QUASI-ISOMETRY CLASSIFICATION OF CERTAIN RIGHT-ANGLED COXETER GROUPS

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Abstract. We investigate the quasi-isometry classification of the right-angled Coxeter groups $W_\Gamma$ which are 1-ended and have triangle-free defining graph $\Gamma$. We begin by characterising those $W_\Gamma$ which split over 2-ended subgroups, and those which are cocompact Fuchsian, in terms of properties of $\Gamma$. This allows us to apply a theorem of Papasoglu [21] to distinguish several quasi-isometry classes. We then carry out a complete quasi-isometry classification of the hyperbolic $W_\Gamma$ with $\Gamma$ a generalised $\Theta$ graph. For this we use Bowditch’s JSJ tree [8] and the quasi-isometries of “fattened trees” introduced by Behrstock–Neumann [5]. Combined with a commensurability classification due to Crisp–Paoluzzi [12], it follows that there are right-angled Coxeter groups which are quasi-isometric but not commensurable. Finally, we generalise the work of Crisp–Paoluzzi [12].

1. Introduction

In this paper we investigate the quasi-isometry classification of certain Coxeter groups. Given a finite simplicial graph $\Gamma$, the associated right-angled Coxeter group $W_\Gamma$ has generating set $S$ the vertices of $\Gamma$, and relations $s^2 = 1$ for all $s \in S$ and $st = ts$ whenever $s, t \in S$ are adjacent vertices. We say that $\Gamma$ is the defining graph of $W_\Gamma$. We assume throughout that $\Gamma$ is finite and $W_\Gamma$ is 2-dimensional, that is, that $\Gamma$ has no triangles.

In order to classify such $W_\Gamma$ up to quasi-isometry, we begin with the number of ends. Denote by $D_\infty$ the infinite dihedral group and by $C_2$ the cyclic group of order 2. Since a finitely generated group is 2-ended if and only if it is virtually $\mathbb{Z}$, all 2-ended $W_\Gamma$ are quasi-isometric to each other. Further, it follows from Theorem 8.7.3 of [14] that if $W_\Gamma$ is 2-dimensional then $W_\Gamma$ is 1-ended if and only if $\Gamma$ is connected and has no separating vertices or edges. For each $T \subseteq S$, the special subgroup $W_T$ of $W_\Gamma$ is the subgroup generated by $T$ (with $W_\emptyset$ trivial). The group $W_T$ is also a right-angled Coxeter group, with defining graph having vertex set $T$ and edge set all edges of $\Gamma$ which have both endpoints in $T$. By Proposition 8.8.2 of [14], if $W_\Gamma$ has infinitely many ends then $W_\Gamma$ has a nontrivial decomposition as a tree of groups, in which each vertex group is a finite or 1-ended special subgroup and each edge group is a finite special subgroup. In light of this, we focus on the case that $W_\Gamma$ is 1-ended in this paper. By Theorem 8.7.2 of [14], if $W_\Gamma$ is 2-dimensional then $W_\Gamma$ is 1-ended if and only if $\Gamma$ is connected and has no separating vertices or edges.

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Having restricted to the 1-ended case, we now discuss several quasi-isometry invariants that have been considered for right-angled Coxeter groups. In [13], we investigated divergence. Up to an equivalence relation which identifies polynomials of the same degree, the rate of divergence is a quasi-isometry invariant [17]. We characterised those $W_\Gamma$ with linear and quadratic divergence by properties of their defining graphs and showed that for every positive integer $d$, there is a $W_{\Gamma_d}$ with divergence polynomial of degree $d$.

A quasi-isometry invariant related to divergence is thickness. Thickness was introduced by Behrstock, Drutu and Mosher in [2], and the order of thickness provides an upper bound on divergence [1]. In [3], Behrstock, Hagen, Sisto and Caprace give an effective characterisation of the right-angled Coxeter groups $W_\Gamma$ (not just the 2-dimensional ones) which are thick, and show that if $W_\Gamma$ is not thick then it is hyperbolic relative to a collection of thick special subgroups. As discussed in [3], this relative hyperbolicity result combined with results from [2] and [16] can also be used for quasi-isometry classification.

By Moussong’s Theorem [14, Theorem 12.3.3], right-angled Coxeter groups are CAT(0) groups. Although the visual boundary is not a quasi-isometry invariant for CAT(0) groups, Charney–Sultan [11] have introduced a new boundary for CAT(0) spaces, called the contracting boundary, which is a quasi-isometry invariant. In Section 5 of [11], the contracting boundary is used to distinguish the quasi-isometry classes of two 1-ended non-hyperbolic right-angled Coxeter groups. Moussong proved that the group $W_\Gamma$ is hyperbolic if and only if $\Gamma$ has no embedded cycles of length four [14, Corollary 12.6.3].

We remark that although every right-angled Artin group is finite index in a right-angled Coxeter group [15], the converse is not true. Thus results on the quasi-isometry classification of right-angled Artin groups (for instance [4, 5, 6]) yield information about right-angled Coxeter groups defined by other natural classes of graphs.

Our first main result, Theorem 1.1 below, considers splittings of $W_\Gamma$ over 2-ended subgroups. Recall that two groups $G$ and $H$ are (abstractly) commensurable if there are finite index subgroups $G_1 < G$ and $H_1 < H$ so that $G_1 \cong H_1$. Our motivation for Theorem 1.1 is Papasoglu’s result [21] that among 1-ended, finitely presented groups that are not commensurable to surface groups, having a splitting over a 2-ended subgroup is a quasi-isometry invariant. We show that whether or not $W_\Gamma$ has such a splitting can be easily identified from $\Gamma$, and that the only such splittings are the “obvious” ones. More precisely, in Section 3 we prove:

**Theorem 1.1.** Assume $W_\Gamma$ is 2-dimensional and 1-ended. Then $W_\Gamma$ splits over a 2-ended subgroup if and only if $\Gamma$ has a pair of nonadjacent vertices $\{s,t\}$ so that $\Gamma \setminus \{s,t\}$ is disconnected. Moreover if $W_\Gamma$ does split over a 2-ended subgroup $H$ then $W_\Gamma = A \star_H B$ where $H$ is conjugate to a special subgroup and $H \cong D_\infty$ or $H \cong D_\infty \times C_2$.

We next identify the $W_\Gamma$ which are cocompact Fuchsian. As in [8], we define a finitely generated group $G$ to be Fuchsian if $G$ is non-elementary and acts properly discontinuously on the hyperbolic plane. The action of $G$ is not required to be faithful, just to have a finite kernel, and the action of $G$ is not required to be orientation-preserving. For example, denote
by $\Lambda_n$ the $n$-cycle of length $n \geq 5$ and put $W_n = W_{\Lambda_n}$. Then $W_n$ is the group generated by reflections in the sides of a right-angled hyperbolic $n$-gon, hence $W_n$ is cocompact Fuchsian. In Section 4 we prove:

**Theorem 1.2.** Let $W_\Gamma$ be 2-dimensional. The following are equivalent:

1. $W_\Gamma$ is cocompact Fuchsian.
2. $W_\Gamma$ is quasi-isometric to $W_n$ for some $n \geq 5$.
3. $\Gamma = \Lambda_n$ for some $n \geq 5$.

Note that if any of the equivalent conditions in the above theorem hold, then $W_\Gamma$ is 1-ended and hyperbolic.

Since commensurability implies quasi-isometry, and $W_n$ for $n \geq 5$ has a finite index surface subgroup, Theorem 1.2 has the following corollary.

**Corollary 1.3.** Let $W_\Gamma$ be 2-dimensional. Then $W_\Gamma$ is commensurable to a surface group if and only if $\Gamma = \Lambda_n$ for some $n \geq 5$.

We can now distinguish several right-angled Coxeter groups up to quasi-isometry. For instance, let $\Xi, \Delta, \Theta$ and $\Pi$ be the graphs in Figure 1.1. Then Papasoglu’s result [21] mentioned above, together with Theorem 1.1 and Corollary 1.3, implies that $W_\Xi$ and $W_\Delta$ are not quasi-isometric, and $W_\Theta$ and $W_\Pi$ are not quasi-isometric. Note that $W_\Theta$ and $W_\Pi$ are hyperbolic, while $W_\Xi$ and $W_\Delta$ are not, so all four of these groups are in distinct quasi-isometry classes.

We next assume that $W_\Gamma$ is hyperbolic and consider the visual boundary. This boundary is, up to homeomorphism, a quasi-isometry invariant of hyperbolic groups. Now the visual boundary of a 1-ended hyperbolic group has no global cut points [9, 24]. A result of Bestvina and Mess [7] then implies that the boundary is locally connected. In [8], Bowditch gives a construction of a canonical JSJ splitting of a 1-ended hyperbolic group with locally connected boundary which is not cocompact Fuchsian (this was part of our motivation for proving Theorem 1.2 above). The JSJ tree for this splitting is a quasi-isometry invariant, as it is defined in terms of the local cut point structure of the boundary.

In Sections 5 and 6 we use Bowditch’s JSJ tree to carry out a quasi-isometry classification of the class of right-angled Coxeter groups with defining graph a *generalised* $\Theta$ graph.

**Definition 1.4.** Let $k \geq 3$ and $n_1, n_2, \ldots, n_k \geq 1$ be integers. Let $\Psi_k$ be the graph with two vertices $a$ and $b$ each of valence $k$ and $k$ edges $e_1, e_2, \ldots, e_k$ connecting $a$ and $b$. That is, $\Psi_k$ is a cage with $k$ bars. The *generalised* $\Theta$ graph $\Theta(n_1, n_2, \ldots, n_k)$ is obtained by, for
each \( 1 \leq i \leq k \), subdividing the edge \( e_i \) of \( \Psi_k \) into \( n_i + 1 \) edges by inserting \( n_i \) new vertices along \( e_i \).

For example, the graph \( \Theta \) in Figure 1.1 above is \( \Theta(1, 2, 2, 3) \). It is easy to verify that if \( \Theta = \Theta(n_1, n_2, \ldots, n_k) \) and \( W_\Theta \) is the associated right-angled Coxeter group, then \( W_\Theta \) is 2-dimensional and 1-ended. Further, \( W_\Theta \) is hyperbolic if and only if (without loss of generality) \( n_i \geq 2 \) for \( 2 \leq i \leq k \). We were motivated to consider this class of defining graphs by the work of Crisp–Paoluzzi [12] on commensurability for the right-angled Coxeter groups \( W_{\Theta(1, n_2, n_3)} \), which is discussed further below. We prove:

**Theorem 1.5.** Let \( \Theta = \Theta(n_1, n_2, \ldots, n_k) \) and \( \Theta' = \Theta(n_1', n_2', \ldots, n_k') \), where \( k, k' \geq 3 \), \( n_1, n_1' \geq 1 \) and \( n_i, n_i' \geq 2 \) for \( i, j \geq 2 \). Assume \( n_1' \geq n_1 \). Let \( W = W_\Theta \) and \( W' = W_{\Theta'} \). Then the following are equivalent:

1. \( W \) and \( W' \) are quasi-isometric.
2. Either:
   - (a) \( n_1 = 1 \), \( n_1' = 1 \) and \( k = k' \); or
   - (b) \( n_1 \geq 2 \), \( n_1' \geq 2 \) and \( k = k' \); or
   - (c) \( n_1 = 1 \), \( n_1' \geq 2 \) and \( k' = 2(k - 1) \).

To prove Theorem 1.5, we first establish in Section 5 that \( W \) and \( W' \) have the same JSJ tree if and only if one of the conditions in (2) holds. This uses several results of Lafont [19]. The key observation is that, generalising Crisp–Paoluzzi [12], the groups \( W \) and \( W' \) may be viewed as fundamental groups of piecewise hyperbolic orbifolds whose universal covers are the universal covers of the spaces considered in [19].

We then complete the proof of Theorem 1.5 in Section 6 by proving that if \( W \) and \( W' \) have the same JSJ tree, they are quasi-isometric. For this, we construct a quasi-isometry from \( W \) to \( W' \) using the quasi-isometries of “fattened trees” introduced by Behrstock–Neumann [5]. While finishing this paper we learned that Malone’s Ph.D. thesis [20] uses similar arguments involving fattened trees, but since [20] is unpublished and that part of our proof is brief, we include it here.

