# Recent results in the theory of twin building lattices [1]

## Summary

**0.1. General subject matter** We introduce a class of metric spaces with remarkable symmetry properties (*buildings*), and a class of finitely generated groups acting on some of them (*Kac-Moody groups*). Then we mention what the viewpoint of geometric group theory enables to prove:

- **Simplicity:** Kac-Moody groups provide a wide class of infinite finitely presented simple groups (Caprace-Rémy).
- **Rigidity:** these groups enjoy strong rigidity properties, e.g., of the type "higher-rank vs hyperbolic spaces" (Caprace-Rémy).
- **Amenability:** they also admit amenable actions on explicit compact spaces (Caprace-Lécureux, Lécureux).
- **Quasi-morphisms:** the existence of some non standard quasi-homomorphisms is well-understood (Caprace-Fujiwara).
- **Quasi-isometry:** Kac-Moody lattices provide infinitely many quasi-isometry classes of simple groups (Caprace-Rémy).

## Plan

**0.2.**

## Contents

## §1. Buildings and Kac-Moody groups

### 1.1. Coxeter complexes

- A *Coxeter group*, say $W$, is a group admitting a presentation: $W = \langle s \in S \mid (st)^{M_{s,t}} = 1 \rangle$ where $M = [M_{s,t}]s,t \in S$ is a *Coxeter matrix* (i.e., symmetric with 1's on the diagonal and other entries in $\mathbf{N}_{\geqslant 2} \cup \{\infty\}$).
- For any Coxeter system $(W,S)$ there is a natural simplicial complex $\Sigma$ on the maximal simplices of which $W$ acts simply transitively: $\Sigma$ is called the *Coxeter complex* of $(W,S)$.

  **Example 1.** *Start with a suitable Euclidean or hyperbolic tiling (dihedral angles are integral submultiples of $\pi$). Then apply Poincaré's theorem: the so-obtained reflection group is a Coxeter group, and the tiling realizes the Coxeter complex.*

- Coxeter complexes will be "slices" in buildings.

### 1.2. Axioms

**Definition 2.** *A building of type $(W,S)$ is a cellular complex, covered by subcomplexes all isomorphic to $\Sigma$ (the apartments) such that:*

*(i) any two simplices (the facets) are contained in an apartment;*

*(ii) given any two apartments $A, A' \simeq \Sigma$, there is a cellular isomorphism $A \simeq A'$ fixing $A \cap A'$.*

**Example 3.** *A tree with all vertices of valency $\geqslant 2$ (resp. a product of such trees) is a building with $W$ equal to the infinite dihedral group $D_\infty$ (resp. $W$ equal to a product of $D_\infty$'s).*

When the group $W$ is a Euclidean reflection group, one says that the building is *affine* or *Euclidean*.

### 1.3. Analogies with Lie groups and exotic examples

- Pick a simple algebraic group over a non-Archimedean field, e.g. $\mathrm{SL}_n(\mathbf{Q}_p)$. Then the group acts on a Euclidean building: this is one of the main results of Bruhat-Tits theory.
- Which other buildings (with an interesting group action) can be exhibited? There exist buildings whose apartments are tilings of hyperbolic spaces.
- For some of them, the full isometry group is a locally compact (totally disconnected, uncountable) abstractly simple group (Haglund-Paulin).
- Some of them are characterized by any group acting discretely and cocompactly: this is a "strong rigidity" (Bourdon).
- This supports the analogy between exotic buildings and symmetric spaces, between Lie groups and full automorphism groups of these new geometries.

### 1.4. Kac-Moody groups

- Kac-Moody groups were constructed by J. Tits to generalize algebraic groups. They share many combinatorial properties with them: they have a (twin) $BN$-pair structure.
- They are defined by a presentation generalizing the generators and relations of $\mathrm{SL}_n$ (using elementary unipotent matrices).
- The defining data for a Kac-Moody group are a field $\mathbf{K}$ and a *generalized Cartan matrix*, i.e., an integral matrix $A = [A_{s,t}]_{s,t \in S}$ such that $A_{s,s} = 2$ for all $s \in S$ and $A_{s,t} \leqslant 0$ for $s \neq t$, with $A_{s,t} = 0$ if and only if $A_{t,s} = 0$.

