GENERICITY OF INFINITE-ORDER ELEMENTS IN HYPERBOLIC GROUPS

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1. Introduction

Let Γ be a finitely generated group and let S be a finite set of generators for Γ. This determines a *word metric* on Γ, given by $d(g, h) =$ the length of the shortest word in S representing $g^{-1}h$, which makes Γ into a discrete, proper metric space. For $r \geq 1$, let $B_S(r)$ be the ball of radius r centred at the identity, with respect to the word metric. We say that the *generic element* of Γ has a particular property $\mathcal{P}$ if

$$\lim_{r\to\infty} \frac{|\{\text{elements in } B_S(r) \text{ with property } \mathcal{P}\}|}{|B_S(r)|} = 1$$

In this note we prove that the generic element of a finitely generated, non-elementary word-hyperbolic group has infinite order (see Section 2.1 below for definitions). Let $E_S(r)$ denote the set of finite-order elements in $B_S(r)$. We will actually prove the following, somewhat stronger, result.

Theorem 1.1. *Let Γ be a finitely generated, non-elementary, word-hyperbolic group, and let S be any finite generating set for Γ. Then there exist positive constants $D = D(\Gamma)$ and $M = M(\Gamma)$ such that*

$$|E_S(r)| \leq M|B_S(\tfrac{r}{2} + D)|$$

In particular,

$$\lim_{r\to\infty} \frac{|E_S(r)|}{|B_S(r)|} = 0 \tag{1}$$

The limit in (1) measures the density of finite-order elements in the group Γ. Theorem 1.1 lies in sharp contrast to the case of virtually nilpotent groups, for which this density is often positive. For example, for the square and triangle reflection groups in the Euclidean plane, the densities in (1) are 1/4 and 2/3 respectively. (See [D] for details). In general this limit is not a quasi-isometry invariant, and in fact, even positivity may not be preserved under a quasi-isometry: for example, every virtually nilpotent group has a torsion-free finite-index subgroup (for which the density in (1) is 0).

Theorem 1.1 is not merely a consequence of the fact that non-elementary word-hyperbolic groups have exponential growth. In fact, there exist examples (see [L]) of finitely generated infinite torsion groups with exponential growth (for which the density in (1) is 1). The proof of Theorem 1.1 relies on the existence of a generalized notion of centre for bounded sets in δ-hyperbolic spaces.

In Section 2 we review definitions and establish some results for later use. In particular, we introduce the concept of quasi-centres of bounded subsets of δ-hyperbolic spaces. In section 3, we obtain a length bound for quasi-centres of orbits of finite-order elements and use that to finish the proof of Theorem 1.1.

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2. Hyperbolic spaces and groups

2.1. **Definitions.** We first recall some basic facts about word-hyperbolic groups given, for example in [GH]or [BH]. Let $\delta > 0$. A geodesic triangle in a metric space is said to be *δ-slim* if each of its sides is contained in the δ-neighbourhood of the other two sides. A geodesic space X is said to be *δ-hyperbolic* if every triangle in X is δ-slim.

Let Γ be a finitely generated group. Its *Cayley graph* with respect to a finite generating set S is the graph whose vertices are elements of Γ and whose edges connect vertices representing group elements that differ by a generator on the left. G is said to be *word-hyperbolic* (or just *hyperbolic*) if its Cayley graph with respect to some generating set, endowed with the word metric, is a δ-hyperbolic metric space. The property of being a hyperbolic group is independent of the choice of finite generating set. A hyperbolic group is said to be *elementary* if it contains a cyclic subgroup of finite index.

2.2. **Growth.** Non-elementary hyperbolic groups have exponential growth. In fact, more is true:

Theorem 2.1. [FN] *Let Γ be a non-elementary hyperbolic group with generating set S. Then there exist $\lambda_1, \lambda_2 > 0$ and $\lambda > 1$ such that for sufficiently high r,*

$$\lambda_1 \cdot \lambda^r < |B_S(r)| < \lambda_2 \cdot \lambda^r$$

2.3. **Quasi-centres.** Let X be a δ-hyperbolic, proper geodesic space and $Y \subset X$ be a non-empty subset of bounded diameter. Y may not have a centre, i.e. a point in X that is equidistant from every point of Y. However, it does have a quasi-centre, a set of uniformly bounded diameter, which can often fulfil the same role as a centre.

For any $x \in X$, let $B(x, r)$ denote the ball of radius r in X, centered at x. Define $\rho_x = \inf\{r > 0 | Y \subseteq B(x, r)\}$ and $\rho = \inf_{x \in X}\{\rho_x\}$.