Finally, we consider commensurability of the class of right-angled Coxeter groups appearing in Theorem 1.5. Let \( \Theta = \Theta(1, n_2, n_3) \) and \( \Theta' = \Theta(1, n_2', n_3') \), where without loss of generality \( 2 \leq n_2 \leq n_3 \) and \( 2 \leq n_2' \leq n_3' \). Crisp–Paoluzzi [12] prove that \( W_\Theta \) is commensurable to \( W_{\Theta'} \) if and only if \( \frac{n_2 - 1}{n_2 - 1} = \frac{n_3 - 1}{n_3' - 1} \). From this and case (a) in Theorem 1.5 above with \( k = k' = 3 \), it follows that:

**Corollary 1.6.** There exist right-angled Coxeter groups which are quasi-isometric but not commensurable.

The only other class of Coxeter groups for which we know of commensurability results are the Coxeter groups generated by reflections in the faces of an \( n \)-simplex in hyperbolic \( n \)-space \( \mathbb{H}^n \). In [18], Johnson et al classify these up to commensurability; all such groups are quasi-isometric to \( \mathbb{H}^n \). In contrast, there are some classes of groups for which quasi-isometry and commensurability are equivalent, for instance fundamental groups of hyperbolic 3-manifolds with boundary [22].
In Section 7, we generalise the results of Crisp–Paoluzzi [12] as follows. Let $W$ and $W'$ be as in the statement of Theorem 1.5 above. We first establish, using similar orbifold-covering arguments to [12], sufficient conditions on $\Theta$ and $\Theta'$ for $W$ and $W'$ to be commensurable (see Proposition 7.2 for a precise statement). We then show in Theorem 7.3 that these sufficient conditions are necessary when $n_1 = n'_1 = 1$ and $k = k' = 4$, and when $n_1 \geq 2$, $n'_1 \geq 2$ and $k = k' = 3$. We conjecture that our sufficient conditions for commensurability are necessary for all $W$ and $W'$ as in Theorem 1.5. Our proofs in Section 7 use intricate combinatorial arguments, which are not needed in the case considered in [12], and which we do not see how to extend to larger values of $k$ and $k'$.

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2. Background

In this section we collect some results and constructions that we use in our proofs. In Section 2.1 we describe orbifolds associated to generalised $\Theta$ graphs, and in Section 2.2 we define an operation called doubling on graphs. Throughout this paper we will use standard results about right-angled Coxeter groups, a reference for which is [14].

2.1. Orbifold for generalised $\Theta$ graphs. In Definition 1.4 above, we introduced generalised $\Theta$ graphs. We now describe an orbifold associated to certain $W$.

Let $\Theta = \Theta(n_1, n_2, \ldots, n_k)$ with $n_i \geq 2$ for $1 \leq i \leq k$ and let $a$ and $b$ be the two valence $k$ vertices of $\Theta$. As in Crisp–Paoluzzi [12], for $n \geq 5$ we may view a right-angled hyperbolic $n$-gon $P_n$ as a reflection orbifold with fundamental group $W_n$. Now for each $1 \leq i \leq k$ form $P_{n_i+3}$ where there is one edge which is not a reflection edge and the other $(n_i + 2)$ edges correspond to the reflections $a, \ldots, b$ going down the $i$th path in $\Theta$ from $a$ to $b$, in order. Then $W$ is the fundamental group of the reflection orbifold $O_\Theta$ obtained by gluing together $P_{n_1+3}, \ldots, P_{n_k+3}$ along their non-reflection edges, as shown in Figure 2.1. Call the common non-reflection edge the branching edge. This is inspired by the construction in [12] for the case $n_1 = n'_1 = 1$ and $k = k' = 3$.

![Figure 2.1. The orbifold associated to $\Theta(2,2,2,3)$](image)

The polygons $P_{n_i+1}$ in $O = O_\Theta$ may be metrised so that they have constant curvature $-1$ and the non-reflection edge in each has the same length. Gluing by an isometry of the common edge then produces a piecewise hyperbolic version of the orbifold $O$ with constant...
curvature $-1$. Thus any orbifold cover of $O$ is also piecewise hyperbolic of constant curvature $-1$. In particular, as observed in [12], the universal cover $\tilde{X}$ of $O$ may be viewed as a collection of convex hyperbolic planar regions glued together along geodesic lines. We call these planar regions pieces and the geodesic lines along which they are glued branching geodesics. By the covering property, each branching geodesic has $k$ pieces glued along it.

We observe that $\tilde{X}$ is an example of the universal cover of a simple, thick 2-dimensional hyperbolic $P$-manifold. In [19], Lafont proved a topological rigidity theorem for such objects by analysing the boundaries of their universal covers. We use some of the results of [19] to construct the JSJ tree associated to $W_\Theta$ in Section 5.2.

2.2. Doubling. We now describe an operation on graphs which guarantees that the group defined by the new graph is commensurable to the group defined by the original graph.

**Definition 2.1 (Double).** Consider a graph $\Gamma$ with vertex set $S$ and let $x \in S$. The double of $\Gamma$ along $x$ is the graph $D_x(\Gamma)$ obtained from $\Gamma$ by doubling over the star of $x$ and then deleting the vertex $x$. (See Figure 2.2.)

![Figure 2.2. The double $D_x(\Theta)$ of $\Theta = \Theta(1,2,2,3)$](image)

**Remark 2.2.** If $\Gamma = \Theta(n_1, n_2, \ldots, n_k)$ is a generalised $\Theta$ graph then we will always denote by $a$ and $b$ the two vertices of this graph of valence $k$, and if $\Gamma = \Theta(1, n_2, \ldots, n_k)$ with $n_i \geq 2$ for $2 \leq i \leq k$, we will always denote by $x$ the unique vertex of this graph which is adjacent to both $a$ and $b$. Note that if $\Theta = \Theta(1, n_2, \ldots, n_k)$ then $D_x(\Theta) = \Theta(n_2, n_2, \ldots, n_k, n_k)$.

Define a homomorphism $\phi : W_\Gamma \to C_2$ by sending $x$ to the generator of $C_2$ and all $s \in S \setminus \{x\}$ to the trivial element. Then $\ker \phi$ is generated by the elements $\{s, xsx \mid s \in S \setminus \{x\}\}$ and is isomorphic to $W_{D_x(\Gamma)}$. Since $\ker \phi$ is an index 2 subgroup, we have:

**Lemma 2.3.** $W_{D_x(\Gamma)}$ is commensurable and therefore quasi-isometric to $W_\Gamma$. \hfill \qed

3. Distinguishing groups with splittings over 2-ended subgroups

In this section, we prove Theorem 3.1 below, which is a refinement of Theorem 1.1 from the introduction.

**Theorem 3.1.** Let $W = W_\Gamma$ be 2-dimensional and 1-ended. The following are equivalent:

1. $W$ splits non-trivially over a 2-ended subgroup.
2. $W = A \ast_H B$ such that $A, B \neq W$ and $H$ is 2-ended.
3. There exist vertices $a$ and $b$ of $\Gamma$ such that:
Lemma 3.2. Suppose a group $H$ has the following properties: up to conjugacy, $H$ has precisely two ends, and if the above conditions hold, then up to conjugacy, $H$ has one of the following forms:

(i) $H = \langle a, b \rangle \cong D_\infty$; or
(ii) $H = \langle a, b, c \rangle \cong D_\infty \times C_2$, where $c$ is a vertex of $\Gamma$ adjacent to both $a$ and $b$ and $\Gamma \setminus \{a, b, c\}$ is disconnected.

Moreover, if the above conditions hold, then up to conjugacy, $H$ is virtually $\langle a, b \rangle$. More precisely, up to conjugacy, $H$ has one of the following forms:

(i) $H = \langle a, b \rangle \cong D_\infty$; or
(ii) $H = \langle a, b, c \rangle \cong D_\infty \times C_2$, where $c$ is a vertex of $\Gamma$ adjacent to both $a$ and $b$ and $\Gamma \setminus \{a, b, c\}$ is disconnected.

We begin by establishing some group-theoretic results that are needed for the proof. The following lemma is proved using elementary group theory.

Lemma 3.2. Suppose a group $G$ is virtually $\mathbb{Z}$. Let $K$ be a subgroup of $G$. Then $K$ is either finite or virtually $\mathbb{Z}$. Further, if $K$ is infinite, then $K$ has finite index in $G$. Since a group is 2-ended if and only if it is virtually $\mathbb{Z}$, it follows that any infinite subgroup of a 2-ended group is 2-ended.

Recall that a reflection $r_i$ in $W_\Gamma$ is a conjugate of an element of $S$. The following proposition establishes the structure of the 2-ended subgroups of $W_\Gamma$ which are generated by reflections.

Proposition 3.3. Let $W = W_\Gamma$ be a 2-dimensional right-angled Coxeter group with generating set $S$. Let $r_1 = g_1 s_1 g_1^{-1}, r_2 = g_2 s_2 g_2^{-1}, \ldots, r_k = g_k s_k g_k^{-1}$ be reduced words which are distinct reflections, where $s_i \in S$. (We allow $g_i = 1$ and do not require that the $s_i$ are distinct.)

If $\langle r_1, r_2, \ldots, r_k \rangle$ with $k \geq 1$ is 2-ended, then one of the following holds:

(1) $k = 2$ and $\langle r_1, r_2 \rangle \cong D_\infty$, in particular $r_1$ and $r_2$ do not commute; or
(2) $k = 3$ and $\langle r_1, r_2, r_3 \rangle \cong D_\infty \times C_2$ and one of the reflections, say $r_2$, commutes with the other two. As a consequence, either $s_1 = s_3$ and $s_2$ is adjacent to $s_1$, or $s_1 \neq s_3$, with $s_2$ adjacent to $s_1$ and $s_3$ and $d_\Gamma(s_1, s_3) = 2$.

Proof. Let $K$ be the subgroup $\langle r_1, r_2, \ldots, r_k \rangle$. If $k = 1$ then $K \cong C_2$ which is not 2-ended, so $k \geq 2$.

We will use standard facts about the action of $W$ on its Davis complex $\Sigma$ (see [14]). Let $H_i \subset \Sigma$ be the wall which is the fixed set of the reflection $r_i$. Each $H_i$ is a closed, convex, totally geodesic subcomplex of $\Sigma$. The two components of $\Sigma \setminus H_i$ are interchanged by $r_i$. Two walls $H_i$ and $H_j$ intersect if and only if the corresponding reflections $r_i$ and $r_j$ generate a finite dihedral subgroup of $W$, and $H_i$ and $H_j$ are disjoint if and only if $\langle r_i, r_j \rangle \cong D_\infty$. Moreover, since $W$ is right-angled, $\langle r_i, r_j \rangle$ is finite if and only if $r_i$ and $r_j$ commute, and if $r_i$ and $r_j$ commute then $s_i$ and $s_j$ are distinct commuting generators (that is, $s_i$ and $s_j$ are adjacent vertices of $\Gamma$).

It follows that if $k = 2$ and $K = \langle r_1, r_2 \rangle$ is 2-ended we are in case (1). Now assume $k \geq 3$. Since $\Gamma$ has no triangles, $\Sigma$ has no triples of pairwise intersecting walls, so we know that there are at most two commuting pairs among $r_1, r_2$ and $r_3$. If $k = 3$, suppose $r_1$ and $r_3$ don’t commute (hence $r_1$ and $r_3$ generate a $D_\infty$). If $r_2$ commutes with each of them, then the triple generates $D_\infty \times C_2$ and we are in one of the cases in (2). (Note that if $s_1 \neq s_3$ then $d_\Gamma(s_1, s_3) = 2$ since $\Gamma$ is triangle-free.)
Now suppose without loss of generality that $r_2$ does not commute with $r_1$. Then using the Ping Pong Lemma, we show that $\langle (r_1r_2)^2, (r_1r_3)^2 \rangle \cong F_2$. For this, define $X^+$ (resp. $X^-$, $Y^+$, $Y^-$) to be the closure of the component of the complement in $\Sigma$ of the wall $r_1r_2H_1$ (respectively, $r_2H_1$, $r_1r_3H_1$, and $r_3H_1$) which does not contain $H_1$. Then it is easy to see that $X^\pm, Y^\pm$ are all disjoint and that
$$(r_1r_2)^2(\Sigma \setminus X^-) \subseteq X^+; \quad (r_2r_1)^2(\Sigma \setminus X^+) \subseteq X^-; \quad (r_1r_3)^2(\Sigma \setminus Y^-) \subseteq Y^+; \quad (r_3r_1)^2(\Sigma \setminus Y^+) \subseteq Y^-.$$
Thus by the Ping Pong Lemma $\langle (r_1r_2)^2, (r_1r_3)^2 \rangle \cong F_2$ and so by Lemma 3.2, $\langle r_1, r_2, r_3 \rangle$ cannot be 2-ended.

