hanoi Towers group are nonatomic.

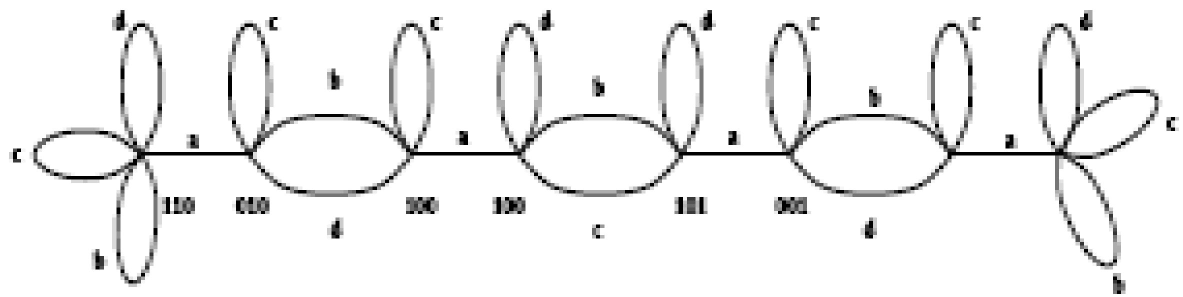
(D'Angeli, Donno, Matter, N., '09: explicit computation of the isomorphism classes and of the invariant measure.)

Schreier graphs of Grigorchuk's group

Γ_2

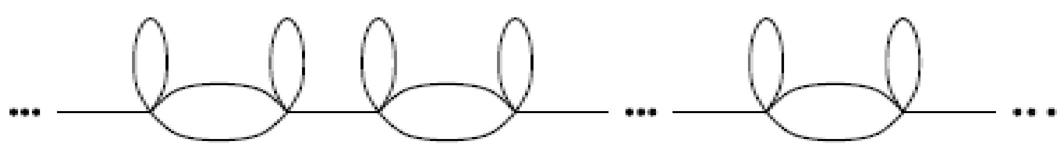


Γ_3

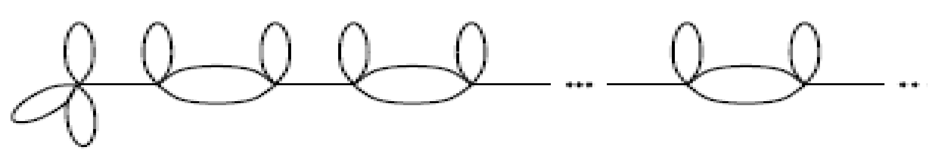


. . .

Γ_ξ, $\xi \neq 111\ldots.$



Γ_ξ, $\xi = 111\ldots.$



Related example:

Thompson's group F doesn't act by automorphisms on any rooted tree, but it does act by homeomorphisms on [0,1], so one can consider Schreier graphs for this action. The corresponding random rooted graph is uniform on $\{\Gamma_\xi \mid \xi \in \partial T\}$. (Savchuk, '10).

Problem. How to characterize groups with uncountably many isomorphism classes of orbital Schreier graphs?

(equivalently, with a nonatomic random weak limit)?

Conjecture. For contracting actions this is equivalent to the limit space being a fractal.

Further evidence: Classification, in terms of limit space, of contracting self-similar groups where the random weak limit has a.s. one end. (Bondarenko, D'Angeli, N., '10).

What is it good for, to have nonatomic RWL? One application.

Abelian Sandpile Model. (Bak-Tang-Wiesenfeld, Dhar, '90).

G a finite connected graph. θ a chip configuration G: $\theta(v)$ is the number of chips at the vertex v. Local rules: if $\theta(v) \geq \deg(v)$, then v is "unstable" and will be "fired":

$$T_v \theta(v) = \theta(v) - \deg(v) \; ; \qquad T_v \theta(w) = \theta(w) + \mathrm{Adj}(v,w).$$

Introduce a vertex s called sink. Then every unstable configuration stabilizes in finite time.

The model is abelian: the result of stabilization is independent of the order in which unstable vertices are toppled.

Once stabilized, we reactivate the game by dropping a new chip at random on $V(G)\setminus\{s\}$. Some stable configurations will be recurrent under this process, they are called critical (they turn out to be "barely stable").

Large scale dynamics

Let θ be a recurrent configuration. It gives rise to an avalanche = the sequence of firings triggered by adding one extra-chip to θ (on some fixed vertex).

A sequence of finite graphs Γ_n, an infinite graph Γ, $\Gamma_n \rightarrow \Gamma$.

On each Γ_n we consider the sink s_n, critical states $\mathrm{Crit}(\Gamma_n)$, and the uniform distribution μ_n on $\mathrm{Crit}(\Gamma_n)$.

Physicists predict that the ASM is critical, i.e., as $n \to \infty$, typical (under μ_n) avalanches are long. (The probability of having a long avalanche decays slowly with n).

Main open problem about ASM: prove criticality for the square lattice.

The only rigorously proven critical case: the binary tree.

One dimensional lattice Z is known to be non-critical.