Definition 2.2. The *quasi-centre* of Y is the set $C(Y) = \{x \in X | \rho_x = \rho\}$.

Note that if x_0 minimizes ρ_x, then x_0 must lie in a compact ball centered at a point in Y, so that the quasi-centre is non-empty. The diameter of $C(Y)$ is uniformly bounded by 2δ (See [B]).

The quasi-centre can be thought of as a generalization of a global fixed point of a finite group acting on a negatively curved Riemannian manifold. Now suppose that X is the Cayley graph of a hyperbolic group and Y is a finite subgroup. $C(Y)$ may not contain any vertices of the Cayley graph, but $C_1(Y)$, the 1-neighbourhood of $C(Y)$, certainly does. Vertices in $C_1(Y)$ conjugate Y into a finite ball.

Theorem 2.3. [B, BH] *Let Γ be a word-hyperbolic group with finite generating set S. Let Y be a finite subgroup of Γ. If x is a vertex in $C_1(Y)$, then for every $g \in Y$, the element $x^{-1}gx$ belongs to $B_S(4\delta + 2)$.*

Theorem 2.3 has the following consequence.

Theorem 2.4. [B, BH] *A hyperbolic group has finitely many conjugacy classes of finite subgroups.*

2.4. **Polygons in δ-hyperbolic spaces.** Let X be a δ-hyperbolic space. A *k-gon* in X is a set of points $A_1, \ldots, A_k$ together with geodesic segments $\gamma_1, \ldots. \gamma_k$, such that γ_i has endpoints A_i and A_{i+1} for $1 \leq i \leq k-1$ and γ_k has endpoints A_k and A_1. The A_i's and the γ_i's are called the *vertices* and the *sides* of the k-gon respectively. We will be lax and denote such a k-gon by $A_1 \ldots A_k$, even though this doesn't completely specify the k-gon. We will denote the side joining A_i and A_{i+1} by A_i and A_{i+1}. A *regular* k-gon is one whose sides have equal lengths.

A *k-gon* in X consists of k-geodesic segments, called its *sides* $\gamma_1, \ldots, \gamma_k$ such that γ_i and γ_{i+1} share an endpoint for $1 \leq i \leq k-1$, and γ_k and γ_1 share an enpoint.

Note: The first one definitely seems better, since in the second, there is the additional fact that any point is the endpoint of at most two of the geodesic segments and secondly, polygons are later refered to by their vertices.

The slim-triangles condition can be used to prove the following "slim-polygons" property.

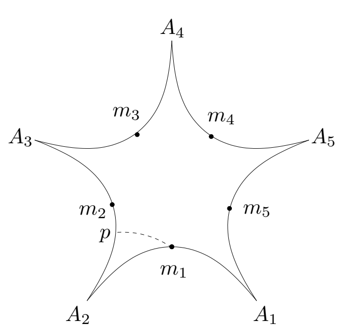
Lemma 2.5. [BH] *Let X be a δ-hyperbolic space. For any k, there is an $\epsilon_k = \epsilon_k(\delta)$ such that given any k-gon in X, each of its sides is contained in the union of the ϵ_k-neighbourhoods of the other $k-1$ sides.*

In fact, we can take $\epsilon_k = (k-2)\delta$. (A better bound is obtained in [BH]). In particular, ϵ_k increases as k increases. We use the slim-polygons property to show that polygons have short diagonals.

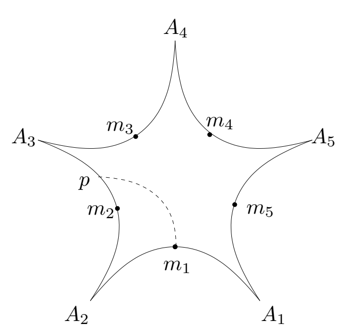
Lemma 2.6. *Let P be a k-gon, $k > 3$, in a δ-hyperbolic space, with side-lengths bounded by some constant l. Then there exists a diagonal of P whose length is no more than $l + 2\epsilon_k$, where ϵ_k is as in Lemma 2.5.*

Proof. Let $P = A_1 A_2 \cdots A_k$. Let m_i be the midpoint of the side $A_i A_{i+1}$, for $i < k$, and let m_k be the midpoint of $A_k A_1$.