Finally assume $k \geq 4$. Then $\langle r_1, r_2, r_3 \rangle$ is an infinite subgroup of a 2-ended group and is therefore 2-ended. So, by the $k = 3$ case, one of the elements, say $r_2$, commutes with the other two, and $r_1$ and $r_3$ don’t commute with each other. Now $\langle r_1, r_3, r_4 \rangle$ is also 2-ended. Similarly, since $r_1$ and $r_3$ don’t commute, we conclude that $r_4$ commutes with each of them. Since $\Gamma$ has no triangles, we now conclude that $r_2$ and $r_4$ do not commute and therefore $r_2$ and $r_4$ generate a $D_\infty$. As a result, $\langle r_1, r_2, r_3, r_4 \rangle \cong D_\infty \times D_\infty$, which is not 2-ended, giving a contradiction. This shows $k \leq 3$.

**Proof of Theorem 3.1.** The implication (2) $\implies$ (1) is by definition. To see that (1) $\implies$ (2), note that any HNN extension admits an epimorphism to $\mathbb{Z}$. Since $W$ is generated by torsion elements, it cannot admit such an epimorphism and so the splitting must be as in (2).

To see (3) $\implies$ (2), suppose the conditions in (3) hold. Then $\langle a, b \rangle \cong D_\infty$ is a 2-ended subgroup of $W_\Gamma$ and the set $S$ of vertices of $\Gamma$ can be written as a disjoint union

$$S = \{a, b\} \sqcup T_1 \sqcup \cdots \sqcup T_n$$

where each $T_j$ is the set of vertices of a component of $\Gamma \setminus \{a, b\}$ and $n \geq 2$. Let $T = \{a, b\} \sqcup T_1$ and $U = \{a, b\} \sqcup T_2 \sqcup \cdots \sqcup T_n$.

Then $W_\Gamma$ has the amalgamated free product decomposition $W_\Gamma = W_T \ast_{\{a, b\}} W_U$ and so $W_\Gamma$ splits over the 2-ended subgroup $\langle a, b \rangle$ as claimed.

We now show that if $W = A \ast_H B$ such that $A, B \neq W$ and $H$ is 2-ended as in (2), then the conditions in (3) hold and $H$ is as described in (i) or (ii) above.

Let $\mathcal{T}$ be the Bass–Serre tree of this splitting. Then the stabilisers of the vertices (respectively, edges) of $\mathcal{T}$ are conjugates of $A$ and $B$ (respectively, $H$). Now, every finite group acting on $\mathcal{T}$ has a global fixed point. Thus for each $s \in S$, the fixed set of $\langle s \rangle \cong C_2$ acting on $\mathcal{T}$ is a (nonempty) subtree which we denote by $\mathcal{T}_s$. Also, if $s$ and $t$ are adjacent vertices of $\Gamma$, then $\langle s, t \rangle \cong C_2 \times C_2$ is finite so $\mathcal{T}_s \cap \mathcal{T}_t$ is nonempty (see [23, Proposition I.6.26]).

We claim that there is an edge of $\mathcal{T}$ which is fixed by two distinct generators $a, b \in S$. To show this, we first define continuous maps from cycles in $\Gamma$ to $\mathcal{T}$ as follows. Choose a vertex $v_s \in \mathcal{T}_s$ for each vertex $s$ of $\Gamma$ and a vertex $v_{st} \in \mathcal{T}_s \cap \mathcal{T}_t$ for each adjacent pair of vertices $s, t$ in $\Gamma$. Given a cycle in $C$ in $\Gamma$, define a map $C \to \mathcal{T}$ as follows: a vertex $s \in C$ maps to $v_s$, the midpoint $x_{st}$ of the edge between consecutive vertices $s$ and $t$ maps to $v_{st}$ and the half-edge between $s$ and $x_{st}$ in $C$ maps to the shortest path in $\mathcal{T}_s$ between $v_s$ and $v_{st}$.

Now as $\Gamma$ has no separating vertices or edges, every vertex of $\Gamma$ is in an embedded cycle. Since $\Gamma$ is connected, the union of the images of these cycles in $\mathcal{T}$ is a connected subtree.
If this union were a single vertex $v$, then $v$ would have to be contained in $\cap_{s \in S}T_s$ and thus would be a global fixed point for the action of $W$. Since no such global fixed point exists, it follows that there is a cycle $C$ whose image in $T$ contains an edge $e$. Since $T$ has no cycles, $e$ is in fact contained in the images of two half-edges of $C$ and since $C$ is embedded in $\Gamma$, these two half-edges are distinct. Thus there are distinct vertices $a$ and $b$ of $\Gamma$ such that $e \in T_a \cap T_b$, i.e. $e$ is fixed by both $a$ and $b$. Set $e_{ab} = e$. This completes the proof of the claim.

We now assume that $H$ is the stabiliser of the edge $e_{ab}$ and consider two mutually exclusive possibilities:

**Case i:** Each fixed set $T_a$ other than $T_a$ and $T_b$ consists of a single vertex. In this case we show that $\Gamma \setminus \{a, b\}$ is disconnected, that $d_T(a, b) \geq 2$ and that $H = \langle a, b \rangle \cong D_\infty$.

Suppose $\Gamma \setminus \{a, b\}$ is connected. Then $\cup_{s \in S \setminus \{a, b\}}T_s$ is connected and therefore consists of a single vertex, say $v$. Moreover, since $a$ and $b$ are both adjacent to vertices in $\Gamma \setminus \{a, b\}$, the vertex $v$ lies in both $T_a$ and $T_b$. This implies that $v$ is a global fixed point for the action of $W$, a contradiction. We conclude that $\{a, b\}$ separates $\Gamma$ into at least two components. Now since $\Gamma$ has no separating edges, it follows that $d_T(a, b) \geq 2$ and consequently $\langle a, b \rangle \cong D_\infty$.

Since $\langle a, b \rangle$ is contained in the stabiliser of $e_{ab}$, we have $\langle a, b \rangle \subseteq H$. We now show that $H = \langle a, b \rangle$. Suppose not. Then there exists a nontrivial $h \in H \setminus \langle a, b \rangle$.

If $h$ has finite order then either $h = gsg^{-1}$ for some $s \in S$ and $g \in W$, or $h = gstg^{-1}$ for some adjacent vertices $s, t \in S$ and $g \in W$ (this follows from Theorem 12.3.4(1) of [14]). In the first case, since $h$ is in the stabiliser of $e_{ab}$, the edge $g^{-1}(e_{ab})$ of $T$ is fixed by $s$. Since we have assumed that $a$ and $b$ are the only generators that fix edges, $s$ must be one of these, say $s = a$. Thus $h = gag^{-1} \neq a$. But then by Proposition 3.3 the subgroup $\langle a, gag^{-1}, b \rangle$ is not 2-ended and so cannot be contained in $H$.

In the second case, since $H$ is 2-ended and $a$ and $b$ do not commute, a similar application of the Ping Pong Lemma to that in the proof of Proposition 3.3 implies that $gstg^{-1}$ must commute with both $a$ and $b$. It follows from Tits’ solution to the word problem [14, Theorem 3.4.2] that both $s$ and $t$ commute with both $a$ and $b$ (and are therefore distinct from $a$ and $b$). However this violates the no-triangles condition.

Now suppose $h$ has infinite order. Since $\langle a, b \rangle$ is an infinite subgroup of the 2-ended group $H$, it has finite index in $H$, so $h^n \in \langle a, b \rangle$ for some $n$. Now $h^n$ is nontrivial and can be written as a word in $a$ and $b$. It follows that $h$ is a word in $a, b$ and generators which commute with each other as well as with $a$ and $b$. Since $\Gamma$ has no triangles, there is at most one other such generator, say $s$. So we may write $h = ws$ where $w \in \langle a, b \rangle$. But then $w^{-1}h = s$ is in $H$ and so $s$ fixes the edge $e_{ab}$, a contradiction. Therefore $h \in \langle a, b \rangle$, contrary to our assumption.

Thus $H = \langle a, b \rangle$.

**Case ii:** At least one fixed set $T_c$ other than $T_a$ and $T_b$ contains an edge. In this case we show that $d_T(a, b) = 2$, with $c$ adjacent to both $a$ and $b$ (possibly after renaming $a, b$ and $c$), that $\Gamma \setminus \{a, b, c\}$ has at least two components and that $H = \langle a, b, c \rangle$.

Let $c_1, c_2, \ldots, c_l \in S \setminus \{a, b\}$ be the distinct elements such that $T_{c_i}$ contains an edge $e_{c_i}$. Since the action of $W$ is transitive on the edges of $T$, there are group elements $g_i$ such that $g_i(e_{c_i}) = e_{ab}$. The stabiliser of $e_{ab}$ then contains $a, b$ and $g_ic_ig_i^{-1}$ for all $i$. Since this stabiliser
is 2-ended, Proposition 3.3 implies that \( l = 1 \). Let \( c = c_1, e_c = e_{c_1} \) and \( g_c = g_1 \), so that \( g_c(e_c) = e_{ab} \).

We claim that \( T_a \cap T_b \cap T_c \) contains an edge. Now the edge \( e_{ab} \) is fixed by \( a, b \) and \( g_c g_c^{-1} \). Thus if \( g_c \) commutes with \( c \), then \( e_{ab} \) is the desired edge. Otherwise assume that \( g_c g_c^{-1} \) is reduced. Now \( \langle a, b, g_c g_c^{-1} \rangle \) is 2-ended so by Proposition 3.3, one of \( a, b \) and \( g_c g_c^{-1} \) commutes with the other two.

If \( g_c g_c^{-1} \) commutes with \( a \) and \( b \), then it follows that \( a \) and \( b \) commute with \( g_c \). Now \( a = g_c^{-1} a g_c \) and \( b = g_c^{-1} b g_c \) fix \( c \) and \( e_c \) is the desired edge. Otherwise assume, without loss of generality, that \( a \) commutes with \( b \) and with \( g_c g_c^{-1} \). Then, as before, \( a \) commutes with \( c \). Thus \( d_T(a, c) = d_T(a, b) = 1 \) and \( d_T(b, c) = 2 \). Thus in this case, \( T_a \cap T_c \) and \( T_a \cap T_b \) are nonempty. Now, given any \( s \in S \setminus \{a, b, c\} \), there is a path \( P \) in \( \Gamma \) connecting \( b \) to \( c \) which passes through \( s \) and avoids \( a \) (since \( a \) doesn’t separate \( \Gamma \)). Since \( T_a \) is a vertex for each \( t \neq a, b, c \), the union of the fixed trees for the vertices along \( P \) strictly between \( b \) and \( c \) is a single vertex and this vertex is in \( T_b \cap T_c \) as well. Since \( s \) was an arbitrary element of \( S \setminus \{a, b, c\} \), we have \( \cup_{s \in S \setminus \{a, b, c\}} T_s \subseteq T_b \cap T_c \).

Since \( T_a \cap T_c \), \( T_a \cap T_b \) and \( T_b \cap T_c \) are nonempty, it follows that \( T_a \cap T_b \cap T_c \) is nonempty. If \( T_b \cap T_c \) consisted of a single point \( v \), then \( T_a \cap T_b \cap T_c = \{v\} = T_b \cap T_c = \cup_{s \in S \setminus \{a, b, c\}} T_s \), making \( v \) a global fixed point, which is a contradiction. Thus \( T_b \cap T_c \) has at least one edge \( e_{bc} \). The stabiliser of \( e_{bc} \) is 2-ended and contains \( \langle g a g^{-1}, b, c \rangle \) for some \( g \). Since \( d_T(b, c) = 2 \), Proposition 3.3 implies that \( g a g^{-1} \) commutes with \( b \) and \( c \). After interchanging the roles of \( a \) and \( c \), we are in the case discussed in the first sentence of the previous paragraph.

This completes the proof of the claim and shows that, up to conjugacy, \( \langle a, b, c \rangle \subseteq H \). After possibly renaming, we may assume that \( d_T(a, b) = 2 \) and that \( c \) is adjacent to both \( a \) and \( b \) in \( \Gamma \) and after conjugating, we may assume that \( H \) is the stabiliser of an edge \( e \in T_a \cap T_b \cap T_c \). The proof that \( H = \langle a, b, c \rangle \) is then similar to Case i above.

Finally, we prove that \( \Gamma \setminus \{a, b, c\} \) has at least two components. Suppose not. Then as in Case i above \( \cup_{s \in S \setminus \{a, b, c\}} T_s \) is a single vertex \( v \), which necessarily lies in the intersection \( T_a \cap T_b \cap T_c \). Let \( e_{ab} \) be an edge incident to \( v \) in \( T_a \cap T_b \). If \( e_{ab} \) is also in \( T_c \), then \( v \) is a global fixed point, a contradiction. Otherwise let \( g \in W \) be such that \( g(e) = e_{ab} \), where \( e \in T_a \cap T_b \cap T_c \). Now the stabiliser of \( e_{ab} \) contains \( g a g^{-1}, g b g^{-1} \) and \( g c g^{-1} \), in addition to \( a \) and \( b \). If \( g a g^{-1} \neq a \), then Proposition 3.3 shows that \( \langle g a g^{-1}, a, b \rangle \) cannot be a subgroup of the edge group. Thus \( g a g^{-1} = a \) and similarly, \( g b g^{-1} = b \), so that the stabiliser of \( e_{ab} \) is \( H g^{-1} = \langle a, b, g c g^{-1} \rangle \), with \( g c g^{-1} \neq c \). Since \( e_{ab} \) is adjacent to \( v \), we have that the stabiliser \( Y \) of \( v \) contains \( \langle g c^{-1}, s \in S \mid s \neq c \rangle \).