Study ASM on Schreier graphs of self-similar groups

Γ_n finite Schreier graphs, Γ_ξ an infinite Schreier graph,

$\xi = x_1 x_2 \dots \in \partial T$.

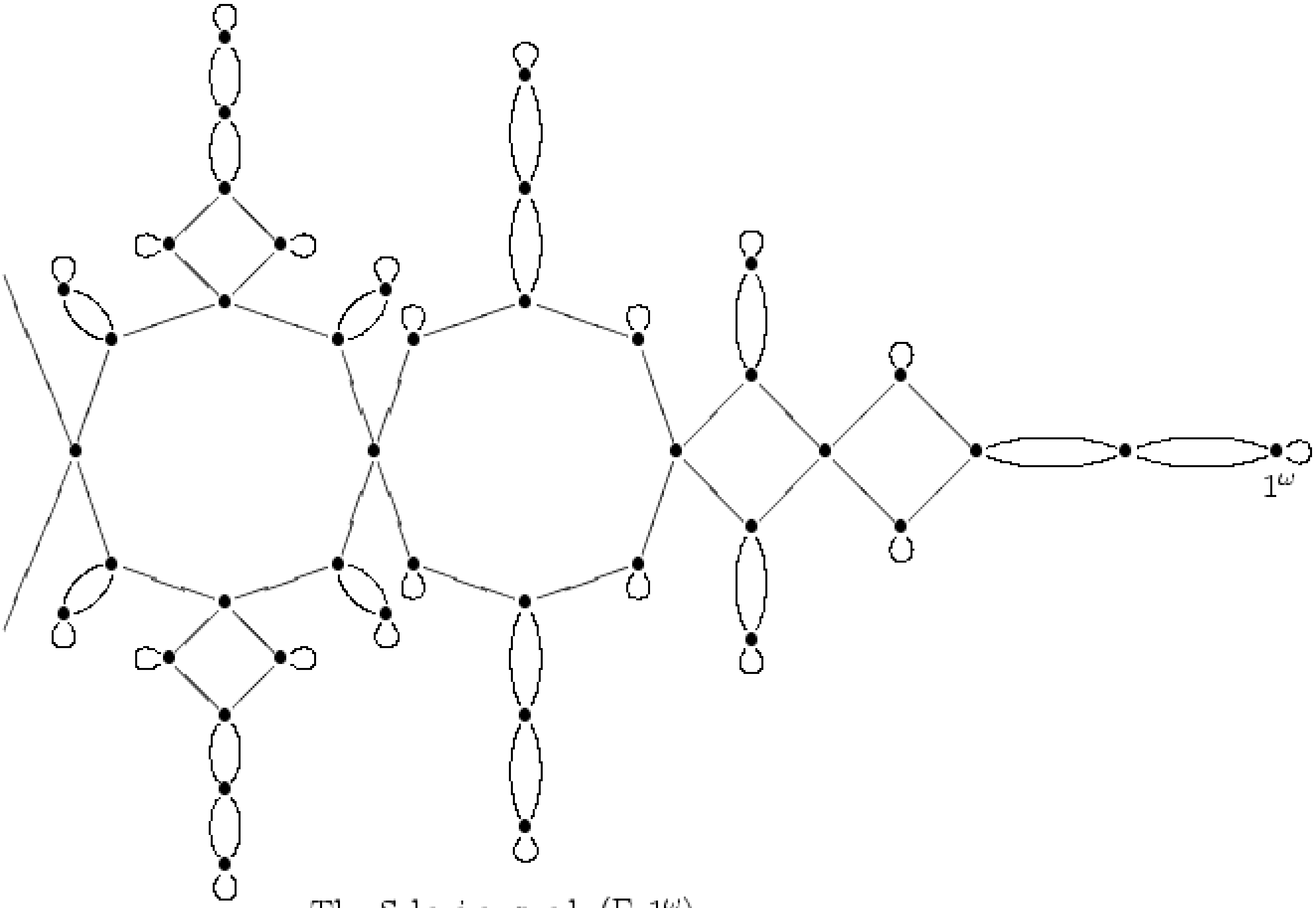
$(\Gamma_n, x_1 \dots x_n) \to (\Gamma_\xi, \xi)$ as $n \to \infty$.

Let θ be a critical configuration on Γ_n. Add to it a grain at $x_1 \dots x_n$ = root of Γ_n and study the behavior of the triggered avalanches, under the uniform distribution on $\mathrm{Crit}(\Gamma_n)$, as $n \to \infty$

Theorem (Matter, N., '10). A large class of examples (1-ended orbital Schreier graphs of contracting self-similar groups) with rigorously proven criticality for the ASM.

In particular, Basilica group provides uncountably many such graphs, all 4-regular, and with quadratic growth.

An uncountable family of 4-regular graphs with 1 end, of quadratic growth, with critical behaviour of ASM. Previously, the only known example of criticality was the binary tree approximated by the binary trees of finite depth.



The Schreier graph $(\Gamma, 1^{\omega})$.