From Lemma 2.5 we know that corresponding to the point m_1 on $A_1 A_2$, there is a point p on another side of P such that $d(m_1, p) \leq \epsilon_k$. There are two possibilities for the position of p. (shown in the figure for $k = 5$):



Case 1: p lies on $A_1 m_k$ or $A_2 m_2$



Case 2: p lies on $m_k A_k$, $m_2 A_3$
or one of the remaining sides

Case 1: Assume, without loss of generality, that p lies on A_2m_2. Then we have:

$$\begin{aligned} d(A_1, A_3) &\leq d(A_1, m_1) + d(m_1, p) + d(p, m_2) + d(m_2, A_3) \\ &\leq d(A_2, m_1) + \epsilon_k + d(p, m_2) + \frac{l}{2} \\ &\leq (d(A_2, p) + \epsilon_k) + \epsilon_k + d(p, m_2) + \frac{l}{2} \\ &\leq d(A_2, m_2) + \frac{l}{2} + 2\epsilon_k \\ &\leq l + 2\epsilon_k \end{aligned}$$

Case 2: Let V be the vertex closest to p, i.e. $d(V, p) \leq \frac{l}{2}$. ($V = A_3$ in the figure). At least one of A_1V and A_2V is a diagonal of P. Further, for $j = 1$ or 2, we have:

$$d(A_j, V) \leq d(A_j, m_1) + d(m_1, p) + d(p, V) \leq l + \epsilon_k$$

In either case we have found a diagonal whose side-length is bounded by $l + 2\epsilon_k$. □

We will use the following consequence of Lemma 2.6:

Lemma 2.7. *Let $P = A_1 \cdots A_k$ be a regular polygon in a δ-hyperbolic space, with side-length s. Then there exits $\eta_k = \eta_k(\delta)$, such that $d(A_j, A_{j+2}) < s + \eta_k$ for some $1 \leq j \leq k$.*

Proof. By applying Lemma 2.6 at most $k-2$ times, P can be cut up into triangles using diagonals of P. At least one of these diagonals is a geodesic joining A_j and A_{j+2} for some j. Since ϵ_k is the largest of $\epsilon_1, \ldots, \epsilon_k$, the length of this diagonal is at most $s + (k-2)(2\epsilon_k)$. Now set $\eta_k = (k-2)(2\epsilon_k)$. □

3. Finishing the proof

3.1. **A length bound for the quasi-centre.** Let g be a finite-order element in Γ. Then vertices in $C_1(\langle g \rangle)$, the 1-neighbourhood of the quasi-centre of $\langle g \rangle$, are approximately half as far from the identity as g:

Proposition 3.1. *Let Γ be a non-elementary word-hyperbolic group, with generating set S. Let g be a finite-order element of Γ and let $x \in C_1(\langle g \rangle)$. Then there exists $D = D(\Gamma, S)$ such that*

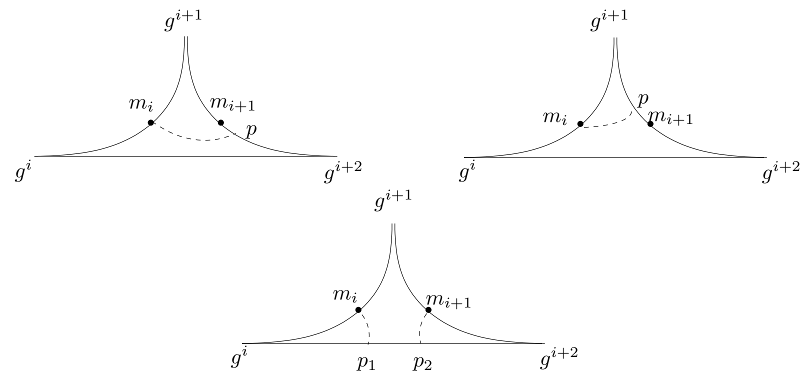
$$d(x, e) \leq \frac{d(g, e)}{2} + D$$

Proof. Let g be of order k. Pick a geodesic γ in the Cayley graph of Γ, joining g to the identity, e. Let P_g denote the regular k-gon with vertices $e, g, \ldots, g^{k-1}$ and sides $\gamma_1 = \gamma, \gamma_2 = g(\gamma), \ldots, \gamma_k = g^{k-1}(\gamma)$. Let m_i denote the midpoint of γ_i. Note that for all i,

$$d(g^i, g^{i+1}) = d(g, e) \text{ and } d(g^i, m_i) = d(g^{i+1}, m_i) = d(g, e)/2$$

We first prove that the set $\{m_i | 1 \leq i \leq k\}$ is bounded, with diameter less than a constant which depends only on k and δ. Since the set is finite, it will be enough to show that $d(m_i, m_{i+1})$ is bounded by such a constant for every i.