To complete the proof, we obtain a contradiction by showing that there is no \( X < W \) such that \( W = X \ast Z Y \), where \( Z = \langle a, b, g c g^{-1} \rangle \) is the stabiliser of the edge \( e_{ab} \). Suppose such an \( X \) exists. By the normal form for amalgamated products, if \( x_1 y_1 \cdots x_n y_n = 1 \), where \( x_i \in X \) and \( y_i \in Y \), then \( x_i, y_i \in Z \) for all \( i \). Assume that \( g c g^{-1} \) is reduced. Then the element \( g \) can be written as \( g = w_1 c w_2 \cdots c w_k \), where \( w_1 \) is possibly trivial and each \( w_i \) is a word in \( S \setminus \{c\} \). Further, \( c \) itself is an alternating product of elements of \( X \) and \( Y \). (This includes the possibility that \( c \in X \).) Consider the equation \( \langle g c^{-1}, g(c)(g^{-1}) = 1 \). Each of the last three parentheses is an alternating product of elements in \( X \) and \( Y \). However, since \( c \) itself is not in \( Z \), not all of these elements can be in \( Z \). This is a contradiction. We have therefore
proved that no such splitting exists. It follows that the original assumption must be wrong and that $\Gamma \setminus \{a, b, c\}$ has at least two components.

\[\square\]

4. Distinguishing cocompact Fuchsian groups

In this section we prove Theorem 1.2 from the introduction, thereby establishing that among 2-dimensional right-angled Coxeter groups, the class of cocompact Fuchsian groups forms a quasi-isometry class. The proof uses the work of Lafont [19].

We will need the following graph-theoretic lemma:

**Lemma 4.1.** Suppose $\Gamma$ is a triangle-free simplicial graph which has no separating vertices or edges. If $\Gamma$ is not a cycle graph or a single edge then $\Gamma$ contains an induced subgraph $\Theta(n_1, n_2, n_3)$ with $n_1, n_2, n_3 \geq 1$.

**Proof.** Since $\Gamma$ has no separating vertices and is not a single edge, it is connected and every vertex of $\Gamma$ is part of an embedded cycle. Since $\Gamma$ is not itself a cycle, it contains at least two distinct (possibly intersecting) cycles, $\alpha$ and $\beta$. We claim that by possibly modifying $\beta$ by an inductive procedure, we may assume that $\alpha$ and $\beta$ intersect and their intersection is connected and contains at least two edges.

To see this, first suppose $\alpha$ and $\beta$ are disjoint and let $\gamma$ be a minimal path connecting $\alpha$ to $\beta$. Suppose the vertices along $\gamma$ are $u_1, \ldots, u_n$, with $u_1$ lying on $\alpha$ and $u_n$ lying on $\beta$. Let $\eta$ be a minimal path in $\Gamma \setminus \{u_n\}$ connecting $u_{n-1}$ to some vertex $w$ on $\beta$ (with $w \neq u_n$), which exists since $u_n$ is not separating. Replace $\beta$ by the cycle formed by concatenating the edge $[u_{n-1}, u_n]$, one of the arcs in $\beta$ from $u_n$ to $w$ and the path $\eta$. Rename this new cycle $\beta$ and note that its distance from $\alpha$ is at most $n - 1$. By inductively performing this procedure, we may assume that $\alpha$ and $\beta$ intersect in at least a vertex.

If the intersection is exactly a vertex, say $u$ and if $v$ denotes a vertex adjacent to $u$ on $\alpha$, then, as before, $v$ can be connected by a minimal path $\eta$ outside $\Gamma \setminus \{u\}$ to some vertex $w$ on $\beta$ (with $w \neq u$) and we may replace $\beta$ by the cycle formed by concatenating the edge $[v, u]$, one of the arcs from $u$ to $w$ in $\beta$ and the path $\eta$. Thus we may assume that $\alpha$ and $\beta$ share at least an edge.

Now suppose the intersection of $\alpha$ and $\beta$ has more than one component. Let $u_1$ and $u_2$ be vertices in the intersection of $\alpha$ and $\beta$, with the property that at least one of the two arcs in $\beta$ connecting $u_1$ and $u_2$ is disjoint from $\alpha$. Replace $\beta$ with this arc, concatenated with an arc in $\alpha$ connecting $u_1$ and $u_2$, to get a pair of cycles $\alpha$ and $\beta$ with connected intersection which contains at least one edge.

If the intersection of $\alpha$ and $\beta$ is exactly one edge, say $[u_1, u_2]$, let $v$ be a vertex adjacent to $u_1$ on $\alpha$. Since $\Gamma$ has no separating edges, there is a minimal path $\eta$ in $\Gamma \setminus [u_1, u_2]$ which connects $v$ to some vertex $w$ on $\beta$ (with $w \notin \{u_1, u_2\}$). We may replace $\beta$ by the cycle formed by concatenating the path $\eta$ with the edges $[v, u_1]$ and $[u_1, u_2]$ and the arc in $\beta$ from $u_2$ to $w$ which does not contain $u_1$. Now $\alpha$ and $\beta$ have connected intersection containing at least the two adjacent edges $[v, u_1]$ and $[u_1, u_2]$.

Since $\Gamma$ has no triangles, $\alpha \setminus (\alpha \cap \beta)$ and $\beta \setminus (\alpha \cap \beta)$ have at least one vertex each. It follows that $\alpha \cup \beta$ is equal to $\Theta(n_1, n_2, n_3)$ for some $n_1, n_2, n_3 \geq 1$. \[\square\]
Proof of Theorem 1.2. The implications (3) \(\implies (2)\), (3) \(\implies (1)\) and (1) \(\implies (2)\) are obvious.

Now suppose \(W_Γ\) is 2-dimensional and \(Γ \neq Λ_n\) for any \(n \geq 5\). Assume by way of contradiction that \(W_Γ\) is quasi-isometric to some \(W_n, n \geq 5\). Then \(W_Γ\) is 1-ended, so that \(Γ\) has no separating vertices or edges. We may assume \(Γ\) is not a single edge, as then \(W_Γ\) would be finite and not quasi-isometric to \(W_n\). Thus by Lemma 4.1, \(Γ\) contains an induced subgraph \(Θ = Θ(n_1, n_2, n_3)\). If more than one \(n_i\) is 1, then \(Θ\) and hence \(Γ\), contains a square. Thus \(W_Γ\) is not hyperbolic and cannot be quasi-isometric to \(W_n\). So we may assume \(n_2, n_3 \geq 2\).

Now \(W_Θ\) is a 1-ended special subgroup of \(W_Γ\), so \(W_Θ\) is a convex subset of \(W_Γ\). It follows that \(∂W_Θ\) is a closed, connected subset of \(∂W_Γ\). Since \(∂W_Γ\) is homeomorphic to \(S^1\), \(∂W_Θ\) is homeomorphic to either \(S^1\) or to a closed interval in \(S^1\).

Assume first that \(n_1 \geq 2\), and let \(\tilde{X}\) denote the universal cover of the orbifold associated to \(W_Θ\), as defined in Section 2.1. Recall that \(\tilde{X}\) consists of convex planar hyperbolic regions called pieces that are glued along branching geodesics. Each branching geodesic has \(k = 3\) pieces incident to it.

By [19, Lemma 3.4], if \(γ\) is a branching geodesic with endpoints \(γ^+, γ^- \in ∂\tilde{X}\) and \(P_1\) and \(P_2\) are distinct pieces incident to \(γ\) with limit sets \(∂P_1\) and \(∂P_2\), then \(∂P_1 \setminus \{γ^+, γ^-\}\) and \(∂P_2 \setminus \{γ^+, γ^-\}\) lie in different connected components of \(∂\tilde{X} \setminus \{γ^+, γ^-\}\). Since each branching geodesic has exactly \(k = 3\) pieces incident to it, it follows that \(∂\tilde{X} \setminus \{γ^+, γ^-\}\) has \(k\) = 3 components and the closure of each component contains both \(ξ^+\) and \(ξ^-\). This is impossible for \(∂W_Θ = ∂\tilde{X}\) homeomorphic to either \(S^1\) or an interval in \(S^1\). Therefore \(W_Γ\) is not quasi-isometric to any \(W_n, n \geq 5\).

Now assume that \(n_1 = 1\). Then in the notation of Remark 2.2, \(D_x(Θ) = Θ(n_2, n_2, n_3, n_3)\) and by Lemma 2.3, \(W_{D_x(Θ)}\) is quasi-isometric to \(W_Θ\). Hence \(∂W_{D_x(Θ)}\) is homeomorphic to \(∂W_Θ\). By similar arguments, applied to the universal cover of the orbifold associated to \(W_{D_x(Θ)}\), that is, with \(k = 4\), we obtain again that \(∂W_Θ\) cannot be homeomorphic to either \(S^1\) or an interval in \(S^1\). Thus in all cases, \(W_Γ\) is not quasi-isometric to any \(W_n, n \geq 5\). □

5. Boundaries and JSJ splittings

In this section we recall Bowditch’s construction of JSJ trees from [8], and compute these trees for right-angled Coxeter groups associated to a generalised \(Θ\) graph.

5.1. Bowditch’s JSJ trees. Given a 1-ended hyperbolic group \(G\) which is not cocompact Fuchsian, Bowditch [8] uses the structure of local cut points of its boundary \(M = ∂G\) to define a canonical JSJ tree associated to \(G\).

To describe the tree, we need to introduce some terminology. Following [8], given \(x \in M\), define the valency \(\text{val}(x)\) of \(x\) to be the number of ends of the locally compact space \(M \setminus \{x\}\). A priori \(\text{val}(x) \in \mathbb{N} \cup \{∞\}\), but Bowditch shows [8, Proposition 5.5] that if \(M\) is the boundary of a 1-ended hyperbolic group, then \(\text{val}(x)\) is finite for all \(x \in M\). The point \(x\) is said to be a local cut point if \(\text{val}(x) \geq 2\).

Now given \(n \in \mathbb{N}\), let \(M(n) = \{x \in M \mid \text{val}(x) = n\}\) and \(M(n+) = \{x \in M \mid \text{val}(x) \geq n\}\). Bowditch defines relations \(\sim\) and \(\approx\) on \(M(2)\) and \(M(3+)\) respectively, as follows. For \(x, y \in M\), let \(N(x, y)\) be the number of components of \(M \setminus \{x, y\}\).
The relation $\sim$. Given $x, y \in M(2)$, let $x \sim y$ if and only if either $x = y$ or $N(x, y) = 2$. The following are some properties of $\sim$.

1. The relation $\sim$ is an equivalence relation on $M(2)$ [8, Lemma 3.1].
2. The $\sim$-equivalence classes are of two types: either they are pairs or they are infinite. Moreover, the infinite ones are Cantor sets [8, Corollary 5.15 and Proposition 5.18].

The relation $\approx$. Given $x, y \in M(3+)$, let $x \approx y$ if $x \neq y$ and $N(x, y) = \text{val}(x) = \text{val}(y) \geq 3$. The following are some properties of $\approx$.

1. If $x \approx y$ and $x \approx z$ then $y = z$ [8, Lemma 3.8]. Thus a priori, a subset of $M(3+)$ is partitioned into pairs of the form $\{x, y\}$ where $x \approx y$.
2. Lemma 5.12 and Proposition 5.13 of [8] show that in fact, every point of $M(3+)$ is in a $\approx$-pair.

Outline of the construction. The JSJ tree is produced as follows. The set $T$ of $\sim$-classes and $\approx$-pairs forms a pretree (i.e. a set with a betweenness condition satisfying certain axioms) which is discrete (i.e. intervals between points are finite).

Definition 5.1. [Betweenness] Given three classes $\eta, \theta, \zeta \in T$, the class $\theta$ is between $\eta$ and $\zeta$ if there exist points $x, y \in \theta \subset M$ such that $\eta$ and $\zeta$ are in different components of $M \setminus \{x, y\}$.

In [10], Bowditch proves that every discrete pretree can be embedded in a discrete median pretree by adding in all possible stars of size at least 3. (A star is a maximal subset $S$ of $T$ with the property that no point of $T$ is between any pair of points in $S$. The size of a star is its cardinality. A pretree $T$ is median if for any three points $x, y, z \in T$, the intervals $[x, y], [y, z]$ and $[z, x]$ have a common point.) Further, any discrete median pretree can be realised as the vertex set of a simplicial tree. The simplicial tree obtained in this way from $T$ is essentially the JSJ tree, although certain additional vertices are added at the midpoints of some of the edges in order to get a cleaner statement about stabilisers.