**Example 4.** *The standard example of such a group is $\Lambda = \mathbf{G}\big(\mathbf{K}[t,t^{-1}]\big)$ for $\mathbf{G}$ a simple matrix group over a field $\mathbf{K}$.*

**1.5. Buildings provided by Kac-Moody theory** The geometric counterpart to the $BN$-pair combinatorics is:

**Fact 5.** *(i) Any Kac-Moody group $\Lambda$ naturally acts on the product $X_- \times X_+$ of two isomorphic buildings $X_\pm$.*

*(ii) The explicit rule for $W$ deduces $[M_{s,t}]_{s,t \in S}$ from $A = [A_{s,t}]_{s,t \in S}$. Precisely: $M_{s,t} = 2, 3, 4, 6$ or $\infty$ according to whether $A_{s,t} \cdot A_{t,s}$ is $0, 1, 2, 3$ or is $\geqslant 4$, respectively.*

Reading backwards $[A_{s,t}]_{s,t \in S} \mapsto [M_{s,t}]_{s,t \in S}$, we can produce buildings (with nice group actions) provided the Weyl group has Coxeter exponents in $\{2;3;4;6;\infty\}$. This is the only condition on the shape of apartments.

[1] Bertrand Rémy (Université Claude Bernard Lyon 1). Talk in Goa, August 14th, 2010.

### 1.6. Affine and exotic buildings

- The case of affine buildings corresponds exactly to the previous examples $\Lambda = \mathbf{G}\big(\mathbf{K}[t,t^{-1}]\big)$, with a concrete matrix interpretation. We say then that $A$ and $\Lambda$ are of *affine* type.

- When $W$ is a Fuchsian group, e.g. $W$ is generated by a right-angled hyperbolic polygon or by a regular triangle of angle $\frac{\pi}{4}$ or $\frac{\pi}{6}$; then $X_\pm$ carries a negatively curved metric.

These are only examples; the general case is a mixture.

**Fact 6.** *(i) Any Kac-Moody group $\Lambda$ over any finite field is finitely generated.*

*(ii) The associated buildings $X_\pm$, for a suitable non-positively curved realization, are locally finite.*

## §2. Simplicity and rigidity

**2.1. Computation of covolume** From now on, $\Lambda$ denotes a Kac-Moody group defined by a generalized Cartan matrix $A = [A_{s,t}]_{s,t\in S}$ and a *finite* field $\mathbf{F}_q$. The full automorphism groups $\mathrm{Aut}(X_\pm)$ are thus locally compact, and as such admit Haar measures.

**Theorem 7** (B. R., 1999)**.** *Assume the Weyl group $W$ of $\Lambda$ is infinite and denote by $W(t) = \sum_{w\in W} t^{\ell(w)}$ its growth series. If $W(\frac{1}{q}) < \infty$, then $\Lambda$ is a lattice of $X_+ \times X_-$; it is never cocompact.*

- When $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t,t^{-1}]\big)$, the condition $W(\frac{1}{q}) < \infty$ is empty since the virtually abelian group $W$ has polynomial growth.

- The diagonal $\Lambda$-action on $X_+ \times X_-$ is always discrete and the real number $W(\frac{1}{q})$ is the covolume for a suitable normalization.

**2.2. Simple groups** The fact that a finitely generated Kac-Moody group can be seen as lattices of some reasonable geometry is the starting point to prove the following simplicity result.

**Theorem 8** (P.-E. Caprace and B.R., 2009)**.** *Let $\Lambda$ be a Kac-Moody group defined over the finite field $\mathbf{F}_q$. Assume that the Weyl group $W$ is infinite and irreducible and that $W(\frac{1}{q}) < \infty$. Then $\Lambda$ is simple (modulo its finite center) whenever the buildings $X_\pm$ are not Euclidean.*

- So whenever $\Lambda$ has no obvious matrix interpretation, it is a simple finitely generated group (and there is a geometric explanation for this).

