EXERCISES FOR LECTURES ON CAT(0) CUBICAL GROUPS

Notation.

X – CAT(0) cubical complex (finite dimensional)

$\hat{\mathcal{H}}(X)$ – hyperplanes of X

$\mathcal{H}(X)$ – halfspaces of X

$\mathfrak{h}$ – a halfspace

$\mathfrak{h}^*$ – the complementary halfpspace of $\mathfrak{h}$

$\hat{\mathfrak{h}}$ – the bounding hyperplane of $\hat{\mathfrak{h}}$

Σ - a *pocset*, poset with an order reversing involution

[pocsets are assumed to be locally finite (intervals are finite) and finite width (lengths of antichains are bounded)]

$(\Omega, \mathcal{S})$ – a discrete wall space

$\mathcal{U}(\Sigma)$ – ultrafilters on Σ

$X^{(0)}(\Sigma)$ – ultrafilters satisfying the Descending Chain Condition (DCC).

$X(\Sigma)$ – CAT(0) cubical complex constructed from Σ

$\rho_\Delta : X(\Sigma) \to X(\Delta)$ – the collapsing map for $\Delta \subset \Sigma$

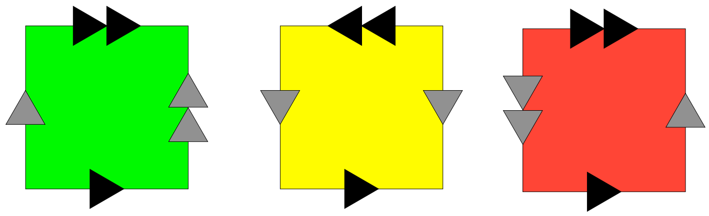
1. CAT(0) cubical complexes

Exercise 1. *Prove that a CAT(0) cubical complex whose links are all complete bipartite graphs is a product of two trees.*

Exercise 2. *Square a surface group.*

Exercise 3. *Show that every RAAG is a cubical group. What is the link of a vertex in the complex you built?*

Exercise 4. *What is this?*



2. Pocsets

Exercise 5. *$\mathcal{H}(X)$ is a locally finite pocset and is finite width (when X is finite dimensional).*

Exercise 6. *Prove or disprove or salvage if possible. A wall space $(\Omega, \mathcal{S})$ is discrete if and only the associated pocset is locally finite.*

Exercise 7. *Suppose that G is finitely generated, $e(G, H) > 1$, let $C(G)$ denote the Cayley graph of G.*

Let $A \subset G$ be the vertex set of the preimage of an unbounded component of $C(G)/H - K$, where K is a compact subset that separates $C(G, H)$ into more than one unbounded component.

Let $\mathcal{S} = \{gA|g \in G\} \cup \{ gA^*|g \in G\}$. Show that $(G, \mathcal{S})$ is a discrete wall space.

3. Ultrafilters: building a CAT(0) cubical complex from a pocset

Example. Suppose that $(\Omega, \mathcal{S})$ is a space with walls whose pocset of subsets is Σ, and $s \in S$. Then

$$\alpha_s = \{A \in \Sigma | s \in A\}$$

Observation 8. *α_s is an ultrafilter. When Σ is discrete, α_s satisfies DCC.*

Exercise 9. *Show that there exists an ultrafilter satisfying DCC (for Σ a locally finite, finite width pocset).*

Exercise 10. *Let $\alpha, \beta, \gamma \in \mathcal{U}(\Sigma)$. Show that*

$$m = (\alpha \cup \beta) \cap (\beta \cup \gamma) \cap (\alpha \cup \gamma)$$

is an ultrafilter. When α, β, γ satisfy DCC, so does m. Can exchange $\cap$ and $\cup$. Interpret m in the case of α, β, γ vertices of a tree, and when they are vertices of the square lattice in the plane.

Exercise 11. *Let $A \in \alpha$, then $(\alpha - \{A\}) \cup \{A^*\}$ is an ultrafilter if and only if A is minimal in α.*

Exercise 12. *If Σ has finite width, then any two DCC ultrafilters are joined by a finite path.*

Exercise 13. *$X^{(2)}$ is simply connected.*

Hint: Consider a minimal 1-skeleton closed path in $X^(1)$. Consider the "switches" are made along the way to produce a shorter close path.

Exercise 14. *X satisfies the Gromov flag link condition.*

We call this construction of a cubical complex from a pocset a *cubulation*.

Exercise 15. *Consider the plane with a finite family of evenly spaced parallel lines. What is the cubical complex associated to this space with walls?*

Exercise 16. *Is there a discrete space with walls constructed from lines in the plane which is yields an infinite dimensional complex? What about the hyperbolic plane?*

Exercise 17. *Suppose that $[G : H] < \infty$ and H acts properly on a CAT(0) cubical complex. Show that G does as well.*

4. The Roller compactification

We can topologize $\mathcal{U}(\Sigma)$ using the Tychonoff topology on the countable product

$$\mathcal{U}(\Sigma) \subset \prod_{\hat{\mathcal{H}}} \{\mathfrak{h}, \mathfrak{h}^*\}$$

Exercise 18. *$\mathcal{U}(\Sigma)$ is closed (and hence compact).*

Exercise 19. *Assume that Σ has finite width. Show that $\mathcal{U}(\Sigma)$ is a compactification of $X^{(0)}(\Sigma)$.*

5. Roller duality

We have to constructions:

$$\text{CAT(0) cubical complex } X \rightsquigarrow \text{pocset of halfspaces } \mathcal{H}(X)$$

$$\text{pocset } \Sigma \rightsquigarrow \text{ cubical complex } X(\Sigma)$$

Exercise 20 (Roller Duality)**.** *These constructions are dual to one another:*

(i) Given a finite width locally finite pocset, Σ, then $\mathcal{H}(X(\Sigma)) \equiv \Sigma$.
(ii) Given a finite dimensional cubical complex X, $X(\mathcal{H}(X)) = X$.