The JSJ tree. The properties of this tree $\Sigma$ are summarised in Theorems 0.1 and 5.28 of [8] and are explained in more detail in Sections 3 and 5 of that paper. The group $G$ acts minimally, simplicially and without edge inversions on $\Sigma$ such that the quotient $\Sigma/G$ is finite.

Vertices. The tree $\Sigma$ has vertices of three types: $V_1(\Sigma), V_2(\Sigma)$ and $V_3(\Sigma)$.

1. The vertices $V_1(\Sigma)$. This set of vertices consists of:
   - the $\sim$-classes in $M(2)$ consisting of exactly two elements;
   - all the $\approx$-pairs; and
   - the extra vertices mentioned at the end of the previous section, which are added at the midpoints of edges between stars and infinite $\sim$-classes.

   For all $v \in V_1(\Sigma)$, the stabiliser $G(v)$ of $v$ is a maximal 2-ended subgroup of $G$. If $v$ comes from a $\sim$ or a $\approx$-pair $\{x, y\}$, its degree in $\Sigma$ is equal to $N(x, y) < \infty$. The degree of the added vertices is clearly 2.
(2) The vertices $V_2(\Sigma)$. This set of vertices is in bijective correspondence with the collection of infinite $\sim$-classes in $M(2)$.

The stabiliser $G(v)$ of a vertex $v \in V_2(\Sigma)$ is a maximal hanging Fuchsian (MHF) subgroup of $G$. This means that there is a properly discontinuous action of $G(v)$ on $\mathbb{H}^2$, without parabolics, such that the quotient $\mathbb{H}^2/G(v)$ is not compact and further, there is an equivariant homeomorphism from the $\sim$-class corresponding to $v$ onto the limit set of the $G$-action on $\partial \mathbb{H}^2$. The degree of a vertex in $V_2(\Sigma)$ is infinite.

(3) The vertices $V_3(\Sigma)$. These are the vertices corresponding to the added stars of size at least three mentioned in the outline of the construction. The stabiliser of a vertex of this type is an infinite non-elementary group which is not a maximal hanging Fuchsian subgroup. The degree of a vertex of this type is infinite.

Edges. The two endpoints of an edge of $\Sigma$ are never of the same type. Vertices $p \in V_1(\Sigma)$ and $q \in V_2(\Sigma)$ are adjacent if and only if they form a star of size 2. By definition, a vertex $q \in V_3(\Sigma)$ corresponds to a star of size at least 3 in $V_1(\Sigma) \cup V_2(\Sigma)$. There is an edge between $q$ and each vertex in this star. The stabiliser of an edge is a 2-ended subgroup.

Remark 5.2. (Quasi-isometry invariance) By the construction above if $G$ and $H$ are quasi-isometric, then since $\partial G$ and $\partial H$ are homeomorphic, the JSJ trees $\Sigma_G$ and $\Sigma_H$ are equal as coloured trees. That is, the JSJ tree is a quasi-isometry invariant.

5.2. JSJ trees for generalised $\Theta$ graphs. In this section we construct the JSJ tree $\Sigma$ associated to $W = W_\Theta$, where $\Theta = \Theta(n_1, \ldots, n_k)$ is a generalised $\Theta$ graph.

Remark 5.3. We may assume that $n_1 \geq 2$: If $\Theta = \Theta(1, n_2, \ldots, n_k)$, then in the notation of Remark 2.2, $D_x(\Theta) = \Theta(n_2, n_2, n_3, n_3, \ldots, n_k, n_k)$. By Lemma 2.3, $W_\Theta$ is quasi-isometric to $W_{D_x(\Theta)}$ and therefore has the same JSJ tree.

For most of this section, we assume that $n_1 \geq 2$. Let $\tilde{X}$ denote the universal cover of the associated orbifold $O$, as defined in Section 2.1. We use Lafont’s analysis in [19] of the boundary $\partial \tilde{X}$ to construct the JSJ tree $\Sigma$ for $W$.

We identify $\partial W = \partial \tilde{X}$ with geodesic rays in $\tilde{X}$ emanating from some base point. We will sometimes abuse notation and use the same letter to denote a geodesic ray and the point it corresponds to in $\partial \tilde{X}$. Recall from Section 2.1 that $\tilde{X}$ consists of convex planar hyperbolic regions called pieces that are glued along branching geodesics. (Note that in [19], a piece is called the lift of a chamber.) Each piece is incident to infinitely many branching geodesics, while each branching geodesic has $k$ pieces incident to it. In this section we think of pieces as being closed, with infinitely many branching geodesics as boundary components.

We first show that the branching geodesics yield $\approx$-pairs (and therefore vertices in $V_1(\Sigma)$):

**Lemma 5.4.** If $\gamma$ is a branching geodesic in $\tilde{X}$, then its endpoints $\{\gamma^+, \gamma^-\}$ in $\partial \tilde{X}$ form a $\approx$-pair. The degree of the corresponding vertex in $\Sigma$ is $k$.

**Proof.** We claim that $\partial \tilde{X} \setminus \{\gamma^+, \gamma^-\}$ has exactly $k$ components. The proof of this fact below is similar to that of item (2) in the proof of Proposition 3.2 in [19]. Recall that $\gamma$ has exactly $k$ pieces incident to it, say $P_1, \ldots, P_k$. Now for $j \geq 1$ inductively define $(P_i)_j$ by
\( (P_i)_1 = P_i \setminus \gamma \).

- \( (P_i)_{j+1} \) is the union of \( (P_i)_j \) and all the pieces of \( \tilde{X} \) incident to \( (P_i)_j \) except those that are also incident to \( \gamma \).

Form the sets \( Y_i = \cup_{j \in \mathbb{N}} (P_i)_j \), for \( 1 \leq i \leq k \). Then as argued in [19, page 777] (the sets defined there differ from ours only in the first step), each \( Y_i \) is path connected and has the property that \( \partial X_i \setminus \{ \gamma^+, \gamma^- \} \) is path connected. Now observe that \( \tilde{X} \setminus \gamma = \bigcup_{1 \leq i \leq k} Y_i \) and the closure of \( Y_i \) is \( Y_i \cup \gamma \). Thus \( \partial \tilde{X} \setminus \{ \gamma^+, \gamma^- \} \) is the union of exactly \( k \) closed path-connected sets whose intersection is precisely \( \{ \gamma^+, \gamma^- \} \), which proves the claim.

Thus \( \{ \gamma^+, \gamma^- \} \) forms a \( \approx \)-pair which separates \( \partial \tilde{X} \) into exactly \( k \) components and therefore yields a vertex in \( V_1(\Sigma) \) of degree \( k \).

Next, we identify some infinite \( \sim \)-classes. Given a piece \( P \), its limit set can be written as a disjoint union \( \partial P = A_P \sqcup B_P \), where \( B_P \) is the set of endpoints of the branching geodesics that are the boundary components of \( P \).

**Lemma 5.5.** Given a piece \( P \) in \( \tilde{X} \), the subset \( A_P \) of \( \partial P \) is a single (infinite) \( \sim \)-class.

Note that the corresponding vertex is in \( V_2(\Sigma) \) and has infinite degree.

**Proof.** Consider a pair of points \( p, q \in A_P \). Then \( \partial \tilde{X} \setminus \{ p, q \} \) consists of two components which are path connected. The proof of this is identical to that of item (2) of Proposition 3.2 of [19], where \( p \) and \( q \) are assumed to be in \( B_P \) instead of \( A_P \). Thus \( p \sim q \). It follows that \( A_P \subset \partial \tilde{X}(2) \), and all the points of \( A_P \) are \( \sim \)-equivalent to each other.

Now if \( p \in A_{P_1} \) and \( q \in A_{P_2} \) where \( P_1 \) and \( P_2 \) are distinct pieces, then \( \partial \tilde{X} \setminus \{ p, q \} \) is connected. The proof is identical to that of item (3) in Proposition 3.2 of [19], where \( p \) and \( q \) are assumed to be in \( B_{P_1} \) and \( B_{P_2} \) instead. It follows that points of \( A_{P_1} \) and \( A_{P_2} \) are not \( \sim \)-equivalent.

Thus given a piece \( P \), the set \( A_P \) is a \( \sim \)-equivalence class. \( \square \)

We claim that Lemmas 5.4 and 5.5 account for all the \( \approx \)-pairs and \( \sim \)-classes. Equivalently, \( V_1(\Sigma) \) (respectively, \( V_2(\Sigma) \)) consists exactly of the vertices described in Lemma 5.4 (respectively, Lemma 5.5).

Every local cut point of \( \partial \tilde{X} \) is in exactly one \( \approx \)-pair or exactly one \( \sim \)-class. If \( \gamma \) is a geodesic ray in \( \tilde{X} \) which passes through finitely many pieces, then it eventually lies in a single piece \( P \). Then \( \gamma \) is in \( \partial P = A_P \sqcup B_P \) and is part of one of the \( \approx \)-pairs or \( \sim \)-classes from Lemmas 5.4 and 5.5.

The remaining points of \( \partial \tilde{X} \) are geodesic rays which pass through infinitely many pieces. We now show that such points are not local cut points and therefore do not yield any additional vertices of \( \Sigma \). In particular, \( \Sigma \) does not have any vertices which are \( \sim \)-pairs.

**Lemma 5.6.** Let \( \gamma \in \partial \tilde{X} \) be a geodesic ray which passes through infinitely many pieces. Then \( \gamma \) is not a local cut point of \( \partial \tilde{X} \).

**Proof.** Since \( \gamma \) passes through infinitely many pieces, it crosses infinitely many branching geodesics. Let \( \{ \eta_i \} \) be the set of branching geodesics crossed by \( \gamma \), indexed by the order in which they are crossed. By (2) in Case 2 of the proof of Proposition 3.1 of [19], if \( U_i \)
denotes the path-connected component of $\partial \tilde{X} \setminus \{\eta_1^+, \eta_1^-\}$ containing $\gamma$, then the collection $\{U_i\}$ forms an open, path-connected, neighbourhood base of $\gamma$ in $\partial \tilde{X}$. Observe that these sets are nested, with $U_i \subset U_j$ if $i > j$.

To show that $\gamma$ is not a local cut point, it is enough to show that for any $n$ and for any $p,q \in U_n \setminus \{\gamma\}$, there is a path in $U_n \setminus \{\gamma\}$ connecting $p$ and $q$. Since $U_n$ is path-connected, there exists a path $\tau$ connecting $p$ and $q$ in $U_n$. If $\tau$ misses the boundary point $\gamma$, we are done. If not we may assume (by excising loops based at $\gamma$ if necessary) that $\tau$ intersects $\gamma$ exactly once. Choose $m > n$ such that $p,q \in U_n \setminus U_m$. Then since $\gamma \in U_m$, the path $\tau$ must enter and exit $U_m$ and therefore meet $\{\eta_m^+, \eta_m^-\}$ at least twice. Let $\alpha$ and $\beta$ be the points in $\tau \cap \{\eta_m^+, \eta_m^-\}$ such that the segment of $\tau$ between $\alpha$ and $\beta$ lies in $U_m$ and contains $\gamma$. If $\alpha = \beta$, then by cutting out the segment of $\tau$ between $\alpha$ and $\beta$ we obtain the desired path between $p$ and $q$. If not, then $\alpha$ and $\beta$ are opposite endpoints of the branching geodesic $\eta_m$. We saw in the proof of Lemma 5.4 that $\partial \tilde{X} \setminus \{\eta_m^+, \eta_m^-\}$ has $k \geq 3$ path components. Thus there is a path component $\tilde{X} \neq U_m$ which does not contain $\eta_m$. By Fact 1 on page 777 of [19], the closure of $\tilde{X}$ is $Y \cup \{\eta_m^+, \eta_m^-\}$ and is path-connected. Thus we may modify $\tau$ by replacing the segment between $\alpha$ and $\beta$ by a path connecting $\alpha$ and $\beta$ in $Y$. Now $Y$ and $U_m$ are in the same path component of $\partial \tilde{X} \setminus \{\eta_m^+, \eta_m^-\}$, since their closures both contain $\{\eta_m^+, \eta_m^-\}$. Thus $Y \subset U_n$. It follows that the new path between $p$ and $q$ is in $U_n \setminus \{\gamma\}$.

This gives the desired path which avoids $\gamma$. \qed

We can now describe the JSJ tree $\Sigma$.

**Proposition 5.7.** Let $W = W_\Theta$ where $\Theta = \Theta(n_1, n_2, \ldots, n_k)$ with $n_1 \geq 1$ and $n_i \geq 2$ for $2 \leq i \leq k$.