- There exist *infinitely many* matrices $A$ such that $\Lambda$ is a *finitely presented, Kazhdan, simple group* for $q >> 1$.

### 2.3. Strategy of proof (simplicity)

- The idea is first to see Kac-Moody groups as analogues of lattices in Lie groups in order to rule out infinite quotients, and finally to stand by decisive differences to rule out finite quotients too.

- The analogy part follows Margulis' strategy for the normal subgroup property: any normal subgroup of a Kac-Moody lattice has finite index (using a criterion by Bader-Shalom).

- What goes wrong for simplicity? The affine example $\Lambda = \mathbf{G}\big(\mathbf{F}_q[t,t^{-1}]\big)$ has a lot of finite quotients.

- This is where we use non-affineness: a strengthening of Tits' alternative for Coxeter groups (Margulis-Noskov-Vinberg) implies that non-affine Coxeter groups are "weakly hyperbolic". This is what we combine, together with a trick on infinite root systems.

### 2.4. Why caring about (T) for simple groups?

- By Peter-Weyl's theorem, *a finitely generated group is residually finite if, and only if, it embeds abstractly in a compact group*. Therefore a finitely generated simple group $\Gamma$ has trivial homomorphic image in any compact group. So if $\Gamma$ acts on a locally finite complex with a global fixed point, the action is actually trivial.

- By Bruhat-Tits fixed point theorem, this implies that if $\Gamma$ acts non-trivially on a $\mathrm{CAT}(0)$ locally finite space, then any orbit is unbounded.

- Now let $Y$ be a proper $\mathrm{CAT}(-1)$-space with $\mathrm{Isom}(Y)$ acting cocompactly. The stabilizer of any $\xi \in \partial_\infty Y$ is amenable (Burger-Mozes). Therefore a non-trivial action of a finitely generated *Kazhdan* simple group $\Gamma$ on $Y$ has no global fixed point in the compactification $Y \cup \partial_\infty Y$.

### 2.5. Super-rigidity

- This is a well-known phenomenon: there are many results disproving the existence of actions of higher-rank lattices, e.g. $\mathrm{SL}_n(\mathbf{Z})$ for $n \geqslant 3$, on the circle (= boundary of $\mathbb{H}^2_{\mathbf{R}}$).

- What makes a Kac-Moody lattice being of higher-rank? For non-affine buildings, Kazhdan's property (T) and the existence of flats of dimension $\geqslant 2$ in the buildings are independent conditions.

**Theorem 9** (P.-E. Caprace and B.R., 2009)**.** *Let $\Lambda$ be a simple Kac-Moody lattice and let $Y$ be a proper $\mathrm{CAT}(-1)$-space with cocompact isometry group. If the buildings $X_\pm$ of $\Lambda$ contain flat subspaces of dimension $\geqslant 2$ and if $\Lambda$ is Kazhdan, then the group $\Lambda$ has no nontrivial action by isometries on $Y$.*

## §3. Amenability, quasi-morphisms and quasi-isometry

**3.1. Amenability, 1** Given a simple Lie real group like $\mathrm{SL}_n(\mathbf{R})$, a certain (Satake) compactification $\overline{X}$ of the associated symmetric space $X = G/K$ provides a geometric parametrization – up to finite index – of maximal amenable subgroups in $G$ (Moore): any point stabilizer in $G$ is an amenable subgroup; conversely, any amenable subgroup of $G$ has a finite index subgroup stabilizing a point in $\overline{X}$.

**Theorem 10** (P.-E. Caprace and J. Lécureux, 2010)**.** *Any locally finite building $X$ admits a compactification providing the same "classification" for amenable subgroups in $G = \mathrm{Isom}(X)$.*

This compactification is not the one given by asymptotic classes of geodesic rays; it is related to the (compact) space of closed subgroups of $G$ and to the combinatorics of infinite root systems.