6. Subpocsets and collapsing

Observation 21. *Suppose $\Delta \subset \Sigma$ is a subpocset. ρ_Δ maps ultrafilters to ultrafilters and preserves DCC.*

Orbit quotients.

Exercise 22. *Consider $\mathbf{Z} \times \mathbf{Z}$ acting on the standard squaring of the plane. What are the orbit quotients?*

Exercise 23. *Consider the standard description of the surface of genus two given as the quotient of the octagon whose edges are identified $ab\bar{a}\bar{b}cd\bar{c}\bar{d}$. Square the surface by putting a vertex in the middle and joining this vertex to the midpoint of each edge. Let X be the universal cover of this surface acted on by the fundamental group of the surface G.*

(i) What are the orbit quotients? Are they locally finite?
(ii) Are the actions on the orbit quotients proper?
(iii) G acts on the product of the orbit quotients. Is this action proper? Is it cocompact?

Projections onto hyperplanes. Let $\hat{\mathfrak{h}} \subset X$ be a hyperplane of X.

$$\hat{\mathcal{H}}' = \{\mathfrak{k} \in X | \hat{\mathfrak{k}} \cap \hat{\mathfrak{h}} \neq \emptyset\}$$

$\mathcal{H}'$ – denote the subpocset of halfspaces associated to $\hat{\mathcal{H}}'$.

Exercise 24. *What is the complex $X(\mathcal{H}')$? Describe the map $X \to X(\mathcal{H}')$.*

Products. $X \cong X_1 \times X_2$ – a product of two cubical complexes.

For $i = 1, 2$, $p_i : X \to X_i$ – natural projections

$\hat{\mathcal{H}}_i = p_i^{-1}(\hat{\mathcal{H}}(X_i))$.

Observation 25. *$\hat{\mathcal{H}}(X)$ decomposes as a disjoint union $\hat{\mathcal{H}}(X) = \hat{\mathcal{H}}_1 \cup \hat{\mathcal{H}}_2$ and every hyperplane in $\hat{\mathcal{H}}_1$ crosses every hyperplane in $\mathcal{H}_2$.*

Exercise 26 (Recognizing Products)**.** *Show that a decomposition of the pocset $\mathcal{H}(X)$ as a disjoint union of transverse pocsets $\mathcal{H}(X) = \mathcal{H}_1 \cup \mathcal{H}_2$, meaning that every element of $\mathcal{H}_1$ is incomparable with every element of $\mathcal{H}_2$, corresponds to a decomposition of X as a product.*

Hint: Roller duality.

Exercise 27 (Product Decomposition Theorem)**.** *Show that every finite dimensional CAT(0) cubical complex admits a canonical decomposition into finitely many irreducibles.*

Hint: Finite dimensionality gives a bound on the number of factors in a decomposition. Consider a maximal non-trivial decomposition...

Pruning

Exercise 28. *Suppose that $G = Aut(X)$ acts on X with finitely many orbits of hyperplanes. Then*

- *there exists a convex, G-invariant subcomplex $Y \subset X$ which has only shallow and essential hyperplanes*
- *show that Y decomposes as a product of two CAT(0) cubical complexes, one of which is finite and the other of which is essential.*

Hint: For the first part, consider collapsing, starting with "outermost" hyperplanes. For the second part, observe that every shallow hyperplane intersects every essential hyperplane.

7. Skewering

Exercise 29. *Let $\mathfrak{h}$ be a halfspace bounded by $\hat{\mathfrak{h}}$ and g a hyperbolic automorphism. Show that g skewers $\hat{\mathfrak{h}}$ if and only if for some $n \neq 0$, we have $g\mathfrak{h} \subset \mathfrak{h}$.*

Exercise 30. *Show that g is peripheral to $\hat{\mathfrak{h}}$ if and only if for some $\mathfrak{h}$ bounded by $\hat{\mathfrak{h}}$, we have $g^n\mathfrak{h}^* \subset \mathfrak{h}$.*

Exercise 31 (Single Skewering)**.** *Let X be essential and let G be a group acting cocompactly on X.*

- *Suppose there exists a single orbit of hyperplanes $G(\hat{\mathfrak{h}})$. Show that there exists a number $N > 0$ (depending only on the dimension of X) such that if $diam(X) > N$, then there exists g skewering $\hat{\mathfrak{h}}$.*
- *Conclude that for every hyperplane $\hat{\mathfrak{h}}$, there exists $g \in G$ skewering $\hat{\mathfrak{h}}$*