1. If $n_1 \geq 2$, the JSJ tree for $W$ is a biregular tree with vertex set $V = V_1 \sqcup V_2$, where each vertex of $V_1$ has valence $k$ and stabiliser a maximal 2-ended subgroup of $W$ and each vertex of $V_2$ has valence countably infinite and stabiliser an MHF subgroup of $W$.

2. If $n_1 = 1$ the JSJ tree for $W$ is a biregular tree with vertex set $V = V_1 \sqcup V_2$, where each vertex of $V_1$ has valence $2(k-1)$ and stabiliser a maximal 2-ended subgroup of $W$ and each vertex of $V_2$ has valence countably infinite and stabiliser an MHF subgroup of $W$.

**Remark 5.8.** The proof of Proposition 5.7 below will show that $\Sigma$ can be constructed as a sort of dual to the decomposition of $\tilde{X}$ into pieces. More precisely, $\Sigma$ can be constructed by putting a vertex on each branching geodesic and on each piece, and then connecting each piece-vertex to every vertex on a boundary component of that piece.

**Proof.** For (1), note that the discussion above implies that $V_1(\Sigma)$ is in bijective correspondence with the set of branching geodesics and $V_2(\Sigma)$ is in bijective correspondence with the set of pieces. Each vertex of $V_1$ has degree $k$ and each vertex of $V_2$ has infinite degree.

We now observe (by Definition 5.1 of betweenness):

1. If $p_1, p_2 \in V_2$ are distinct, then there exists $q \in V_1$ between $p_1$ and $p_2$.

2. If $q_1, q_2 \in V_1$ are distinct, then there exists $p \in V_2$ between $q_1$ and $q_2$.

It follows that a star in $V_1 \cup V_2$ cannot have two vertices of $V_1$ or two vertices of $V_2$. Thus all stars have at most two vertices and $V_3$ is empty.
The edges of $\Sigma$ are exactly the stars of size 2. It is easy to see that $q \in V_1$ and $p \in V_2$ form a star if and only if the branching geodesic corresponding to $q$ is a boundary component of the piece corresponding to $p$.

The statement about stabilisers comes from Bowditch’s construction. Statement (2) follows from Remark 5.3.

6. Quasi-isometry for generalised $\Theta$ graphs

In this section we prove Theorem 1.5 from the introduction, obtaining the complete quasi-isometry classification of hyperbolic right-angled Coxeter groups defined by generalised $\Theta$ graphs. In particular, the proof shows that the JSJ tree is a complete invariant for this class.

Proof of Theorem 1.5. If (1) holds, i.e. $W$ and $W'$ are quasi-isometric, then by Remark 5.2 they have the same JSJ tree. The description of the tree in Proposition 5.7 then implies that one of the three conditions in (2) holds.

Assume that (2)(b) holds. To show that $W$ and $W'$ are quasi-isometric, we will use the quasi-isometries of “fattened trees” introduced by Behrstock–Neumann in [5]. As in [5], denote by $T_0$ the tree with all vertices of valence 3 and all edges of length 1 and let $X_0$ be the “fattening” of $T_0$ where each edge $E$ is replaced by a strip isometric to $E \times [-\frac{1}{2}, \frac{1}{2}]$ and each vertex is replaced by an equilateral triangle of side-length 1 around the boundary of which the strips corresponding to incoming edges are attached.

Let $\gamma$ be a branching geodesic of the universal cover $\tilde{X}$ of the orbifold for $W$ (see Section 2.1). Fix $\Phi_\gamma : \gamma \to \partial_0 X_0$ an $M$-bi-Lipschitz homeomorphism to a boundary component of $X_0$. There are only finitely many isometry classes of pieces of $\tilde{X}$. Thus by Theorem 1.2 of [5], there is a $K > 0$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ such that for every piece $P$ of $\tilde{X}$ with $\gamma$ as a boundary component, $\Phi_\gamma$ extends to a $\phi(M)$-bi-Lipschitz homeomorphism $\Phi_P : P \to X_0$ which is $K$-bi-Lipschitz on every boundary component of $P$ other than $\gamma$. Without loss of generality, we may assume that $K \geq M$ and $\phi(M) \geq M$.

Now let $P$ be a piece of $\tilde{X}$ with $\gamma$ as a boundary component and let $Q$ be a piece of $\tilde{X}$ which is adjacent to $P$ and does not intersect $\gamma$. Again by Theorem 1.2 of [5], the $K$-bi-Lipschitz homeomorphism induced by $\Phi_P$ from the branching geodesic $P \cap Q$ to a boundary component of $X_0$ extends to a $\phi(K)$-bi-Lipschitz homeomorphism $\Phi_Q : Q \to X_0$ which is $K$-bi-Lipschitz on every boundary component of $Q$. Without loss of generality, we may assume that $\phi(K) \geq \phi(M)$.

Applying Theorem 1.2 of [5] (and the fact that there are only finitely many isometry types of pieces) repeatedly, we inductively construct a $\phi(K)$-bi-Lipschitz homeomorphism $\Phi$ from $\tilde{X}$ to the space $X$ defined as follows. The space $X$ consists of one copy of $X_0$ for each piece $P$ of $\tilde{X}$, with $\Phi|_P$ equal to the map $\Phi_P$ constructed above. The boundary components of the copies of $X_0$ in $X$ are glued together according to the adjacency relation between pieces of $\tilde{X}$. That is, if $P_1$ and $P_2$ are disjoint then $\Phi(P_1)$ and $\Phi(P_2)$ are disjoint, while if $P_1$ and $P_2$ are adjacent along a branching geodesic $\eta$, then $\Phi(P_1)$ and $\Phi(P_2)$ are identified along $\Phi(\eta)$. We carry out the same construction for $\tilde{X}'$, the universal cover of the orbifold for $W'$, obtaining a $\phi'(K')$-bi-Lipschitz homeomorphism $\Phi'$ from $\tilde{X}'$ to $X'$. 

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Now since each branching geodesic of the universal covers $\tilde{X}$ and $\tilde{X}'$ is contained in exactly $k = k'$ distinct pieces, each image of a branching geodesic in either $X$ or $X'$ is contained in exactly $k = k'$ distinct copies of $X_0$. It follows that $X = X'$. It is now immediate that $\tilde{X}$ and $\tilde{X}'$ are bi-Lipschitz homeomorphic, hence $W$ and $W'$ are quasi-isometric.

Suppose (2)(c) holds. Then, in the notation of Remark 2.2, $D_x(\Theta) = \Theta(n_2, n_2, \ldots, n_k, n_k)$, and $W$ is commensurable hence quasi-isometric to $W_{D_x(\Theta)}$ by Lemma 2.3. Since $W_{D_x(\Theta)}$ is quasi-isometric to $W'$ by the implication (2)(b) $\Rightarrow$ (1), it follows that $W$ and $W'$ are quasi-isometric.

Finally, if (2)(a) holds, then clearly $D_x(\Theta)$ and $D_{x'}(\Theta')$ satisfy the conditions of (2)(b), where $x$ and $x'$ are the unique vertices in the first branches of $\Theta$ and $\Theta'$ respectively. Then by Lemma 2.3 and implication (2)(b) $\Rightarrow$ (1), $W$ and $W'$ are quasi-isometric. $\Box$

7. Commensurability for Generalised $\Theta$ Graphs

In this section we begin by proving Proposition 7.2 below, which establishes sufficient conditions for the commensurability of pairs of hyperbolic right-angled Coxeter groups defined by generalised $\Theta$ graphs, and generalises a result of Crisp–Paoluzzi [12]. Further, we conjecture:

Conjecture 7.1. The converse of Proposition 7.2 is true.

In Theorem 1.1 of [12], Crisp–Paoluzzi prove Proposition 7.2 and its converse in the case $n_1 = n'_1 = 1$ and $k = k' = 3$. In Theorem 7.3 below, we prove the converse of Proposition 7.2 in the additional cases $n_1 = n'_1 = 1$ and $k = k' = 4$, and $n_1 \geq 2, n'_1 \geq 2$ and $k = k' = 3$. It follows that the converse to Proposition 7.2 holds whenever $k = k' = 3$ (see Corollary 7.4). Before proving Theorem 7.3, we will describe a general set-up, in order to indicate where the difficulty lies in proving the converse in general.

Proposition 7.2. Let $W$ and $W'$ be as in the statement of Theorem 1.5. If any of the following conditions is satisfied, then $W$ and $W'$ are commensurable.

1. $n_1 = n'_1 = 1$, $k = k'$ and after possibly permuting the $n_i$ and $n'_j$,

$$\frac{n_2 - 1}{n'_2 - 1} = \frac{n_3 - 1}{n'_3 - 1} = \cdots = \frac{n_k - 1}{n'_k - 1}.$$ 

2. $n_1 \geq 2$, $n'_1 \geq 2$, $k = k'$ and after possibly permuting the $n_i$ and $n'_j$,

$$\frac{n_1 - 1}{n'_1 - 1} = \frac{n_2 - 1}{n'_2 - 1} = \cdots = \frac{n_k - 1}{n'_k - 1}.$$ 

3. $n_1 = 1$, $n'_1 \geq 2$, $k' = 2(k - 1)$ and after possibly permuting the $n_i$ and $n'_j$,

$$\frac{n_2 - 1}{n'_2 - 1} = \frac{n_2 - 1}{n'_3 - 1} = \frac{n_3 - 1}{n'_4 - 1} = \cdots = \frac{n_k - 1}{n'_{k-1} - 1} = \frac{n_k - 1}{n'_{k'} - 1}.$$ 

Proof. Suppose (2) holds and let $\mathcal{O} = \mathcal{O}(n_1, n_2, \ldots, n_k)$ be the orbifold for $W$ (see Section 2.1). Generalising Section 3.1 of [12], we observe that for any positive integer $R$ there is a degree $R$ orbifold covering $R\mathcal{O} \to \mathcal{O}$ obtained by unfolding $R$ times along mirrors of type
Let $W$ and $W'$ be as in the statement of Theorem 1.5. Suppose $W$ and $W'$ are commensurable.

(1) If $n_1 = n'_1 = 1$ and $k = k' = 4$ then, after possibly permuting the $n_i$ and $n'_i$,
\[
\frac{n_2 - 1}{n'_2 - 1} = \frac{n_3 - 1}{n'_3 - 1} = \frac{n_4 - 1}{n'_4 - 1}.
\]

(2) If $n_1 \geq 2$, $n'_1 \geq 2$ and $k = k' = 3$ then, after possibly permuting the $n_i$ and $n'_i$,
\[
\frac{n_1 - 1}{n'_1 - 1} = \frac{n_2 - 1}{n'_2 - 1} = \frac{n_3 - 1}{n'_3 - 1}.
\]

We defer the proof of this theorem to the end of this section. It is now easy to conclude:

**Corollary 7.4.** The converse of Proposition 7.2 is true when $k = k' = 3$.

**Proof.** Suppose $k = k' = 3$. If $W$ and $W'$ are commensurable, then they are quasi-isometric, so as $k = k'$ either 2(a) or 2(b) from Theorem 1.5 holds. If 2(a) holds, then [12, Theorem 1.1] says that the desired ratios are equal. If 2(b) holds then Theorem 7.3(2) yields the result.

We now describe the set-up for our proof of Theorem 7.3. This set-up holds for any $W$ and $W'$ as in Theorem 1.5, and generalises the argument in Section 3.3 of [12]. Suppose $W$ and $W'$ are commensurable. Then $W$ and $W'$ are quasi-isometric, and so after doubling if necessary, we may assume without loss of generality that $k = k'$. That is, either 2(a) or 2(b) of Theorem 1.5 holds.

First, suppose 2(b) of Theorem 1.5 holds, so that $n_1, n'_1 \geq 2$ and $k = k'$. Since $W$ and $W'$ are virtually torsion-free, we may suppose that they contain isomorphic torsion free finite index subgroups. Let $X$ and $X'$ be the corresponding finite-sheeted orbifold covers of $O$
Lemma 7.5. If \( f \) under \( p \) the entry does not necessarily induce a bijection of types. We now use \( f \) closures of components of \( X \) component of \( X \) of a component of \( X \) contains exactly one branching edge in its boundary.

The following lemma records some elementary facts about this set-up.