**3.2. Amenability, 2** Roughly speaking, a $G$-action on a space $S$ is called *amenable* if there is a sequence of maps $\{\mu_n : S \to \mathcal{M}^1(G)\}_{n\geqslant 0}$ (to the probability measures on $G$) such that $\lim_{n\to\infty} \| \mu_n(g.x) - g_*\mu_n(x) \| = 0$ uniformly on compact subsets of $G \times X$. In the previous situation, the group $G = \mathrm{Isom}(X)$ itself is not amenable in general, but we have the following.

**Theorem 11** (J. Lécureux, 2010)**.** *For any building $X$, any proper action by a locally compact group on the above compactification $\overline{X}$ is amenable.*

Admitting an amenable action on a compact space is an important property in analytic group theory (Novikov conjecture and others). It also provides theoretical resolutions in bounded cohomology and boundary maps in rigidity theory.

**3.3. Quasi-homomorphisms, 1**

- Recall that a *quasi-character* for a group $G$ is a map $f : G \to \mathbf{R}$ such that $\sup_{g,h\in G} | f(gh) - f(g) - f(h) | < \infty$.
- The set of all quasi-characters of $G$ is denoted by $\mathrm{QH}(G)$.
- The set of non-trivial quasi-characters is by definition $$\widetilde{\mathrm{QH}(G)} = \frac{\mathrm{QH}(G)}{\mathrm{Hom}(G,\mathbf{R}) \oplus \ell^\infty(G)}.$$
- Quasi-homomorphisms are related to rigidity questions; higher-rank lattices in Lie groups don't have non-trivial quasi-characters (Burger-Monod).
- As we saw for rigidity, it is not clear what to require to consider that a building is of higher rank. The next result shows that many buildings are not of higher-rank with respect to quasi-homomorphims.

**3.4. Quasi-homomorphisms, 2**

**Theorem 12** (P.-E. Caprace and K. Fujiwara, 2010)**.** *Let $(W,S)$ be an infinite, irreducible, non-affine Coxeter system and let $X$ be a building of type $(W,S)$. Let $G$ be a group acting on $X$ by automorphisms so that at least one of the following conditions is satisfied:*

(i) *The $G$-action on $X$ is Weyl-transitive.*

(ii) *For some apartment $\mathbb{A} \subset X$, the stabilizer $\mathrm{Stab}_G(\mathbb{A})$ acts cocompactly on $\mathbb{A}$.*

*Then $\widetilde{\mathrm{QH}(G)}$ is infinite-dimensional.*

Combined with Kac-Moody theory, this implies that, up to isomorphism, there exist infinitely many finitely presented simple groups of strictly positive stable commutator length.

**3.5. Quasi-isometry** Let $G$ be a locally compact group admitting a finitely generated lattice $\Gamma$. This implies that $G$ admits a compact generating subset, say $\widehat{\Sigma}$; we denote by $d_{\widehat{\Sigma}}$ the word metric associated with $\widehat{\Sigma}$. Similarly, we fix a finite generating set $\Sigma$ for $\Gamma$ and denote by $d_\Sigma$ the associated word metric. The lattice $\Gamma$ is called *undistorted* in $G$ if $d_\Sigma$ is quasi-isometric to the restriction of $d_{\widehat{\Sigma}}$ to $\Gamma$. This amounts to saying that the inclusion of $\Gamma$ in $G$ is a quasi-isometric embedding from $(\Gamma, d_\Sigma)$ to $(G, d_{\widehat{\Sigma}})$.

**Theorem 13** (P.-E. Caprace and B.R., 2010)**.** *Any Kac-Moody lattice $\Lambda < \mathrm{Aut}(X_+) \times \mathrm{Aut}(X_-)$ is undistorted.*

Again, combined with simplicity results from Kac-Moody theory, this implies that there exist infinitely many pairwise non-quasi-isometric finitely presented simple groups.