By Lemma 2.7, there exists some j such that $d(g^j, g^{j+2}) < d(g, e) + \eta_k$. Hence, for every i, we have $d(g^i, g^{i+2}) = d(g^j, g^{j+2}) < d(g, e) + \eta_k$. For each i, pick a geodesic joining g^i and g^{i+2}, and consider the triangle $g^i g^{i+1} g^{i+2}$. Since this triangle is δ-slim, each of the points m_i and m_{i+1} is δ-close to a side not containing it. Without loss of generality we may assume that one of the following three cases occurs:



Case 1: We show that $d(p, m_{i+1}) < \delta$ as well. By the triangle inequality we have:

$$\begin{aligned} d(p, m^{i+1}) + d(m^{i+1}, g^{i+1}) &\leq d(g^{i+1}, m_i) + d(m_i, p) \\ &\implies d(p, m_{i+1}) + \frac{d(g,e)}{2} < \frac{d(g,e)}{2} + \delta \implies d(p, m_{i+1}) < \delta \end{aligned}$$

So $d(m_i, m_{i+1}) < 2\delta$.

Case 2: Once again $d(m_i, m_{i+1}) < 2\delta$. The proof is similar to that of Case 1.

Case 3: The triangle inequality implies that $d(g^i, p_1) > \frac{d(g,e)}{2} - \delta$ and $d(g^{i+2}, p_2) > \frac{d(g,e)}{2} - \delta$ (since $d(g^i, m_i) = d(g^{i+2}, m_{i+1}) = \frac{d(g,e)}{2}$), so that

$$\begin{aligned} d(p_1, p_2) &= d(g^i, g^{i+2}) - d(g^i, p_1) - d(g^{i+2}, p_2) \\ &< d(g,e) + \eta_k - \frac{d(g,e)}{2} + \delta - \frac{d(g,e)}{2} + \delta \\ &= \eta_k + 2\delta \end{aligned}$$

So $d(m_i, m_{i+1}) < \eta_k + 4\delta$.

Thus the diameter of the set $\{m_i | 1 \leq i \leq k\}$ is at most $k(\eta_k + 4\delta)$. This means each of its elements (in particular, m_1) is "approximately" equidistant from the vertices $e, g, \dots g^{k-1}$. More precisely, for all i, we have

$$d(g^i, m_1) \leq d(g^i, m_i) + d(m_i, m_1) \leq \frac{d(g,e)}{2} + k(\eta_k + 4\delta)$$

This implies $\rho \leq \rho_{m_1} \leq d(e,g)/2 + k(\eta_k + 4\delta)$. Now by Lemma 2.4 there are only finitely many conjugacy classes of finite-order elements. Let

$$D = 1 + \max\{k(\eta_k + 4\delta) | k \text{ is the order of an element in } \Gamma\}$$

Note that D depends only on Γ and S. If $x \in C_1(\langle g \rangle)$, then

$$d(x,e) \leq \rho + 1 \leq \frac{d(g,e)}{2} + k(\eta_k + 4\delta) + 1 \leq \frac{d(g,e)}{2} + D$$

□

3.2. **Proof of Theorem 1.1.** For every $g \in E_S(r)$, pick any $x_g \in C_1(\langle g \rangle)$. Let D be as in Proposition 3.1, i.e. $d(x_g, e) \leq \frac{d(g,e)}{2} + D$. We now have a map

$$\phi : E_S(r) \to B_S(\frac{r}{2} + D)$$
$$g \mapsto x_g$$

Given $x \in B_S(\frac{r}{2} + D)$, $\phi^{-1}(x)$ consists of elements h in Γ, such that $x \in C_1(\langle h \rangle)$. By Lemma 2.3 above, $x^{-1}hx$ is an element of $B_S(4\delta + 2)$, a finite ball. Then if $M = |B_S(4\delta + 2)|$, there are at most M choices for $x^{-1}hx$, and hence for h, so that $|\phi^{-1}(x)| \leq M$. It follows that

$$|E_S(r)| \leq M|B_S(\frac{r}{2} + D)|$$

Note that M depends only on Γ and S.

Now since Γ is non-elementary, Lemma 2.1 guarantees the existence of $\lambda > 1$ and $\lambda_1, \lambda_2 > 0$, such that for sufficiently high r, $\lambda_1 \lambda^r < |B_S(r)| < \lambda_2 \lambda^r$. So

$$F(\Gamma, S) = \lim_{r\to\infty} \frac{|E_S(r)|}{|B_S(r)|} \leq \lim_{r\to\infty} \frac{M|B_S(\frac{r}{2} + D)|}{|B_S(r)|} \leq \lim_{r\to\infty} \frac{\lambda_2 \lambda^{\frac{r}{2}+D}}{\lambda_1 \lambda^r} = 0$$

□

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