**Lemma 7.5.** If \( P \) and \( P' \) are the matrices defined above, then

1. For all \( 1 \leq i, j \leq k \), we have \( p_{ij} = 0 \iff p'_{ij} = 0 \).
2. Permuting the \( m_i \) corresponds to permuting the rows of \( P \) and of \( P' \) (by the same permutation) and permuting the \( m'_{ij} \) corresponds to permuting the columns of \( P \) and of \( P' \) (by the same permutation).
3. For all \( 1 \leq i, j \leq k \), if \( p_{ij}, p'_{ij} \neq 0 \) then
   \[
   \frac{m_i}{m'_{ij}} = \frac{p'_{ij}}{p_{ij}}.
   \]
4. There is an integer \( |P| > 0 \) such that each row sum and each column sum of \( P \) is equal to \( |P| \). Similarly, there is an integer \( |P'| > 0 \) such that each row sum and each column sum of \( P' \) is equal to \( |P'| \).
5. Given a minor \[
\begin{bmatrix}
p_{ij} & p_{it} \\
p_{nj} & p_{nt}
\end{bmatrix}
\] of \( P \) such that none of its entries are 0, we have
   \[
   \begin{bmatrix}
p'_{ij} & p'_{it} \\
p'_{nj} & p'_{nt}
\end{bmatrix}
   \]
   is the corresponding minor of \( P' \).

**Proof.** Statements (1) and (2) are immediate from the definitions of \( P \) and \( P' \). For (3) we use Lemma 2.1 of [12], which states that if \( S \) is a compact surface, with or without boundary,
which is tiled by $|S|$ right-angled $(m + 4)$-gons, then the Euler characteristic of $S$ is given by $\chi(S) = -\frac{m|S|}{4}$. Now assume $p_{ij}, p'_{ij} \neq 0$ and let $S$ be the (non-empty) union of all subsurfaces of $X$ of type $m_i$ which are mapped by $f$ to subsurfaces of $X'$ of type $m'_j$. Let $S' = f(S)$ and note that $S'$ is a union of subsurfaces of $X$ of type $m'_j$. Then $S$ is tiled by $p_{ij}$ tiles which are right-angled $(m_i + 4)$-gons and $S'$ is tiled by $p'_{ij}$ tiles which are right-angled $(m'_j + 4)$-gons. Since $\chi(S) = \chi(S')$ we have that $\frac{m_i}{4}p_{ij} = \frac{m'_j}{4}p'_{ij}$ and the statement follows.

To prove (4) note that the total number of tiles of each given type is equal to the number of branching edges in $X$ (or $X'$). Let $|P|$ (respectively, $|P'|$) denote this number, which is equal to the degree of the cover $X \to O$ (respectively, $X' \to O'$). Since the total number of tiles of type $m_i$ is the sum of the $i$th row of $P$, we see that the row sums of $P$ are $|P|$. To see that the column sums of $P$ are $|P|$, note that for each type $m'_j$, at each branching edge, exactly one tile maps to a subsurface of $X'$ tiled by $(m'_j + 4)$-gons. So the number of tiles in $X$ which map to type $m'_j$ is just the number of branching edges in $X$, namely $|P|$. This number is also equal to the $j$th column sum of $P$.

Similarly, considering $f^{-1}$ instead of $f$, one sees that the row and column sums of $P'$ are all equal to the number of branching edges in $X'$, that is, to $|P'|$, the degree of the cover.

Finally, to prove (5), we use (3):

$$\frac{p'_{ij}p'_n}{p_{ij}p_n} = \frac{m_i m_n}{m'_j m'_l} = \frac{m_i m_n}{m'_i m'_j} = \frac{p'_{il}p'_{nj}}{p_{il}p_{nj}}.$$  

\[\square\]

Now suppose 2(a) of Theorem 1.5 holds, that is, $n_1 = n'_1 = 1$ and $k = k'$. Let $x$ be the unique vertex in the first branch of $\Theta$. We may construct an orbifold $O$ with fundamental group $W$ so that $O$ has $O_{D_x(\Theta)}$ as a double-sheeted cover, as follows. Recall that $D_x(\Theta) = \Theta(n_2, n_2, n_3, n_3, \ldots, n_k, n_k)$. The orbifold $O_{D_x(\Theta)}$ then has $2(k - 1)$ tiles, with two tiles of each type $(n_2 - 1)$ through $(n_k - 1)$. Then the quotient $O = O_\Theta$ formed by identifying each pair of tiles of the same type in $O_{D_x(\Theta)}$ is an orbifold with fundamental group $W$. Thus $O$ has $(k - 1)$ types of tiles. Similarly one may construct $O' = O'_\Theta$ as an orbifold with fundamental group $W'$ and $(k' - 1) = (k - 1)$ types of tiles.

From here on, we have the same set-up as above and Lemma 7.5 still holds, the only differences being that now $P$ and $P'$ are $(k - 1) \times (k - 1)$ matrices, $m_i = n_{i+1} - 1$ and $m'_j = n'_{j+1} - 1$.

**Remark 7.6** (Crisp-Paoluzzi result). The case considered in [12] is $n_1 = n'_1 = 1$ and $k = k' = 3$. In this case $P$ and $P'$ are $2 \times 2$ matrices and by Lemma 7.5 (4), we have

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \text{and} \quad P' = \begin{bmatrix} A' & B' \\ B' & A' \end{bmatrix},$$

for some non-negative integers $A, B, A', B'$. By Lemma 7.5(1) and (4), at least one of the pairs $A, A'$ and $B, B'$ are positive. By interchanging $m_1$ and $m_2$ if necessary and using Lemma 7.5(2), we may assume that $A, A' > 0$. Then by Lemma 7.5(3), we have

$$\frac{n_2 - 1}{n'_2 - 1} = \frac{m_1}{m'_1} = \frac{A'}{A} = \frac{m_2}{m'_2} = \frac{n_3 - 1}{n'_3 - 1}.$$
This proves the converse of Proposition 7.2 in this case, and is equivalent to the proof in [12].

In both cases in Theorem 7.3, the matrices $P$ and $P'$ are $3 \times 3$, and to prove Theorem 7.3, we wish to show that after possibly permuting the $n_i$ and $n'_i$, the following equality holds: $m_1/m'_1 = m_2/m'_2 = m_3/m'_3$. By Lemma 7.5(2) and (3), this is equivalent to showing that after possibly permuting the rows and columns of $P$ and performing the same operations on the rows and columns of $P'$, we have

\begin{equation}
\frac{p'_{11}}{p_{11}} = \frac{p'_{22}}{p_{22}} = \frac{p'_{33}}{p_{33}}.
\end{equation}

We remark that the two cases in addition to [12] in which we can prove the converse to Proposition 7.2 are exactly those where $P$ and $P'$ are $3 \times 3$ matrices. We do not know how to extend our arguments to larger matrices. We end with the proof of Theorem 7.3.

**Proof of Theorem 7.3.** Let $f : X \to X'$, $P$ and $P'$ be as described above. We begin by using Lemma 7.5(4) to write $P$ and $P'$ in a special form. Let $A$ denote the smallest entry of $P$. Assume (after possibly permuting rows and columns) that $p_{11} = A$. Then we may write $p_{22} = A + S$ and $p_{33} = A + T$, with $S, T \geq 0$. If we set $p_{23} = B$ and $p_{32} = C$, we may write the remaining entries of $P$ in terms of $A, B, C, S$, and $T$. To begin, we evaluate $p_{12}$ using the fact that the first row sum is equal to the third column sum (Lemma 7.5(4)):

\[ A + p_{12} + p_{13} = p_{13} + B + A + T \implies p_{12} = B + T. \]

This means that $|P| = p_{12} + p_{22} + p_{32} = B + T + A + S + C$ and the rest of the entries can easily be determined to obtain $P$ as on the left below. The corresponding form for $P'$, on the right, was obtained by putting $p'_{11} = A'$, $p'_{22} = A' + S'$, $p'_{33} = A' + T'$, $p'_{23} = B'$ and $p'_{32} = C'$.

\begin{equation}
\begin{bmatrix}
A & B + T & C + S \\
C + T & A + S & B \\
B + S & C & A + T
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A' & B' + T' & C' + S' \\
C' + T' & A' + S' & B' \\
B' + S' & C' & A' + T'
\end{bmatrix}.
\end{equation}

Each entry of $P$ and $P'$ is nonnegative by definition, so $A, A', B, B', C, C' \geq 0$. By construction, $S, T \geq 0$. However, $f$ may not take the smallest entry of $P$ to the smallest entry of $P'$, so one or both of $S'$ and $T'$ could be negative.

**Case 0.** For clarity, we first tackle the special case in which all of $A, B, C, S, T, A', B', C', S'$ and $T'$ are positive. In this case, we may write

\begin{equation}
A' = aA, \quad B' = bB, \quad C' = cC, \quad S' = sS, \quad T' = tT,
\end{equation}

where $a, b, c, s, t > 0$.

We will prove this case by contradiction as follows. In Step 1 below, we define a homogeneous system of linear equations associated to this case and show that it has a “positive” solution, that is, a vector whose coordinates are all positive is a solution to the system. In Steps 2 and 3 we show that if the conclusion of the theorem fails, then regardless of the relative sizes of $a, b, c, s$ and $t$, there exists an equation in the system whose coefficients are either all positive or all negative. This implies that that equation and hence the system, cannot have a positive solution, which is a contradiction.
In more detail, in Step 2 we show that if the conclusion of the theorem fails, each equation in the system is nontrivial. Further, for each equation in the system, we list a set of orders on $a, b, c, s$ and $t$, such that if the order on $a, b, c, s$ and $t$ (arranged from largest to smallest) is in the set, then the coefficients of the corresponding equation all have the same sign. Then in Step 3, we show that the union of the sets from Step 2 is the collection of all possible orders on $a, b, c, s$ and $t$, showing that no matter what the values of $a, b, c, s$ and $t$ are, one obtains the contradiction described in the previous paragraph.

Step 1. Defining the associated system of equations with a positive solution. Now consider the minor of $P$ below (together with the corresponding minor of $P'$):

$$
\begin{bmatrix}
A & C + S \\
C + T & B
\end{bmatrix}
\quad \quad
\begin{bmatrix}
A' & C' + S' \\
C' + T' & B'
\end{bmatrix}
$$

By Lemma 7.5(5) and the definitions (7.3) above, we have

$$
\begin{pmatrix}
aA \\
A
\end{pmatrix}
\begin{pmatrix}
bB \\
B
\end{pmatrix}
= \begin{pmatrix}
cC + sS \\
C + S
\end{pmatrix}
\begin{pmatrix}
cC + tT \\
C + T
\end{pmatrix}
$$

which rearranges to

$$(ab - c^2)C^2 + (ab - ct)CT + (ab - cs)CS + (ab - st)ST = 0.$$
and the conclusion holds. The last two cases are similar. Thus the equations are all non-trivial.

If the conclusion of the theorem fails, then in particular, $a$, $b$, $c$, $s$ and $t$ are not all equal and they can be ordered from smallest to largest (possibly with some equalities). Next we show that for any possible order on $a$, $b$, $c$, $s$ and $t$, at least one of the equations in the system has coefficients that are all positive or all negative. To do this, we record, in Column 3 of Table 1, a set of orders on $a$, $b$, $c$, $s$ and $t$ which guarantee that all the corresponding (non-zero) coefficients have the same sign. Then we show that the union of all these sets is the set of all orders on $a$, $b$, $c$, $s$ and $t$.

The sets in Column 3 of Table 1 are written using the following notation. Given numbers $s_1,\ldots,s_l,t_1,\ldots,t_r$, let $$\{s_1,\ldots,s_l\geq t_1,\ldots,t_r\}$$ denote the set of orders on $s_1,\ldots,s_l,t_1,\ldots,t_r$ (possibly with equalities) in which each of $s_1,\ldots,s_l$ is bigger than or equal to each of $t_1,\ldots,t_r$. So for example, $$\{s_1,s_2\geq t\}$$ denotes the set $$\{s_1\geq s_2\geq t, s_2\geq s_1\geq t\}.$$ The cardinality of $$\{s_1,\ldots,s_l\geq t_1,\ldots,t_r\}$$ is the number of orders in the set, so in the example, the cardinality is 2.

The notation $$\{s_1,\ldots,s_l\geq t_1,\ldots,t_r\}$$ allows the possibility that all the numbers are equal. Each of the sets listed in Column 3 guarantees that the coefficients are all $\geq 0$ or all $\leq 0$. Since we have already established that each equation is non-trivial, we conclude that all the non-zero coefficients have the same sign. (The set listed in each row is not necessarily maximal.)

**Step 3. Showing that the union of the sets obtained in Step 2 includes all orders.** In fact, we will only need to consider the first six rows of Table 1 to arrive at the contradiction. Each of the twelve sets in the first six rows of the Column 3 of Table 1 has cardinality 12 and the reader may easily verify that they are pairwise disjoint except for 6 pairs, whose intersections all have cardinality 4:

- $\{a,b,s\geq c,t\}$ and $\{a,b\geq c,s,t\}$
- $\{a,c,s\geq b,t\}$ and $\{a,c\geq b,s,t\}$
- $\{b,c,s\geq a,t\}$ and $\{b,c\geq a,s,t\}$

Lastly, the intersection of any three of these twelve sets is empty. Their union therefore has cardinality $12 \times 12 - 6 \times 4 = 120$. Since the total number of possible orders on five numbers is 120, the above union contains all possible orders, giving us the desired contradiction.

This completes the proof in the case that all of $A$, $B$, $C$, $S$, $T$, $A'$, $B'$, $C'$, $S'$ and $T'$ are positive. We now discuss the proof in general, dividing it into two main cases, depending on whether or not $P$ has entries equal to 0.

**I. All the entries of $P$ (and therefore $P'$) are positive.** This means that $A$, $B$, $C$, $A'$, $B'$, $C'$, $S'$ and $T'$ are positive. We now discuss the proof in general, dividing it into two main cases, depending on whether or not $P$ has entries equal to 0.

- **Case 1:** $S', T' \geq 0$. There are 15 cases in addition to Case 0 above, as each of $S$, $S'$, $T$ and $T'$ could be either 0 or positive. They are proved using the same strategy. We discuss a few key cases:
<table>
<thead>
<tr>
<th>Minor</th>
<th>Coefficients</th>
<th>$t &gt; 0$</th>
<th>$t &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} A &amp; C+S \ C+T &amp; B \end{bmatrix}$</td>
<td>$ab-c^2$ $ab-ct$ $ab-cs$ $ab-st$</td>
<td>${a,b \geq c,s,t}$</td>
<td>${a,b \geq c,s}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} A &amp; B+T \ B+S &amp; C \end{bmatrix}$</td>
<td>$ac-b^2$ $ac-bs$ $ac-bt$ $ac-st$</td>
<td>${a,c \geq b,s,t}$</td>
<td>${a,c \geq b,s}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} A+S &amp; B \ C &amp; A+T \end{bmatrix}$</td>
<td>$a^2-bc$ $as-bc$ $at-bc$ $st-bc$</td>
<td>${b,c \geq a,s,t}$</td>
<td>${b,c \geq a,s}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} A &amp; C+S \ B+S &amp; A+T \end{bmatrix}$</td>
<td>$a^2-bc, at-bc$ $a^2-cs, at-cs$ $a^2-bs, at-bs$ $a^2-s^2, at-s^2$</td>
<td>${b,c,s \geq a,t}$</td>
<td>${b,c,s \geq a}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} B+T &amp; C+S \ A+S &amp; B \end{bmatrix}$</td>
<td>$b^2-ac$ $bt-ac$ $b^2-cs$ $bt-cs$ $b^2-as$ $bt-as$ $b^2-s^2$ $bt-s^2$</td>
<td>${a,c,s \geq b,t}$</td>
<td>${a,c,s \geq b}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} C+T &amp; A+S \ B+S &amp; C \end{bmatrix}$</td>
<td>$c^2-ab$ $ct-ab$ $c^2-as$ $ct-as$ $c^2-bs$ $ct-bs$ $c^2-s^2$ $ct-s^2$</td>
<td>${a,b,s \geq c,t}$</td>
<td>${a,b,s \geq c}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} A &amp; B+T \ C+T &amp; A+S \end{bmatrix}$</td>
<td>$a^2-bc, as-bc$ $a^2-ct, as-ct$ $a^2-bt, as-bt$ $a^2-t^2, as-t^2$</td>
<td>${b,c,t \geq a,s}$</td>
<td>${a,s \geq b,c,t}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} C+T &amp; B \ B+S &amp; A+T \end{bmatrix}$</td>
<td>$b^2-ac$ $bs-ac$ $b^2-ct$ $bs-ct$ $b^2-at$ $bs-at$ $b^2-t^2$ $bs-t^2$</td>
<td>${a,c,t \geq b,s}$</td>
<td>${b,s \geq a,c,t}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} B+T &amp; C+S \ C &amp; A+T \end{bmatrix}$</td>
<td>$c^2-ab$ $cs-ab$ $c^2-at$ $cs-at$ $c^2-bt$ $cs-bt$ $c^2-t^2$ $cs-t^2$</td>
<td>${a,b,t \geq c,s}$</td>
<td>${c,s \geq a,b,t}$</td>
</tr>
</tbody>
</table>

**Table 1.** The coefficients of equations defined by minors of $P$, and sets of orders which guarantee that the coefficients all have the same sign.
(i) \((S = S' = 0; T, T' > 0)\). Here \(s = 0\) and \(t > 0\). The coefficients of the equations in the system are obtained by entirely dropping the coefficients containing \(s\) in Table 1. The corresponding sets are obtained by deleting \(s\) from the sets in Column 3. The first three rows yield the sets \(\{a, b \geq c, t\}, \{c, t \geq a, b\}, \{a, c \geq b, t\}, \{b, t \geq a, c\}, \{b, c \geq a, t\}, \{a, t \geq b, c\}\), whose union is all the orders on \(a, b, s\) and \(t\).

The cases \((S, S' > 0; T = T' = 0)\) and \((S = S' = T = T' = 0)\) are similar.

(ii) \((S, T, T' > 0, S' = 0)\). Here \(s = 0\) and \(t > 0\), but unlike in (i), \(S\) occurs among the variables. The coefficients are obtained by setting \(s = 0\) in the coefficients in the table. The sets are obtained by taking only those in which \(s\) occurs on the right of \(\geq\) and dropping \(s\). For example the first row would have coefficients \(ab - c^2, ab - ct\) and \(ab\) and the set would be \(\{a, b \geq c, t\}\). (Since \(ab > 0\), we do not get the companion set \(\{c, t \geq a, b\}\).) The sets thus obtained from the first six rows of the table cover all possible orders on \(a, b, c\) and \(t\).

The cases \((S > 0; S' = T = T' = 0), (T > 0; T' = S = S' = 0), (T, S, S' > 0; T' = 0)\) and \((S, T > 0; S' = T' = 0)\) are similar, with minor variations. In all these cases \(s, t \geq 0\).

(iii) \((S, T' > 0; S' = T = 0)\). Here \(s = 0\) but \(t\) is not defined. The variables could have instances of \(S\) and \(T'\). We must consider orders on \(a, b\) and \(c\). Considering row 4 of the table, we obtain coefficients \(a^2 - bc, a^2\) and \(a\) and the set \(\{a \geq b, c\}\). Similarly, rows 5 and 6 yield the sets \(\{b \geq a, c\}\) and \(\{c \geq a, b\}\). Their union covers all orders on \(a, b\) and \(c\).

(iv) The remaining six cases are analogues of cases in (i) and (ii), if one considers \(f^{-1}\) instead of \(f\).

Case 2: One of \(S'\) and \(T'\) is negative. Assume without loss of generality that \(S' \geq 0\) and \(T' < 0\). The proof proceeds as in Case 1, with the only change being that now \(t < 0\). The fourth column of Table 1 shows the sets which guarantee that the corresponding equations have all-positive or all-negative coefficients when \(a, b, c, s > 0\) and \(t < 0\). It is enough to consider the first six rows. It is easy to see that the union of the sets listed covers all possible orders on \(a, b, c, s\) and \(t\). The analogues of the other cases in Case 1 above can be handled similarly.

Case 3: \(S', T' < 0\). We may further assume that \(A' > B', C'\). (If not, then interchange columns 2 and 3 in \(P\) and \(P'\), so that \(A + S, A + T, A' + S'\) and \(A' + T'\) are replaced by \(A + \bar{S}, A + \bar{T}, A' + \bar{S}'\) and \(A' + \bar{T}'\), where \(\bar{S} = B - A\) and \(\bar{T} = C - A\) are nonnegative (since \(A\) is the smallest value of \(P\)) and at least one of \(\bar{S}' = B' - A'\) and \(\bar{T}' = C' - A'\) is nonnegative. This case is then covered in either Case 1 or Case 2.)

Now assume without loss of generality that \(0 \leq S \leq T\). If \(S' \leq T' < 0\), then consider \(f^{-1}\) instead of \(f\), so that the roles of \(P\) and \(P'\) are interchanged. In \(P'\), we have \(A' + S' \leq A' + T' < A'\). Write this as \(U \leq U + Q < U + R\), with \(U = A' + S', Q = T' - S' \geq 0\) and \(R = -S' \geq 0\). Then the corresponding entries of \(P\) are \(U', U' + Q'\) and \(U' + R'\), with \(Q' = T - S \geq 0\) and \(R' = -S \leq 0\). Again, this is covered by Case 2 (considering \(f^{-1}\)).
Finally, if $0 \leq S \leq T$ and $T'=S'<0$ and $A'>B',C'$, then we have $A \leq A+S \leq B+T,C+T$ and $A'>A'+S'>B'+T',C'+T'$. Using these inequalities, we conclude that

$$
\frac{A'}{A} > \frac{B'+T'}{B+T} \quad \text{and} \quad \frac{A'+S'}{A+S} > \frac{C'+T'}{C+T} \quad \Rightarrow \quad \left( \frac{A'}{A} \right) \left( \frac{A'+S'}{A+S} \right) > \left( \frac{B'+T'}{B+T} \right) \left( \frac{C'+T'}{C+T} \right)
$$

However, this is a contradiction, because, applying Lemma 7.5(5) to the top left minor of $P$, we see that the two sides of the last inequality are actually equal. This completes the proof of the theorem when all the entries of $P$ and $P'$ are positive.

**II. Some entries of $P$ (and therefore $P'$) are 0.** Suppose $P$ contains a row with two 0s. We may assume that $p_{11} = A = 0$ and $p_{22} = B + T = 0$. Thus $B = T = 0$ and $P$ reduces to

$$
P = \begin{bmatrix}
0 & 0 & C + S \\
C & S & 0 \\
S & C & 0
\end{bmatrix}
$$

and $P'$ has the same form, with primes added to each letter. Moreover, $C, C', S, S' \geq 0$, as they are all entries of $P$ or $P'$.

If either $C$ or $S$ is 0, then $P$ (and $P'$) rearrange to diagonal matrices such that each diagonal entry is $S$ (and $S'$) or $C$ (and $C'$) so that the conclusion of the theorem holds. If neither of them is 0, then $C, C', S, S' > 0$ and using the bottom left minor, we see that

$$
\frac{\left(\frac{C'}{C}\right)^2}{\left(\frac{S'}{S}\right)^2} \quad \Rightarrow \quad \frac{C'}{C} = \frac{S'}{S} \quad \Rightarrow \quad \frac{S'}{S} = \frac{C'}{C}.
$$

The conclusion follows, since $P$ rearranges to a diagonal matrix with diagonal entries either $C + S$ or $S$.

The case in which $P$ has a column with two 0s is similar. Thus we may assume that $P$ has at most three 0s, no two of which are in the same row or column. If it has exactly three 0s, then we may assume that $p_{11} = p_{22} = p_{33} = 0$, so that $A = A + S = A + T = 0 \Rightarrow S = T = 0$. Lemma 7.5 (1) implies that $p'_{11} = p'_{22} = p'_{33} = 0$ as well, so that $S' = T' = 0$. Now $P$ and $P'$ reduce to

$$
P = \begin{bmatrix}
0 & B & C \\
C & 0 & B \\
B & C & 0
\end{bmatrix} \quad \text{and} \quad P' = \begin{bmatrix}
0 & B' & C' \\
C' & 0 & B' \\
B' & C' & 0
\end{bmatrix}
$$

which rearrange to diagonal matrices in which diagonal entries are $B$ (or $B'$) and the conclusion follows.

If $P$ has exactly two 0s and they are not contained in a common row or column, we may assume that $A = A + S = 0$, which implies that $S = 0$ and $p_{33} = T > 0$. Now Lemma 7.5 (1) implies that $A' = A' + S' = 0$ and $p'_{33} > 0$. Consequently, $S' = 0$ and $T' = p'_{33} > 0$

This is similar to Case 1(i), except here $a = 0$ in addition to $s = 0$. It is enough to consider the last two rows of the table. The coefficients are obtained by deleting the coefficients involving either $a$ or $s$ from the list and the sets are obtained by omitting $a$ and $s$ from the corresponding sets. Thus one obtains $\{c,t \geq b\}, \{b \geq c, t\}, \{b, t \geq c\}$ and $\{c \geq b, t\}$, which clearly cover all possible orders on $b, c$ and $t$.

Finally, if $P$ has exactly one 0, then $A = 0$ and therefore $A' = 0$. Then $S, T, S', T' > 0$ (as they are non-zero entries of $P$ or $P'$). This is similar to the previous case. Here $a = 0$
and \( b, c, s, t > 0 \). The sets are obtained from Rows 3, 8 and 9 of the table by deleting all the instances of \( a \), and they cover all possible orders on \( b, c, s \) and \( t \).

\[\square\]

**References**


